A trivariate near-best blending quadratic quasi-interpolant

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A trivariate near-best blending quadratic quasi-interpolant

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Abstract

In this paper, we construct a new trivariate spline quasi-interpolation operator. It is expressed as blending sum of univariate and bivariate C^1 quadratic spline quasi-interpolants and it is of near-best type, i.e. it has a small infinity norm and the coefficients functionals defining it are determined by minimizing an upper bound of the operator infinity norm, derived from the Bernstein-Bézier coefficients of its Lebesgue function.

Keywords: B-spline; Box spline; Quasi-interpolation; Blending operator

1 Introduction

A quasi-interpolant Q is a linear operator of the type $Q : C(\mathbb{R}^s) \longrightarrow \mathcal{S}(\phi)$, where $\mathcal{S}(\phi)$ is the space spanned by the integer translates of a non-negative compactly supported function ϕ. It is supposed that they form a convex partition of unity. These operators are constructed to be exact on the space $\mathbb{P}(\phi)$ of polynomials of maximal total degree included in $\mathcal{S}(\phi)$ and have the following form

$$Qf = \sum_{i \in \mathbb{Z}^s} \lambda\left(f(\cdot + i)\right) \phi(\cdot - i),$$

λ being a general linear functional, usually a point, derivative or integral linear functional. Operators defined in this way (or modified to approximate functions defined on an interval, from uniform or irregularly spaced knots) have been used to solve problems in many different areas, like science and engineering, and also to develop numerical schemes useful in practice (see e.g. [24, 25, 26] and [10, 16], respectively).

Furthermore, approximating noisy data requires the use of adapted methods. Specific types of quasi-interpolants have been proposed in the literature to diminish as much as possible the increase of noise present in the data. They are based on the minimization of the infinity norm $\|Q\|_\infty$ of the operator Q. If λ is the linear functional given by

$$\lambda f = \sum_{j \in J} c_j f(\cdot - j),$$

J being a finite subset of $\mathbb{Z}^s$, then, for $\|f\|_\infty \leq 1$

$$\|Q\|_\infty \leq \sum_{j \in J} |c_j| =: \|c\|_1,$$

where $c := \{c_j, j \in J\}$. Then, the upper bound $\|c\|_1$ is minimized instead of $\|Q\|_\infty$, subject to the linear constraints yielding the exactness of Q on the space $\mathbb{P}(\phi)$ and the quasi-interpolant associated with the solution of this minimization problem is called near best quasi-interpolant. As far as the authors know, the first systematic study on

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the construction of such operators appears in [4], where univariate quasi-interpolants are based on a point or derivative linear functional (see [5] for the nonuniform case). The bivariate case was considered in [1, 3, 6] by using C^2-quartic B-splines on the four-direction mesh, H-spline and a Ω-spline, respectively (the case of quadratic box spline appears in [17] and the use of cubic multi-box spline is considered in [19]). The extension to the three-dimensional case is done in [8, 13, 18, 20].

In order to obtain a better upper bound to be minimized it is possible to bound the Lebesgue function associated with Q from the Bernstein-Bézier coefficients of ϕ. This approach has been considered in [2, 9].

In this paper, we deal with the construction of a new near-best trivariate spline quasi-interpolation operator by blending 1D and 2D C^1 quadratic spline quasi-interpolants and minimizing an objective function constructed from the Bernstein-Bézier coefficients of the Lebesgue function of the resulting operator. In particular, in Section 2, we introduce the univariate and bivariate spline spaces, quasi-interpolation operators in such spaces and we define the blending trivariate operator. In Section 3, we study the problem of the construction of near-best quasi-interpolants, by defining the objective function characterizing the minimization problem and providing the explicit solution. Finally, in Section 4, some conclusions are presented.

2 Spline spaces and quasi-interpolation operators

Let B be the quadratic B-spline supported on the interval $\left[-\frac{3}{2},\frac{3}{2}\right]$. It is a C^1 quadratic B-spline on the real line having knots at the half-integers. Its Bernstein-Bézier (BB-) coefficients in every sub-interval of its support appear in Figure 1 (see e.g. [12]).

Let M be the quadratic box spline on the four-directional triangulation τ of the plane generated by the directions $d_1 := (1,0)$, $d_2 := (0,1)$, $d_3 := d_1 + d_2$, $d_4 := d_2 - d_1$. It is a $C^1\left(\mathbb{R}^2\right)$ function whose restriction to every triangle in τ is a quadratic polynomial (see e.g. [11, 12]). Figure 1 shows the support of the box spline and provides the BB-coefficients of $8 \cdot M$ in the triangles of τ included in the polygon with vertices $(0,0)$, $(1,-1)$, $\left(\frac{3}{2},-\frac{1}{2}\right)$, $\left(\frac{3}{2},\frac{1}{2}\right)$, and $(1,1)$. The BB-coefficients relative to the other triangles in the support of M are determined by the symmetries of the octagon.

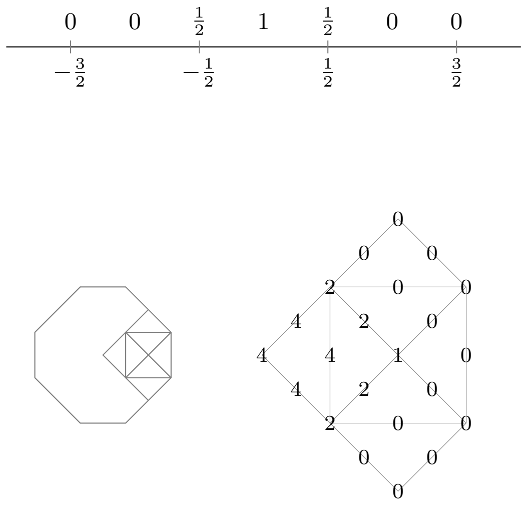

Figure 1: (Left) BB-coefficients of the quadratic B-spline B. (Right) BB-coefficients of the box spline $8 \cdot M$.

