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RANDOM-FIELD SOLUTIONS OF WEAKLY HYPERBOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH POLYNOMIALLY BOUNDED COEFFICIENTS

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Abstract. We study random-field solutions of a class of stochastic partial differential equations, involving operators with polynomially bounded coefficients. We consider linear equations under suitable hyperbolicity hypotheses, and we provide conditions on the initial data and on the stochastic term, namely, on the associated spectral measure, so that these kind of solutions exist in suitably chosen functional classes. We also give a regularity result for the expected value of the solution.

1. Introduction

We consider linear stochastic partial differential equations (SPDEs in the sequel) of the general form

$$L(t, x, \partial_t, \partial_x)u(t, x) = \gamma(t, x) + \sigma(t, x)\dot{\Xi}(t, x), \tag{1.1}$$

where L is a linear partial differential operator that contains partial derivatives in time ($t \in \mathbb{R}$) and space ($x \in \mathbb{R}^d$, $d \geq 1$), γ, σ are real-valued functions, subject to certain regularity conditions, Ξ is a random noise term that will be described in detail in Section 2, and u is an unknown stochastic process called *solution* of the SPDE. Since the sample paths of the solution u are in general not in the domain of the operator L, in view of the singularity of the random noise, we rewrite (1.1) in its corresponding integral (i.e., *weak*) form and look for *mild solutions of* (1.1), that is, stochastic processes $u(t, x)$ satisfying

$$u(t, x) = v_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y)\gamma(s, y)dyds + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y)\sigma(s, y)\dot{\Xi}(s, y)dyds, \tag{1.2}$$

where:

- v_0 is a deterministic term, taking into account the initial conditions;
- Λ is a suitable kernel, associated with the fundamental solution of the partial differential equation (PDE in the sequel) $Lu = 0$;
- the first integral in (1.2) is of deterministic type, while the second is a stochastic integral.

Note that both integrals in (1.2) contain a slight abuse of notation, since $\Lambda(t, s, x, y)$ is, in general, a distribution with the respect to the variables $(x, y) \in \mathbb{R}^{2d}$. Given the commonly wide usage of such so-called *distributional integrals*, we will also adopt here this notation in the representation of our class of mild solutions to (1.1).

The kind of solution u we can construct for equation (1.1) depends on the approach we employ to make sense of the stochastic integral appearing in (1.2).

A first approach consists in the Da Prato- Zabczyk theory of stochastic integration, see [23], where a Brownian motion, with values in a Hilbert space, is associated with the random noise, and then the stochastic integral is defined as an infinite sum of Itô integrals with respect to one-dimensional Brownian motions. This leads to the so-called *function(al-spaces)-valued solutions* of (1.1), i.e. solutions involving random functions taking values in suitable functional spaces. A general theory of existence and uniqueness of function-valued solutions for (semi)linear SPDEs is presented in the recent paper [6]. In general, this kind of solution cannot be evaluated in the spatial argument (usually, it is a random element in the t (that is, *time*) parameter, taking values in a $L^p(\mathbb{R}^d)$-modeled Banach or Hilbert space).

An alternative approach, due to [11, 20, 39], focuses instead on the concept of stochastic integral with respect to a martingale measure. That is, the stochastic integral in (1.2) is defined through the martingale measure derived from the random noise $\dot{\Xi}$. Here one obtains a so-called *random-field solution*, that is, a solution u defined as a map associating a random variable to each $(t, x) \in [0, T_0] \times \mathbb{R}^d$, where $T_0 > 0$ is the time horizon of the

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solution of the equation. In many cases, the theory of integration with respect to processes taking values in functional spaces, and the theory of integration with respect to martingale measures, lead to the same solution u (in some sense) of an SPDE, see [22] for a precise comparison.

In the recent paper [7], the existence of a random-field solution to (1.1) in the case of linear strictly hyperbolic SPDEs with (t, x)-dependent coefficients, uniformly bounded with respect to x, has been shown. The main tools used for achieving this objective, namely, pseudodifferential and Fourier integral operators, come from microlocal analysis. To our knowledge, that was the first time that their full potential has been rigorously applied within the theory of random-field solutions to hyperbolic SPDEs. Other applications of these operators in the context of S(P)DEs can be found in [38], where S(P)DEs are investigated in the framework of function-valued solutions by means of pseudodifferential operators, and in [31], where a program for employing Fourier integral operators in stochastic structural analysis is described. We are not aware of any other systematic application of microlocal and Fourier integral operators techniques to the analysis of hyperbolic SPDEs with unbounded coefficients.

