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MANIFOLDS COVERED BY LINES AND EXTREMAL RAYS

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Abstract. Let X be a smooth complex projective variety and let $H \in \mathrm{Pic}(X)$ be an ample line bundle. Assume that X is covered by rational curves with degree one with respect to H and with anticanonical degree greater than or equal to $(\dim X - 1)/2$. We prove that there is a covering family of such curves whose numerical class spans an extremal ray in the cone of curves $\mathrm{NE}(X)$.

Introduction

Let X be a smooth complex projective variety which admits a morphism with connected fibers $\varphi\colon X \to Z$ onto a normal variety Z such that the anticanonical bundle $-K_X$ is φ-ample, $\dim X > \dim Z$ and $\rho_X = \rho_Z + 1$ (*i.e.* an elementary extremal contraction of fiber type).
It is well known, by fundamental results of Mori theory, that through every point of X there is a rational curve contracted by φ. The numerical classes of these curves lie in an extremal ray of the cone $\mathrm{NE}(X)$. By taking a covering family of such curves one obtains a *quasi-unsplit* family of rational curves, *i.e.* a family such that the irreducible components of all the degenerations of curves in the family are numerically proportional to a curve in the family. It is very natural to ask if the converse is also true:

> Given a covering quasi-unsplit family V of rational curves, is there an extremal elementary contraction which contracts all curves in the family or, in other words, does the numerical class of a curve in the family span an extremal ray of $\mathrm{NE}(X)$?

As proved in [8] (see also [10] and [14]) there is always a rational fibration, defined on an open set of X, whose general fibers are proper, which contracts a general curve in V. More precisely, a general fiber is an equivalence class with respect to the relation induced by the closure $\mathcal{V}$ of the family V in the Chow scheme of X in the following way: two points x and y are equivalent if there exists a connected chain of cycles in $\mathcal{V}$ which joins x and y.

By a careful study of this fibration and of its indeterminacy locus, a partial answer to this question has been given in [6, Theorem 2]; namely, if the dimension of a general equivalence class is greater than or equal to the dimension of the variety minus three then the numerical class of a general curve in the family spans an extremal ray of $\mathrm{NE}(X)$.

Before the results in [6] a special but very natural situation in which the question arises has been studied in [5]. In that paper manifolds covered by rational curves

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of degree one with respect to an ample line bundle H were considered, and it was proved that a covering family of such curves (we will call them lines, by abuse) of anticanonical degree greater than or equal to $\frac{\dim X+2}{2}$ spans an extremal ray (see also [4, Theorem 2.4]).

Recently, in [15, Theorem 7.3], the extremality of a covering family V of lines was proved under the weaker assumption that the anticanonical degree of such curves, denoted by abuse of notation by $-K_X \cdot V$, is greater than or equal to $\frac{\dim X+1}{2}$.

The goal of the present paper is to prove the following

Theorem. *Let (X, H) be a polarized manifold with a dominating family of rational curves V such that $H \cdot V = 1$. If $-K_X \cdot V \geq \frac{\dim X-1}{2}$, then $[V]$ spans an extremal ray of* $\mathrm{NE}(X)$.

The main idea is, as in [15], to combine the ideas and tecniques of [5], especially taking into consideration a suitable adjoint divisor $K_X + mH$ and studying its nefness, with those of [6], in particular regarding the existence of special curves in the indeterminacy locus of the rational fibration associated to V.

1. Background material

Let X be a smooth projective variety defined over the field of complex numbers. A *contraction* $\varphi\colon X \to Z$ is a proper surjective map with connected fibers onto a normal variety Z.
If the canonical bundle K_X is not nef, then the negative part of the cone $\mathrm{NE}(X)$ of effective 1-cycles is locally polyhedral, by the Cone Theorem. By the Contraction Theorem, to every face in this part of the cone is associated a contraction.
Unless otherwise stated, we will reserve the name *extremal face* for a face contained in $\overline{\mathrm{NE}}(X) \cap \{a \in \mathrm{N}_1(X) \mid K_X \cdot a < 0\}$, and we will call *extremal contraction* the contraction of such a face.
An extremal contraction associated to an extremal face of dimension one, *i.e.* to an extremal ray, is called an *elementary contraction*; an extremal ray τ is called *numerically effective*, and the associated contraction is said to be of *fiber type*, if $\dim Z < \dim X$; otherwise the ray is called *non nef* and the contraction is *birational*.
If the codimension of the exceptional locus of an elementary birational contraction is equal to one, the ray and the contraction are called *divisorial*, otherwise they are called *small*.
A Cartier divisor which is the pull-back of an ample divisor A on Z is called a *supporting divisor* of the contraction φ.
If the anticanonical bundle of X is ample, X is called a Fano manifold. For a Fano manifold, the *index*, denoted by r_X, is defined as the largest natural number r such that $-K_X = rH$ for some (ample) divisor H on X.
Throughout the paper, unless otherwise stated, we will use the word *curve* to denote an irreducible curve.

Definition 1.1. A *family of rational curves* is an irreducible component $V \subset \mathrm{Ratcurves}^n(X)$ (see [14, Definition 2.11]). Given a rational curve we will call a *family of deformations* of that curve any irreducible component of $\mathrm{Ratcurves}^n(X)$ containing the point parametrizing that curve. We will say that V is *unsplit* if it is proper. We define $\mathrm{Locus}(V)$ to be the set of points of X through which there is a curve among those parametrized by V; we say that V is a *covering family* if

$\mathrm{Locus}(V) = X$ and that V is a *dominating family* if $\overline{\mathrm{Locus}(V)} = X$.
We denote by V_x the subscheme of V parametrizing rational curves passing through $x \in \mathrm{Locus}(V)$ and by $\mathrm{Locus}(V_x)$ the set of points of X through which there is a curve among those parametrized by V_x.
By abuse of notation, given a line bundle $L \in \mathrm{Pic}(X)$, we will denote by $L \cdot V$ the intersection number $L \cdot C_V$, with C_V any curve among those parametrized by V.

Proposition 1.2. ([14, IV.2.6]) *Let V be an unsplit family of rational curves on X. Then*

(a) $\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) \geq \dim X - K_X \cdot V - 1$;
(b) *every irreducible component of* $\mathrm{Locus}(V_x)$ *has dimension* $\geq -K_X \cdot V - 1$.

This last proposition, in case V is the unsplit family of deformations of a rational curve of minimal anticanonical degree in an extremal face of $\mathrm{NE}(X)$, gives the *fiber locus inequality*:

Proposition 1.3. ([12, Theorem 0.4], [19, Theorem 1.1]) *Let φ be a Fano–Mori contraction of X. Denote by E the exceptional locus of φ and by F an irreducible component of a non-trivial fiber of φ. Then*

$$\dim E + \dim F \geq \dim X + \ell - 1,$$

where $\ell := \min\{-K_X \cdot C \mid C \text{ is a rational curve in } F\}$. If φ is the contraction of an extremal ray τ, then $\ell(\tau) := \ell$ is called the length of the ray.