From the B-spline B and the box spline M, we consider the spaces $\mathcal{B}_1 := \text{span}\,\{B(\cdot - k) : k \in \mathbb{Z}\}$ and $\mathcal{B}_2 := \text{span}\,\left\{M(\cdot - i_1, \cdot - i_2) : (i_1, i_2) \in \mathbb{Z}^2\right\}$. They contain the spaces of univariate and bivariate quadratic polynomials, respectively (cf. [11, p. 53], [12, p. 19]).

The main goal is to construct a near-best trivariate quasi-interpolation operator (QIO) from some univariate and bivariate QIOs properly chosen. Firstly, we consider the univariate Schoenberg QIO $\overline{S}$ defined by

$$\overline{S}f(z) := \sum_{k\in\mathbb{Z}} f(k)\, B(z-k), \tag{2.1}$$

and the operator $\overline{Q}$ defined by

$$\overline{Q}f(z) := \sum_{k\in\mathbb{Z}} \left(\sum_{\ell=-2}^{2} a_\ell f(k-\ell) \right) B(z-k), \tag{2.2}$$

where the coefficients a_ℓ are chosen to produce an operator exact on $\mathbb{P}_2(\mathbb{R})$, the space of univariate quadratic polynomials. The operator $\overline{S}$ given in (2.1) is exact only on $\mathbb{P}_1(\mathbb{R})$, the space of univariate linear polynomials. Regarding $\overline{Q}$, it is well-known that it is possible to define a QIO exact on $\mathbb{P}_2(\mathbb{R})$ using only three coefficients instead of five like in (2.2) (see e.g. [21]). Explicitly, $a_{-2} = a_2 = 0$, $a_{-1} = a_1 = \frac{1}{8}$, and $a_0 = -\frac{5}{2}$. However, some oversampling is allowed in order to be able to reduce the infinity norm of the operator. We have introduced the minimum number of freedom degrees and suppose that $a_{-2} = a_2$ and $a_{-1} = a_1$ to produce an even fundamental function. For the sequence of coefficients we will write $a := (a_0; a_1; a_2)$.

Lemma 1 *The quasi-interpolant (QI) $\overline{Q}f(z)$ given by (2.2) can be written as*

$$\overline{Q}f(z) = \sum_{k\in\mathbb{Z}} f(k)\, L_B(z-k),$$

where the fundamental function L_B is the linear combination of integer translates of the B-spline B given by the expression

$$L_B(z) := \sum_{\ell=-2}^{2} a_\ell B(z-\ell). \tag{2.3}$$

Moreover, the operator $\overline{Q}$ is exact on $\mathbb{P}_2(\mathbb{R})$ if and only if

$$a_0 + 2a_1 + 2a_2 = 1 \quad \text{and} \quad a_1 + 4a_2 = -\frac{1}{8}.$$

Proof. The first claim is derived easily. Regarding the exactness, we will use a general result in [7, p. 274] that implies in the quadratic case the exactness on $\mathbb{P}_2(\mathbb{R})$ of the differential operator $\mathcal{D}_3$ given by

$$D_3 f := \sum_{k\in\mathbb{Z}} \left(f(k) - \frac{1}{8} f''(k) \right) B(\cdot - z).$$

It is obvious to prove that $\overline{Q}$ reproduces the monomial $m_0(z) := 1$ if and only if $a_0 + 2a_1 + 2a_2 = 1$. This constraint on the coefficients defining $\overline{Q}$ also yields that $\overline{Q}$ reproduces the monomial $m_1(z) := z$. As far as monomial $m_2(z) := z^2$ is concerned, $\overline{Q}$ provides equality

$$\overline{Q}m_2(z) = \sum_{k\in\mathbb{Z}} k^2 B(z-k) + 2(a_1 + 4a_2).$$

Since

$$D_3 m_2(z) = \sum_{k\in\mathbb{Z}} k^2 B(z-k) - \frac{1}{4},$$

then $\overline{Q}$ will reproduce m_2 if and only if $a_1 + 4a_2 = -\frac{1}{8}$, and the proof is complete.

Now, let S be the Schoenberg QIO associated with the box spline M (see e.g. [22]), which is defined by

$$Sf(x,y) := \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2)\, M(x-i_1, y-i_2). \tag{2.4}$$

It is exact on the space of bilinear polynomials. Once again, in order to obtain a trivariate operator with small infinity norm, we consider the QIO Q defined by

$$Qf(x,y) := \sum_{(i_1,i_2)\in\mathbb{Z}^2} \left(\sum_{(j_1,j_2)\in J} c_{j_1,j_2} f(i_1-j_1, i_2-j_2) \right) M(x-i_1, y-i_2), \tag{2.5}$$

where

$$J := \{(0,0),(1,0),(0,1),(-1,0),(0,-1),(2,0),(0,2),(-2,0),(0,-2),(1,1),(-1,1),(-1,-1),(1,-1)\},$$

and $c := \{(c_{j_1,j_2}),\ (j_1,j_2)\in J\}$ is a lozenge sequence [2] such that Q is exact on $\mathbb{P}_2(\mathbb{R}^2)$, the space of bivariate polynomials of total degree two, i.e.

$$c_{0,1} = c_{-1,0} = c_{0,-1} = c_{1,0},\ c_{0,2} = c_{-2,0} = c_{0,-2} = c_{2,0},\ c_{-1,1} = c_{-1,-1} = c_{1,-1} = c_{1,1}.$$

We will write $c = (c_{0,0}; c_{1,0}; c_{2,0}, c_{1,1})$.