In the present paper we deal with the existence of a random-field solution to hyperbolic SPDES of the form (1.1) with (t, x)-dependent coefficients admitting, at most, a polynomial growth as $|x| \to \infty$. As customary for the classes of the associated deterministic PDEs, we are interested in both the smoothness, as well as the decay/growth at spatial infinity of the solutions. Here we also obtain an analog of such *global regularity* properties, employing suitable *weighted Sobolev spaces*, namely, the so-called Sobolev-Kato spaces $H^{z,\zeta}(\mathbb{R}^d)$, $z, \zeta \in \mathbb{R}$. The results proved in this paper expand the theory developed in [7] to the cases of operators L which are not strictly hyperbolic, and whose coefficients are not uniformly bounded.

More precisely, here we treat *weakly hyperbolic equations* of arbitrary order $m \in \mathbb{N}$ of the form (1.1), whose coefficients are defined on the whole space $\mathbb{R}^d$, with

$$L = D_t^m - \sum_{j=1}^{m} A_j(t, x, D_x) D_t^{m-j}, \qquad A_j(t, x, D) = \sum_{|\alpha| \le j} a_{\alpha j}(t, x) D_x^\alpha, \tag{1.3}$$

where $m \ge 1$, $a_{\alpha j} \in C^\infty([0, T], C^\infty(\mathbb{R}^d))$ for $|\alpha| \le j$, $j = 0, \ldots, m$, and, for all $k \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^d$, there exists a constant $C_{jk\alpha\beta} > 0$ such that

$$|\partial_t^k \partial_x^\beta a_{\alpha j}(t, x)| \le C_{jk\alpha\beta} \langle x \rangle^{|\alpha| - |\beta|},$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $0 \le |\alpha| \le j$, $1 \le j \le m$. The hyperbolicity of L means that the symbol $\mathcal{L}_m(t, x, \tau, \xi)$ of the SG-principal part of L, defined here below, satisfies

$$\mathcal{L}_m(t, x, \tau, \xi) := \tau^m - \sum_{j=1}^{m} \sum_{|\alpha| = j} a_{\alpha j}(t, x) \xi^\alpha \tau^{m-j} = \prod_{j=1}^{m} \Big(\tau - \tau_j(t, x, \xi) \Big), \tag{1.4}$$

with $\tau_j(t, x, \xi)$ real-valued, $\tau_j \in C^\infty([0, T]; S^{1,1}(\mathbb{R}^d))$, $j = 1, \ldots, m$. The latter means that, for any $\alpha, \beta \in \mathbb{N}_0^d$, $k \in \mathbb{N}_0$, there exists a constant $C_{jk\alpha\beta} > 0$ such that

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta \tau_j(t, x, \xi)| \le C_{jk\alpha\beta} \langle x \rangle^{1 - |\alpha|} \langle \xi \rangle^{1 - |\beta|},$$

for $(t, x, \xi) \in [0, T] \times \mathbb{R}^{2d}$, $j = 1, \ldots, m$ (see Section 3 below for the definition of the so-called SG-classes of symbols $S^{m,\mu}(\mathbb{R}^d)$, $(m, \mu) \in \mathbb{R}^2$, and the corresponding class of pseudodifferential operators). The real solutions $\tau_j = \tau_j(t, x, \xi)$, $j = 1, \ldots, m$, of the equation $\mathcal{L}_m(t, x, \tau, \xi) = 0$ with respect to τ are usually called *characteristic roots* of the operator L. We will focus on weakly hyperbolic operators with characteristics of constant multiplicities. Of course, the strictly hyperbolic operators with coefficients of polynomial growth in x are covered, too, as a special case of the constant multiplicites ones, when the maximum multiplicity l of the characteristics satisfies $l = 1$. Postponing to Definition 3.8 below their precise characterization, we give here an example.

