Manifolds polarized by vector bundles

DOI:10.1007/s10231-006-0005-2

MANIFOLDS POLARIZED BY VECTOR BUNDLES.

MARCO ANDREATTA AND CARLA NOVELLI

Abstract. Let X be a complex projective manifold of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank r. Let also $\tau = \tau(X, \mathcal{E}) = \min\{t \in \mathbb{R} : K_X + t \det \mathcal{E}$ is nef$\}$ be the nef value of the pair $(X, \mathcal{E})$. In the paper we classify the pairs $(X, \mathcal{E})$ such that $\tau(X, \mathcal{E}) \geq \frac{n-2}{r}$.

1. Introduction.

Let X be a smooth complex projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank r on X. We assume $n \geq 3$, the case of curves and surfaces being well known. The pair $(X, \mathcal{E})$ is usually called a *polarized variety* (i.e. X is a variety with a polarization given by $\mathcal{E}$); the name comes from the case $r = 1$ and $\mathcal{E}$ very ample, i.e. $\mathcal{E}$ is a hyperplane in a given embedding of X.

We want to classify polarized varieties, or better find suitable assumptions under which it is possible to give a classification of the pairs $(X, \mathcal{E})$. For instance a famous theorem of S. Mori ([13]) says that if $\mathcal{E} = TX$ then X is the projective space; and this is true even more generally when $\mathcal{E}$ is just a subsheaf of the tangent bundle ([5]).

For this purpose, in the spirit of Mori theory, one can define the following numerical invariant:

$$\tau = \tau(X, \mathcal{E}) = \min\{t \in \mathbb{R} : K_X + t \det \mathcal{E} \text{ is nef}\}.$$

Assume first of all that τ is a positive number; equivalently we are assuming that X is not minimal in the sense of the Minimal Model Program or of the Mori theory: i.e. K_X is not nef.

τ is called the *nef value* (or the threshold value) of the pair $(X, \mathcal{E})$ and it has some very nice features which we recall now (for further details we refer to [11, Theorem 4.1.1]).

First of all, by the Kawamata's rationality theorem, τ is a rational number. Moreover in the Mori-Kleiman cone $\overline{NE(X)} \subset N_1(X)$ the divisor $K_X + \tau \det \mathcal{E}$ defines a face $F(\mathcal{E}) := \{C \in \overline{NE(X)} : (K_X + \tau \det \mathcal{E}).C = 0\}$ which stays in the polyhedral part of the cone, $\overline{NE(X)}_{K_X<0}$, and which is therefore generated by a finite number of extremal rays $R_i = \mathbb{R}[C_i]$ where C_i is a rational curve. Recall that the *length* of an extremal ray $R \subset \overline{NE(X)}_{K_X<0}$ is the integer defined as $l(R) =$min $\{-K_X.C : [C] \in R\}$. By a theorem of Mori $l(R) \leq n + 1$.

Secondly, by the Kawamata-Shokurov base point free theorem, a high multiple of the divisor $K_X + \tau \det \mathcal{E}$ is spanned by global sections and therefore it defines a map $\varphi : X \to Z$ into a normal projective variety with connected fibers. The map

Key words and phrases. ample vector bundle – extremal rays – adjunction theory; 14J60, 14J40, 14E30.

φ is called the *nef value morphism* (relative to $(X, \mathcal{E})$). Note that by construction $-K_X$ is φ-ample, therefore φ is a Fano-Mori contraction (see [4]) and it contracts all curves in F.

The program is then the following. Find suitable assumption on $\tau(X, \mathcal{E})$ under which it is possible to describe the face $F(\mathcal{E}) \subset N_1(X)$ and (or) the nef value morphism $\varphi_{\mathcal{E}} : X \to Z$. Subsequently, with the use of the Minimal Model Program, classify under this assumption the pairs $(X, \mathcal{E})$.

In this paper we succesfully develope the above program for all $\tau \geq \frac{n-2}{r}$.

If $r = 1$ the program has a classical start up, it was carried out in modern time by A.J. Sommese and T. Fujita and with different generalizations by many others including the first author; for a complete survey we refer the reader to [6].

If $1 = \tau(\geq \frac{n-2}{r})$ the program was developed by many authors in the following series of papers: [16], [21], [9], [17], [1], [12] and [2].

