Rational curves and bounds on the Picard number of Fano manifolds

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RATIONAL CURVES AND BOUNDS ON THE PICARD NUMBER OF FANO MANIFOLDS

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Abstract. We prove that Generalized Mukai Conjecture holds for Fano manifolds X of pseudoindex $i_X \geq (\dim X + 3)/3$. We also give different proofs of the conjecture for Fano fourfolds and fivefolds.

1. Introduction

Let X be a Fano manifold, *i.e.* a smooth complex projective variety whose anticanonical bundle $-K_X$ is ample. The *index* of a Fano manifold X is defined as

$$r_X := \max\{m \in \mathbb{N} \mid -K_X = mL \text{ for some line bundle } L\},$$

while the *pseudoindex* of X is defined as

$$i_X := \min\{m \in \mathbb{N} \mid -K_X \cdot C = m \text{ for some rational curve } C \subset X\}.$$

We denote by ρ_X the Picard number of X, *i.e.* the dimension of the $\mathbb{R}$-vector space $\mathrm{N}_1(X)$ of 1-cycles modulo numerical equivalence.

In 1988, Mukai [9] proposed the following conjecture:

Conjecture 1.1. *Let X be a Fano manifold of dimension n. Then $\rho_X(r_X - 1) \leq n$, with equality if and only if $X = (\mathbb{P}^{r_X-1})^{\rho_X}$.*

The first step towards the conjecture was made in 1990 by Wiśniewski; in [12], where the notion of pseudoindex was introduced, he proved that if $i_X > (\dim X + 2)/2$ then $\rho_X = 1$; moreover, if $r_X = (\dim X + 2)/2$ then either $\rho_X = 1$ or $X = (\mathbb{P}^{r_X-1})^2$.
The problem was reconsidered in 2002 by Bonavero, Casagrande, Debarre and Druel; in [2] they proposed the following more general conjecture:

Conjecture 1.2. *Let X be a Fano manifold of dimension n. Then $\rho_X(i_X - 1) \leq n$, with equality if and only if $X = (\mathbb{P}^{i_X-1})^{\rho_X}$.*

In [2] Conjecture (1.2) was proved for Fano manifolds of dimension four (in lower dimension the result can be read off from the classification), for homogeneous manifolds, and for toric Fano manifolds of pseudoindex $i_X \geq (\dim X + 3)/3$ or dimension ≤ 7. The toric case was completely settled later by Casagrande in [5].

As to the general case, in 2004, Andreatta, Chierici and Occhetta in [1] proved Conjecture (1.2) for Fano manifolds of dimension five and for Fano manifolds of pseudoindex $i_X \geq (\dim X + 3)/3$ admitting a special covering family of rational curves (an unsplit family, see Definition (2.1)). They also found some sufficient condition for the existence of such a family.

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In this paper we reconsider the results of [1], and we are able to remove the extra assumption on the existence of the special family, proving that Conjecture (1.2) holds for Fano manifolds X of pseudoindex $i_X \geq (\dim X + 3)/3$; this is done in section (4).
In the last section of the paper we also provide a considerably shorter and simplified proof of Conjecture (1.2) for Fano manifolds of dimension 4 and 5.

The key result is Theorem (4.2), which is based on an extension of classical estimates of the dimension of the locus of irreducible curves of a family through a point to the locus of limits of this curves passing through a point (Proposition (3.4)).
In order to prove this result we need to recall the construction of the scheme $\mathrm{Chain}(\mathcal{U})$, associated to a proper covering family $\mathcal{V}$ of cycles. This is the content of section (3), while section (2) contains the basic definitions about families of rational curves and their properties which are of frequent use in the paper.

2. Families of rational curves

Definition 2.1. A *family of rational curves* V on X is an irreducible component of the scheme $\mathrm{Ratcurves}^n(X)$ (see [6, Definition II.2.11]).
Given a rational curve we will call a *family of deformations* of that curve any irreducible component of $\mathrm{Ratcurves}^n(X)$ containing the point parametrizing that curve.
We define $\mathrm{Locus}(V)$ to be the set of points of X through which there is a curve among those parametrized by V; we say that V is a *covering family* if $\mathrm{Locus}(V) = X$ and that V is a *dominating family* if $\overline{\mathrm{Locus}(V)} = X$.
By abuse of notation, given a line bundle $L \in \mathrm{Pic}(X)$, we will denote by $L \cdot V$ the intersection number $L \cdot C$, with C any curve among those parametrized by V.
We will say that V is *unsplit* if it is proper; clearly, an unsplit dominating family is covering.
We denote by V_x the subscheme of V parametrizing rational curves passing through a point x and by $\mathrm{Locus}(V_x)$ the set of points of X through which there is a curve among those parametrized by V_x. If, for a general point $x \in \mathrm{Locus}(V)$, V_x is proper, then we will say that the family is *locally unsplit*; by Mori's Bend and Break arguments, if V is a locally unsplit family, then $-K_X \cdot V \leq \dim X + 1$.
If X admits dominating families, we can choose among them one with minimal degree with respect to a fixed ample line bundle, and we call it a *minimal dominating family*; such a family is locally unsplit.