Definition 1.4. We define a *Chow family of rational curves* $\mathcal{W}$ to be an irreducible component of $\mathrm{Chow}(X)$ parametrizing rational and connected 1-cycles.
We define $\mathrm{Locus}(\mathcal{W})$ to be the set of points of X through which there is a cycle among those parametrized by $\mathcal{W}$; notice that $\mathrm{Locus}(\mathcal{W})$ is a closed subset of X ([14, II.2.3]). We say that $\mathcal{W}$ is a *covering family* if $\mathrm{Locus}(\mathcal{W}) = X$.

Definition 1.5. If V is a family of rational curves, the closure of the image of V in $\mathrm{Chow}(X)$, denoted by $\mathcal{V}$, is called the *Chow family associated to* V.

Remark 1.6. If V is proper, *i.e.* if the family is unsplit, then V corresponds to the normalization of the associated Chow family $\mathcal{V}$.

Definition 1.7. Let $\mathcal{V}$ be the Chow family associated to a family of rational curves V. We say that V (and also $\mathcal{V}$) is *quasi-unsplit* if every component of any reducible cycle in $\mathcal{V}$ is numerically proportional to V.

Definition 1.8. Let $\mathcal{W}$ be a Chow family of rational curves on X and $Z \subset X$. We define $\mathrm{Locus}(\mathcal{W})_Z$ to be the set of points $x \in X$ such that there exists a cycle Γ among those parametrized by $\mathcal{W}$ with $\Gamma \cap Z \neq \emptyset$ and $x \in \Gamma$.
We define $\mathrm{ChLocus}(\mathcal{W})_Z$ to be the set of points $x \in X$ such that there exists a chain of cycles among those parametrized by $\mathcal{W}$ connecting x and Z. Notice that, a priori $\mathrm{ChLocus}(\mathcal{W})_Z$ is a countable union of closed subsets of X.

Notation: If $T \subset X$ we will denote by $\mathrm{N}_1(T, X) \subset \mathrm{N}_1(X)$ the vector subspace generated by numerical classes of curves in T; we will denote by $\mathrm{NE}\,(T, X) \subset \mathrm{NE}(X)$ the subcone generated by numerical classes of curves in T.
The notation $\langle \dots \rangle$ will denote a linear subspace, while the notation $\langle \dots \rangle_c$ will denote a subcone.

Lemma 1.9. ([14, Proposition IV.3.13.3], [1, Lemma 4.1]) *Let $T \subset X$ be a closed subset and let $\mathcal{W}$ be a Chow family of rational curves. Then every curve contained in $\mathrm{ChLocus}(\mathcal{W})_T$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in T and irreducible components of cycles among those parametrized by $\mathcal{W}$ which intersect T.*

Lemma 1.10. (Cf. [5, Proof of Lemma 1.4.5], [17, Lemma 1]) *Let $T \subset X$ be a closed subset and let V be a quasi-unsplit family of rational curves. Then every curve contained in $\mathrm{ChLocus}(\mathcal{V})_T$ is numerically equivalent to a linear combination with rational coefficients*

$$\lambda C_T + \mu C_V,$$

where C_T is a curve in T, C_V is a curve among those parametrized by V and $\lambda \geq 0$.

Corollary 1.11. (Cf. [9, Corollary 2.2 and Remark 2.4]) *Let Σ be an extremal face of $\mathrm{NE}(X)$ and denote by F a fiber of the contraction associated to Σ. Let V be a quasi-unsplit family numerically independent from curves whose numerical class is in Σ. Then*

$$\mathrm{NE}\left(\mathrm{ChLocus}(\mathcal{V})_F, X\right) = \langle \Sigma, [V] \rangle_c,$$

i.e. *the numerical class in X of a curve in $\mathrm{ChLocus}(\mathcal{V})_F$ is in the subcone of $\mathrm{NE}(X)$ generated by Σ and $[V]$.*

Lemma 1.12. *Let D be an effective divisor on X and L a nef divisor. If $(L+D)|_D$ is nef then $L + D$ is nef.*

Proof. Assume that γ is an effective curve on X such that $(L + D) \cdot \gamma < 0$. By the nefness of L we have $D \cdot \gamma < 0$, hence $\gamma \subset D$. But $L + D$ is nef on D, a contradiction. □

2. Rationally connected fibrations

Let X be a smooth complex projective variety and let $\mathcal{W}$ be a covering Chow family of rational curves.

Definition 2.1. The family $\mathcal{W}$ defines a relation of rational connectedness with respect to $\mathcal{W}$, which we shall call *rc($\mathcal{W}$)-relation* for short, in the following way: x and y are in rc($\mathcal{W}$)-relation if there exists a chain of cycles among those parametrized by $\mathcal{W}$ which joins x and y.

To the rc($\mathcal{W}$)-relation we can associate a fibration, at least on an open subset ([7], [14, IV.4.16]); we will call it *rc($\mathcal{W}$)-fibration*.

In the notation of [6], by [10, Theorem 5.9] there exists a closed irreducible subset of $\mathrm{Chow}(X)$ such that, denoting by Y its normalization and by $Z \subset Y \times X$ the restriction of the universal family, we have a commutative diagram

(2.1.1)
$$\begin{array}{ccc} Z & \xrightarrow{e} & X \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle q} & \\ Y & & \end{array}$$

where p is the projection onto the first factor and e is a birational morphism whose exceptional locus E does not dominate Y. Moreover, a general fiber of q is irreducible and is a rc($\mathcal{W}$)-equivalence class.
Let B be the image of E in X; note that $\dim B \leq \dim X - 2$, as X is smooth.

If we consider a (covering) Chow family $\mathcal{V}$, associated to a quasi-unsplit dominating family V, then by [6, Proposition 1, (ii)] B is the union of all rc($\mathcal{V}$)-equivalence classes of dimension greater than $\dim X - \dim Y$.

Moreover we have the following

Lemma 2.2. *Let V be a quasi-unsplit dominating family of rational curves on a smooth complex projective variety X. Let B be the indeterminacy locus of the rc($\mathcal{V}$)-fibration $q\colon X \dashrightarrow Y$, let D be a very ample divisor on $q(X \setminus B)$ and let $\widehat{D} := \overline{q^{-1}D}$. Then*

1. *$\widehat{D} \cdot V = 0$;*
2. *if $C \not\subset B$ is a curve not numerically proportional to $[V]$, then $\widehat{D} \cdot C > 0$;*
3. *if $\widehat{D} \cdot C > 0$ for every curve $C \subset B$ not numerically proportional to $[V]$, then $[V]$ spans an extremal ray of* $\mathrm{NE}(X)$.