Building on the above quoted papers, in [14] M. Ohno classified the pairs for $\tau \geq \frac{n-1}{r}$ and $\tau \geq 1$. After the paper was written we found a preprint of Ohno, [15], were he also consider the case $\frac{n-1}{r} > \tau \geq \frac{n-2}{r}$ and $\tau \geq 1$. Note that in our paper the assumption $\tau \geq 1$ is not needed, the proofs are different and in general much shorter.

2. Notations, preliminaries and a starting point.

We use the standard notation from algebraic geometry. In particular we use the language of the minimal model program and it is compatible with that of [11] to which we refer. We just recall the following two facts that we will use in the proofs. Let, as in the introduction, $R \subset \overline{NE(X)}_{K_X<0}$ be an extremal ray, $l(R)$ its length and $\varphi_R : X \to Z$ the Fano-Mori contraction which contracts all curves in R. Let then $E = E(\varphi)$ be the exceptional locus of φ_R (if φ_R is of fiber type then $E := X$); let S be an irreducible component of a (non trivial) fiber F.

Proposition 2.1. [19] *The following formula holds*

$$\dim S + \dim E \geq \dim X + l(R) - 1.$$

Proposition 2.2. [11, Proposition 5.1.6], [1, Proposition 1.4.1] *If φ_R is divisorial (i.e. it is birational with exceptional locus of dimension $n-1$) then the exceptional locus is a prime divisor.*

If φ_R is of fiber type (i.e. $\dim X > \dim Z$) and $\dim Z \leq 2$ then it is equidimensional and Z is smooth.

Our starting point will be the following result.

Theorem 2.3. *In the above notation let $R = R_i$ for any extremal ray in the face $F(\mathcal{E})$ and let $C \subset X$ be any rational curve such that $l(R) = -K_X.C$ and $[C] \in R$. Then*

$$\tau(X, \mathcal{E}) \leq \frac{l(R)}{r} \left(\leq \frac{n+1}{r} \right).$$

Moreover

1) equality holds if and only if $\det \mathcal{E}.C = r$, and if V is a family of rational curves (i.e. a closed irreducible component $V \subset Hom(\mathbb{P}^1, X)$) which contains $f : \mathbb{P}^1 \to C \subset X$ then it is unsplit (i.e. its image in Chow is proper).

2) If equality holds and X is rationally chain connected with respect to V (i.e. for all $x_1, x_2 \in X$ there exists a chain of rational curves parametrized by morphisms from V which joins x_1 and x_2), which is equivalent to assume $\rho(X) = 1$, then there exists a (uniquely defined) line bundle L over X such that $\deg f^*L = 1$ and $\mathcal{E} \cong \oplus^r L$.

Proof. Assume by contradiction that $\tau(X, \mathcal{E}) > \frac{l(R)}{r}$. Then

$$0 = (K_X + \tau \det \mathcal{E}).C > K_X.C + \frac{l(R)}{r} \det \mathcal{E}.C = K_X.C \left(1 - \frac{\det \mathcal{E}.C}{r}\right).$$

This implies that $\det \mathcal{E}.C < r$ which is a contradiction since $\mathcal{E}$ is ample.

In the same way one proves that equality holds iff $\det \mathcal{E}.C = r$.

The rest of the theorem follows from [5, Proposition 1.2]. $\square$

Remark 2.4. The assumption that the base field is the complex number is used in the proof of 2). It would be nice to have a proof of it over an arbitrary algebraically closed field.

Note also that part 2) will be used to reduce the general case to the case $r = 1$.

3. Classification of $(X, \mathcal{E})$ with $\tau(\mathcal{E}) \geq \frac{n-2}{r}$.

Proposition 3.1. *If $\frac{n+1}{r} \leq \tau$ then $(X, \mathcal{E}) = (\mathbb{P}^n, \oplus^r \mathcal{O}_{\mathbb{P}^n}(1))$.*

Proof. Now and in the rest of the paper we will let R be any ray in the face $F(\mathcal{E}) := \{C \in \overline{NE(X)} : (K_X + \tau \det \mathcal{E}).C = 0\}$. By theorem 2.3 we have that $l(R) = n + 1$. Then we have, by [7], that $X = \mathbb{P}^n$ and, by theorem 2.3, that $\mathcal{E} = \oplus^r L$ for a line bundle L over X. Therefore $\tau(X, L) = n + 1$ and we reduce our proposition to the known case $r = 1$. $\square$