Lemma 2 *The spline Qf in (2.5) can be written as*

$$Qf(x,y) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2)\, L_M(x-i_1, y-i_2),$$

where the fundamental function L_M is expressed as the linear combination of the integer translates of M given by

$$L_M(x,y) := \sum_{(j_1,j_2)\in J} c_{j_1,j_2} M(x-j_1, y-j_2). \tag{2.6}$$

Moreover, Q is exact on $\mathbb{P}_2(\mathbb{R}^2)$ if and only if

$$c_{0,0} + 4c_{1,0} + 4c_{2,0} + 4c_{1,1} = 1 \quad \textit{and} \quad c_{1,0} + 4c_{2,0} + 2c_{1,1} = -\frac{1}{8}.$$

Proof. As in the proof of Lemma 1, it is straightforward to prove the first claim. With respect to the exactness of Q, we will use a method similar to that described in Lemma 1. Now the starting point is the differential operator D exact on $\mathbb{P}_2(\mathbb{R}^2)$ given by

$$Df := \sum_{i\in\mathbb{Z}^2} \left(f(i) - \frac{1}{8}\Delta f(i) \right) M(\cdot - i),$$

where Δ stands for the Laplacian of f. It is obtained from a general method described in [14]. As in the univariate case, the operator Q reproduces the monomial $m_{0,0}(x,y) := 1$ if and only if $c_{0,0} + 4c_{1,0} + 4c_{2,0} + 4c_{1,1} = 1$, and under this constraint the monomials $m_{1,0}(x,y) := x$ and $m_{0,1}(x,y) := y$ are automatically reproduced since for α equal to $(1,0)$ or $(0,1)$ it holds

$$Qm_\alpha(x,y) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} m_\alpha(i_1,i_2)\, M(x-i_1, y-i_2) = Dm_\alpha(x,y) = m_\alpha(x,y).$$

Moreover, after some calculations, for $m_{2,0}(x,y) := x^2$ it follows that

$$Qm_{2,0}(x,y) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} \left(i_1^2 + 2(c_{1,0} + 2c_{1,1} + 4c_{2,0}) \right) M(x-i_1, y-i_2).$$

Therefore, it is equal to

$$x^2 = Dm_{2,0}(x,y) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} \left(i_1^2 - \frac{1}{4}\right) M(x-i_1, y-i_2)$$

if and only if $c_{1,0} + 2c_{1,1} + 4c_{2,0} = -\frac{1}{8}$.

Once again, both equalities imply that also the monomials $m_{0,2}(x,y) := y^2$ and $m_{1,1}(x,y) := xy$ are automatically reproduced, and the proof is complete.

Now, we define trivariate extensions of the operators above.

Definition 3 *Once defined in (2.1) and (2.4) the univariate and bivariate Schoenberg operators and the QIOs $\overline{Q}$ and Q in (2.4) and (2.5), we consider the trivariate extensions of these operators. They are given by*

$$\overline{S}f(x,y,z) = \sum_{k\in\mathbb{Z}} f(x,y,k)\, B(z-k), \tag{2.7}$$

$$\overline{Q}f(x,y,z) = \sum_{k\in\mathbb{Z}} \left(\sum_{\ell=-2}^{2} a_\ell f(x,y,k-\ell)\right) B(z-k) = \sum_{k\in\mathbb{Z}} f(x,y,k)\, L_B(z-k), \tag{2.8}$$

$$Sf(x,y,z) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2,z)\, M(x-i_1,y-i_2), \tag{2.9}$$

$$\begin{aligned} Qf(x,y,z) &= \sum_{(i_1,i_2)\in\mathbb{Z}^2} \left(\sum_{(j_1,j_2)\in J} c_{j_1,j_2} f(i_1-j_1, i_2-j_2, z)\right) M(x-i_1,y-i_2) \qquad (2.10)\\ &= \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2,z)\, L_M(x-i_1,y-i_2). \end{aligned}$$

We are now in position to define the type of operator we are interested in (see [9, 15, 20]).

Definition 4 *From the operators given by (2.7), (2.8), (2.9) and (2.10), the trivariate blending operator R is defined as*

$$R := S\overline{Q} + Q\overline{S} - S\overline{S}. \tag{2.11}$$

The operator R is a linear map into the tensor product spline space spanned by the trivariate piecewise polynomial functions $M(\cdot - i_1, \cdot - i_2)\, B(\cdot - k)$, $(i_1,i_2,k) \in \mathbb{Z}^3$. The QI Rf provided by the operator in (2.11) can be expressed from the fundamental functions relative to the operators $\overline{Q}$ and Q given in (2.3) and (2.6).

