Connections between the geometry of a projective variety and of an ample section

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(Article begins on next page)

Connections between the geometry of a projective variety and of an ample section.

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Abstract

Let X be a smooth complex projective variety and let $Z = (s = 0)$ be a smooth submanifold which is the zero locus of a section of an ample vector bundle $\mathcal{E}$ of rank r with $\dim Z = \dim X - r$.
We show with some examples that in general the Kleiman-Mori cones $\overline{NE(Z)}$ and $\overline{NE(X)}$ are different. We then give a necessary and sufficient condition for an extremal ray in $\overline{NE(X)}$ to be also extremal in $\overline{NE(Z)}$.
We apply this result to the case $r = 1$ and Z a Fano manifold of high index; in particular we classify all X with an ample divisor which is a Mukai manifold of dimension ≥ 4.
In the last section we prove a general result in case Z is a minimal variety with $0 \leq \kappa(Z) < \dim Z$.

1 Introduction

Let X be a smooth complex projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank r on X such that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus, $Z = (s = 0)$, is a smooth submanifold of the expected dimension $\dim Z = \dim X - r = n - r$.
If $\dim Z \geq 3$, by Sommese's version of Weak Lefschetz theorem, the natural inclusion $N_1(Z) \to N_1(X)$ and the natural restriction map $N^1(X) \to N^1(Z)$ are isomorphisms, while if $\dim Z = 2$ the map $N_1(Z) \to N_1(X)$ is surjective, and the map $N^1(X) \to N^1(Z)$ is injective.
Denote by $\overline{NE(X)} \subset N_1(X)$ the *Kleiman–Mori cone* of X, that is the closure of the cone of effective 1-cycles and assume by simplicity that $\dim Z \geq 3$ (but we will consider also the case $\dim Z = 2$); in this case we can view $\overline{NE(Z)}$ as a subset of $\overline{NE(X)}$.
In the papers [2] and [3] we studied the relation between the cones $\overline{NE(Z)}$ and $\overline{NE(X)}$; the present paper adds some new results and some applications. In the spirit of Mori philosophy this is a first step to compare the geometric structure of a variety with the one of its ample sections.
In section two we recall an example given in [2] and we present a new one in which the two cones are different; in both of them $\mathcal{E}$ is very ample, and in the first Z is a Fano variety, so also these conditions are not sufficient to give $\overline{NE(Z)} = \overline{NE(X)}$.

The problem then is to find any other conditions under which $\overline{NE(Z)} = \overline{NE(X)}$, or under which at least parts of the cones are the same, in particular the parts of the cone negative with respect to the intersection with the canonical bundles, K_X and K_Z, which are known to be polyhedral by Mori theory.
In section three we gather some known results in this direction. In particular we give a necessary and sufficient condition for an extremal ray of X to be an extremal ray of Z too.
In section four we restrict to the case $r = 1$, i.e. $\mathcal{E}$ is an ample line bundle; in this case we prefer even to change notation and therefore we will denote by L the ample line bundle $\mathcal{E}$.
We study the case in which $Z \in |L|$ is a Fano manifold of high index; namely $-K_Z = rH_Z$ with H_Z ample on Z and r an integer, called the index, such that $r \geq \frac{\dim Z}{2}$. In this set up, if H_Z is spanned by global sections, we prove that $\overline{NE(Z)} = \overline{NE(X)}$, apart from the case in which $Z \simeq \mathbb{P}^1 \times \mathbb{P}^3$, which gives rise to one of the example in section two. Then we classify all the pairs (X, L) in which Z is a Mukai variety (i.e. a Fano manifold of index $\dim Z - 2$) of dimension ≥ 4. This last problem was first studied in [6], here we improve significantly their classification.
In the last section we study the case in which Z does not have negative extremal rays, but the canonical bundle of Z is not ample. In this case we prove that K_Z is semiample and X admits a fibration in Fano varieties which extends the pluricanonical map of Z. If moreover $r = 1$ the effective cones of the general fibers of X and Z coincide, so this can be viewed as a relative version of the results in section three. Finally we apply our results to the case in which Z is a surface with Kodaira dimension 0 or 1, not necessarily minimal, giving a different proof of some of the results obtained in [15] and [16].
We use frequently some basic notations and theorems of the so called Mori theory.
After this paper was written, we found out that some results and examples similar to the ones in section two and three were recently obtained in [12].