Proposition 3.2. *Assume $\frac{n}{r} \leq \tau < \frac{n+1}{r}$ and let $a := \det \mathcal{E}.C - r$. Then the pair $(X, \mathcal{E})$ is one of the following.*
1) $X = \mathbb{P}^n$, $a \geq 1$ and $an \leq r$. If $r \leq n$ then $\mathcal{E}$ is either $T\mathbb{P}^n$ or $\mathcal{O}_{\mathbb{P}^n}(2) \oplus (\oplus^{(r-1)} \mathcal{O}_{\mathbb{P}^n}(1))$.
2) $X = \mathbb{Q}^n$ and $\mathcal{E} = \oplus^r \mathcal{O}_{\mathbb{Q}^n}(1)$.
3) X is a scroll over a smooth curve R (i.e. X is the projectivization of a rank n vector bundle on a smooth curve R, $\pi : \mathbb{P}(F) \to R$, and $\mathcal{E}_{|F} = \oplus^r \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ for every fiber F of π).

Proof. By theorem 2.3 we have that $l(R) \geq n$.

If $l(R) = n + 1$ then by [7] we have that $X = \mathbb{P}^n$. Moreover $\frac{n}{r} \leq \tau = \frac{-K_X.C}{\det \mathcal{E}.C} = \frac{n+1}{r+a} < \frac{n+1}{r}$ gives the bounds on a.

If $r \leq n$ then $a = 1, r = n$ and thus $\tau = 1$ and the theorem follows from [17].

If $l(R) = n$ and $\rho(X) = 1$ by theorem 2.3 we have that $\mathcal{E} = \oplus^r L$ for a line bundle L over X such that $\deg f^*L = 1$. Therefore $\tau(X, L) = n$ and we reduce our proposition to the known case $r = 1$. This gives the case 2) of the proposition.

Let $l(R) = n$ and $\rho(X) > 1$; by propositions 2.1 and 2.2 the map $\varphi_R : X \to Z$ is onto a smooth curve. If F is a general fiber the pair $(F, \mathcal{E}_{|F})$ is $(\mathbb{P}^{n-1}, \oplus^r \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ by proposition 3.1. Using the same argument as in section (3.3) of [9] we see that this is true for every fiber F. $\square$

Proposition 3.3. *Assume $\frac{n-1}{r} \leq \tau < \frac{n}{r}$ and let $a := \det \mathcal{E}.C - r$. Then the pair $(X, \mathcal{E})$ is one of the following.*

a) $\rho(X)=1$ and
1) $X=\mathbb{P}^n$, a is a positive integer and $\frac{n-1}{2}a \le r < na$. In particular if $r \le n-1$ (for instance if $\tau \ge 1$) then either $a=1, r \ge \frac{n-1}{2}$ and $\mathcal{E}=\mathcal{O}_{\mathbb{P}^n}(2) \oplus (\oplus^{(r-1)}\mathcal{O}_{\mathbb{P}^n}(1))$ or $a=2, r=n-1, \tau=1$ and the possible $\mathcal{E}$ are described in [18].
2) X is a Fano manifold, $-K_X.C=n$ for every minimal (rational) curve, $a \ge 1$ and $a(n-1) \le r$, $\tau \le \frac{1}{a}$. In particular if $r \le n$ then $X=\mathbb{Q}^n$ and $\mathcal{E}$ is uniform with splitting type $(2,1,...,1)$ (and, for $r=n-1$, it is described by [18] and [20]).
3) there exists an ample line bundle L over X such that $-K_X=L^{\otimes(n-1)}$ (i.e. X is a del Pezzo manifold) and $\mathcal{E} \cong L^{\oplus r}$.
b) $\rho(X)>1$ and
4) X is a scroll over a smooth curve R (i.e. X is the projectivization of a rank n vector bundle on a smooth curve R, $\pi: \mathbb{P}(F) \to R$), $a \ge 1$ and $a(n-1) \le r$. If $r \le (n-1)$ then for every fiber F of π the pair $(F, \mathcal{E}_{|F})$ is as in 1) of proposition 3.2.
5)X is a hyperquadric fibration over a smooth curve R (i.e. X is a section of a divisor of relative degree 2 in a $(n+1)$-dimensional scroll over R) and for every smooth fiber F the pair $(F, \mathcal{E}_{|F})$ is as in 2) of proposition 3.2.
6) X is a $\mathbb{P}^{n-2}$-bundle over a smooth surface S, locally trivial in the complex topology, and $\mathcal{E}_{|F}=\oplus^r\mathcal{O}_{\mathbb{P}^{n-2}}(1)$ for every fiber F of π (see also the following remark).
Or
7) X is the blow-up of $\mathbb{P}^3$ in one point, $\pi: Bl_x\mathbb{P}^3 \to \mathbb{P}^3$, and $\mathcal{E}=\oplus^r\ (\pi^*(\mathcal{O}_{\mathbb{P}^3}(2))-[\pi^{-1}(x)])$ (this is actually a particular case of 6)).