Example 1.1. An example of a *weakly SG-hyperbolic operator L with roots of constant multiplicities* is given by the square of the so-called *SG-wave operator* $A = D_t^2 - \langle x \rangle^2 \langle D \rangle^2 = -\partial_t^2 + (1 + |x|^2)(\Delta - 1)$, that is

$$L = (D_t^2 - \langle x \rangle^2 \langle D \rangle^2)^2 = D_t^4 - 2\langle x \rangle^2 \langle D \rangle^2 D_t^2 + \langle x \rangle^4 \langle D \rangle^4 + \mathrm{Op}(p), \qquad x \in \mathbb{R}^d,$$

$p \in S^{3,3}(\mathbb{R}^d)$, where, for $c \in S^{m,\mu}(\mathbb{R}^d)$, $\mathrm{Op}(c)$ denotes the pseudodifferential operator with symbol c, see Section 3. The SG-principal symbol of L is $L_4(x, \tau, \xi) = (\tau^2 - \langle x \rangle^2 \langle \xi \rangle^2)^2$, with real-valued roots $\tau_\pm(x, \xi) = \pm \langle x \rangle \langle \xi \rangle$, both of *multiplicity 2*, *separated* at infinity, in the sense that $\tau_+(x, \xi) - \tau_-(x, \xi) \gtrsim \langle x \rangle \langle \xi \rangle$. Similarly, A is an example of strictly SG-hyperbolic operator, with characteristic roots $\tau_\pm$, both of multiplicity equal to 1.

We study SPDEs of the form (1.1), (1.3), (1.4), where Ξ is an $\mathcal{S}'(\mathbb{R}^d)$-valued Gaussian process, white in time and coloured in space, with correlation measure Γ and spectral measure μ, see Section 2 for a precise definition. We derive conditions on the coefficients of L, on the right-hand side terms γ and σ, and on the spectral measure μ (hence, on Ξ), such that there exists a unique random-field (mild) solution to the corresponding Cauchy problem. Namely, we will prove that

(WH)$_{\mathbf{CMR}}$ if L is weakly SG-hyperbolic with constant multiplicities, and

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(1+|\xi+\eta|^2)^{m-l}}\mu(d\xi)<\infty,$$

where l is the maximum multiplicity of the characteristic roots,

then, under suitable assumptions on the coefficients γ,σ and on the Cauchy data, there exists a random-field solution to the SPDE (1.1). Notice that the more general are the assumptions on L (i.e., the larger is l), the smallest is the class of the stochastic noises that we can allow to get a random-field solution. Our main result reads as follows (see Sections 3 and 4, and Theorem 4.1 below, for the precise definitions and statement).

Theorem. Let us consider the Cauchy problem

$$\begin{cases} Lu(t,x)=\gamma(t,x)+\sigma(t,x)\dot{\Xi}(t,x), & (t,x)\in(0,T]\times\mathbb{R}^d,\\ D_t^j u(0,x)=u_j(x), & x\in\mathbb{R}^d,\ 0\le j\le m-1,\end{cases}\tag{1.5}$$

for an SPDE associated with an operator of the form (1.3), satisfying the hyperbolicity hypothesis **(WH)$_{\mathbf{CMR}}$**. Assume also that L is of Levy type, and let $u_j\in H^{z+m-j-1,\zeta+m-j-1}(\mathbb{R}^d)$, $0\le j\le m-1$, with $z\in\mathbb{R}$ and $\zeta>d/2$. Finally, assume that $\gamma\in C([0,T];H^{z,\zeta}(\mathbb{R}^d))$, $\sigma\in C([0,T],H^{0,\zeta})$, $s\mapsto\mathcal{F}\sigma(s)=\nu_s\in L^2([0,T],\mathcal{M}_b(\mathbb{R}^d))$.

Then, for some time horizon $0<T_0\le T$, there exists a random-field solution u of (1.5). Moreover, $\mathbb{E}[u]\in C([0,T_0],H^{z+m-l,\zeta+m-l}(\mathbb{R}^d))$.