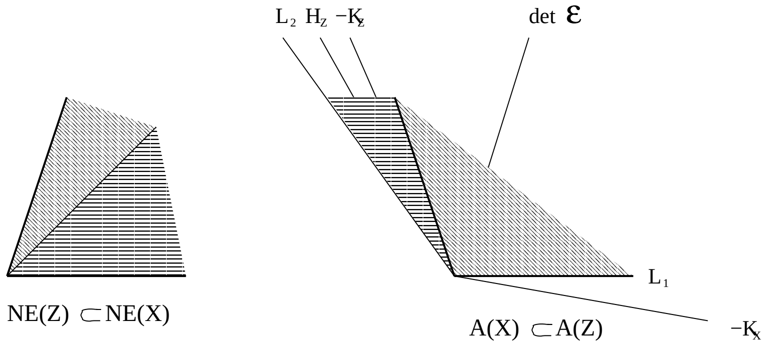
2 Examples

As we said in the introduction, it is not always true that $NE(X) = NE(Z)$. In [2] and [3] we gave some examples where the inclusion $NE(X) \supset NE(Z)$ is strict.
Let us recall here the example 4.10 in [2], which in turn generalizes an example of L. Bădescu (see also [17, Example 4.2]): consider the sequence

$$0 \longrightarrow \oplus^n \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{G} := \oplus^n (\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(s-a)) \longrightarrow \oplus^n \mathcal{O}_{\mathbb{P}^1}(s) \longrightarrow 0$$

which is exact in view of [4, Remark 1, p.170] and choose a, s in such a way that $0 < a - s < a$.
The construction in [11, B.5.6] applies and gives $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ as the zero set of a general section of the ample vector bundle $\mathcal{E} = \oplus^n \xi_{\mathcal{G}}$ on $X = \mathbb{P}(\mathcal{G})$.

Note that $\mathcal{E}$ is actually very ample and that for $n = 1$ it is a line bundle. The cones are described in the following picture



In particular there is a ray in common (the one associated to the curve contracted by the projections $p_1 : Z \to \mathbb{P}^1$).
In general the other rays are different, that is the contraction to the second factor of Z, $p_2 : Z \to \mathbb{P}^{n-1}$, cannot be extended to X.
Let L_i be the pull back through the projection p_i of the hyperplane bundle; the contraction p_2 is supported by $K_Z + H_Z = bL_2 \quad (b > 0)$ and $H_Z = 2L_1 + (n - r + b)L_2$ is an ample line bundle on Z which is not the restriction of an ample line bundle on X.

Another example was given in [3, section 4], producing $Z = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(1) \oplus \mathcal{O})$ as a section of an ample vector bundle on $X = \mathbb{P}^k \times \mathbb{P}^2 \quad (k \geq 3)$.
In this case the ray corresponding to the projection on $\mathbb{P}^2$ is common to $NE(Z)$ and $NE(X)$, while the ray in $NE(Z)$ corresponding to the blow-down on $\mathbb{P}^3$ lies in the interior of $NE(X)$.

A third example, suggested by Massimiliano Mella, is the following: let X be the blow up of the product $\mathbb{P}^r \times \mathbb{P}^1$ at a point x.
The cone of curves of X has three rays: $NE(X) =< s, f, e >$ where s is the class of the strict trasform of $\{x\} \times \mathbb{P}^1$, f is the class of the strict transform of a line through x in the $\mathbb{P}^r$ which contains x and e is the class of a line in the exceptional divisor.
Let A be a general section of a very ample line bundle. If A has sufficiently high degree then it does not contain any effective curve whose numerical class is s or f.
The first assertion is clear, since there is only a curve in the numerical class of s; to see the second consider the fiber of $p : X \to \mathbb{P}^1$ which contains the exceptional divisor. This fiber is reducible and consists of two components, namely the exceptional divisor, which is $\mathbb{P}^r$, and a $\mathbb{P}^1$-bundle over $\mathbb{P}^{r-1}$.
Any curve in X whose numerical class is f has to be a fiber of this last $\mathbb{P}^1$-bundle. But any section of a sufficiently high degree very ample line bundle does not contain such curves.

So $NE(A)$ is a subcone of $NE(X)$ and there is only one ray in common, the one generated by e.

3 Comparing the cones

We begin with a result which in some cases solves the problem of determining whether an extremal ray of X is an extremal ray of Z.

Theorem 3.1. *If $R_X = \mathbb{R}_+[C]$ is an extremal ray of X such that $-(K_X + \det \mathcal{E}) \cdot C > 0$, then there is a curve on Z whose numerical class is $[\lambda C]$.*
In particular, if $N_1(X) = N_1(Z)$ (for instance if $\dim Z \geq 3$) then an extremal ray of X is also an extremal ray of Z if and only if it stays in the semi-space defined by $\{x \in N_1(X) : -(K_X + \det \mathcal{E}) \cdot x > 0\}$.
In particular, if Z is not minimal there exists at least one extremal ray which is in common between $\overline{NE(X)}$ and $\overline{NE(Z)}$.