8) there exist a smooth variety X' and a morphism $\varphi: X \to X'$ expressing X as blow-up of X' at a finite set of points B and an ample vector bundle $\mathcal{E}'$ on X' such that $\mathcal{E} \otimes ([\varphi^{-1}(B)])=\varphi^*\mathcal{E}'$ and $K_{X'}+\tau \det \mathcal{E}'$ is ample.
Moreover $\mathcal{E}_{|E}=\oplus^r\mathcal{O}_{\mathbb{P}^{n-1}}(1)$, where E is any irreducible component of the exceptional locus of φ.
The pair $(X', \mathcal{E}')$ is called the first reduction of *$(X, \mathcal{E})$.*

Proof. By theorem 2.3 we have that $l(R) \ge n-1$.

If $l(R)=n+1$ then by [7] we have that $X=\mathbb{P}^n$. If $a=0$ we can apply theorem 2.3 and $\mathcal{E}=\oplus^r\mathcal{O}_{\mathbb{P}^n}(1)$ which is a contradiction.

Since $r+a=\det \mathcal{E}.C=\frac{l(R)}{\tau}=\frac{n+1}{\tau}$ we have that $\frac{n-1}{2}a \le r < na$. In particular if $r \le n-1$ then $a=1,2$. If $a=1$ then $r \ge \frac{n-1}{2}$ and $\mathcal{E}$ is uniform with splitting type $(2,1,...,1)$, therefore $\mathcal{E}=\mathcal{O}_{\mathbb{P}^n}(2) \oplus (\oplus^{(r-1)}\mathcal{O}_{\mathbb{P}^n}(1))$. If $a=2$ then $r=n-1, \tau=1$ and the possible $\mathcal{E}$ are described in [18].

If $l(R)=n$ and $\rho(X)=1$ we can assume again by theorem 2.3 that $a \ge 1$. Moreover $r+a=\frac{n}{\tau} \le \frac{nr}{n-1}$ implies $a(n-1) \le r$, $\tau \le \frac{1}{a}$. If $r \le n$ then $a=1$ and therefore $-K_X.C=n$ and $\det \mathcal{E}.C=n$ or $\det \mathcal{E}.C=n+1$. In the first case $\tau=1$, $r=n-1$ and we conclude using [18] and [20]. In the second, since n and $n+1$ are relatively prime, we can find an ample line bundle H such that $H.C=1$ and therefore such that $\tau(X,H)=n$. We can now apply the known case $r=1$.

If $l(R)=n-1$ and $\rho(X)=1$, then we are in the assumption of theorem 2.3. Then $\mathcal{E} \cong L^{\oplus r}$ for a line bundle L over X such that $\deg f^*L=1$ and we are in the case 3) of the proposition.

If $l(R)=n$ and $\rho(X)>1$ then, as in the proof of proposition 3.2, it is straightforward to see that the map φ gives X the structure of a scroll over a smooth curve. The rest of point 4) follows from proposition 3.2 applied to the pair $(F, \mathcal{E}_{|F})$.