We will also formulate a similar result (see Theorem 4.14) concerning the case of involutive roots, that is:

(WH)$_{\mathbf{IR}}$ if L is weakly SG-hyperbolic with involutive roots and

$$\int_{\mathbb{R}^d}\mu(d\xi)<\infty,$$

then, under suitable assumptions on the coefficients γ,σ and on the Cauchy data, there exists a random-field solution to the SPDE (1.1). An operator with involutive characteristics is given in the next Example 1.2.

Example 1.2. An example of a *weakly hyperbolic operator L with involutive roots of non-constant multiplicities* is given by

$$L=(D_t+tD_{x_1}+D_{x_2})(D_t-(t-2x_2)D_{x_1}),\qquad x\in\mathbb{R}^2,$$

see [30]. Indeed, the SG-principal part admits the roots $\tau_1(t,x,\xi)=-t\xi_1-\xi_2$ and $\tau_2(t,x,\xi)=(t-2x_2)\xi_1$, which are real-valued but *not always separated*. In fact, τ_1 and τ_2 coincide in the set $\{(t,x,\xi)\mid\xi_2=2(x_2-t)\xi_1\}\subset[0,T]\times\mathbb{R}^{2d}$. Nevertheless, by straightforward computations we find

$$\begin{aligned}[D_t+tD_{x_1}+D_{x_2},D_t-(t-2x_2)D_{x_1}]&=[tD_{x_1},D_t]-[D_t,(t-2x_2)D_{x_1}]-[tD_{x_1}+D_{x_2},(t-2x_2)D_{x_1}]\\&=2\mathrm{i}D_{x_1}-(tD_{x_1}+D_{x_2})((t-2x_2)D_{x_1})+(t-2x_2)D_{x_1}(-tD_{x_1}-D_{x_2})\\&=2\mathrm{i}D_{x_1}-2\mathrm{i}D_{x_1}=0,\end{aligned}$$

which is (3.12) with $a_{12}(t)=b_{12}(t)=c_{12}(t)\equiv 0$.

Notice that the condition on the spectral measure to be satisfied in **(WH)$_{\mathbf{IR}}$** corresponds to the *limit case* $l=m$ of the condition given in **(WH)$_{\mathbf{CMR}}$**. Notice also that, of course, when $m=1$ or $l=m$, the two conditions coincide.

The main tools for proving existence and uniqueness of solutions to (1.1) will be Fourier integral operators with symbols in the so-called SG classes. Such symbol classes have been introduced in the '70s by H.O. Cordes (see, e.g. [13]) and C. Parenti [32] (see also R. Melrose [27]). To construct the fundamental solution of (1.1) we need, on one hand, to perform compositions between pseudo-differential operators and Fourier integral operators of SG type, using the theory developed in [14], and, on the other hand, compositions between Fourier integral operators of SG type with possibly different phase functions. The latter can be achieved using the composition results obtained in [4], with the aim of employing them to study the SPDEs treated in the present paper. The proofs of the results achieved in this paper follow an approach similar to the one adopted for the applications treated in [4, 15, 18]. To provide a presentation as self-contained as possible, for the convenience of the reader, we provide (at different levels of detail) various preliminaries from the existing literature. The paper is organized as follows.

In Section 2 we recall some notions about stochastic integration with respect to martingale measures and the corresponding concept of random-field solution to an SPDE. Since, in contrast to the classical references [20, 39], here we have to deal with integrands of the form $\Lambda(t,s,x,y)\sigma(s,y)$ with (t,x) fixed, we directly present here the conditions that Λ and σ have to satisfy to let the stochastic integral with respect to a martingale measure in (1.2) be well-defined.

In Section 3 we give a brief summary of the main tools, coming from microlocal analysis, that we use for the construction of the fundamental solution operator and of its kernel $\Lambda(t,s,x,y)$. The results presented in this section come mainly from [4, 13, 14, 15, 18]. A few more results from microlocal analysis that we need are collected in an Appendix.