Lemma 5 *It holds*

$$Rf(x,y,z) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} \sum_{k\in\mathbb{Z}} f(i_1,i_2,k)\, L(x-i_1, y-i_2, z-k),$$

where

$$\begin{aligned} L(x,y,z) &:= M(x,y)\, L_B(z) + L_M(x,y)\, B(z) - M(x,y)\, B(z) \qquad (2.12)\\ &= L_M(x,y)\, B(z) + M(x,y)\, (L_B(z) - B(z)). \qquad (2.13) \end{aligned}$$

Moreover, R reproduces the monomials 1, x, y, z, x^2, y^2, z^2, xy, xz, yz, x^2z, xz^2, y^2z, yz^2, xyz, and xyz^2.

Proof. The proof of the first statement is trivial. For the second one, see [20, Theorem 4].

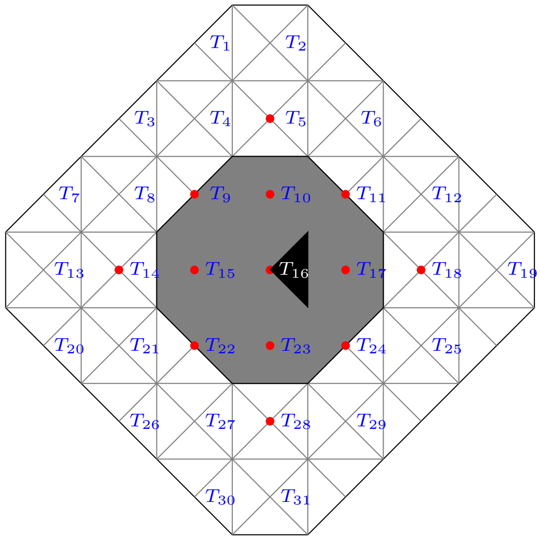

Figure 2: The supports Ω and ω of L_M and M, respectively, the T-like triangles T_i, $1 \leq i \leq 31$, and the points associated with the lozenge sequence c.

3 Near-best trivariate quadratic quasi-interpolation

The infinity norm of the QIO R is provided by the maximum of the associated Lebesgue function

$$\Lambda(x,y,z) := \sum_{(i_1,i_2)\in\mathbb{Z}^2} \sum_{k\in\mathbb{Z}} |L(x-i_1, y-i_2, z-k)|. \tag{3.1}$$

Since we are dealing with a uniform partition of the three dimensional space, Λ is a 1-periodic function, so that to determine its maximum value it is sufficient to consider its restriction to the cube $\left[-\frac{1}{2},\frac{1}{2}\right]^3$. Moreover, due to the symmetries of B and M and those of the coefficients in (2.2) and (2.5), the maximum is attained in a subset of the prism $P := T \times I$ with triangular horizontal sections, where T is the triangle defined in Section 2 and I is the interval $\left[-\frac{1}{2},\frac{1}{2}\right]$ (see [2, Lemma 3]).

Considering the complex structure of Λ, which also depends on a and c, it is very difficult to determine its maximum in P and the points at which it is reached. Therefore, we look for a good upper bound of $\|R\|_\infty$ by examining carefully the contribution of every term $L(x-i_1, y-i_2, z-k)$ to $\Lambda(x,y,z)$, $(x,y,z) \in P$.

According to (2.13), the fundamental function L in (2.12) is decomposed into two terms. The first one, $L_M(x,y)\,B(z)$, is supported on $\Omega \times \left[-\frac{3}{2},\frac{3}{2}\right]$, Ω being the octagon with vertices

$$\left(\frac{7}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{7}{2}\right),\left(-\frac{1}{2},\frac{7}{2}\right),\left(-\frac{7}{2},\frac{1}{2}\right),\left(-\frac{7}{2},-\frac{1}{2}\right),\left(-\frac{1}{2},-\frac{7}{2}\right),\left(\frac{1}{2},-\frac{7}{2}\right) \quad \text{and} \quad \left(\frac{7}{2},-\frac{1}{2}\right).$$

However, the second term, $M(x,y)\,(L_B(z)-B(z))$, is supported on $\omega \times \left[-\frac{7}{2},\frac{7}{2}\right]$, where ω is the octagon included in Ω with vertices

$$\left(\frac{3}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{3}{2}\right),\left(-\frac{1}{2},\frac{3}{2}\right),\left(-\frac{3}{2},\frac{1}{2}\right),\left(-\frac{3}{2},-\frac{1}{2}\right),\left(-\frac{1}{2},-\frac{3}{2}\right),\left(\frac{1}{2},-\frac{3}{2}\right) \quad \text{and} \quad \left(\frac{3}{2},-\frac{1}{2}\right).$$

Both octagons are represented in Figure 2, where also the points associated with the lozenge sequence c and the T-like triangles involved in (3.1) are shown.

To facilitate the calculation of the upper bound to the Lebesgue function, we will consider the polynomials defining the B-spline B on $\left[-\frac{7}{2},\frac{7}{2}\right]$ and the box spline M on Ω instead of $\left[-\frac{3}{2},\frac{3}{2}\right]$ and ω, respectively.

It is straightforward to prove the following result (see Figure 1).