Let $l(R) = n-1$ and $\rho(X) > 1$; if $\varphi_R : X \to Z$ is of fiber type then by propositions 2.1 and 2.2 it is onto either a smooth curve or a smooth surface. If F is a general fiber the pair $(F, \mathcal{E}_{|F})$ is $(\mathbb{Q}^{n-1}, \oplus^r \mathcal{O}_{\mathbb{Q}^{n-1}}(1))$ by proposition 3.2 in the first case and $(\mathbb{P}^{n-2}, \oplus^r \mathcal{O}_{\mathbb{P}^{n-2}}(1))$ by proposition 3.1 in the second. In the first case, using the same arguments as in section (3.3) of [9], we see that this is true for every fiber F and then that $\varphi_R : X \to Z$ is a hyperquadric fibration. Also in the second case, using this time the argument in 2.2 of [1], one can see that this is true for every fiber F and then that $\varphi_R : X \to Z$ is a $\mathbb{P}^{n-2}$-bundle, locally trivial in the complex topology.

We are therefore left with the case $l(R) = n-1$, $\rho(X) > 1$ and $\varphi_R : X \to Z$ birational (if the last assumption holds it is usually said that the ray R is not nef). Theorem 1.1 of [3] says that if $\varphi_R : X \to Z$ then Z is smooth and φ_R is the blow-up of Z at a point. Moreover, if E denotes the exceptional locus of φ_R, by proposition 3.1 $(E, \mathcal{E}_{|E}) = (\mathbb{P}^{n-1}, \oplus^r \mathcal{O}_{\mathbb{P}^{n-1}}(1))$.

Also the same proof as the one of lemma 4.2 in [3] proves that if a ray R of the face $F(\mathcal{E}) := \{C \in \overline{NE(X)} : (K_X + \tau \det \mathcal{E}).C = 0\}$ with $\tau \geq \frac{n-1}{r}$ is non nef then all rays in the face are non nef with the only exception given by the blow-up of $\mathbb{P}^3$ in one point, $\pi : Bl_x\mathbb{P}^3 \to \mathbb{P}^3$, and $\mathcal{E} = \oplus^r(\pi^*(\mathcal{O}_{\mathbb{P}^3}(2)) - [\pi^{-1}(x)])$.

All this leads to the case 7) and to the last case of the proposition. $\square$

Remark 3.4. In part 2) of the proposition X should be the quadric also if $r > n$.

Remark 3.5. In part 6) of the proposition X is not necessarily a scroll, as example 2.3 of [1] shows. The example in particular says also that the assumption $\rho(X) = 1$ is necessary in part 2) of theorem 2.3.

Proposition 3.6. *Assume $\frac{n-2}{r} \leq \tau < \frac{n-1}{r}$ and let $a := \det \mathcal{E}.C - r$.*

Then either the pair $(X, \mathcal{E})$ is one of the following:

a) $\rho(X) = 1$ and

1) $X = \mathbb{P}^n$, a is a positive integer and $\frac{n-2}{3}a \leq r < \frac{n-1}{2}a$. In particular if $r \leq n-2$ (for instance if $\tau \geq 1$), then $a = 1, 2, 3$ and $\mathcal{E}$ is a decomposable bundle.

2) X is a Fano manifold, $-K_X.C = n$ for every minimal rational curve, a is a positive integer and $\frac{n-2}{2}a \leq r < (n-1)a$.

3) X is a Fano manifold, $-K_X.C = n-1$ for every minimal rational curve, a is a positive integer and $(n-2)a \leq r$. If $r \leq n-2$ (for instance if $\tau \geq 1$) then $K_X + \det \mathcal{E} = 0$.

4) There exists an ample line bundle L over X such that $-K_X = L^{\otimes(n-2)}$ (i.e. X is a Mukai manifold) and $\mathcal{E} \cong \oplus^r L$.

b) $\rho(X) > 1$ and

5) X is a scroll over a smooth curve R, $\frac{n-2}{2}a \leq r < (n-1)a$ and for every fiber F the pair $(F, \mathcal{E}_{|F})$ is as in 1) of proposition 3.3. In particular if $r \leq n-2$ (for instance if $\tau \geq 1$) then either $\mathcal{E}_{|F} = \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus (\oplus^{(r-1)}\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ or $r = n-2$, $\tau = 1$ and the possible $\mathcal{E}_{|F}$ are described in [18].

6) X is a Fano fibration over a smooth curve R and $r \geq a(n-2)$. In particular if $r \leq n-2$ (for instance if $\tau \geq 1$) then for the general fiber F the pair $(F, \mathcal{E}_F)$ is as in 2) of proposition 3.3.