In Section 4 we focus on the hyperbolic SPDE (1.1), (1.3), (1.4), and prove our main theorem under the assumption of weak SG-hyperbolicity with constant multiplicities $\textbf{(WH)}_{\textbf{CMR}}$ (see Theorem 4.1). We recall the construction of the equivalent systems performed in [4, 15, 18] and of their fundamental solution. The latter are crucial results, since all the classes of hyperbolic equations that we consider here can be reduced to a first order hyperbolic system. We also state a similar result under the assumption of weak SG-hyperbolicity with involutive roots $\textbf{(WH)}_{\textbf{IR}}$ (see Theorem 4.14). Finally, we mention that the results illustrated in Section 4 about the structure of the kernel $\Lambda(t,s,x,y)$ appearing in (1.2) are employed also in [6], where we look for function-valued solutions to the semilinear SPDEs

$$L(t,x,\partial_t,\partial_x)u(t,x) = \gamma(t,x,u(t,x)) + \sigma(t,x,u(t,x))\dot{\Xi}(t,x) \tag{1.6}$$

associated with (1.1).

1.1. **Notation.** Throughout this article, we let $\langle a\rangle := (1+|a|^2)^{1/2}$ for all $a \in \mathbb{R}^d$, and we denote $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, $\mathbb{R}^d_* := \mathbb{R}^d\setminus\{0\}$. Also, α and β will generally denote multiindeces, with their standard arithmetic operations. As usual, we will denote partial derivatives with ∂, and set $D = -i\partial$, i being the imaginary unit, which is convenient when dealing with Fourier transformations. We will denote by $C^m(X)$, $C^m_0(X)$, $C_b(X)$, $\mathcal{S}(X)$, $\mathcal{D}(X)$, $\mathcal{S}'(X)$, $\mathcal{S}'(X)_\infty$, and $\mathcal{D}'(X)$, the m-times continuously differentiable functions, the m-times continuously differentiable functions with compact support, the continuous and bounded functions, the Schwartz functions, the test functions space $C^\infty_0(X)$, the tempered distributions, the tempered distributions with rapid decrease and the distributions on some finite or infinite-dimensional space X, respectively. We recall that a distribution $\Theta \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the space $\mathcal{S}'(\mathbb{R}^d)_\infty$ if, for every k, $\langle\cdot\rangle^k\Theta$ is a bounded distribution on $\mathbb{R}^d$, i.e. it belongs to the dual space of $\{\varphi \in C^\infty(\mathbb{R}^d) | \forall\alpha \in \mathbb{N}^d\ \partial^\alpha\varphi \in L^1(\mathbb{R}^d)\}$. It can be shown that $\mathcal{S}'(\mathbb{R}^d)_\infty = \mathcal{O}'_C(\mathbb{R}^d)$, where $\mathcal{O}'_C$ is the widest class of distributions such that the convolution with elements of $\mathcal{S}'$ is well-defined. One of its characterizations, which is useful for us, is the following: $\Theta \in \mathcal{O}'_C(\mathbb{R}^d)$ if and only if, for every $\chi \in \mathcal{D}(\mathbb{R}^d)$, we have $\Theta * \chi \in \mathcal{S}(\mathbb{R}^d)$. For more details, see [36] and the recent paper [5]. Usually, $C > 0$ will denote a generic constant, whose value can change from line to line without further notice. When operator composition is considered, we will usually insert the symbol $\circ$ when the notation $\mathrm{Op}(b)$ and/or $\mathrm{Op}_\varphi(a)$, for pseudodifferential and Fourier integral operators, respectively, are adopted for both factors, as well as in some situations where parameter-dependent operators occurs, for the sake of clarity. When at least one of the operators involved in the product of composition is denoted by a single capital letter, and when no confusion can occur, we will, as customary, omit the symbol $\circ$ completely, and just write, e.g., PQ, RD_t, etc. Finally, $A \asymp B$ means that the estimates $A \lesssim B$ and $B \lesssim A$ hold true, where $A \lesssim B$ means that $|A| \le c\cdot|B|$, for a suitable constant $c > 0$.

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2. Stochastic integration with respect to a martingale measure.