In the case of ample divisor we have a slightly better result:

Theorem 3.2. *Assume that $\dim Z \geq 3$ and that $r = 1$, i.e. that Z is a section of an ample line bundle L. Then the extremal rays of X which stay in the closed semi-space defined by $\{x \in N_1(X) : -(K_X + L) \cdot x \geq 0\}$ are in the boundary of $\overline{NE(Z)}$ as well.*

Corollary 3.3. *Let Z be a section of an ample line bundle L on X. If $mK_Z = \mathcal{O}_Z$ for some $m > 0$, then X is Fano, $K_Z = \mathcal{O}_Z$ and $NE(X) = NE(Z)$.*

In the proofs we need two technical result which we recall now for the reader's convenience.

Proposition 3.4. *[24] Let R be an extremal ray on X and let φ the associated contraction. Let $E = E(\varphi)$ be the exceptional locus of φ (if φ is of fiber type then $E := X$) and let S be an irreducible component of a (non trivial) fiber F. We define the length of the ray to be the positive integer*

$$l = \min\{-K_X \cdot C : C \text{ is a rational curve in } S\}.$$

The following formula holds

$$\dim S + \dim E \geq \dim X + l - 1.$$

Proposition 3.5. *[2, Prop. 2.18]. Let X, $\mathcal{E}$ and Z be as in the introduction. Let Y be a subvariety of X of dimension $\geq r$. Then $\dim Z \cap Y \geq \dim Y - r$.*

Proof. of 3.1.
Let $R_X = \mathbb{R}_+[C]$ be an extremal ray of X such that $-(K_X + \det \mathcal{E}) \cdot C > 0$ and let $\varphi : X \to W$ be the associated contraction; we have that $l(R_X) \geq \det \mathcal{E} \cdot C + 1 \geq r + 1$.
Let S be an irreducible component of a non trivial fiber of φ; the theorem will

follow if we prove that $\dim Z \cap S \geq 1$; since in this case there is a curve in R_X which lies in Z.
If φ is birational then $\dim S \geq l(R_X) \geq r+1$ by the inequality 3.4 and by 3.5 $\dim Z \cap S \geq 1$. If φ is of fiber type then $\dim S \geq l(R_X) - 1 \geq r$ by the inequality 3.4. If $\dim S \geq r+1$ we conclude again by 3.5. Therefore we can assume by contradiction that $\dim S = r$, $\dim Z \cap S = 0$ and thus that $\det \mathcal{E} \cdot C = r$, $-K_X \cdot C = r+1$. In particular we have $-(K_X + \det \mathcal{E}) \cdot C = 1$.
By Fujita's characterization of scrolls, see [10, Lemma 2.12], X is a $\mathbb{P}^r$-bundle over W; in particular $\rho(X) = \rho(W) + 1$. Since $\dim Z \cap F = 0$ and $\mathcal{E}_{|F} = \oplus^r \mathcal{O}_{\mathbb{P}^r}(1)$, Z is isomorphic to W and this is a contradiction with the Lefschetz theorem.
The condition in the second part of the theorem is clearly necessary. □

Proof. of 3.2. We repeat the above proof for an extremal ray of X, $R_X = \mathbb{R}_+[C]$, such that $-(K_X + L) \cdot C \geq 0$ and with $\varphi : X \to W$ its associated contraction. If $\dim F \cap Z > 0$ then the theorem follows as above, thus we can assume that $\dim F = 1$ and $\dim F \cap Z = 0$ for all irreducible components of non trivial fibers of φ. Thus φ is the blow up of a smooth subvariety or φ is a conic bundle by the results in [1]. Moreover, since $-(K_X + L) \cdot C \geq 0$, $L \cdot C \leq -K_X \cdot C$ which is 1 in the first case and 1 or 2 in the second. The result in this case follows from [14, Lemma]. □

Remark 3.6. *The last theorem and corollary should be true also in the case $r > 1$. However it seems difficult to prove in the general case the technical results from [1] and [14] used in the above proof.*

Remark 3.7. *To decide whether an extremal ray of Z is an extremal ray of X seems to be more difficult and at the moment we are not able to give a complete answer. In the following we simply recall two partial results proved in [3] and [20].*

Theorem 3.8. *Let $R_Z = \mathbb{R}_+[C]$ be an extremal ray of Z. Then R_Z is extremal also in X if one of the following holds:*

a) [3, Theorem 3.2]) There is an ample line bundle H on X and a positive real number τ such that $\overline{NE(Z)}$ is contained in the semi-space $K_X + \det \mathcal{E} + \tau H \geq 0$ and $(K_X + \det \mathcal{E} + \tau H) \cdot C = 0$.

b) [20, Proposition 5]) There is a component $V_X \subset Hom(\mathbb{P}^1, X)$ which contains C and such that:
1) $Locus(V_X) = X$.
2) V_X is proper (i.e. an unsplit family).