Definition 2.2. Let U be an open dense subset of X and $\pi\colon U \to Z$ a proper surjective morphism to a quasi-projective variety; we say that a family of rational curves V is a *horizontal dominating family with respect to* π if $\mathrm{Locus}(V)$ dominates Z and curves parametrized by V are not contracted by π. If such families exist, we can choose among them one with minimal degree with respect to a fixed ample line bundle and we call it a *minimal horizontal dominating family* with respect to π; such a family is locally unsplit.

Remark 2.3. By fundamental results in [8], a Fano manifold admits dominating families of rational curves; also horizontal dominating families with respect to proper morphisms defined on an open set exist, as proved in [7]. In the case of Fano manifolds with "minimal" we will mean minimal with respect to $-K_X$, unless otherwise stated.

Definition 2.4. We define a *Chow family of rational 1-cycles* $\mathcal{W}$ to be an irreducible component of $\mathrm{Chow}(X)$ parametrizing rational and connected 1-cycles.
We define $\mathrm{Locus}(\mathcal{W})$ to be the set of points of X through which there is a cycle among those parametrized by $\mathcal{W}$; notice that $\mathrm{Locus}(\mathcal{W})$ is a closed subset of X ([6, II.2.3]). We say that $\mathcal{W}$ is a *covering family* if $\mathrm{Locus}(\mathcal{W}) = X$.
If V is a family of rational curves, the closure of the image of V in $\mathrm{Chow}(X)$, denoted by $\mathcal{V}$, is called the *Chow family associated to* V.

Remark 2.5. If V is proper, *i.e.* if the family is unsplit, then V corresponds to the normalization of the associated Chow family $\mathcal{V}$.

Definition 2.6. Let V be a family of rational curves and let $\mathcal{V}$ be the associated Chow family. We say that V (and also $\mathcal{V}$) is *quasi-unsplit* if every component of any reducible cycle parametrized by $\mathcal{V}$ has numerical class proportional to the numerical class of a curve parametrized by V.

Definition 2.7. Let $V^1, \dots, V^k$ be families of rational curves on X and $Y \subset X$.
We define $\mathrm{Locus}(V^1)_Y$ to be the set of points $x \in X$ such that there exists a curve C among those parametrized by V^1 with $C \cap Y \neq \emptyset$ and $x \in C$. We inductively define $\mathrm{Locus}(V^1, \dots, V^k)_Y := \mathrm{Locus}(V^k)_{\mathrm{Locus}(V^1,\dots,V^{k-1})_Y}$. Notice that, by this definition, we have $\mathrm{Locus}(V)_x = \mathrm{Locus}(V_x)$. Analogously we define $\mathrm{Locus}(\mathcal{W}^1, \dots, \mathcal{W}^k)_Y$ for Chow families $\mathcal{W}^1, \dots, \mathcal{W}^k$ of rational 1-cycles.

Notation: If Γ is a 1-cycle, then we will denote by $[\Gamma]$ its numerical equivalence class in $\mathrm{N}_1(X)$; if V is a family of rational curves, we will denote by $[V]$ the numerical equivalence class of any curve among those parametrized by V.
If $Y \subset X$, we will denote by $\mathrm{N}_1(Y, X) \subseteq \mathrm{N}_1(X)$ the vector subspace generated by numerical classes of curves of X contained in Y; moreover, we will denote by $\mathrm{NE}\,(Y, X) \subseteq \mathrm{NE}(X)$ the subcone generated by numerical classes of curves of X contained in Y. We will denote by $\langle \dots \rangle$ the linear span.

We will make frequent use of the following dimensional estimates:

Proposition 2.8. *(*[6, IV.2.6]*) Let V be a family of rational curves on X and $x \in \mathrm{Locus}(V)$ a point such that every component of V_x is proper. Then*

(a) $\dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) \geq \dim X - K_X \cdot V - 1$*;*
(b) *every irreducible component of* $\mathrm{Locus}(V_x)$ *has dimension* $\geq -K_X \cdot V - 1$.

Definition 2.9. We say that k quasi-unsplit families $V^1, \dots, V^k$ are numerically independent if in $\mathrm{N}_1(X)$ we have $\dim\langle [V^1], \dots, [V^k] \rangle = k$.

Lemma 2.10. *(Cf.* [1, Lemma 5.4]*) Let $Y \subset X$ be an irreducible closed subset and $V^1, \dots, V^k$ numerically independent unsplit families of rational curves such that $\langle [V^1], \dots, [V^k] \rangle \cap NE(Y, X) = \underline{0}$. Then either* $\mathrm{Locus}(V^1, \dots, V^k)_Y = \emptyset$ *or*

$$\dim \mathrm{Locus}(V^1, \dots, V^k)_Y \geq \dim Y + \sum -K_X \cdot V^i - k.$$

Remark 2.11. As pointed out by the referee, we need to assume the irreducibility of Y; otherwise in the statement we have to replace $\dim Y$ with $\dim Y_0$ where Y_0 is an irreducible component of Y of minimal dimension.