Let us consider a distribution-valued Gaussian process $\{\Xi(\phi);\ \phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with mean zero and covariance functional given by

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} \big(\phi(t) * \tilde{\psi}(t)\big)(x)\, \Gamma(dx)dt, \tag{2.1}$$

where $\tilde{\psi}(t,x) := \psi(t,-x)$, $*$ is the convolution operator and Γ is a nonnegative, nonnegative definite, tempered measure on $\mathbb{R}^d$. Then [36, Chapter VII, Théorème XVIII] implies that there exists a nonnegative tempered measure μ on $\mathbb{R}^d$ such that $\mathcal{F}\mu = \widehat{\mu} = \Gamma$. $\mathcal{F}$ and $\widehat{\ }$ denote the Fourier transform given, for functions $f \in L^1(\mathbb{R}^d)$, by

$$(\mathcal{F}f)(\xi) = \widehat{f(\xi)} := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)dx. \tag{2.2}$$

In (2.2), $x \cdot \xi$ denotes the inner product in $\mathbb{R}^d$, and the Fourier transform is extended to tempered distributions $T \in \mathcal{S}'(\mathbb{R}^d)$ by the relation $\langle \mathcal{F}T, \phi\rangle = \langle T, \mathcal{F}\phi\rangle$, for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. By Parseval's identity, the right-hand side of (2.1) can be rewritten as

$$\mathbb{E}[\Xi(\phi)\Xi(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} [\mathcal{F}\phi(t)](\xi) \cdot \overline{[\mathcal{F}\psi(t)](\xi)}\, \mu(d\xi)dt.$$

The tempered measure Γ is usually called *correlation measure*. The tempered measure μ such that $\Gamma = \widehat{\mu}$ is usually called *spectral measure*.

In this section we consider the SPDE (1.1) and its mild solution (1.2): this is the way in which we understand (1.1); we provide conditions to show that each term on the right-hand side of (1.2) is meaningful.
In fact, we call *(mild) random-field solution to* (1.1) an $L^2(\Omega)$-family of random variables $u(t,x)$, $(t,x) \in [0,T]\times\mathbb{R}^d$, jointly measurable, satisfying the stochastic integral equation (1.2).

We want to give a precise meaning to the stochastic integral in (1.2) by defining

$$\int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)dsdy := \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)M(ds,dy), \tag{2.3}$$

where, on the right-hand side, we have a stochastic integral with respect to the martingale measure M related to Ξ. As explained in [21], by approximating indicator functions with C_0^∞-functions, the process Ξ can indeed be extended to a worthy martingale measure $M = (M_t(A);\ t \in \mathbb{R}_+, A \in \mathscr{B}_b(\mathbb{R}^d))$, where $\mathscr{B}_b(\mathbb{R}^d)$ denotes the bounded Borel subsets of $\mathbb{R}^d$. The natural filtration generated by this martingale measure will be denoted in the sequel by $(\mathscr{F}_t)_{t\geq 0}$. The stochastic integral with respect to the martingale measure M of stochastic processes f and g, indexed by $(t,x) \in [0,T]\times\mathbb{R}^d$ and satisfying suitable conditions, is constructed by steps (see [11, 20, 39]), starting from the class $\mathcal{E}$ of simple processes, and making use of the pre-inner product (defined for suitable f, g)

$$\langle f,g\rangle_0 = \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^d} \big(f(s) * \tilde{g}(s)\big)(x)\, \Gamma(dx)ds\Big] = \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^d} [\mathcal{F}f(s)](\xi) \cdot \overline{[\mathcal{F}g(s)](\xi)}\, \mu(d\xi)ds\Big], \tag{2.4}$$

with corresponding semi-norm $\|\cdot\|_0$, as follows.

(1) For a *simple process*

$$g(t,x;\omega) = \sum_{j=1}^m 1_{(a_j,b_j]}(t) 1_{A_j}(x) X_j(\omega) \in \mathcal{E},$$

(with $m \in \mathbb{N}$, $0 \le a_j < b_j \le T$, $A_j \in \mathscr{B}_b(\mathbb{R}^d)$, X_j bounded, and $\mathscr{F}_{A_j}$-measurable random variable for all $1 \le j \le n$) the stochastic integral with respect to M is given by

$$(g\cdot M)_t := \sum_{j=1}^m \big(M_{t\wedge b_j}(A_j) - M_{t\wedge a_j}(A_j)\big)X_j,$$

where $x \wedge y := \min\{x,y\}$. One can show, by applying the definition, that the fundamental isometry

$$\mathbb{E}\Big[(g\cdot M)_t^2\Big] = \|g\|_0^2 \tag{2.5}$$

holds true for all $g \in \mathcal{E}$.