Proof. See [6, Proof of Proposition 1]. □

Corollary 2.3. [6, Proposition 3]. *Let V be a quasi-unsplit dominating family of rational curves on a smooth complex projective variety X; denote by B the indeterminacy locus of the rc($\mathcal{V}$)-fibration and by f_V the dimension of the general rc($\mathcal{V}$)-equivalence class.*
If $[V]$ does not span an extremal ray of $\mathrm{NE}(X)$*, then B is not empty. In particular there exist rc($\mathcal{V}$)-equivalence classes of dimension $\geq f_V + 1$.*

We now give a lower bound on the dimension of $\mathrm{ChLocus}(\mathcal{V})_S$, depending on the position of the subvariety S with respect to the indeterminacy locus of the rc($\mathcal{V}$)-fibration.

Lemma 2.4. *Let V be a quasi-unsplit dominating family of rational curves on a smooth complex projective variety X; denote by B the indeterminacy locus of the rc($\mathcal{V}$)-fibration and by f_V the dimension of the general rc($\mathcal{V}$)-equivalence class.*
Let $S \subset X$ be an irreducible subvariety such that $[V] \notin \mathrm{NE}\,(S, X)$. Then there exists an irreducible X_S contained in $\mathrm{ChLocus}(\mathcal{V})_S$ *such that*

1. *if $S \not\subset B$, then $\dim X_S \geq \dim S + f_V$;*
2. *if $S \subset B$, then $\dim X_S \geq \dim S + f_V + 1$.*

Moreover, X_S is not rc($\mathcal{V}$)-connected.

Proof. We refer to diagram (2.1.1). Given any $T \subset Z$ we will set $Z_T := p^{-1}(p(T))$. Let $S' \subset Z$ be an irreducible component of $e^{-1}(S)$ which dominates S via e.
By our assumptions on $\mathrm{NE}\,(S, X)$ we have that S' meets any fiber of $p|_{Z_{S'}}$ in points so, up to replace $Z_{S'}$ with $S' \times_{p(S')} Z_{S'}$, we can assume that S' is a section of $p|_{Z_{S'}}$.
Let Z' be an irreducible component of $Z_{S'}$ which contains S'. We have

$$\dim Z' \geq \dim p(S') + f_V = \dim S' + f_V \geq \dim S + f_V. \tag{2.4.1}$$

Moreover, notice that $S = e(S') \subset e(Z') \subset e(Z_{S'}) \subset \mathrm{ChLocus}(\mathcal{V})_S$.

Assume that $S \not\subset B$. Then $Z' \not\subset E$, hence the map $e|_{Z'}\colon Z' \to X$ is generically finite. Therefore, in view of (2.4.1), $\dim e(Z') = \dim Z' \geq \dim S + f_V$; moreover, since $S \subset e(Z')$ we have that $e(Z')$ is not rc($\mathcal{V}$)-connected.

Assume now that $S \subset B$. Assertion (2) will follow once we prove that the general fiber G of $e|_{\overline{Z}}$ has dimension strictly smaller than the general fiber of $e|_{S'}$ for at least one irreducible component $\overline{Z}$ of $Z_{S'}$ which dominates $p(S')$. In fact, recalling also (2.4.1), in this case we will have

$$\dim e(\overline{Z}) = \dim \overline{Z} - \dim G > (\dim S' + f_V) - (\dim S' - \dim S) = f_V + \dim S.$$

Claim. Let G be an irreducible component of a fiber of $e|_{Z_{S'}}$, let $z \in G$ be any point and let $z' := p^{-1}(p(z)) \cap S'$ be the intersection of the fiber of p containing z with S'; then there exists an irreducible component F of the fiber F' of $e|_{S'}$ containing z' such that $p(G) \subseteq p(F)$.

To prove the claim, recall that, since $e(Z_G) \subset \mathrm{ChLocus}(\mathcal{V})_{e(z)}$, the image via e of any curve in $Z_G \cap S'$ – which is irreducible, being a section over $p(G)$ – must be a point, otherwise it would be a curve contained in $S \cap \mathrm{ChLocus}(\mathcal{V})_{e(z)}$, which is a contradiction, since curves in S are numerically independent from $[V]$.
Therefore $Z_G \cap S'$ is contained in a fiber F' of $e|_{S'}$. To prove the claim we take as F the irreducible component of F' containing $Z_G \cap S'$.

Let $S^1 \subset S'$ be the proper closed subset on which $e|_{S'}$ is not equidimensional and let $S^2 \subset S'$ be the proper closed subset of points in which the fiber of $e|_{S'}$ is not locally irreducible. Recalling that $p|_{S'}$ is a finite map we see that $p\,(S^1 \cup S^2)$ is a proper closed subset of $p(S')$.
Let $y \in p(S') \setminus p\,(S^1 \cup S^2)$ be a general point; in particular there is only one irreducible component F of the fiber F' of $e|_{S'}$ passing through $z' = p^{-1}(y) \cap S'$ and $\dim F = \dim S' - \dim S$.
Notice that $\dim e(Z_F) > f_V$, otherwise a one parameter family of fibers of p meeting F would have the same image in X (Cf. [6, End of proof of Proposition 1], where $e(Z_F) = \mathrm{Locus}(V_{e(F)})$).
This implies that, for an irreducible component $\overline{Z}_F$ of Z_F we have $\dim e(\overline{Z}_F) > f_V$.
Taking as $\overline{Z}$ an irreducible component of $Z_{S'}$ containing $\overline{Z}_F$ we have that, for every point $z \in p^{-1}(y) \cap \overline{Z}$ and any irreducible component G of the fiber of $e|_{\overline{Z}}$ passing through z we have $p(G) \subseteq p(F)$, hence $\dim G < \dim F = \dim S' - \dim S$; the same inequality then holds for the general fiber by semicontinuity of the local dimension.
Noticing that S is contained in $\mathrm{ChLocus}_{e(\overline{Z})}(\mathcal{V})$ the last assertion follows. □

Remark 2.5. Both the bounds in Lemma (2.4) are sharp. An example for the second one is given by [6, Example 2]: in that example $B \simeq \mathbb{P}^2 \times \mathbb{P}^1$; taking as S a fiber of the projection onto $\mathbb{P}^2$ we have equality in (2).