4 Adjunction

The pairs (X, L) such that $Z \in |L|$ is a Fano manifold of index $r \geq \dim Z - 2$ were studied by adjunction theory; in particular, the case of projective space

and quadrics was considered in [5], the del Pezzo varieties were studied in [18] and the Mukai varieties were the object of the recent paper [6].
The next result we propose is an improvement of [6, Theorem 2.1], obtained combining the proof of that result with theorem 3.2.

Theorem 4.1. *Let Z be a section of an ample line bundle L on X with* $\dim Z \geq 3$. *Assume that $-K_Z = qH_Z + B_Z$, where q is a positive integer, H_Z is ample and spanned on Z and B is nef on X (for instance $B = \mathcal{O}_X$). Assume also that $K_X + (q+1)L$ is nef and that $k(K_X + (q+1)L) \geq 2$. Then H is ample and X is Fano. Moreover $NE(X) = NE(Z)$.*

Proof. By assumption $-qH = K_X + L + B$ and therefore $q(L - H) = K_X + (q+1)L + B$.
Our assumptions, together with Kawamata base point freeness, imply that $L - H$ is semiample and that $k(L - H) \geq 2$. Therefore by [21, Theorem 7.65] we have that $H^1(X, H - L) = 0$.
This vanishing, together with the long exact sequence associated to the sequence

$$0 \to H - L \to H \to H_Z \to 0,$$

gives that H is spanned on Z. Since Z is ample this implies that H is spanned out of a finite set of points, therefore H is nef, $-K_X = L + qH + B$ is and X is a Fano manifold. Assume now that H is not ample, that is there exists an extremal ray $R_X = \mathbb{R}_+[C]$ on which H is zero. We consider the contraction associated to R, φ and we conclude as in the proof of 3.2. Note that since $H \cdot C = 0$ we have that $(L + B) \cdot C = -K_X \cdot C$ and therefore we are in the assumptions of theorem 3.2. $\square$

The next result shows that in some cases we can avoid the assumptions on $K_X + (q+1)L$. We recall that if Z is a Fano manifold its index is the largest natural number r such that $-K_Z = rH_Z$ for some (ample) Cartier divisor.

Theorem 4.2. *Let Z be a Fano variety of dimension ≥ 4 and index r, i.e. $-K_Z = rH_Z$. Assume that H_Z is spanned and that Z is a section of an ample line bundle L on X.*
If $r \geq \frac{\dim Z}{2}$ then X is a Fano variety and $NE(X) = NE(Z)$ unless either $Z = \mathbb{P}^1 \times \mathbb{P}^3$ and X is a projective bundle over $\mathbb{P}^1$ or $Z = \mathbb{P}^1 \times V$, with V a del Pezzo threefold of Picard number one and X is a del Pezzo fibration over $\mathbb{P}^1$.

Proof. If $D := K_X + (r+1)L$ is nef and $\kappa(D) \geq 2$ the result follows from theorem 4.1.
So let us assume first that D is not nef, i.e. that $D \cdot C < 0$ for some effective curve C; in this case there exists an extremal ray $R = \mathbb{R}_+[\Gamma]$ such that $D \cdot R < 0$. In particular $-(K_X + L) \cdot R = (-D + rL) \cdot \Gamma > 0$, so that, by theorem 3.2, R is extremal for $NE(Z)$.
Let $\Gamma_Z \subset Z$ be a minimal extremal curve in R; that is a rational curve in R for which $-K_Z \cdot \Gamma_Z$ is minimal. By Mori theory it is known that $-K_Z \cdot \Gamma_Z \leq \dim Z + 1$; moreover equality holds if and only if $Z \simeq \mathbb{P}^{\dim Z}$, by a recent result

in [9]. We can assume that the last is not the case (otherwise $X \simeq \mathbb{P}^n$ and the theorem is obvious) so $-K_Z \cdot \Gamma_Z \leq \dim Z$.
Since $D = r(L - H)$ we have that $(L - H) \cdot \Gamma_Z < 0$, hence $H \cdot \Gamma_Z \geq 2$.
Therefore

$$2r \leq rH_Z \cdot \Gamma_Z = -K_Z \cdot \Gamma_Z \leq \dim Z,$$

forcing $r \leq \frac{n-1}{2}$.
We are thus left with the case $r = \frac{n-1}{2}$; in this case Z has an extremal ray of length $= \dim Z$; note also that $L \cdot \Gamma_Z = 1$, so $K_Z + (n-1)L_Z$ is not ample.
By the first step of adjunction theory, see [8] section 7.2, either Z is a quadric or Z is a projective bundle over a curve, but the first is impossible since $r = \frac{n-1}{2} \neq \dim Z$.
In the second case Z, being a Fano variety and a projective bundle over a curve can be only a product $\mathbb{P}^1 \times \mathbb{P}^{n-2}$ or the blow up of $\mathbb{P}^{n-1}$ along a codimension two linear subspace; in the first case we have $r = 2$, so $\dim Z = 4$, while in the second we have $r = 1$ and $\dim Z = 2$ against our assumptions.
The description of X now follows from [2, Proposition 4.9 and Remark 4.11].