(2) Since the pre-inner product (2.4) is well-defined on elements of $\mathcal{E}$, if now we define $\mathcal{P}_0$ as the completion of $\mathcal{E}$ with respect to $\langle\cdot,\cdot\rangle_0$, then, for all the elements g of the Hilbert space $\mathcal{P}_0$, we can construct the stochastic integral with respect to M as an $L^2(\Omega)$-limit of simple processes via the isometry (2.5). So, $\mathcal{P}_0$ turns out to be the space of all integrable processes (with respect to M). Moreover, it has recently been shown (see Lemma 2.2 in [35]) that $\mathcal{P}_0 = L^2_p([0,T]\times\Omega,\mathcal{H})$, where here $L^2_p(\ldots)$ stands for the predictable stochastic processes in $L^2(\ldots)$ and $\mathcal{H}$ is the Hilbert space which is obtained by completing the Schwartz functions with respect to the inner product $\langle\cdot,\cdot\rangle_0$. Thus, $\mathcal{P}_0$ consists of predictable processes which may contain tempered distributions in the x-argument (whose Fourier transforms are functions, $\mathbb{P}$-almost surely).

Now, to give a meaning to the integral (2.3), we need to impose conditions on the distribution Λ and on the coefficient σ such that $\Lambda\sigma \in \mathcal{P}_0$. In [7], sufficient conditions for the existence of the integral on the right-hand side of (2.3) have been given, in the case that σ depends on the spatial argument y, assuming that the spatial Fourier transform of the function σ is a complex-valued measure with finite total variation. Namely, we assume that, for all $s \in [0,T]$,

$$|\mathcal{F}\sigma(\cdot,s)| = |\mathcal{F}\sigma(\cdot,s)|(\mathbb{R}^d) = \sup_{\pi}\sum_{A\in\pi}|\mathcal{F}\sigma(\cdot,s)|(A) < \infty,$$

where π is any partition on $\mathbb{R}^d$ into measurable sets A, and the supremum is taken over all such partitions. Let, in the sequel, $\nu_s := \mathcal{F}\sigma(\cdot,s)$, and let $|\nu_s|_{\mathrm{tv}}$ denote its total variation. We summarize these conditions in the following theorem.

Theorem 2.1. *Let Δ_T be the simplex given by $0 \le t \le T$ and $0 \le s \le t$. Let, for $(t,s,x) \in \Delta_T \times \mathbb{R}^d$, $\Lambda(t,s,x)$ be a deterministic function with values in $\mathcal{S}'(\mathbb{R}^d)_\infty$, and let σ be a function in $L^2([0,T],C_b(\mathbb{R}^d))$ such that:*

- ***(A1)*** *the function $(t,s,x,\xi) \mapsto [\mathcal{F}\Lambda(t,s,x)](\xi)$ is measurable, the function $s \mapsto \mathcal{F}\sigma(s) = \nu_s \in L^2([0,T],\mathcal{M}_b(\mathbb{R}^d))$, and, moreover,*

$$\int_0^T \left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}|[\mathcal{F}\Lambda(t,s,x)](\xi+\eta)|^2\mu(d\xi)\right)|\nu_s|^2_{\mathrm{tv}}\,ds < \infty; \tag{2.6}$$

- ***(A2)*** *Λ and σ are as in **(A1)** and*

$$\lim_{h\downarrow 0}\int_0^T\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\sup_{r\in(s,s+h)}|[\mathcal{F}(\Lambda(t,s,x)-\Lambda(t,r,x))](\xi+\eta)|^2\mu(d\xi)\right)|\nu_s|^2_{\mathrm{tv}}\,ds = 0.$$