A key fact underlying our strategy to obtain bounds on the Picard number, based on [6, Proposition II.4.19], is the following:

Lemma 2.12. *([1, Lemma 4.1]) Let $Y \subset X$ be a closed subset, $\mathcal{V}$ a Chow family of rational 1-cycles. Then every curve contained in $\mathrm{Locus}(\mathcal{V})_Y$ is numerically equivalent to a linear combination with rational coefficients of a curve contained in Y and of irreducible components of cycles parametrized by $\mathcal{V}$ which meet Y.*

Corollary 2.13. *Let V^1 be a locally unsplit family of rational curves, and $V^2, \ldots, V^k$ unsplit families of rational curves. Then, for a general $x \in \mathrm{Locus}(V^1)$,*

(a) $\mathrm{N}_1(\mathrm{Locus}(V^1)_x, X) = \langle [V^1] \rangle$;
(b) *either* $\mathrm{Locus}(V^1, \ldots, V^k)_x = \emptyset$ *or* $\mathrm{N}_1(\mathrm{Locus}(V^1, \ldots, V^k)_x, X) = \langle [V^1], \ldots, [V^k] \rangle$.

3. Chains of rational curves

Let X be a smooth complex projective variety. Let V be a dominating family of rational curves on X and denote by $\mathcal{V}$ the associated Chow family, with universal family $\mathcal{U}$:

$$
\begin{tikzcd}
\mathcal{U} \arrow[r, "q"] \arrow[d, "p"'] & X \\
\mathcal{V} &
\end{tikzcd}
$$

Definition 3.1. Let $Y \subset X$ be a closed subset; define $\mathrm{ChLocus}_m(\mathcal{V})_Y$ to be the set of points $x \in X$ such that there exist cycles $\Gamma_1, \ldots, \Gamma_m$ with the following properties:

- Γ_i belongs to the family $\mathcal{V}$;
- $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$;
- $\Gamma_1 \cap Y \neq \emptyset$ and $x \in \Gamma_m$,

i.e. $\mathrm{ChLocus}_m(\mathcal{V})_Y = \mathrm{Locus}(\mathcal{V}, \ldots, \mathcal{V})_Y$, with $\mathcal{V}$ appearing m times, is the set of points that can be joined to Y by a connected chain of at most m cycles belonging to the family $\mathcal{V}$.
Considering among cycles parametrized by $\mathcal{V}$ only irreducible ones, in the same way one can define $\mathrm{ChLocus}_m(V)_Y$.

Define a relation of *rational connectedness with respect to* $\mathcal{V}$ on X in the following way: two points x and y of X are in $\mathrm{rc}(\mathcal{V})$-relation if there exists a chain of cycles in $\mathcal{V}$ which joins x and y, *i.e.* if $y \in \mathrm{ChLocus}_m(\mathcal{V})_x$ for some m. In particular, X is *$rc(\mathcal{V})$-connected* if for some m we have $X = \mathrm{ChLocus}_m(\mathcal{V})_x$.

The family $\mathcal{V}$ defines a proper prerelation in the sense of [6, Definition IV.4.6]; to this prerelation it is associated a proper proalgebraic relation $\mathrm{Chain}(\mathcal{U})$ (see [6, Theorem IV.4.8]) and the $\mathrm{rc}(\mathcal{V})$-relation just defined is nothing but the set theoretic relation $\langle \mathcal{U} \rangle$ associated to $\mathrm{Chain}(\mathcal{U})$. We briefly recall this construction for the reader's convenience. See [6, IV.4] or [4, Appendix] for details.
Define $\mathrm{Chain}_1(\mathcal{V})$ to be the fiber product $\mathcal{U} \times_{\mathcal{V}} \mathcal{U}$, with projections q_1 and q_2 on X, which give rise to a morphism $q_1 \times q_2 \colon \mathrm{Chain}_1(\mathcal{V}) \longrightarrow X \times X$.
Denoting by $\pi_i \colon (\mathrm{Chain}_1(\mathcal{V}))^N \to \mathrm{Chain}_1(\mathcal{V})$ the projection onto the i-th factor and by $q_{1,i}$ (respectively $q_{2,i}$) the composition of q_1 (respectively q_2) with π_i, inductively

define $\mathrm{Chain}_{m+1}(\mathcal{V}) := \mathrm{Chain}_1(\mathcal{V}) \times_X \mathrm{Chain}_m(\mathcal{V})$, as in the following diagram

$$\begin{tikzcd} \mathrm{Chain}_{m+1}(\mathcal{V}) \arrow[r] \arrow[d] & \mathrm{Chain}_m(\mathcal{V}) \arrow[d, "q_{1,1}"] \\ \mathrm{Chain}_1(\mathcal{V}) \arrow[r, "q_2"'] & X \end{tikzcd}$$

Finally set $\mathrm{Chain}(\mathcal{U}) := \bigcup_m \mathrm{Chain}_m(\mathcal{U})$.