We can thus assume that D is nef and that $\kappa(D) \leq 1$.
If $\kappa(D) = 0$ we have

$$-K_X \equiv (r+1)L.$$

In this case X is a Fano variety and $-(K_X + L) \equiv rL$ is ample. We can apply theorem 3.2 to get $NE(X) = NE(Z)$ and we are done.
If $\kappa(D) = 1$ then any extremal ray of Z not contracted by D has fibers of dimension ≤ 1; this implies that $r = 2$ and thus that $\dim Z = 4$. By [23, 1.4] Z is $\mathbb{P}^1 \times \mathbb{P}^3$ or $\mathbb{P}^1 \times V$, with V a del Pezzo threefold with Picard number one.
Let $\varphi : X \to C$ be the contraction associated to $K_X + 3L$; first of all note that, since Z is a Fano variety, then $C \simeq \mathbb{P}^1$; moreover for a general fiber F we have $-K_F = 3L_F$, so that F is a del Pezzo variety and $L_F = \mathcal{O}_F(1)$. In particular $Z \simeq \mathbb{P}^1 \times \mathbb{P}^3$ cannot be an ample section of such an X. $\square$

We are now in the position to give a list of pairs (X, L) as above with $Z \in |L|$ a Mukai variety of dimension $\dim Z \geq 4$, strengthening the results in [6]; note that, apart from case 6 (d), all cases are effective.

Theorem 4.3. *Let Z be a section of an ample line bundle L on a manifold X of dimension n. Assume that $\dim Z \geq 4$ and that Z is a Mukai manifold, i.e. $-K_Z = (n-3)H_Z$ for some ample line bundle H_Z on Z.*
Then the triple (X, L, H) is one of the following:

1. *X is a Mukai manifold, that is $-K_X = (n-2)L$, with $\rho = 1$ and $L = H$.*
2. *X is a del Pezzo manifold, that is $-K_X = (n-1)H$, with $\rho = 1$ and $L = 2H$.*
3. *$X = \mathbb{Q}^n$, an hyperquadric in $\mathbb{P}^{n+1}$, $L = \mathcal{O}_{\mathbb{Q}^n}(3)$ and $H = \mathcal{O}_{\mathbb{Q}^n}(1)$.*
4. *$X = \mathbb{P}^n$, $L = \mathcal{O}_{\mathbb{P}^n}(4)$ and $H = \mathcal{O}_{\mathbb{P}^n}(1)$.*

5. $\dim X = 6$ *and*
   (a) $X = \mathbb{P}^6$, $L = \mathcal{O}_{\mathbb{P}^6}(1)$ *and* $H = \mathcal{O}_{\mathbb{P}^6}(2)$.
   (b) $X = \mathbb{P}^3 \times \mathbb{P}^3$ *and* $L = H = \mathcal{O}(1,1)$, *that is* $Z = \mathbb{P}(T_{\mathbb{P}^3})$.
6. $\dim X = 5$ *and*
   (a) $X = \mathbb{Q}^5$, *an hyperquadric in* $\mathbb{P}^6$, $L = \mathcal{O}_{\mathbb{Q}^5}(1)$ *and* $H = \mathcal{O}_{\mathbb{Q}^5}(2)$.
   (b) $X = \mathbb{P}^5$, $L = \mathcal{O}_{\mathbb{P}^5}(2)$ *and* $H = \mathcal{O}_{\mathbb{P}^5}(2)$.
   (c) X *is a projective bundle over* $\mathbb{P}^1$ *and* $Z = \mathbb{P}^1 \times \mathbb{P}^3$ *(see section two; X is not necessarily Fano and $NE(X) \neq NE(Z)$).*
   (d) X *is a del Pezzo fibration over* $\mathbb{P}^1$ *and* $Z = \mathbb{P}^1 \times V$, *with* V *a del Pezzo threefold of Picard number one.*
   (e) $X = \mathbb{P}^2 \times \mathbb{P}^3$, $L = \mathcal{O}(1,2)$ *and* $H = \mathcal{O}(1,1)$.
   (f) X *is a Mukai 5-fold, i.e.* $-K_X = 3L$, *(with* $\rho = 2$*) and* $H = L$. *According to [25] they are:*
      i. $X = \mathbb{P}^2 \times \mathbb{Q}^3$, $L = H = \mathcal{O}(1,1)$.
      ii. $X = \mathbb{P}(T^3_{\mathbb{P}})$, $L = H = \mathcal{O}(1,1)$.
      iii. $X = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(2))$, $L = H = \xi + p^*\mathcal{O}_{\mathbb{P}^3}(1)$.