3. Blowing-down

In this section we consider the following situation, which will show up in the proof of Theorem (4.3):

Lemma 3.1. *Let (X, H) be a polarized manifold with a dominating family of rational curves V such that $H \cdot V = 1$. Denote by f_V the dimension of the general rc($\mathcal{V}$)-equivalence class and assume that there exists an extremal face Σ in* NE(X) *whose associate contraction $\sigma\colon X \to X'$ is a smooth blow-up along a disjoint union of subvarieties T_i of dimension $\leq f_V$ such that $E_i \cdot V = 0$ for every exceptional divisor E_i and $H \cdot l_i = 1$ if l_i is a line in a fiber of σ. Finally denote by V' a family of deformation of $\sigma(C)$, with C a general curve parametrized by V. Then*

(1) $-K_{X'} \cdot V' = -K_X \cdot V$;
(2) *there exists an ample line bundle H' on X' such that $H' \cdot V' = 1$;*

(3) *if C' is a curve parametrized by V' such that $T_i \cap C' \neq \emptyset$, then $C' \subset T_i$;*
(4) $\rho_{X'} > 1$*;*
(5) *if $[V']$ spans an extremal ray of* $\mathrm{NE}(X')$*, then $[V]$ spans an extremal ray of* $\mathrm{NE}(X)$.

Proof. It is enough to prove the statement in case $\dim \Sigma = 1$, *i.e.* $\sigma\colon X \to X'$ is the blow-up of X' along a smooth subvariety T associated to the extremal ray Σ. In fact, if $\dim \Sigma > 1$, the contraction of Σ factors through elementary contractions, each one satisfying the assumptions in the statement.

Denote by E the exceptional locus of σ. Since $E \cdot V = 0$ the first assertion in the statement follows from the canonical bundle formula for blow-ups.

Moreover, the fact that $E \cdot V = 0$ also implies that any $\mathrm{rc}(\mathcal{V})$-equivalence class meeting E is actually contained in E. Therefore, if F is a non-trivial fiber of σ, then $\mathrm{ChLocus}(\mathcal{V})_F \subseteq E$. By Lemma (2.4)

$$\dim \mathrm{ChLocus}(\mathcal{V})_F \geq f_V + \dim F \geq \dim X - 1,$$

hence $E = \mathrm{ChLocus}(\mathcal{V})_F$ and $\dim T = f_V$; in particular, applying Corollary (1.11) we get that $\mathrm{NE}\,(E, X) = \langle [V], \Sigma \rangle_c$.

The line bundle $(H+E)|_E$ is nef and it is trivial only on Σ, since $(H+E)\cdot\Sigma = 0$ and $(H+E)\cdot V = 1$. Then $H+E$ is nef by Lemma (1.12).
Notice also that $H+E$ is trivial only on Σ. Indeed, let γ be an effective curve on X such that $(H+E)\cdot\gamma = 0$. Due to the ampleness of H we have $E\cdot\gamma < 0$, hence $\gamma \subset E$. This implies that $[\gamma] \in \Sigma$. Therefore $H + E = \sigma^*H'$, with H' an ample line bundle on X'. By the projection formula $H' \cdot V' = 1$, hence part (2) in the statement is proved.

Now, let C' be a curve parametrized by V' meeting T and assume by contradiction that C' is not contained in T; denote by $\widetilde{C}'$ its strict transform. Then

$$1 = H' \cdot C' = \sigma^*H' \cdot \widetilde{C}' = (H+E)\cdot \widetilde{C}' \geq 2,$$

which is a contradiction. It follows that every curve parametrized by V' which meets T is contained in it; so we get part (3) in the statement.

As to part (4), assume by contradiction that $\rho_{X'} = 1$. This implies that X' is $\mathrm{rc}(\mathcal{V}')$-connected, but this is impossible as, in view of part (3), we cannot join points of T and points outside of T with curves parametrized by V'.

Finally, to prove part (5) assume that $[V']$ spans an extremal ray of X' and let B be the indeterminacy locus of the $\mathrm{rc}(\mathcal{V})$-fibration. We claim that $E \cap B = \emptyset$.

Assume by contradiction that this is not the case; then E meets (and hence contains) an $\mathrm{rc}(\mathcal{V})$-equivalence class G of dimension $\dim G \geq f_V + 1$. Take a fiber F of σ meeting G. Then $\dim F + \dim G > \dim E$. On the other hand, $\dim(F \cap G) = 0$ as $[V] \notin \Sigma$. So we get a contradiction.

Let A be a supporting divisor of the contraction associated to $[V']$. The pull-back σ^*A defines a two-dimensional face Π of $\overline{\mathrm{NE}}(X)$ containing Σ and $[V]$. Let $\widehat{D}$ be as in Lemma (2.2); by the same lemma $\widehat{D}\cdot\Sigma > 0$ and $\widehat{D}\cdot V = 0$.
Assume that Π is not spanned by Σ and $[V]$; in this case there exists a class $c \in \overline{\mathrm{NE}}(X)$ belonging to Π such that $E\cdot c > 0$ and $\widehat{D}\cdot c < 0$.
Let $\{C_n\}$ be a sequence of effective one cycles such that the limit of $\mathbb{R}_+[C_n]$ is $\mathbb{R}_+c$;

by continuity, for some n_0 we have $E \cdot C_n > 0$ and $\widehat{D} \cdot C_n < 0$ for $n \geq n_0$, hence $C_n \subset B$, and $E \cap C_n \neq \emptyset$ for $n \geq n_0$, contradicting $E \cap B = \emptyset$. □

4. Main theorem

First of all we consider polarized manifolds (X, H) with a quasi-unsplit dominating family of rational curves V proving that if, for m large enough, the adjoint divisor $K_X + mH$ defines an extremal face containing $[V]$, then $[V]$ spans an extremal ray of X.

Proposition 4.1. *Let (X, H) be a polarized manifold which admits a quasi-unsplit dominating family of rational curves V; denote by f_V the dimension of a general rc($\mathcal{V}$)-equivalence class.*
If, for some integer m such that $m + f_V \geq \dim X - 3$, the divisor $K_X + mH$ is nef and it is trivial on $[V]$, then $[V]$ spans an extremal ray of $\mathrm{NE}(X)$.