With this language x and y are $\mathrm{rc}(\mathcal{V})$-equivalent if, for some m, the point (x, y) is in the image of $q_{1,1} \times q_{2,m}\colon \mathrm{Chain}_m(\mathcal{V}) \longrightarrow X \times X$. The variety X is then $\mathrm{rc}(\mathcal{V})$-connected if for some m the morphism $q_{1,1} \times q_{2,m}\colon \mathrm{Chain}_m(\mathcal{V}) \longrightarrow X \times X$ is dominant (hence onto, by the properness of $\mathcal{V}$).

To the proper prerelation defined by $\mathcal{V}$ it is associated a fibration, which we will call the *rc($\mathcal{V}$)-fibration*:

Theorem 3.2. *(*[6, IV.4.16]*, Cf.* [3]*) Let X be a normal and proper variety and $\mathcal{V}$ a proper prerelation; then there exists an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi\colon X^0 \to Z^0$ such that*

- *$\langle \mathcal{U} \rangle$ restricts to an equivalence relation on X^0;*
- *$\pi^{-1}(z)$ is a $\langle \mathcal{U} \rangle$-equivalence class for every $z \in Z^0$;*
- *$\forall\, z \in Z^0$ and $\forall\, x, y \in \pi^{-1}(z)$, $x \in \mathrm{ChLocus}_m(\mathcal{V})_y$ with $m \leq 2^{\dim X - \dim Z} - 1$.*

Clearly X is $\mathrm{rc}(\mathcal{V})$-connected if and only if $\dim Z^0 = 0$.

Proposition 3.3. *Let V be a minimal dominating family of rational curves and denote by $\mathcal{V}$ the associated Chow family. Assume that $\dim \mathrm{Locus}(V)_x \geq s$, for a general $x \in X$ and some integer s; then for every $x \in X$ every irreducible component of $\mathrm{Locus}(\mathcal{V})_x$ has dimension $\geq s$.*

Proof. Consider the morphism $q_1 \times q_2\colon \mathrm{Chain}_1(\mathcal{V}) \longrightarrow X \times X$; by [3, Lemme 2] (or [4, Lemma 1.14]) we know that $\mathrm{Chain}_1(\mathcal{V})$ is irreducible. Denote by $\mathcal{C}^1$ the image $(q_1 \times q_2)(\mathrm{Chain}_1(\mathcal{V})) \subset X \times X$.
Let $p : \mathcal{C}^1 \to X$ be the restriction of the first projection; the inverse image of a point x_0 via p consists of the points which belong to a cycle in $\mathcal{V}$ containing x_0, hence $p^{-1}(x_0) = \mathrm{Locus}(\mathcal{V})_{x_0}$. By the minimality assumption, through a general point $x \in X$ there are no reducible cycles, hence $\dim p^{-1}(x) \geq s$. The statement now follows by the semicontinuity of the local dimension of a fiber ([10, Corollary 3, pag. 51]), which ensures that the dimension of every irreducible component of every fiber of p has dimension $\geq s$. $\square$

Corollary 3.4. *Let V be a minimal dominating family of rational curves and denote by $\mathcal{V}$ the associated Chow family. Then every irreducible component of $\mathrm{Locus}(\mathcal{V})_x$ has dimension $\geq -K_X \cdot V - 1$.*

Given $\mathcal{V}^1, \ldots, \mathcal{V}^k$ Chow families of rational 1-cycles, it is possible to define, as above, a relation of $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-connectedness, to which it is associated a fibration, which we will call $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-fibration. The variety X will be called *rc($\mathcal{V}^1, \ldots, \mathcal{V}^k$)-connected* if the target of the fibration is a point.

For such varieties we have the following application of Lemma (2.12):

Proposition 3.5. *(Cf.* [1, Corollary 4.4]*) If X is rationally connected with respect to some Chow families of rational 1-cycles $\mathcal{V}^1, \dots, \mathcal{V}^k$, then $\mathrm{N}_1(X)$ is generated by the classes of irreducible components of cycles in $\mathcal{V}^1, \dots, \mathcal{V}^k$.*
In particular, if $\mathcal{V}^1, \dots, \mathcal{V}^k$ are quasi-unsplit families, then $\rho_X \le k$ and equality holds if and only if $\mathcal{V}^1, \dots, \mathcal{V}^k$ are numerically independent.

A straightforward consequence of the above proposition is the following:

Corollary 3.6. *If X is rationally connected with respect to Chow families of rational 1-cycles $\mathcal{V}^1, \dots, \mathcal{V}^k$ and D is an effective divisor, then D cannot be trivial on every irreducible component of every cycle parametrized by $\mathcal{V}^1, \dots, \mathcal{V}^k$.*

4. Large pseudoindex

In this section we will prove a bound on the Picard number of Fano manifolds which are rationally connected with respect to a special Chow family. Then we will show that Conjecture (1.2) holds for Fano manifolds X of pseudoindex $i_X \ge (\dim X + 3)/3$.