Proposition 6 *The restrictions b_k of the B-spline B to the sub-intervals $\left[k-\frac{9}{2}, k-\frac{7}{2}\right]$, $1 \leq k \leq 7$, are given by $b_1(z) = b_2(z) = 0$, $b_3(z) = \frac{1}{8}(3+2z)^2$, $b_4(z) = \frac{3}{4} - z^2$, $b_5(z) = \frac{1}{8}(3-2z)^2$, $b_6(z) = b_7(z) = 0$. Moreover, those of $L_B(z) - B(z)$ are*

$$
\begin{aligned}
&Q_1(z) = a_{-2}b_1(z+2) - b_1(z), \ Q_2(z) = a_{-1}b_1(z+1) + a_{-2}b_2(z+2) - b_2(z), \\
&Q_3(z) = a_0 b_1(z) + a_{-1}b_2(z+1) + a_{-2}b_3(z+2) - b_3(z), \\
&Q_4(z) = a_1 b_1(z-1) + a_0 b_2(z) + a_{-1}b_3(z+1) - b_4(z), \\
&Q_5(z) = a_2 b_1(z-2) + a_1 b_2(z-1) + a_0 b_3(z) - b_5(z), \\
&Q_6(z) = a_2 b_2(z-2) + a_1 b_3(z-1) - b_6(z), \ Q_7(z) = a_2 b_3(z-2) - b_7(z).
\end{aligned}
$$

A similar result is easily stated regarding the restrictions of M and L_M to the T-like triangles.

Proposition 7 *The nonzero restrictions p_i of M to the triangles T_i, $i \in \{9, 10, 15, 16, 17, 22, 23\}$, are given by the following polynomials:*

$$
\begin{aligned}
&p_9(x,y) = \frac{1}{4}\left(4+4x+x^2-4y-2xy+y^2\right), \ p_{10}(x,y) = \frac{1}{8}\left(7-4x-2x^2-8y+4xy+2y^2\right), \\
&p_{15}(x,y) = \frac{1}{8}\left(5+4x-4y^2\right), p_{16}(x,y) = \frac{1}{2}\left(1-x^2-y^2\right), p_{17}(x,y) = \frac{1}{8}\left(9-12x+4x^2\right), \\
&p_{22}(x,y) = \frac{1}{4}\left(4+4x+x^2+4y+2xy+y^2\right), p_{23}(x,y) = \frac{1}{8}\left(7-4x-2x^2+8y-4xy+2y^2\right).
\end{aligned}
$$

If q_i stands for the restriction of L_M to T_i, then it holds

$$
\begin{aligned}
&q_1(x,y) = c_{0,2}p_1(x,y-2), \quad q_2(x,y) = c_{0,2}p_2(x,y-2), \quad q_3(x,y) = c_{-1,1}p_1(x+1,y-1), \\
&q_4(x,y) = c_{0,1}p_1(x,y-1) + c_{-1,1}p_2(x+1,y-1) + c_{0,2}p_3(x,y-2), \\
&q_5(x,y) = c_{1,1}p_1(x-1,y-1) + c_{0,1}p_2(x,y-1) + c_{0,2}p_4(x,y-2), \\
&q_6(x,y) = c_{1,1}p_2(x-1,y-1) + c_{0,2}p_5(x,y-2), \quad q_7(x,y) = c_{-2,0}p_1(x+2,y), \\
&q_8(x,y) = c_{-1,0}p_1(x+1,y) + c_{-2,0}p_2(x+2,y) + c_{-1,1}p_3(x+1,y-1), \\
&q_9(x,y) = c_{0,0}p_1(x,y) + c_{-1,0}p_2(x+1,y) + c_{0,1}p_3(x,y-1) + c_{-1,1}p_4(x+1,y-1) + c_{0,2}p_6(x,y-2), \\
&q_{10}(x,y) = c_{1,0}p_1(x-1,y) + c_{0,0}p_2(x,y) + c_{1,1}p_3(x-1,y-1) + c_{0,1}p_4(x,y-1) + c_{-1,1}p_5(x+1,y-1) \\
&\qquad + c_{0,2}p_7(x,y-2), \\
&q_{11}(x,y) = c_{2,0}p_1(x-2,y) + c_{1,0}p_2(x-1,y) + c_{1,1}p_4(x-1,y-1) + c_{0,1}p_5(x,y-1), \\
&q_{12}(x,y) = c_{2,0}p_2(x-2,y) + c_{1,1}p_5(x-1,y-1), \quad q_{13}(x,y) = c_{-2,0}p_3(x+2,y), \\
&q_{14}(x,y) = c_{-1,-1}p_1(x+1,y+1) + c_{-1,0}p_3(x+1,y) + c_{-2,0}p_4(x+2,y) + c_{-1,1}p_6(x+1,y-1), \\
&q_{15}(x,y) = c_{0,-1}p_1(x,y+1) + c_{-1,-1}p_2(x+1,y+1) + c_{0,0}p_3(x,y) + c_{-1,0}p_4(x+1,y) + c_{-2,0}p_5(x+2,y) \\
&\qquad + c_{0,1}p_6(x,y-1) + c_{-1,1}p_7(x+1,y-1), \\
&q_{16}(x,y) = c_{1,-1}p_1(x-1,y+1) + c_{0,-1}p_2(x,y+1) + c_{1,0}p_3(x-1,y) + c_{0,0}p_4(x,y) + c_{-1,0}p_5(x+1,y) \\
&\qquad + c_{1,1}p_6(x-1,y-1) + c_{0,1}p_7(x,y-1) \\
&q_{17}(x,y) = c_{1,-1}p_2(x-1,y+1) + c_{2,0}p_3(x-2,y) + c_{1,0}p_4(x-1,y) + c_{0,0}p_5(x,y) + c_{1,1}p_7(x-1,y-1), \\
&q_{18}(x,y) = c_{2,0}p_4(x-2,y) + c_{1,0}p_5(x-1,y), \quad q_{19}(x,y) = c_{2,0}p_5(x-2,y), \quad q_{20}(x,y) = c_{-2,0}p_6(x+2,y), \\
&q_{21}(x,y) = c_{-1,-1}p_3(x+1,y+1) + c_{-1,0}p_6(x+1,y) + c_{-2,0}p_7(x+2,y), \\
&q_{22}(x,y) = c_{0,-2}p_1(x,y+2) + c_{0,-1}p_3(x,y+1) + c_{-1,-1}p_4(x+1,y+1) + c_{0,0}p_6(x,y) + c_{-1,0}p_7(x+1,y), \\
&q_{23}(x,y) = c_{0,-2}p_2(x,y+2) + c_{1,-1}p_3(x-1,y+1) + c_{0,-1}p_4(x,y+1) + c_{-1,-1}p_5(x+1,y+1) \\
&\qquad + c_{1,0}p_6(x-1,y) + c_{0,0}p_7(x,y), \\
&q_{24}(x,y) = c_{1,-1}p_4(x-1,y+1) + c_{0,-1}p_5(x,y+1) + c_{2,0}p_6(x-2,y) + c_{1,0}p_7(x-1,y), \\
&q_{25}(x,y) = c_{1,-1}p_5(x-1,y+1) + c_{2,0}p_7(x-2,y), \quad q_{26}(x,y) = c_{-1,-1}p_6(x+1,y+1), \\
&q_{27}(x,y) = c_{0,-2}p_3(x,y+2) + c_{0,-1}p_6(x,y+1) + c_{-1,-1}p_7(x+1,y+1), \\
&q_{28}(x,y) = c_{0,-2}p_4(x,y+2) + c_{1,-1}p_6(x-1,y+1) + c_{0,-1}p_7(x,y+1),
\end{aligned}
$$