7) X is a fibration over a smooth curve R; for the general fiber F we have $\mathcal{E}_{|F} = \oplus^r L$, where $(n-2)L = -K_F$ (i.e. F is a del Pezzo manifold).

8) X is a $\mathbb{P}^{n-2}$-fibration over a smooth surface S and $r \geq n-2$. In particular if $r \leq n-2$ (for instance if $\tau \geq 1$) then X is a $\mathbb{P}^{n-2}$-bundle and for every fiber F either $\mathcal{E}_{|F} = T\mathbb{P}^{n-2}$ or $\mathcal{E}_{|F} = \mathcal{O}_{\mathbb{P}^{n-2}}(2) \oplus (\oplus^{(r-1)}\mathcal{O}_{\mathbb{P}^{n-2}}(1))$.
9) X is a hyperquadric fibration over a smooth surface S and for the general fiber F the pair $(F, \mathcal{E}_{|F}) = (\mathbb{Q}^{n-2}, \oplus^r \mathcal{O}_{\mathbb{Q}^{n-2}}(1))$.
10) X is a fibration over a threefold T with at most isolated rational and Gorenstein singularities and for all fibers F over a smooth point the pair $(F, \mathcal{E}_{|F}) = (\mathbb{P}^{n-3}, \oplus^r \mathcal{O}_{\mathbb{P}^{n-3}}(1))$.

Or
11) f_R is the blow up of a smooth variety either in a point or along a smooth curve with exceptional locus E. In the first case if $r \leq n-2$ (for instance if $\tau \geq 1$) then $r = n-2$ and $\mathcal{E}_{|E} = \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus (\oplus^{(r-1)}\mathcal{O}_{\mathbb{P}^{n-1}}(1))$. In the second case, $(F, \mathcal{E}_{|F}) = (\mathbb{P}^{n-2}, \oplus^r \mathcal{O}_{\mathbb{P}^{n-2}}(1))$ for all fibers $F \subset E$.
12) f_R is a divisorial contraction whose exceptional locus, E, satisfies one of the following:
i) $(E, E_E; \mathcal{E}_{|E}) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(-2); \oplus^r \mathcal{O}_{\mathbb{P}^{n-1}}(1))$;
ii) $(E, E_E; \mathcal{E}_{|E}) = (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(-1); \oplus^r \mathcal{O}_{\mathbb{Q}^{n-1}}(1))$.

Proof. By theorem 2.3 we have that $l(R) \geq n-2$.

If $l(R) = n+1$ then by [7] we have that $X = \mathbb{P}^n$ and the rest is straightforward.

Assume first that $\rho(X) = 1$.

If $l(R) = n$ then $\tau = \frac{n}{r+a}$ and this implies $\frac{n-2}{2}a \leq r < a(n-1)$.

If $l(R) = n-1$ then $\tau = \frac{n-1}{r+a}$ and this implies $(n-2)a \leq r$ and $a > 0$. If $r \leq (n-2)$ then $a = 1$, $r = n-2$, $\tau = 1$ and therefore $K_X + \det \mathcal{E} = 0$.

If $l(R) = n-2$ then we are in the assumption of theorem 2.3. In particular $\mathcal{E} \cong L^{\oplus r}$ for a line bundle L over X such that $\deg f^*L = 1$ and we are in the case 4) of the proposition.

Assume then that $\rho(X) > 1$ and let $\varphi := \varphi_R : X \to Z$ be the map associated to the ray R. Since $\rho(X) > 1$ then $\dim Z > 0$.

If $l(R) = n$ then, as in the proof of proposition 3.2, it is straightforward to see that the map φ gives X the structure of a scroll over a smooth curve and that the pair $(F, \mathcal{E}_{|F})$ is as in 1) of proposition 3.3.

Let $l(R) = n-1$ and assume φ is of fiber type; then by propositions 2.1 and 2.2 it is onto either a smooth curve or a smooth surface. In the first case if F is a general fiber the pair $(F, \mathcal{E}_{|F})$ is as in 2) of proposition 3.3. In the second case if F is a general fiber then $F = \mathbb{P}^{n-2}$ and the pair $(F, \mathcal{E}_{|F})$ is as in 1) of proposition 3.2. In particular if $r \leq n-2$ then $r = n-2$ and $\mathcal{E}_{|F} = T\mathbb{P}^{n-2}$ or $\mathcal{O}_{\mathbb{P}^{n-2}}(2) \oplus (\oplus^{r-1}\mathcal{O}_{\mathbb{P}^{n-2}}(1))$. Using this time the argument in 2.2 of [1], one can see that this is true for every fiber F and then that $\varphi_R : X \to Z$ is a $\mathbb{P}^{n-2}$-bundle, locally trivial in the complex topology.