Proof. The case $\rho := \rho(X) = \rho(Z) = 1$ is straightforward since in this case H is ample on X and we have the equality $-K_X = (n-3)H + L$ (for more details one can look at the proof of proposition 3.1 in [6]). We can thus assume $\rho := \rho(X) = \rho(Z) > 1$.
By the theorems in [22] applied to Z, we have that

$$n - 3 \leq \frac{\dim Z + 2}{2} = \frac{n+1}{2}$$

with equality if and only if $Z = \mathbb{P}^3 \times \mathbb{P}^3$.
Since the last cannot be an ample section of any projective manifold, by [8, Corollary 5.2.4], we have that $n \leq 6$.
If $n = 6$ then, by [25] we have the following possibilities for Z:

1. $Z = \mathbb{P}^2 \times \mathbb{Q}^3$.
2. $Z = \mathbb{P}(T^3_{\mathbb{P}})$, $H_Z = \mathcal{O}(1,1)$.
3. $Z = \mathbb{P}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(2))$.

Only $Z = \mathbb{P}(T^3_{\mathbb{P}})$ is an ample divisor in a smooth X. In fact, the first case is ruled out by [8, Corollary 5.2.4] and the third case is ruled out by [10].
Note that H_Z is spanned so we can apply theorem 4.1 which gives that H is ample. By [2, Theorem 4.1] each contraction on $Z = \mathbb{P}(T^3_{\mathbb{P}})$ lift to a $\mathbb{P}^3$-bundle on X. It is straightforward to prove now that $X = \mathbb{P}^3 \times \mathbb{P}^3$ and that $L = H = \mathcal{O}(1,1)$.

Let then $n = 5$. By [23, 2.6] H_Z is spanned unless $Z = \mathbb{P}^1 \times V_1$, where V_1 is a del Pezzo threefold of degree one; in this case we repeat the final part of the proof of theorem 4.2 and we get that X is a del Pezzo fibration over $\mathbb{P}^1$ (it cannot be a projective bundle or a quadric fibration because $-K_{V_1}$ is not spanned).
So we can assume that H_Z is spanned and apply theorem 4.2; in particular, if X is not as in 6.(c) or in 6.(d) then X is a Fano variety and $NE(X) = NE(Z)$. It follows that the line bundle H which restricts to H_Z is ample. Note that, by inequality 3.4, Z and X cannot have small contractions and note also that $-K_X = 2H + L$.
Assume that $H \cdot C_1 \neq L \cdot C_1$; in particular $l(R_1) \geq 4$. We claim that there are two fibers F_1 and F_2 of the contractions of R_1 and R_2 which have nonempty intersection. If both the contractions are of fiber type this is clear. Otherwise, if R_1 is a birational contraction, its exceptional locus E_1 is an effective divisor, and has positive intersection with at least one extremal ray (this is a general fact on Fano manifolds). Since $E_1 \cdot R_1 < 0$ we have $E_1 \cdot R_2 > 0$ and the claim is proved.
Let F_1 and F_2 the two fibers with a point in common; we have $\dim(F_1) + \dim(F_2) \leq 5$, $\dim(F_1) \geq l(R_1) - 1 \geq 3$ and $\dim(F_2) \geq l(R_2) - 1 = 2$.
Therefore both the contractions are of fiber type and the preceding inequalities are true for any fiber.
In particular, by [10, Lemma 2.12], φ_1 is a $\mathbb{P}^3$-bundle and φ_2 is a $\mathbb{P}^2$-bundle.
It is straightforward to prove now that $X = \mathbb{P}^2 \times \mathbb{P}^3$ and that $L = \mathcal{O}(1,2)$ and $H = \mathcal{O}(1,1)$.
Therefore we can assume now that $L = H$, that is X is a Mukai 5-fold with $\rho = 2$ and the result follows again from [25]. $\square$

Remark 4.4. *In case L is assumed to be very ample, the case $Z = \mathbb{P}^1 \times V$, with V a del Pezzo threefold of Picard number one can be excluded in both theorem 4.2 and theorem 4.3, by [7, Proposition 0.1].*

5 Ample sections without extremal rays

Theorem 5.1. *Let $X, \mathcal{E}$ and Z be as in the introduction. Assume that Z is minimal and has Kodaira dimension $0 \leq \kappa(Z) < \dim Z$. Then:*

1. *$K_X + \det \mathcal{E}$ is nef (in particular, by the Base Point Free Theorem, it is semiample) but not big, i.e. $K_X + \det \mathcal{E}$ is a good supporting divisor of a Fano-Mori contraction $\Phi : X \longrightarrow Y$ of fiber type.*

2. *K_Z is semiample and Φ extends the pluricanonical map $\varphi_{|mK_Z|}$ for $m \gg 0$.*