We start with a technical result:

Lemma 4.1. *Let X be a Fano manifold of pseudoindex i_X, let $Y \subset X$ be a closed irreducible subset of dimension $\dim Y > \dim X - i_X$ and let W be an unsplit non dominating family of rational curves such that $[W] \notin NE(Y, X)$.*
Then $\mathrm{Locus}(W) \cap Y = \emptyset$.

Proof. If the intersection were nonempty, by Lemma (2.10) we would have

$$\dim \mathrm{Locus}(W)_Y \ge \dim Y - K_X \cdot W - 1 > \dim X - 1,$$

so W would be a dominating family, a contradiction. □

Theorem 4.2. *Let X be a Fano manifold of Picard number ρ_X and pseudoindex i_X, and let V be a minimal dominating family of rational curves for X. Assume that X is rc($\mathcal{V}$)-connected and that $3i_X > -K_X \cdot V > \dim X + 1 - i_X$. Then $\rho_X = 1$.*

Proof. Since X is rc($\mathcal{V}$)-connected, for some integer m the morphism $\mathrm{Chain}_m(\mathcal{V}) \longrightarrow X \times X$ is onto; equivalently, $X = \mathrm{ChLocus}_m(\mathcal{V})_x$ for every $x \in X$. Let x be a general point; we will show that every irreducible component of a $\mathcal{V}$-cycle in a connected m-chain passing though x is numerically proportional to V. The statement will then follow by repeated applications of Lemma (2.12).
Since $-K_X \cdot V < 3i_X$, any reducible $\mathcal{V}$-cycle Γ has two irreducible components, hence either both of them are numerically proportional to V or neither of them is numerically proportional to V.
Assume by contradiction that there exist m-chains through x, $\Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_m$, with $x \in \Gamma_1$ and $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$, such that, for some $j \in \{1, \dots, m\}$ the irreducible components Γ_j^1 and Γ_j^2 of Γ_j are not numerically proportional to $\mathcal{V}$.
Let $j_0 \in \{1, \dots, m\}$ be the minimum integer for which such a chain exists; by the generality of x we have $j_0 \ge 2$. If $j_0 = 2$ set $x_1 = x$, otherwise let x_1 be a point in $\Gamma_{j_0-1} \cap \Gamma_{j_0-2}$. Since $\Gamma_{j_0-1} \subset \mathrm{Locus}(\mathcal{V})_{x_1}$ there is an irreducible component Y of $\mathrm{Locus}(V)_{x_1}$ which meets Γ_{j_0}.
By Corollary (3.4) we have $\dim Y \ge -K_X \cdot V - 1 > \dim X - i_X$; moreover, since j_0 was minimal, every cycle parametrized by $\mathcal{V}$ passing through x_1 is numerically proportional to $\mathcal{V}$, hence $\mathrm{N}_1(Y, X) = \langle [V] \rangle$ by Lemma (2.12).

Let γ be a component of Γ_{j_0} meeting Y. Denote by W a family of deformations of γ; then the family W is unsplit, as $-K_X \cdot V < 3i_X$ and it is not dominating, by the minimality of V. We now get the desired contradiction by Lemma (4.1). □

Construction 4.3. Let X be a Fano manifold; let V^1 be a minimal dominating family of rational curves on X and consider the associated Chow family $\mathcal{V}^1$.
If X is not $\mathrm{rc}(\mathcal{V}^1)$-connected, let V^2 be a minimal horizontal dominating family with respect to the $\mathrm{rc}(\mathcal{V}^1)$-fibration, $\pi_1\colon X \dashrightarrow Z^1$. If X is not $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2)$-connected, we denote by V^3 a minimal horizontal dominating family with respect to the the $\mathrm{rc}(\mathcal{V}^1, \mathcal{V}^2)$-fibration, $\pi_2\colon X \dashrightarrow Z^2$, and so on. Since $\dim Z^{i+1} < \dim Z^i$, for some integer k we have that X is $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-connected.
Notice that, by construction, the families $V^1, \ldots, V^k$ are numerically independent.

Lemma 4.4. *Let X be a Fano manifold of pseudoindex $i_X \geq 2$ and let $V^1, \ldots, V^k$ be families of rational curves as in Construction (4.3). Then*

$$\sum_{i=1}^{k}(-K_X \cdot V^i - 1) \leq \dim X.$$

In particular, $k(i_X - 1) \leq \dim X$, and equality holds if and only if $X = (\mathbb{P}^{i_X-1})^k$.