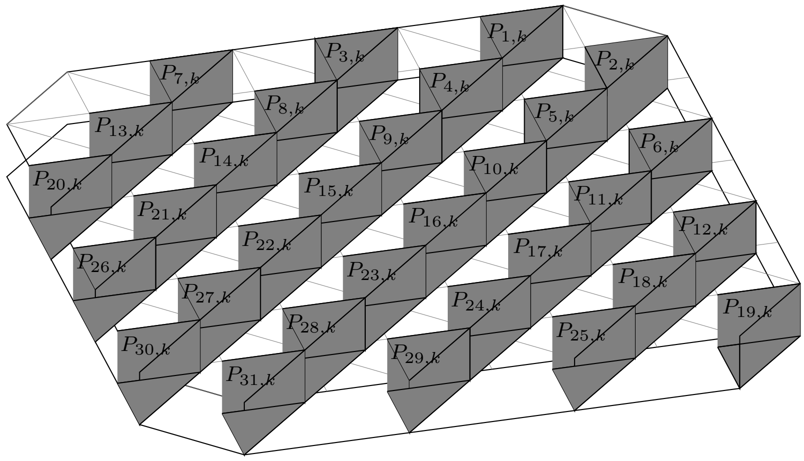

Figure 3: Prisms $P_{j,k}$ of the k-th level of the set $\Omega \times \left[-\frac{7}{2}, \frac{7}{2}\right]$.

$q_{29}(x,y) = c_{0,-2}p_5(x,y+2) + c_{1,-1}p_7(x-1,y+1)$, $\quad q_{30}(x,y) = c_{0,-2}p_6(x,y+2)$,
$q_{31}(x,y) = c_{0,-2}p_7(x,y+2)$.

Once determined the polynomial structure of B, $L_B - B$, M and L_M, it only remains to restrict the fundamental function of the operator R to every P-like prism $P_{j,k}$, $1 \le j \le 31$, $1 \le k \le 7$ (see Figure 3) and to translate the resulting functions as indicated below to produce the terms whose supports contain the prism P. Concerning the translation in the z-direction, we go through the set $\Omega \times \left[-\frac{7}{2}, \frac{7}{2}\right]$ from interval $\left[-\frac{7}{2}, -\frac{5}{2}\right]$ to interval $\left[\frac{5}{2}, \frac{7}{2}\right]$. Regarding the translation in the x and $y-$ directions, the centers $\gamma_i := (\gamma_{i,1}, \gamma_{i,2})$ form the subset $\Gamma := \{\gamma_i, 1 \le i \le 31\}$ given by

$$\begin{aligned}\Gamma = \{&(1,-3),(0,-3),(2,-2),(1,-2),(0,-2),(-1,-2),(3,-1),(2,-1),(1,-1),(0,-1),\\ &(-1,-1),(-2,-1),(3,0),(2,0),(1,0),(0,0),(-1,0),(-2,0),(-3,0),(3,1),(2,1),\\ &(1,1),(0,1),(-1,1)(-2,1),(2,2),(1,2),(0,2),(-1,2),(1,3),(0,3)\}.\end{aligned}$$

For every $1 \le j \le 31$ and $1 \le k \le 7$, the trivariate function

$$S_{j,k}(x,y,z) := q_j(x-\gamma_{j,1}, y-\gamma_{j,2})\, b_k(z+k-4) + p_j(x-\gamma_{j,1}, y-\gamma_{j,2})\, Q_k(z+k-4)$$

is the restriction of $L(\cdot - \gamma_{j,1}, \cdot - \gamma_{,j,2})$ to the prism P. Therefore, we get

$$\Lambda(x,y,z) \le \sum_{\substack{1\le j\le 31\\ 1\le k\le 7}} |S_{j,k}(x,y,z)| =: U(x,y,z), \ (x,y,z) \in P. \tag{3.2}$$