Let $l(R) = n-2$ and assume φ is of fiber type; then, by propositions 2.1 and 2.2, φ is onto either a smooth curve or a smooth surface or a threefold. If F is a general fiber the pair $(F, \mathcal{E}_{|F})$ is a del Pezzo manifold (F, L) with $\mathcal{E}_{|F} = \oplus^r L$ by 3) of proposition 3.3 in the first case, and $(\mathbb{Q}^{n-2}, \oplus^r \mathcal{O}_{\mathbb{Q}^{n-2}}(1))$ by 2) of proposition 3.2 in the second case. If $\dim Z = 3$ then it is well known that Z has rational and Gorenstein singularities. Moreover in our case they are also isolated: to prove this take a general hyperplane section S in Z and consider the map $\varphi_{|\varphi^{-1}(S)} : \varphi^{-1}(S) \to S$. By proposition 1.3 of [1] this map is elementary and therefore, by proposition

2.2, S is smooth, thus Z has isolated singularities. For the general fiber F the pair $(F, \mathcal{E}_{|F})$ is $(\mathbb{P}^{n-3}, \oplus^r \mathcal{O}_{\mathbb{P}^{n-3}}(1))$ by proposition 3.1. As in the proof of [2, Theorem 5.1], the same holds for all fibers F over smooth points.

We are left with the case $l(R) = n-1, n-2$, $\rho(X) > 1$ and $\varphi_R : X \to Z$ birational. In the first case Theorem 1.1 of [3] says that Z is smooth and φ_R is the blow-up of Z at a point. Moreover, if E denotes the exceptional locus of φ_R, by adjunction $\det \mathcal{E}_{|E} = \mathcal{O}_{\mathbb{P}^{n-1}}(r+a)$; in particular, if $r \le n-2$, then $r = n-2$ and $\mathcal{E}_{|E} = \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus (\oplus^{(r-1)} \mathcal{O}_{\mathbb{P}^{n-1}}(1))$.

In the second case Theorem 5.3 of [3] says that φ_R is divisorial and, if E denotes the exceptional locus of φ_R: either $\varphi_R(E)$ is a point, $(E, -E_E) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(2))$, by adjunction $\det \mathcal{E}_{|E} = \mathcal{O}_{\mathbb{P}^{n-1}}(r)$, therefore $\mathcal{E}_{|E} = \oplus^r \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ by theorem 2.3; or $\varphi_R(E)$ is a point, $(E, -E_E) = (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1))$, where $\mathbb{Q}^{n-1}$ is a possibly singular hyperquadric, by adjunction $\det \mathcal{E}_{|E} = \mathcal{O}_{\mathbb{Q}^{n-1}}(r)$, therefore $\mathcal{E}_{|E} = \oplus^r \mathcal{O}_{\mathbb{Q}^{n-1}}(1)$ by theorem 2.3; or Z is smooth and φ_R is the blow-up along a smooth curve $\varphi_R(E) \subset Z$, and for all fibers $F \subset E$ by adjunction $\det \mathcal{E}_{|F} = \mathcal{O}_{\mathbb{P}^{n-2}}(r)$, therefore $(F, \mathcal{E}_{|F}) = (\mathbb{P}^{n-2}, \oplus^r \mathcal{O}_{\mathbb{P}^{n-2}}(1))$.

Remark 3.7. Assume that $K_X + \tau \det \mathcal{E}$ is big (and nef by the definition of τ). Then all rays in the face F defined by $K_X + \tau \det \mathcal{E}$ are not nef. If $\tau \ge \frac{n-2}{r}$ then they are described in proposition 3.3 7),8) and proposition 3.6 11),12). If moreover $\dim X \ge 4$ the exceptional loci of the rays in the face F are disjoint and therefore the map Φ associated to $K_X + \tau \det \mathcal{E}$ contracts them to different points and disjoint curves. The last statement follows from Theorem 2.4 in [10]. This allows to define the second reduction of the pair $(X, \mathcal{E})$ in the spirit of section 7. of [6].