Proof. In Construction (4.3) at the i-th step, denoted by x_i a general point in $\mathrm{Locus}(V^i)$, the dimension of the quotient drops at least by $\dim \mathrm{Locus}(V^i)_{x_i}$, which, by part (b) of Proposition (2.8), is greater than or equal to $-K_X \cdot V^i - 1$. It follows that

$$\dim X \geq \sum_{i=1}^{k} \dim \mathrm{Locus}(V^i)_{x_i} \geq \sum_{i=1}^{k}(-K_X \cdot V^i - 1) \geq k i_X - k = k(i_X - 1).$$

If $\dim X = k(i_X - 1)$, then for any i we have $-K_X \cdot V^i = i_X$, so V^i is an unsplit family and $\dim \mathrm{Locus}(V^i)_{x_i} = i_X - 1$, hence the family V^i is covering by part (a) of Proposition (2.8). We can now apply [11, Theorem 1] to conclude. □

Theorem 4.5. *Let X be a Fano manifold of Picard number ρ_X and pseudoindex $i_X \geq (\dim X + 3)/3$. Then $\rho_X(i_X - 1) \leq \dim X$ and equality holds if and only if $X = (\mathbb{P}^{i_X-1})^{\rho_X}$.*

Proof. Let $V^1, \ldots, V^k$ be families of rational curves such as in Construction (4.3); by Lemma (4.4) we have that $k(i_X - 1) \leq \dim X$. If for some j the family V^j is not unsplit we have $-K_X \cdot V^j \geq 2i_X$, so, again by Lemma (4.4), this can happen for at most one j and implies $k = 1$.

If all the families V^i are unsplit, then we have $\rho_X = k$ by Proposition (3.5). Moreover, if $k(i_X - 1) = \dim X$, by Lemma (4.4) we have $X = (\mathbb{P}^{i_X-1})^{\rho_X}$.

We can thus assume that V^1 is not unsplit and X is $\mathrm{rc}(\mathcal{V}^1)$-connected.
By the minimality of V^1 we have $-K_X \cdot V^1 \leq \dim X + 1 < 3i_X$; on the other hand, since V^1 is not unsplit, we have

$$-K_X \cdot V^1 \geq 2i_X \geq 2\,\frac{\dim X + 3}{3} > 2\,\frac{\dim X}{3} \geq \dim X + 1 - i_X,$$

so we can apply Theorem (4.2) to conclude. □

5. Low dimensions

In this section we will present different proofs of Conjecture (1.2) for Fano manifolds of dimension four and five, which are simpler and shorter than the original ones.

Theorem 5.1. *Let X be a Fano manifold of Picard number ρ_X, pseudoindex i_X and dimension 4. Then $\rho_X(i_X - 1) \leq 4$. Moreover, equality holds if and only if $X = \mathbb{P}^4$, or $X = \mathbb{P}^2 \times \mathbb{P}^2$, or $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Clearly we can assume $i_X \geq 2$. Let $V^1, \dots, V^k$ be families of rational curves as in Construction (4.3); by Lemma (4.4) we get $k(i_X - 1) \leq 4$, hence $k \leq 4$.
If for some j the family V^j is not unsplit, then $-K_X \cdot V^j \geq 2i_X \geq 4$, hence, by Lemma (4.4), this can happen for at most one j and implies $k \leq 2$ and $i_X = 2$.

If all the families V^i are unsplit, then $\rho_X = k$ by Proposition (3.5). Moreover, if $k(i_X - 1) = 4$, we have $X = (\mathbb{P}^{i_X-1})^{\rho_X}$ by Lemma (4.4).

We can thus assume that one of these families, say V^j, is not unsplit. By part (a) of Corollary (2.13), we have $\mathrm{N}_1(\mathrm{Locus}(V^j)_{x_j}, X) = \langle [V^j] \rangle$ for a general point $x_j \in \mathrm{Locus}(V^j)$.

If $j = 2$, then, for a general point $x_2 \in \mathrm{Locus}(V^2)$, we have $X = \mathrm{Locus}(V^2, V^1)_{x_2}$ by Lemma (2.10). Therefore, by part (b) of Corollary (2.13), we obtain that $\mathrm{N}_1(X) = \langle [V^1], [V^2] \rangle$, so $\rho_X = 2$.

Assume now that $j = 1$, *i.e.* V^1 is not unsplit. We claim that X is $\mathrm{rc}(\mathcal{V}^1)$-connected.
Notice that, by the minimality of V^1, we can assume that X has no dominating families of rational curves of anticanonical degree $< 2i_X = 4$. If X is not $\mathrm{rc}(\mathcal{V}^1)$-connected, since a general fiber of π_1 contains $\mathrm{Locus}(V^1)_{x_1}$ which, by part (b) of Proposition (2.8), has dimension at least three, then $\dim Z^1 = 1$. By part (b) of Proposition (2.8), for a general point $x_2 \in \mathrm{Locus}(V^2)$, we get

$$2 \leq -K_X \cdot V^2 \leq \dim \mathrm{Locus}(V^2)_{x_2} + 1 \leq \dim Z^1 + 1 = 2,$$

hence V^2 has anticanonical degree 2, and so it is dominating by part (a) of the same proposition, a contradiction which proves the claim.