Proof. Assume by contradiction that $[V]$ does not span an extremal ray in $\mathrm{NE}(X)$. This implies that $K_X + mH$ defines an extremal face Σ of dimension at least two, containing $[V]$. By [15, Lemma 7.2] there exists an extremal ray $\vartheta \in \Sigma$ whose exceptional locus is contained in the indeterminacy locus B of the rc($\mathcal{V}$)-fibration. Since $(K_X + mH) \cdot \vartheta = 0$, the length $\ell(\vartheta)$ is greater than or equal to m.
Let F be a non-trivial fiber of the contraction associated to ϑ; since this contraction is small, being $\dim B \leq \dim X - 2$, then $\dim F \geq m + 1$ by Proposition (1.3).
By part (2) of Lemma (2.4), the dimension of $\mathrm{ChLocus}(\mathcal{V})_F$ is

$$\dim \mathrm{ChLocus}(\mathcal{V})_F \geq \dim F + f_V + 1.$$

As the rc($\mathcal{V}$)-equivalence classes are either contained in B or have empty intersection with it, $\mathrm{ChLocus}(\mathcal{V})_F \subset B$. Therefore we get

$$\dim X - 2 \geq \dim B \geq \dim \mathrm{ChLocus}(\mathcal{V})_F \geq f_V + m + 2 \geq \dim X - 1,$$

which is a contradiction. □

As the last preparatory step, we consider the following special case.

Lemma 4.2. *Let V be a quasi-unsplit dominating family of rational curves on a smooth complex projective variety X. Denote by f_V the dimension of a general rc($\mathcal{V}$)-equivalence class. Assume that there exists an extremal ray ϑ, independent from $[V]$, whose associated contraction has a fiber F such that $\dim F + f_V \geq \dim X$. Then $\dim F + f_V = \dim X$ and $\mathrm{NE}(X) = \langle [V], \vartheta \rangle_c$. In particular $\rho_X = 2$.*

Proof. By part (1) of Lemma (2.4) we have

$$\dim X \geq \dim \mathrm{ChLocus}(\mathcal{V})_F \geq f_V + \dim F,$$

hence $\dim F + f_V = \dim X$ and $\mathrm{ChLocus}(\mathcal{V})_F = X$; so the assertion follows by Corollary (1.11). □

Theorem 4.3. *Let (X, H) be a polarized manifold with a dominating family of rational curves V such that $H \cdot V = 1$. If $-K_X \cdot V \geq \frac{\dim X - 1}{2}$, then $[V]$ spans an extremal ray of* $\mathrm{NE}(X)$.

Proof. Let B be the indeterminacy locus of the rc($\mathcal{V}$)-fibration $q \colon X \dashrightarrow Y$, let D be a very ample divisor on $q(X \setminus B)$ and let $\widehat{D} := \overline{q^{-1}D}$. Denote by m the anticanonical degree of V and by f_V the dimension of a general rc($\mathcal{V}$)-equivalence class. Notice that, since V is a dominating family, we have $m \geq 2$.

By Proposition (1.2) $\dim \operatorname{Locus}(V_x) \geq -K_X \cdot V - 1 = m-1$; since a general fiber of the rc($\mathcal{V}$)-fibration contains $\operatorname{Locus}(V_x)$ for every point x in it, we have $f_V \geq m-1$.

If $K_X + mH$ is nef, then the assertion follows by Proposition (4.1); therefore we can assume that $K_X + mH$ is not nef.
Let ϑ be an extremal ray such that $(K_X + mH)\cdot\vartheta < 0$ and let φ_ϑ be the associated contraction. Notice that ϑ has length $\ell(\vartheta) \geq m+1$, hence every non-trivial fiber of φ_ϑ has dimension $\geq m$ by Proposition (1.3). On the other hand, by Lemma (4.2) we can confine to assume that all fibers of φ_ϑ have dimension $\leq m+1$.
In particular this implies that, denoted by C_ϑ a minimal degree curve whose numerical class belongs to ϑ, we have $H \cdot C_\vartheta = 1$. Indeed, if this were not the case, we would have $\ell(\vartheta) \geq 2m+1$, hence every non-trivial fiber of φ_ϑ would have dimension $\geq 2m > m+1$, by Proposition (1.3) and the fact that $m \geq 2$.

If the Picard number of X is one the theorem is clearly true, so we can assume that $\rho_X \geq 2$. Now we split up the proof in two cases, according to the value of ρ_X: first we consider the case $\rho_X = 2$ and then the general one.

Case (a) $\rho_X = 2$.

The proof is based on different arguments, depending on the dimension of the fibers of the contraction associated to the extremal ray ϑ.

Case (a1) The contraction φ_ϑ admits an $(m+1)$-dimensional fiber F.

Consider $X_F := \operatorname{ChLocus}(\mathcal{V})_F$. We have, by Corollary (1.11), that $\operatorname{NE}(X_F, X) = \langle [V], \vartheta \rangle_c$ and, by Lemma (2.4), that

$$\dim X_F \geq \dim F + f_V \geq (m+1) + (m-1) \geq \dim X - 1.$$

If $X_F = X$, then the statement is proved. So we can assume that an irreducible component $\overline{X}_F$ of X_F is a divisor and thus that $f_V = m-1$. Notice that $\overline{X}_F \cdot V = 0$, otherwise we would have $X_F = X$.
Consider now the intersection number of X_F with curves whose numerical class belongs to ϑ; since $\rho_X = 2$ and $\overline{X}_F \cdot V = 0$ we cannot have also $\overline{X}_F \cdot \vartheta = 0$.

Let us show that we cannot have $\overline{X}_F \cdot \vartheta < 0$, too.
Assume by contradiction that this is the case. Then $\operatorname{Exc}(\vartheta) \subset \overline{X}_F$, so φ_ϑ is divisorial by Proposition (1.3). By the same proposition, recalling that we are assuming that all the fibers of φ_ϑ have dimension $\leq m+1$, every non-trivial fiber has dimension $m+1$.
Then φ_ϑ is the blow-up of a smooth variety X' along a smooth center T by [2, Theorem 4.1 (iii)]. The dimension of the center is

$$\dim T = (n-1) - (m+1) \leq m - 1 = f_V.$$

We can thus apply part (4) of Lemma (3.1) and we get $\rho_X = \rho_{X'} + 1 > 2$, reaching a contradiction.

Therefore $\overline{X}_F \cdot \vartheta > 0$, hence $(\overline{X}_F)|_{\overline{X}_F}$ is nef and thus, by Lemma (1.12), $\overline{X}_F$ is nef. As $\overline{X}_F \cdot V = 0$ and $\rho_X = 2$, $\overline{X}_F$ is the supporting divisor of an elementary contraction of X whose associated extremal ray is spanned by $[V]$.

Case (a2) The contraction φ_ϑ is equidimensional with m-dimensional fibers.

By Proposition (1.3), φ_ϑ is of fiber type and $\ell(\vartheta) = m+1$. Hence, by [11, Lemma 2.12], X is a projective bundle over a smooth variety Y, *i.e.* $X = \mathbb{P}_Y(\mathcal{E})$, where $\mathcal{E} = (\varphi_\vartheta)_* H$.