□

References

[1] M. Andreatta, E. Ballico and J.A. Wiśniewski, *Vector bundles and adjunction*, Internat. J. Math., **3,** (1992), 331–340.

[2] M. Andreatta and M. Mella, *Contractions on a manifold polarized by an ample vector bundle*, Transactions of the A.M.S., **349,** (1997), 4669–4683.

[3] M. Andreatta and G. Occhetta, *Special rays in the Mori cone of a projective variety*, Nagoya Math. J., **168,** (2002), 127–137.

[4] M. Andreatta and J.A. Wiśniewski, *On contractions of smooth varieties*, J. Algebraic Geometry, **7,** (1998), 253–312.

[5] M. Andreatta and J.A. Wiśniewski, *On manifolds whose tangent bundle contains an ample subbundle*, Invent. Math., **146,** (2001), 209–217.

[6] M.C. Beltrametti and A.J. Sommese, *The adjunction theory of complex projective varieties*, volume 16 of Exp. Math., de Gruyter, Berlin, 1995.

[7] K. Cho, Y. Miyaoka and N.I. Shepherd Barron, *Characterization of projective space and applications to complex symplectic manifolds*, Higher dimensional birational geometry, Kyoto, 1997, Adv. Stud. Pure Math., **35,** Math. Soc. Japan, Tokyo (2002), 1–88.

[8] T. Fujita, *On polarized manifolds whose adjoint bundles are not semipositive*, Algebraic Geometry, Senday, 1985, Adv. Stud. Pure Math., **10,** North-Holland, Amsterdam (1987), 167–178.

[9] T. Fujita, *On adjoint bundle of ample vector bundles*, Proc. of the Conference in Alg. Geometry, Bayreuth (1990).

[10] T. Fujita, *On Kodaira energy and reduction of polarized manifolds*, Manuscripta Math., **76,** (1992) 59–84.

[11] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the Minimal Model Problem*, Algebraic Geometry, Sendai, 1985, Adv. Stud. Pure Math., **10,** North-Holland, Amsterdam (1987) 283–360.

[12] H. Maeda, *Nefness of adjoint bundles for ample vector bundles*, Le Matematiche (Catania), **50,** (1995) 73–82.

[13] S. Mori, *Projective manifolds with ample tangent bundle*, Ann. of Math. **110,** (1979) 593–606.

[14] M. Ohno, *On nef values of determinants of ample vector bundles*, Free resolutions of coordinate rings of projective varieties and related topics, Kyoto, 1998, Surikaisekikenkyusho Kokyuroku **1078,** (1999) 75–85.

[15] M. Ohno, *Classification of generalized polarized manifolds by their nef values*, e-print ArXiv Math. AG/0503119.

[16] T. Peternell, *A characterization of $\mathbb{P}^n$ by vector bundles*, Math. Z., **205,** (1990) 487–490.

[17] T. Peternell, *Ample vector bundles on Fano manifolds*, Internat. J. Math., **2,** (1991) 311–322.

[18] T. Peternell, M. Szurek and J.A. Wiśniewski, *Fano manifolds and vector bundles*, Math. Ann., **294,** (1992) 151–165.

[19] J.A. Wiśniewski, *On contractions of extremal rays of Fano manifolds*, J. Reine Angew. Math, **417,** (1991) 141–157.

[20] J.A. Wiśniewski, *A report on Fano manifolds of middle index and $b_2 \geq 2$*, Projective geometry with applications, 19–26, Lecture Notes in Pure and Appl. Math., **166,** Dekker, New York, 1994.

[21] Y.G. Ye and Q. Zhang, *On ample vector bundle whose adjunction bundles are not numerically effective*, Duke Math. Journal, **60,** (1990) 671-687.

Marco AndreattaDipartimento di Matematica, Università degli Studi di Trento, Via Sommarive, 14, I-38050 Povo (TN), Italy, e-mail: andreatt@science.unitn.it, Fax: +39 0461 881624, and Carla NovelliDipartimento di Matematica, Università degli Studi di Trento, Via Sommarive, 14, I-38050 Povo (TN), Italy, e-mail: novelli@science.unitn.it, Fax: +39 0461 881624,