Therefore X is $\mathrm{rc}(\mathcal{V}^1)$-connected and we can apply Theorem (4.2) to get $\rho_X = 1$. □

Theorem 5.2. *Let X be a Fano manifold of Picard number ρ_X, pseudoindex i_X and dimension 5. Then $\rho_X(i_X - 1) \leq 5$. Moreover, equality holds if and only if either $X = \mathbb{P}^5$ or $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Clearly we can assume $i_X \geq 2$. Let $V^1, \dots, V^k$ be families of rational curves as in Construction (4.3); by Lemma (4.4) we get $k(i_X - 1) \leq 5$, hence $k \leq 5$.
If V^j is not unsplit for some j, then $-K_X \cdot V^j \geq 2i_X \geq 4$, hence, by Lemma (4.4) this can happen for at most one j and implies $k \leq 3$. Notice that, if $-K_X \cdot V^j \geq \dim X + 1$, then $-K_X \cdot V^j = \dim X + 1$ and $k = j = 1$; moreover, if $j \neq 1$ then $i_X = 2$.

If all the families V^i are unsplit, then $\rho_X = k$ by Proposition (3.5). Moreover, if $k(i_X - 1) = 5$, we have $X = (\mathbb{P}^{i_X-1})^{\rho_X}$ by Lemma (4.4).

We can thus assume that one of these families, say V^j, is not unsplit. By part (a) of Corollary (2.13), we have $\mathrm{N}_1(\mathrm{Locus}(V^j)_{x_j}, X) = \langle [V^j] \rangle$ for a general point $x_j \in \mathrm{Locus}(V^j)$.

If $j = 3$, then, for a general point $x_3 \in \mathrm{Locus}(V^3)$, we have $X = \mathrm{Locus}(V^3, V^2, V^1)_{x_3}$ by Lemma (2.10). Therefore, by part (b) of Corollary (2.13), we obtain that $\mathrm{N}_1(X) = \langle [V^1], [V^2], [V^3] \rangle$, so $\rho_X = 3$.

Assume now that $j = 2$. We claim that X is $\mathrm{rc}(V^1, \mathcal{V}^2)$-connected.
In fact, by part (b) of Proposition (2.8), we have $\dim \mathrm{Locus}(V^2)_{x_2} \geq 3$ for a general point $x_2 \in \mathrm{Locus}(V^2)$; therefore a general fiber of the $\mathrm{rc}(V^1, \mathcal{V}^2)$-fibration has dimension at least $\dim \mathrm{Locus}(V^2, V^1)_{x_2}$, which is at least four by Lemma (2.10). This implies $\dim Z^2 \leq 1$, and thus, if X were not $\mathrm{rc}(V^1, \mathcal{V}^2)$-connected, we would have $\dim \mathrm{Locus}(V^3)_{x_3} = 1$ for a general point $x_3 \in \mathrm{Locus}(V^3)$. Hence, by part (b) of Proposition (2.8), $-K_X \cdot V^3 = 2 = i_X$, so V^3 would be unsplit and, by part (a) of the same proposition, covering, against the minimality of V^2.

Consider an irreducible component of $\mathrm{Locus}(V^2, V^1)_{x_2}$ of maximal dimension. By Lemma (2.10) this dimension is ≥ 4, hence either $X = \mathrm{Locus}(V^2, V^1)_{x_2}$ and $\rho_X = 2$ by part (b) of Corollary (2.13), or this component is a divisor, call it D.
If $D \cdot V^1 > 0$ then, being V^1 covering, we have $X = \mathrm{Locus}(V^1)_D$, and $\rho_X = 2$ by Lemma (2.12) and part (b) of Corollary (2.13).
Assume now that $D \cdot V^1 = 0$. By Lemma (4.1), D cannot meet components of reducible cycles of $\mathcal{V}^2$ whose classes are not contained in the plane spanned by $[V^1]$ and $[V^2]$ in $N_1(X)$. So, if there were such a reducible cycle $\Gamma = \Gamma_1 + \Gamma_2$, we would have $D \cdot \Gamma_i = 0$, hence also $D \cdot V^2 = 0$, and D would be trivial on every component of every cycle in V^1 and $\mathcal{V}^2$, against Corollary (3.6).
It follows that the class of every reducible cycle of $\mathcal{V}^2$ is contained in the plane spanned by $[V^1]$ and $[V^2]$ and $\rho_X = 2$ by Proposition (3.5).

Finally assume that $j = 1$, *i.e.* V^1 is not unsplit. Notice that, by the minimality of V^1, we can assume that X has no dominating families of rational curves of anticanonical degree $< 2i_X$.