Notice that Y has Picard number one and is covered by rational curves – the images of the curves parametrized by V – therefore Y is a Fano manifold.
By the canonical bundle formula for projective bundles we have

$$K_X + (m+1)H = \varphi_\vartheta^*(K_Y + \det \mathcal{E}).$$

In particular, if C_V is a curve among those parametrized by V, by the projection formula we can compute

$$(K_Y + \det \mathcal{E}) \cdot (\varphi_\vartheta)_*(C_V) = (K_X + (m+1)H) \cdot C_V = 1.$$

It follows that $(K_Y + \det \mathcal{E}) \cdot \varphi_\vartheta(C_V) = 1$ and that $K_Y + \det \mathcal{E}$ is the ample generator of $\mathrm{Pic}(Y)$. The ampleness of $\mathcal{E}$ implies that $\det \mathcal{E} \cdot \varphi_\vartheta(C_V) \geq m+1$; therefore $-K_Y \cdot \varphi_\vartheta(C_V) \geq m$, hence the index r_Y of Y is greater than or equal to m.

If $r_Y = m$, denoted by l a rational curve of minimal degree in Y, then $\det \mathcal{E} \cdot l = m+1$; moreover, the splitting type of $\mathcal{E}$, which is ample and of rank $m+1$, on rational curves of minimal degree is uniform of type $(1, \dots, 1)$.
We can thus apply [3, Proposition 1.2], so we obtain that $X \simeq \mathbb{P}^m \times Y$. It follows that the curves of V are contained in the fibers of the first projection and that $[V]$ spans an extremal ray.

Therefore we are left with $r_Y \geq m+1$. Recalling that $\dim Y = \dim X - m \leq m+1$, by the Kobayashi–Ochiai Theorem ([13]) we get that Y is a projective space or a hyperquadric.

Assume by contradiction that $[V]$ does not span an extremal ray of X.
By part (3) of Lemma (2.2) there exists a curve $C \subset B$, whose numerical class is not proportional to $[V]$, such that $\widehat{D} \cdot C \leq 0$. Actually, since $\rho_X = 2$ and $\widehat{D} \cdot V = 0$, we have $\widehat{D} \cdot C < 0$.
By part (2) of Lemma (2.4), there exists $X_C \subset \mathrm{ChLocus}(\mathcal{V})_C$ which is not $\mathrm{rc}(\mathcal{V})$-connected such that $\dim X_C \geq f_V + \dim C + 1 \geq m+1$.
By Lemma (1.10) $\widehat{D}$ has non positive intersection number with every curve in X_C and it is trivial only on curves which are numerically proportional to $[V]$.
Since $\widehat{D} \cdot \vartheta > 0$, we have that φ_ϑ does not contract curves in X_C, hence $\dim Y \geq \dim X_C \geq m+1$ and so $\dim Y = \dim X_C = m+1$.
Since X_C is not $\mathrm{rc}(\mathcal{V})$-connected, for every point c of X_C, the intersection X_c of the $\mathrm{rc}(\mathcal{V})$-equivalence class containing c with X_C has dimension $= m$. In particular X_C is the union of a one parameter family of $\mathrm{rc}(\mathcal{V})$-connected subvarieties X_c.

We claim that there exists a line l in Y which is not contained in $\varphi_\vartheta(X_c)$ for any $c \in C$. Notice that, since φ_ϑ does not contract curves in X_C, through a general point y in Y there is a finite number of such subvarieties.
If $Y \simeq \mathbb{P}^{m+1}$, a line joining y with a point outside the union of these subvarieties has the required property.
Assume now that $Y \simeq \mathbb{Q}^{m+1}$; for any $y \in \mathbb{Q}^{m+1}$ the locus of the lines through y is a quadric cone $\mathbb{Q}^m_y$ with vertex y. Therefore, if every line through y is contained in $\varphi_\vartheta(X_c)$ for some $c \in C$, then $\mathbb{Q}^m_y$ is an irreducible component of $\varphi_\vartheta(X_c)$; since X_c moves in a one-dimensional family, for the general point $y \in \mathbb{Q}^{m+1}$, the general line through y has the required property.

The splitting type of $\mathcal{E}$ on this line is one of the following: $(2, 1, \dots, 1)$ if $Y \simeq \mathbb{Q}^{m+1}$ and either $(3, 1, \dots, 1)$ or $(2, 2, 1, \dots, 1)$ if $Y \simeq \mathbb{P}^{m+1}$. Recalling that $m \geq 2$ we have that, among the summands of $\mathcal{E}_l$ there is at least one $\mathcal{O}_{\mathbb{P}^1}(1)$.

Consider $\mathbb{P}_l(\mathcal{E}|_l)$; its cone of curves is generated by the class of a line in a fiber of the projection onto l and the class of a minimal section C_0. By the discussion above we have that $H \cdot C_0 = 1$. Moreover, $\varphi_\vartheta^*(K_Y + \det \mathcal{E}) \cdot C_0 = 1$, hence $[C_0] = [V]$; in particular $\widehat{D}$ is nef on $\mathbb{P}_l(\mathcal{E}|_l)$.
Consider an irreducible curve in $\mathbb{P}_l(\mathcal{E}|_l) \cap X_C$; by our choice of l, this curve is not contained in a $\mathrm{rc}(\mathcal{V})$-equivalence class contained in X_C, so it is negative with respect to $\widehat{D}$, a contradiction. The case $\rho_X = 2$ is thus completed.

Case (b) $\rho_X > 2$.

Notice that, in view of Corollary (2.3), we can confine to assume that $B \neq \emptyset$; moreover, by part (3) of Lemma (2.2), we can also assume the existence of a curve $C \subset B$ such that $[C]$ is not proportional to $[V]$ and $\widehat{D} \cdot C \leq 0$.

We claim that $K_X + (m+1)H$ is nef.
Assume by contradiction that $K_X + (m+1)H$ is not nef. Let τ be a ray such that $(K_X + (m+1)H) \cdot \tau < 0$, denote by C_τ a rational curve of minimal anticanonical degree in τ and by φ_τ the contraction associated to τ.
Notice that τ has length $\ell(\tau) \geq m+2$, hence every non-trivial fiber of φ_τ has dimension $\geq m+1$ by Proposition (1.3).
On the other hand φ_τ cannot have fibers of dimension $> m+1$, otherwise, by Lemma (4.2), we would have $\rho_X = 2$. Therefore every non-trivial fiber of φ_τ has dimension $m+1$.
In view of Proposition (1.3), we thus get that φ_τ is of fiber type and that the length of τ is $\ell(\tau) = m+2$; this last fact gives $H \cdot C_\tau = 1$. Let us consider W_τ to be a minimal degree covering family of curves whose numerical class belongs to τ.
Since B is not empty, there are $\mathrm{rc}(\mathcal{V})$-equivalence classes of dimension $\geq f_V + 1 \geq m$; let G be one of these classes. Notice that since φ_τ is equidimensional with $(m+1)$-dimensional fibers, we have $f_W = m+1$. By part (1) of Lemma (2.4) we have

$$\dim \mathrm{ChLocus}(\mathcal{W}_\tau)_G \geq \dim G + f_W = 2m + 1 \geq \dim X,$$

so by Lemma (1.9) we deduce $\rho_X = 2$, a contradiction which proves the nefness of $K_X + (m+1)H$.