Now, we decompose the prism P into the following three tetrahedra. If the vertices of P are $V_{1,1} = \left(0,0,\frac{1}{2}\right)$, $V_{1,2} = \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$, $V_{1,3} = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$, $V_{2,1} = \left(0,0,-\frac{1}{2}\right)$, $V_{2,2} = \left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$ and $V_{2,3} = \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$, the tetrahedra are $\mathbb{T}_1 := [V_{1,1}, V_{2,1}, V_{2,2}, V_{2,3}]$, $\mathbb{T}_2 := [V_{1,1}, V_{1,2}, V_{2,2}, V_{2,3}]$ and $\mathbb{T}_3 := [V_{2,3}, V_{1,1}, V_{1,2}, V_{1,3}]$ (see Figure 4).

Let $\mathbb{T} := [V_1, V_2, V_3, V_4]$ be one of the tetrahedra above, and let $\lambda := (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be the barycentric coordinates of a point (x,y,z) with respect to $\mathbb{T}$, i.e. it holds

$$(x,y,z) = \sum_{v=1}^{4} \lambda_v V_v, \ |\lambda| := \sum_{v=1}^{4} \lambda_v = 1.$$

Since every function $S_{j,k}(x,y,z)$ in the upper bound (3.2) is a trivariate quartic polynomial on $\mathbb{T}$ that can be represented in terms of the Bernstein polynomials

$$\mathbb{B}_\alpha^{4,\mathbb{T}}(\lambda) := \frac{4!}{\alpha!}\lambda^\alpha = \frac{4!}{\alpha_1!\alpha_2!\alpha_3!\alpha_4!}\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4},$$

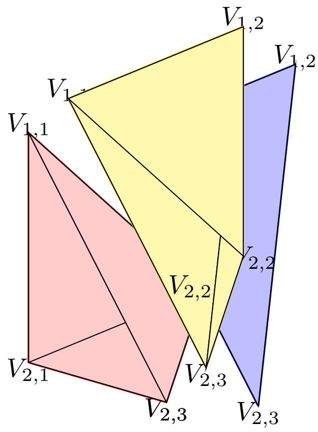

Figure 4: Decomposition of prism P into tetrahedra.

where the length $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ of the multi-index $\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in (\mathbb{N} \cup \{0\})^4$ is equal to four. Therefore, there exist coefficients $g^{\mathbb{T}}_{\alpha,j,k} := g^{\mathbb{T}}_{\alpha,j,k}(a, c)$ such that

$$S_{j,k}(x, y, z) = \sum_{|\alpha|=4} g^{\mathbb{T}}_{\alpha,j,k} \mathbb{B}^{4,\mathbb{T}}_{\alpha}(\lambda).$$

Thus, since $\left(\mathbb{B}^{4,\mathbb{T}}_{\alpha}\right)_{|\alpha|=4}$ is a non-negative partition of unity, for all $(x, y, z) \in \mathbb{T}$, it follows from (3.2) that

$$U(x, y, z) \le \sum_{\substack{1\le j\le 31\\1\le k\le 7}} \sum_{|\alpha|=4} \left|g^{\mathbb{T}}_{\alpha,j,k}\right| \mathbb{B}^{4,\mathbb{T}}_{\alpha}(\lambda) = \sum_{|\alpha|=4} \left(\sum_{\substack{1\le j\le 31\\1\le k\le 7}} \left|g^{\mathbb{T}}_{\alpha,j,k}\right| \right) \mathbb{B}^{4,\mathbb{T}}_{\alpha}(\lambda) \le \max_{|\alpha|=4} \sum_{\substack{1\le j\le 31\\1\le k\le 7}} \left|g^{\mathbb{T}}_{\alpha,j,k}\right|.$$

When this construction is carried out for $\mathbb{T}_1$, $\mathbb{T}_2$ and $\mathbb{T}_3$, the following result is obtained.

Proposition 8 *Under the conditions above, the function*

$$F(a, b) := \max_{1\le \ell\le 3} \max_{|\alpha|=4} \sum_{\substack{1\le j\le 31\\1\le k\le 7}} \left|g^{\mathbb{T}_\ell}_{\alpha,j,k}(a, c)\right| \tag{3.3}$$

is an upper bound to the infinity norm of the trivariate blending QIO R.

Hence, we state the following minimization problem in order to determine a near-best blending QIO.

Problem 9 *Minimize the objective function $F(a, c)$ given in (3.3) on*

$$\mathcal{A} := \left\{c \mid Q \textit{ is exact on } \mathbb{P}_2\left(\mathbb{R}^2\right)\right\} \times \left\{a \mid \overline{Q} \textit{ is exact on } \mathbb{P}_2(\mathbb{R})\right\}$$

Every coefficient $g_{a,j,k}$ is a linear function of a and c, hence $|g_{a,j,k}|$ is also a convex function. Thus, F is a convex function on $\mathcal{A}$ since it is the maximum of a set of convex functions, and the existence of a solution for Problem 9 is guaranteed (see e.g. [23]). If the minimum value of F is attained at a point $(c, a) \in \mathcal{A}$, then corresponding operator R is said to be a near-best QIO.

The nonlinear minimization Problem 9 can be solved by converting it into a linear programming one with inequality and equality constraints.

An arduous work that uses the symbolic computation software *Mathematica* allows to prove the following result.