If X is not $\mathrm{rc}(\mathcal{V}^1)$-connected, since a general fiber of π_1 contains $\mathrm{Locus}(V^1)_{x_1}$ which, by part (b) of Proposition (2.8), has dimension at least three, then $\dim Z^1 \leq 2$. It follows that, by part (b) of Proposition (2.8),

$$-K_X \cdot V^2 \leq \dim \mathrm{Locus}(V^2)_{x_2} + 1 \leq \dim Z^1 + 1 \leq 3,$$

for a general point $x_2 \in \mathrm{Locus}(V^2)$; hence V^2 has anticanonical degree $< 2i_X$, so it can not be dominating. This also implies $\dim Z^1 = 2$.

For a general point $x_1 \in \mathrm{Locus}(V^1)$, we get $\dim \mathrm{Locus}(V^1, V^2)_{x_1} \geq 4$ by Lemma (2.10). Since $D := \mathrm{Locus}(V^1, V^2)_{x_1}$ is contained in $\mathrm{Locus}(V^2)$ and V^2 is not dominating we have $D = \mathrm{Locus}(V^2)$. In particular D dominates Z^1 and so meets a general fiber of π_1, which is $\mathrm{Locus}(V^1)_{x_1}$, for some x_1, by dimensional reasons. It follows that $X = \mathrm{Locus}(\mathcal{V}^1)_D$.
By part (b) of Corollary (2.13), we have $\mathrm{N}_1(D, X) = \langle [V^1], [V^2] \rangle$; hence, by Lemma (4.1), D cannot meet reducible cycles of $\mathcal{V}^1$ whose classes are not contained in the plane spanned by $[V^1]$ and $[V^2]$. Therefore $\rho_X = 2$ by Lemma (2.12).

Finally we deal with the case in which X is $\mathrm{rc}(\mathcal{V}^1)$-connected; let x be a general point. Since x is general and V^1 is minimal we have $\overline{\mathrm{Locus}(V^1)_x} = \mathrm{Locus}(V^1)_x$ and $\mathrm{N}_1(\mathrm{Locus}(V^1)_x, X) = \langle [V^1] \rangle$ by part (a) of Corollary (2.13).

If $\mathrm{Locus}(V^1)_x = X$, then $\rho_X = 1$. So we can suppose that $\dim \mathrm{Locus}(V^1)_x < 5$, and thus, by part (b) of Proposition (2.8), $-K_X \cdot V^1 < 3i_X$ and $i_X = 2$; in particular every reducible cycle parametrized by $\mathcal{V}^1$ has two irreducible components.

If every irreducible component of a $\mathcal{V}^1$-cycle in a connected m-chain though x is numerically proportional to V^1 then $\rho_X = 1$ by repeated applications of Lemma (2.12).

We can thus assume that there exist m-chains through x, $\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m$, with $x \in \Gamma_1$ and $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$, such that, for some $j \in \{1, \dots, m\}$ the irreducible components Γ_j^1 and Γ_j^2 of Γ_j are not numerically proportional to $\mathcal{V}^1$.

Let $j_0 \in \{1, \dots, m\}$ be the minimum integer for which such a chain exists; by the generality of x we have $j_0 \geq 2$. If $j_0 = 2$ set $x_1 = x$, otherwise let x_1 be a point in $\Gamma_{j_0-1} \cap \Gamma_{j_0-2}$. Since $\Gamma_{j_0-1} \subset \mathrm{Locus}(\mathcal{V}^1)_{x_1}$ there is an irreducible component Y of $\mathrm{Locus}(V^1)_{x_1}$ which meets Γ_{j_0}. By Corollary (3.4) we have $\dim Y \geq -K_X \cdot V^1 - 1 \geq 3$, and, by Lemma (2.12), $\mathrm{N}_1(Y, X) = \langle [V^1] \rangle$.

Let γ be a component of Γ_{j_0} meeting Y and denote by W a family of deformations of γ; then the family W is unsplit, as $-K_X \cdot V^1 < 3i_X$, and it is not covering, by the minimality of V^1.

By Lemma (2.10) we have $\dim \mathrm{Locus}(W)_Y \geq 4$, hence $\mathrm{Locus}(W) = \mathrm{Locus}(W)_Y$; by part (b) of Corollary (2.13) we get $\mathrm{N}_1(\mathrm{Locus}(W)_Y, X) = \langle [V^1], [W] \rangle$.

Denote by G the divisor $\mathrm{Locus}(W)$; since G meets Y and does not contain it, being x general, we have $G \cdot V^1 > 0$; therefore $X = \mathrm{Locus}(\mathcal{V}^1)_G$. Since $G \cdot V^1 > 0$ we also have that, if $\Gamma_1 + \Gamma_2$ is a reducible cycle parametrized by $\mathcal{V}^1$, then $G \cdot \Gamma_i > 0$ for at least one $i = 1, 2$.

On the other hand, in view of Lemma (4.1), G must be trivial on every irreducible component of a cycle in $\mathcal{V}^1$ not contained in the plane spanned by $[V^1]$ and $[W]$. Therefore there are no such cycles, and $\rho_X = 2$ by Proposition (3.5). □

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