Recall now that the extremal ray ϑ which we fixed at the beginning of the proof has length $\ell(\vartheta) \geq m+1$ and is generated by a curve C_ϑ such that $H \cdot \vartheta = 1$, therefore $(K_X + (m+1)H) \cdot \vartheta = 0$ and $K_X + (m+1)H$ is not ample.
Let Σ be the extremal face contracted by $K_X + (m+1)H$. We now consider separately two cases, depending on the existence in Σ of a fiber type extremal ray.

Case (b1) There exists a fiber type extremal ray ϱ in Σ.

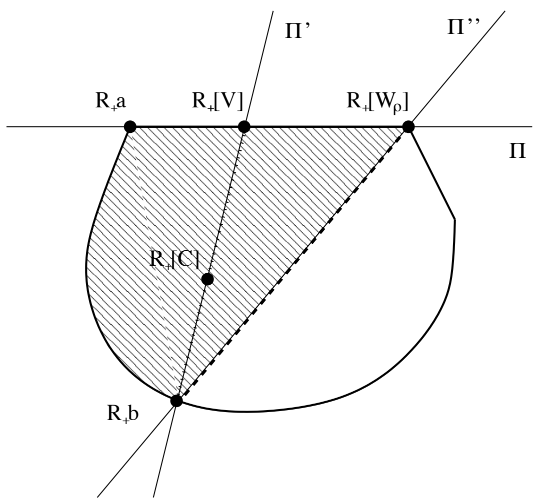
Let φ_ϱ be the contraction associated with ϱ and denote by W_ϱ a minimal degree covering family of curves whose numerical class belongs to ϱ.
By part (2) of Lemma (2.4), there exists an irreducible $X_C \subset \mathrm{ChLocus}(\mathcal{V})_C$ such that $\dim X_C \geq f_V + 2$.
According to Lemma (1.10), every curve in X_C can be written as $\alpha[C] + \beta[V]$ with $\alpha \geq 0$; in particular, since $\widehat{D} \cdot V = 0$ by Lemma (2.2), it follows that $\widehat{D}$ is not positive on any curve contained in X_C. By the same lemma $\widehat{D} \cdot W_\varrho > 0$, hence $[W_\varrho] \notin \mathrm{NE}\,(X_C, X)$. Therefore part (1) of Lemma (2.4) gives

$$\dim \mathrm{ChLocus}(\mathcal{W}_\varrho)_{X_C} \geq \dim X_C + f_{W_\varrho} \geq f_V + 2 + m \geq \dim X,$$

where f_{W_ϱ} is the dimension of the general $\mathrm{rc}(\mathcal{W}_\varrho)$-equivalence class.
Therefore, by applying twice Lemma (1.10), we get that the class of every curve in X can be written as

$$\lambda(\alpha[C] + \beta[V]) + \mu[W_\varrho] \tag{4.3.1}$$

with $\alpha, \lambda \geq 0$ and $\alpha[C] + \beta[V] \in \mathrm{NE}\,(X_C, X)$.
This has some very important consequences: first of all, since we are assuming $\rho_X > 2$, this implies that $\rho_X = 3$; in particular $[C]$ is not contained in the plane Π in $\mathrm{N}_1(X)$ spanned by $[W_\varrho]$ and $[V]$. Moreover the intersection of Π with $\mathrm{NE}(X)$ is a face of $\mathrm{NE}(X)$.
We have to prove that $\Pi \cap \overline{\mathrm{NE}}(X) = \langle [V], [W_\varrho] \rangle_c$. If this is not the case, then there exists a class a such that $\Pi \cap \overline{\mathrm{NE}}(X) = \langle a, [W_\varrho] \rangle_c$ and $\widehat{D} \cdot a < 0$.
Denote by $b \in \mathrm{N}_1(X)$ a class, not proportional to $[V]$, lying in the intersection of $\partial\overline{\mathrm{NE}}(X)$ with the plane $\Pi' = \mathrm{N}_1(X_C, X)$ and by Π'' the plane spanned by $[W_\varrho]$ and b.
Formula (4.3.1), traslated in geometric terms, says that $\mathrm{NE}(X)$ is contained in the intersection of half-spaces determined by Π and by Π'' as in the figure below, which shows a cross-section of $\overline{\mathrm{NE}}(X)$.



Let $\{C_n\}$ be a sequence of effective one cycles such that the limit of $\mathbb{R}_+[C_n]$ is $\mathbb{R}_+ a$; by continuity, for some n_0 we have $\widehat{D} \cdot C_n < 0$ for $n \geq n_0$, hence $C_n \subset B$ for $n \geq n_0$, and all the above arguments apply to C_n, for $n \geq n_0$. In particular, defining b_n and Π''_n as above, we get that, for $n \geq n_0$, $\mathrm{NE}(X)$ is contained in the intersection of half-spaces determined by Π and by Π''_n. Since $\Pi''_n \to \Pi$ as $\mathbb{R}_+[C_n] \to \mathbb{R}_+ a$ and $\rho_X = 3$ we get a contradiction.

Case (b2) Every ray in Σ is birational.

Let η be any ray in Σ. By Proposition (1.3), for every non-trivial fiber of its associated contraction φ_η we have $\dim F \geq \ell(\eta) \geq m+1$. Recalling that, by Lemma (4.2), we can assume $\dim F \leq m+1$, we have $\dim F = m+1 = \ell(\eta)$. This also implies that, if C_η is a minimal degree curve whose numerical class is contained in η we have $H \cdot C_\eta = 1$.
By Proposition (1.3), φ_η is a divisorial contraction, hence, by [2, Theorem 4.1

(iii)], is the blow-up of a smooth variety along a smooth center T of dimension $(n-1)-(m+1) \leq m-1$.