3. *The general fiber F of the contraction Φ is a Fano manifold of pseudoindex $\geq r$ with $-K_F = \det \mathcal{E}_F$ and $K_F|_{Z \cap F} = \mathcal{O}_F|_{Z \cap F}$; in particular, if $mK_Z = \mathcal{O}_Z$, then X is a Fano manifold with $-K_X = \det \mathcal{E}$ and $K_Z = \mathcal{O}_Z$.*

Proof. Assume by contradiction that $K_X + \det \mathcal{E}$ is not nef. Then there exists an extremal ray $\mathbb{R}_+[C] \in \overline{NE(X)}$ such that $(K_X + \det \mathcal{E}) \cdot C < 0$; then, by

3.1, there is a curve $\Gamma \subset Z$ such that $[\Gamma] \in \mathbb{R}_+[C]$. In particular $K_Z \cdot \Gamma = (K_X + \det \mathcal{E}) \cdot \Gamma < 0$, against the minimality of Z.
On the other hand $K_X + \det \mathcal{E}$ is not ample, otherwise $K_Z = (K_X + \det \mathcal{E})_{|Z}$ would be ample, and so $K_X + \det \mathcal{E}$ is a good supporting divisor for an extremal face in $\overline{NE(X)}$.
By the Kawamata-Shokurov base point free theorem there exists a positive integer m such that $m(K_X + \det \mathcal{E})$ is spanned by global sections. Therefore also $m(K_X + \det \mathcal{E})_{|Z} = mK_Z$ is spanned by global sections.
Let $\Phi : X \longrightarrow Y$ be the map defined by $m(K_X + \det \mathcal{E})$ and $\varphi : Z \longrightarrow Z'$ the map defined by mK_Z. Taking $m \gg 0$ we can assume that they both have connected fibers, that Y and Z' are normal and that the following diagram is commutative.

$$\begin{tikzcd} X \arrow[d, "\Phi"'] & Z \arrow[l, "i"'] \arrow[d, "\varphi"] \arrow[dl, "\Phi_{|Z}"'] \\ Y & Z' \arrow[l, "\pi"] \end{tikzcd}$$

Since $\kappa(Z) < \dim Z$, the map φ is of fiber type; by the above diagram, $\Phi_{|Z}$ is of fiber type and we claim that Φ itself is of fiber type.
Assume by contradiction that Φ is birational and let E be the exceptional locus. By 3.4, $\dim F \geq r$ for all non trivial fibers; so, by 3.5, $\dim F \cap Z \geq 0$ and therefore $\Phi(E) \subseteq \Phi_{|Z}(Z)$.
On the other hand, since $\Phi_{|Z}$ is of fiber type, Z is contained in E and thus $\Phi_{|Z}(Z) \subseteq \Phi(E)$.
Then $\Phi(E) = \Phi_{|Z}(Z)$ and $\dim \Phi(E) = \dim \Phi_{|Z}(Z) < \dim Z = n - r$.
Since $\dim Y = n$, it is possible to find a subvariety $W' \subset Y$ such that $\dim W' = r$ and $W' \cap \Phi(E) = \emptyset$. But Φ is an isomorphism away from E, so $W := \Phi^{-1}(W')$ is a subvariety of X of dimension r such that its intersection with E is empty. Therefore $Z \cap W = \emptyset$, but this is a contradiction since $\dim Z \cap W \geq \dim W - r = 0$.
Let F' be any fiber of Φ; note that $\dim F' \cap Z \geq \dim X - \dim Y - r \geq 1$. So we can apply Lefschetz theorem to F' and $\mathcal{E}_{F'}$, obtaining $H^0(Z \cap F', \mathbb{Z}) \cong H^0(F', \mathbb{Z}) \cong \mathbb{Z}$. Using the Universal Coefficient Theorem we get $H_0(Z \cap F') \cong \mathbb{Z}$ and so $\Phi_{|Z}$ has connected fibers.
Therefore φ and $\Phi_{|Z}$ are morphisms with connected fibers onto normal varieties which contract curves in the same ray. This implies that π is an isomorphism.
Let F be the general fiber of Φ. We have that $K_F = K_X|_F = -\det \mathcal{E}|_F$; then F is a Fano manifold of pseudoindex $\geq r$. Moreover, if we consider the restriction of F to Z, we have $K_F|_{Z \cap F} = (K_X + \det \mathcal{E}|_F)_{Z \cap F} = \mathcal{O}_F|_{Z \cap F}$.
In particular, if $\mathcal{O}_Z = mK_Z = m(K_X + \det \mathcal{E})_Z$, then $\mathcal{O}_X = m(K_X + \det \mathcal{E})$ and thus $-K_X$ is ample. So X is a Fano manifold with $-K_X = \det \mathcal{E}$ and $K_Z = \mathcal{O}_Z$. $\square$

Remark 5.2. *If Z is a minimal variety then it is conjectured that $\kappa(Z) \geq 0$ (see [13] and [19]). The conjecture is true for minimal surfaces or threefolds*

If $\dim Z = 2$ we can study the case $0 \leq \kappa(Z) < \dim Z = 2$ without the assumption of minimality; this was done in [15] and [16], here we give a different proof. In higher dimensions, even for $\dim Z = 3$, this is much more difficult.