Proposition 10 *The minimum value of $F(a,c)$ on $\mathcal{A}$ is equal to $\frac{532653}{393376} \approx 1.35406$. It is attained uniquely at $a = \left(\frac{12907}{12293}; \frac{823}{98344}; -\frac{3279}{98344}\right)$ and $c = \left(\frac{50881}{49172}; \frac{441}{12293}; -\frac{441}{12293}, -\frac{1709}{196688}\right)$.*

Proof. By replacing the solutions $(a_0, a_1, a_2) = \left(\frac{1}{16}(17-24x), x, -\frac{1}{32}(1+8x)\right)$ and $(c_{0,0}, c_{1,0}, c_{2,0}, c_{1,1}) = \left(y, z, \frac{1}{16}(-5+4y+8y), \frac{1}{16}(9-8y-24z)\right)$ of the linear systems in Lemmas 1 and 2 into the objective function of Problem 9, we get an unconstrained minimization problem with objective function

$$\widetilde{F}(x,y,z) = \max_{1\le \ell \le 3} \max_{|\alpha|=4} \sum_{\substack{1\le j\le 31 \\ 1\le k\le 7}} \left| G^{\mathbb{T}_\ell}_{\alpha,j,k}(x,y,z) \right|,$$

where

$$G^{\mathbb{T}_\ell}_{\alpha,j,k}(x,y,z) := g^{\mathbb{T}_\ell}_{\alpha,j,k}\left(\frac{1}{16}(17-24x), x, -\frac{1}{32}(1+8x), y, z, \frac{1}{16}(-5+4y+8y), \frac{1}{16}(9-8y-24z)\right)$$

Function $\widetilde{F}$ has been constructed from the BB-coefficients relative to the three tedrahedra in which the prism P has been decomposed of the integer translates of the fundamental function associated with the operator R. A large number of BB-coefficients are equal to zero because the B-spline B is nonzero only on the interval $\left[-\frac{3}{2}, \frac{3}{2}\right]$, and the box spline M is zero on $\Omega \setminus \omega$. It is also possible to eliminate the repetitions of coefficients that occur in the three tetrahedra. An explicit calculation shows that $\widetilde{F}$ is defined from 34 expressions depending on a maximum of 121 terms, so

$$\widetilde{F}(x,y,z) = \max_{1\le p\le 34} \frac{1}{\widetilde{c}_p} \sum_{n=1}^{121} \widetilde{d}_{p,n} \left| \widetilde{\phi}_{p,n} x + \widetilde{\varphi}_{p,n} y + \widetilde{\psi}_{p,n} z \right|,$$

for integers $\widetilde{\phi}_{p,n}$, $\widetilde{\varphi}_{p,n}$, $\widetilde{\psi}_{\alpha,n}$, and $\widetilde{c}_p, \widetilde{d}_{p,n} \in \mathbb{N}$. Therefore, the minimization of $\widetilde{F}$ is equivalent to the following linear programming problem:

$$\begin{array}{ll} \text{Minimize} & \mu \\ \text{such that} & \left\{ \begin{array}{ll} \displaystyle\sum_{n=1}^{121} \widetilde{d}_{p,n}(u_{p,n}+v_{p,n}) - \widetilde{c}_p \mu \le 0, & 1 \le p \le 34, \\ \widetilde{\phi}_{p,n}(X_1 - X_2) + \widetilde{\varphi}_{p,n}(Y_1 - Y_2) + \widetilde{\psi}_{p,n}(Z_1 - Z_2) - u_{p,n} + v_{p,n} = 0, & 1 \le n \le 121, \\ u_{p,n}, v_{p,n}, X_1, X_2, Y_1, Y_2, Z_1, Z_2, \mu \ge 0. & \end{array} \right. \end{array}$$

The solution of this problem is then determined with the symbolic calculation software, and the minimum value $\mu = \frac{532653}{393376}$ is reached at

$$X_1 = \frac{823}{98344},\ X_2 = 0,\ Y_1 = \frac{50881}{49172},\ Y_2 = 0,\ Z_1 = \frac{441}{12293} \quad \text{and} \quad Z_2 = 0,$$

i.e. $x = \frac{823}{98344}$, $y = \frac{50881}{49172}$ and $z = \frac{441}{12293}$. Analyzing the $\widetilde{F}$ function in a neighbourhood of $\left(\frac{823}{98344}, \frac{50881}{49172}, \frac{441}{12293}\right)$ it is concluded that it is the unique point at which the minimum is attained.

If $Q_{a,c}$ denotes the operator given by the solution of Problem 9, then the evaluation of its Lebesgue function at the points resulting in dividing the subset $\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$ into $20\times 20\times 20$ equal parts provides the value 1.34899 as a lower bound to $\|Q_{a,c}\|_\infty$. This shows that the proposed construction has allowed to improve the result in [9], where the near-best blending QIO is obtained by minimizing an objective function also established from the BB-coefficients of the B-spline and the box spline, and has a uniform norm equal to $\frac{11}{8} \simeq 1.375$.

4 Conclusions

In this paper, we have constructed a new trivariate near-best spline quasi-interpolation operator by blending 1D and 2D C^1 quadratic spline quasi-interpolants and minimizing an upper bound of its infinity norm. It is derived from the Bernstein-Bézier coefficients of its Lebesgue function. In particular, the new operator has a smaller norm with respect to the blending quasi-interpolant obtained in [9]. Such a technique can also be generalized by considering different spline spaces.

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