Let E be the exceptional divisor of φ_η. By part (2) of Lemma (2.4), there exists an irreducible $X_C \subset \mathrm{ChLocus}(\mathcal{V})_C$ with $\dim X_C \geq f_V + 2$.
By Lemma (1.10) $\widehat{D}$ has non positive intersection number with every curve in X_C. If $E \cap X_C \neq \emptyset$, then there is a fiber F of φ_η meeting X_C. Counting dimensions, we find that $\dim(F \cap X_C) \geq 1$, which is a contradiction as $\widehat{D} \cdot \eta > 0$. So $E \cap X_C = \emptyset$, whence $E \cdot V = 0$.
Therefore E contains $\mathrm{rc}(\mathcal{V})$-equivalence classes and $\dim T \geq f_V$, since φ_η is finite-to-one on $\mathrm{rc}(\mathcal{V})$-equivalence classes. Recalling that $f_V \geq m-1$ we derive $\dim T = f_V = m-1$.

Assume that $\dim \Sigma \geq 2$ and let E_1, E_2 be the exceptional loci of two different extremal rays η_1, η_2 in Σ; since the fibers of the contractions φ_{η_1} and φ_{η_2} have dimension $m+1$ and $2(m+1) > \dim X$ we have that $E_1 \cap E_2 = \emptyset$.
Therefore the contraction $\sigma\colon X \to X'$ of the face Σ verifies the assumptions of Lemma (3.1), hence there exists an ample line bundle H' on X' and an unsplit dominating family V' on X' such that $H' \cdot V' = 1$ and $-K_{X'} \cdot V' = -K_X \cdot V \geq \frac{\dim X' - 1}{2}$.
Denote by $f_{V'}$ the dimension of the general $\mathrm{rc}(\mathcal{V}')$-equivalence class. Since a general fiber of the $\mathrm{rc}(\mathcal{V}')$-fibration contains $\mathrm{Locus}(V'_{x'})$, we have $f_{V'} \geq \dim \mathrm{Locus}(V'_{x'}) - 1 \geq m-1$.
Consider the adjoint divisor $K_{X'} + mH'$; if it is nef, or an extremal ray ϑ' such that $(K_{X'} + mH') \cdot \vartheta' < 0$ has a fiber of dimension $\geq m+2$, then $[V']$ spans an extremal ray by Proposition (4.1) or by Lemma (4.2), so $[V]$ spans an extremal ray by Lemma (3.1).

Let us show that the remaining case does not happen.
Assume that there is an extremal ray ϑ' such that $(K_{X'} + mH') \cdot \vartheta' < 0$ and every fiber of the associated contraction has dimension $\leq m+1$. In particular we have $H' \cdot \vartheta' = 1$, otherwise we would have $\ell(\vartheta') \geq 2m+1$, hence every non-trivial fiber of the associated contraction would have dimension $\geq 2m > m+1$ by Proposition (1.3). Moreover, we have $(K_{X'} + (m+1)H') \cdot \vartheta' \leq 0$, since $\ell(\vartheta') \geq m+1$.
On the other hand, recalling that $\sigma^* H' = H + \sum E_i$ and that $\sigma^* K_{X'} = K_X - \sum (m+1) E_i$, we have

$$\sigma^*(K_{X'} + (m+1)H') = K_X + (m+1)H,$$

so, by the projection formula, $K_{X'} + (m+1)H'$ is ample on X', a contradiction. □

Corollary 4.4. *Let (X, H) be a polarized manifold of dimension at most five, with a dominating family of rational curves V such that $H \cdot V = 1$. Then $[V]$ spans an extremal ray of* $\mathrm{NE}(X)$.

5. An example

In the paper [5], an application of the results about extremality of families of lines was a relative version of a theorem proved in [18], which was the first step towards a conjecture of Mukai for Fano manifolds.
This conjecture states that, for a Fano manifold X, denoted by ρ_X its Picard

number and by r_X its index, we have

$$\rho_X(r_X - 1) \leq \dim X.$$

More precisely, in [18, Theorem B] it was proved that, if $r_X \geq \frac{\dim X}{2} + 1$, then $\rho_X = 1$ unless $X \simeq \mathbb{P}^{\dim X/2} \times \mathbb{P}^{\dim X/2}$, while in [5, Theorem 3.1.1] it was proved that a fiber type contraction $\varphi\colon X \to Y$ supported by $K_X + mL$ with $m \geq \frac{\dim X}{2} + 1$ is elementary, unless $X \simeq \mathbb{P}^{\dim X/2} \times \mathbb{P}^{\dim X/2}$.

In the last few years some progress has been made towards Mukai conjecture; in particular it was recently proved in [16, Theorem 3] that it holds for a Fano manifold with (pseudo)index greater than or equal to $\frac{\dim X}{3} + 1$.
It is therefore natural to ask if the corresponding relative statement is true, namely, given a fiber type contraction $\varphi\colon X \to Y$, corresponding to an extremal face Σ, supported by $K_X + mL$ with $m \geq \frac{\dim X}{3} + 1$ is it possible to find a bound on the dimension of Σ?

The answer to this question is negative, as we will show with an example in which $m = \frac{\dim X}{2}$; it follows that [5, Theorem 3.1.1] cannot be improved.

Example 5.1. *Let Z be a smooth variety of dimension $k+2$, denote by Y the product $Z \times \mathbb{P}^k$ and by p_1, p_2 the projections onto the factors. Let $\{z_i\}_{i=1,\dots,t}$ be points of Z and denote by F_i the fibers of p_1 over z_i.*
Let $\sigma\colon X \to Y$ be the blow-up of Y along the union of F_i. The canonical bundle of X is

$$K_X = \sigma^* K_Y + (k+1)\sum_{i=1}^{t} E_i = \sigma^*(p_1^* K_Z + p_2^* K_{\mathbb{P}^k}) + (k+1)\sum_{i=1}^{t} E_i; \tag{5.1.1}$$

denoting by $\mathcal{H} := (p_2 \circ \sigma)^* \mathcal{O}_{\mathbb{P}^k}(1)$ and by $L' := \mathcal{H} - \sum E_i$, we can rewrite formula (5.1.1) *as*

$$K_X + (k+1)L' = \sigma^*(p_1^* K_Z).$$

It is easy to check that L' is $(p_1 \circ \sigma)$-ample. Let $A \in \mathrm{Pic}(Z)$ be an ample line bundle such that $K_Z + (k+1)A$ is ample; then $L := L' + \sigma^*(p_1^* A)$ is an ample line bundle on X; moreover $L \cdot l = 1$ for a line l in the strict transform of a fiber F of p_1 not contained in the center of σ.
The contraction $p_1 \circ \sigma$ is supported by $K_X + (k+1)L = K_X + \frac{\dim X}{2} L$ and contracts a face of dimension $t+1$.

Remark 5.2. The difference between the relative and the absolute case is given by the existence of minimal horizontal dominating families of rational curves for proper morphisms defined on a open subset of a Fano manifold (for the definition and the references see [1, Remark 6.4]). Such families do not exist in general in the relative case.

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