Theorem 5.3. *Let $X, \mathcal{E}$ and Z be as in the introduction. Assume that $r \geq 2$ and that Z is a surface of Kodaira dimension $\kappa(Z) = 0$ or an elliptic surface of Kodaira dimension $\kappa(Z) = 1$. Then $(X, \mathcal{E})$ is one of the following:*

1. *$X = \mathbb{P}_W(\mathcal{F})$, where $\mathcal{F}$ is an ample vector bundle of rank $n-1$ over a smooth surface W with $\kappa(W) = \kappa(Z)$ and $\mathcal{E} = \Phi^*\mathcal{V} \otimes \xi$, where ξ is the tautological line bundle on X, $\mathcal{V}$ is a vector bundle of rank $n-2$ on W and $\Phi : X \longrightarrow W$ is the bundle projection. In this case Z is not minimal and $\Phi|_Z : Z \longrightarrow W$ is a birational morphism, but not an isomorphism.*

2. *There exist a birational morphism $\Phi : X \longrightarrow X'$ expressing X as a projective manifold X' blown up at a finite set B of points (possibly empty) and an ample vector bundle $\mathcal{E}'$ of rank $n-2$ on X' such that $\mathcal{E} = \Phi^*\mathcal{E}' \otimes [-\Phi^{-1}(B)]$ and $K_{X'} + \det \mathcal{E}'$ is nef. In this case the triplet $(X', \mathcal{E}', Z' := \Phi(Z))$ is as in the introduction, with $r \geq 2$, and Z' is a minimal surface with $\kappa(Z) = \kappa(Z')$.*

   Moreover:
   if $\kappa(Z) = 0$, then X' is a Fano manifold with $-K_{X'} = \det \mathcal{E}'$ and Z' is a $K3$ surface dominated by Z via the birational morphism $\Phi|_Z$;
   if $\kappa(Z) = 1$, then X' is endowed with a morphism $\Phi : X' \longrightarrow Y$ onto a smooth curve Y, whose general fiber F is a projective manifold of dimension $n-1$ satisfying the condition $K_F + \det \mathcal{E}'_F = \mathcal{O}_F$; Φ induces on Z' the elliptic fibration.

Proof. If Z is minimal the proof follows easily from 5.1.
Assume therefore that Z is not minimal; each of the extremal rays of Z corresponds to the contraction of a (-1)-curve. Let H be an ample line bundle on X and let τ be the nefvalue of H_Z, i.e. the minimum real number such that $K_Z + \tau H_Z$ is nef. Then $K_X + \det \mathcal{E} + \tau H$ is nef but not ample and it is zero exactly on the curves of a face F which is extremal both in Z and in X.
Let $\Phi : X \longrightarrow W$ be the contraction associated to a ray in F and let $\varphi : Z \longrightarrow Z'$ the contraction of the (-1)-curve corresponding to the ray. By [3, Prop. 3.8], Φ can be either birational or of fiber type; moreover in the last case it is an adjunction theoretic scroll onto W and $\Phi_{|Z} = \varphi$. Since $\dim W = 2$ we can apply [8, Proposition 14.1.3] which says that Φ is actually a $\mathbb{P}^{n-2}$-bundle and we are in the case 1. of the Theorem. Suppose now that Φ is birational. By [3, Proof of Theorem 1.2] $X' = \Phi(X)$ is smooth, Φ is the blow-up of a point B on X' and $\Phi_{|Z} = \varphi$. Moreover $\mathcal{E}_F \cong \oplus^r \mathcal{O}_{\mathbb{P}^{r-1}}(1)$, where F is a fiber of Φ. Then, by [3, Lemma 2.9] there exists an ample vector bundle $\mathcal{E}'$ of rank $r = n-2$ on X' such that $\mathcal{E} \otimes \Phi^{-1}(B) = \Phi^*\mathcal{E}'$ and Z' is a section of $\mathcal{E}'$. If Z' is not minimal

we can thus repeat the above arguments.
We observe that the case of a fiber type contraction on X' cannot happen now and in any further steps and therefore we are in the case 2. of the Theorem. This claim can be proved exactly as in the last part of the proof of Theorem (1.4) in [3]. $\square$

Remark 5.4. *Lanteri and Maeda in [16] showed that the elliptic fibration $\Phi|_{Z'} : Z' \longrightarrow Y$ has actually no multiple fibers and the genus of the curve Y is $g(Y) = h^{1,0}(Z)$.*

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