Then $\Lambda\sigma \in \mathcal{P}_0$. In particular, the stochastic integral on the right-hand side of (2.3) is well-defined and

$$\mathbb{E}\Big[((\Lambda(t,\cdot,x,*)\sigma(\cdot,*))\cdot M)^2_t\Big] \le \int_0^t\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}|[\mathcal{F}\Lambda(t,s,x)](\xi+\eta)|^2\mu(d\xi)\right)|\nu_s|^2_{\mathrm{tv}}\,ds.$$

The reason for the assumption that $\Lambda(t) \in \mathcal{S}'(\mathbb{R}^d)_\infty$ is that, in this case, the Fourier transform in the second spatial argument is a smooth function of slow growth and the convolution of such a distribution with any other distribution in $\mathcal{S}'(\mathbb{R}^d)$ is well-defined, see [36, Chapter VII, §5]. A necessary and sufficient condition for $T \in \mathcal{S}'(\mathbb{R}^d)_\infty$ is that each regularization of T with a C_0^∞-function is a Schwartz function. This will be true in our application, due to Proposition 3.9 and the fact that the Fourier transform is a bijection on the Schwartz functions.

The meaning of assumption **(A1)** can be easily realized by formally computing, using (2.5), the definition of convolution between a distribution ν_s and a function $\mathcal{F}\Lambda(t,s,x,\cdot)$, together with Minkowski's integral inequality. In fact,

$$\begin{aligned}
&\|\Lambda(t,\cdot,x,*)\sigma(\cdot,*)\|_0^2\\
&= \int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Lambda(t,s,x,y)\Lambda(t,s,x,y-z)\sigma(s,y)\sigma(s,y-z)dy\Gamma(dz)ds\\
&= \int_0^t\int_{\mathbb{R}^d}\left|[\mathcal{F}\Lambda(t,s,x,\cdot)\sigma(s,\cdot)](\xi)\right|^2\mu(d\xi)ds\\
&= \int_0^t\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}[\mathcal{F}\Lambda(t,s,x,\cdot)](\xi-\eta)\nu_s(d\eta)\right|^2\mu(d\xi)ds\\
&\le \int_0^t\left(\int_{\mathbb{R}^d}\left(\int_{R^d}|[\mathcal{F}\Lambda(t,s,x,\cdot)](\xi-\eta)|^2\mu(d\xi)\right)^{1/2}|\nu_s(d\eta)|_{\mathrm{tv}}\right)^2ds
\end{aligned}$$

$$\leq \int_0^t \Big(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |[\mathcal{F}\Lambda(t,s,x,\cdot)](\xi+\eta)|^2 \mu(d\xi) \Big) |\nu_s|^2_{\text{tv}}\, ds. \tag{2.7}$$

We see that ν_s must have finite total variation for almost all $s \in [0,T]$ so that the previous term is finite. The condition $\sigma(s) \in C_b(\mathbb{R}^d)$ for every $s \in [0,T]$ in fact follows directly from the fact that ν_s has finite total variation for all $s \in [0,T]$ by considering the inverse Fourier transform (in the distributional sense) of ν_s and recalling that the Fourier transform of a measure with finite total variation is uniformly continuous, see [8, Chapter 5, §26].

The formal computations in (2.7) become rigorous defining the stochastic integral of $\Lambda\sigma$ as the limit in $\mathcal{P}_0$ of an appoximating sequence: in the proof of Theorem 2.1 we approximate $\Lambda \in \mathcal{S}'_\infty$ with a sequence in $\{\Lambda_j\} \in \mathcal{S}$, then we approximate the sequence $\{\Lambda_j\}$ by a sequence of step functions with respect to time $\Lambda_{j,k} \in \mathcal{P}_0$; assumption **(A2)** provides the convergence of $\{\Lambda_{j,k}\sigma\}_k$ to $\Lambda_j\sigma$ and of $\{\Lambda_j\sigma\}_j$ to $\Lambda\sigma$ in $\mathcal{P}_0$. For a complete proof of Theorem 2.1 see [7, Theorem 2.3].

Remark 2.2. In the particular case $\sigma = \sigma(s)$, that is, if σ does not depend on the spatial argument, then

$$\mathcal{F}\sigma(s) = (2\pi)^d \sigma(s)\delta_0,$$

where δ_0 is the Dirac delta distribution with total variation 1. By the same computations as in (2.7), we arrive at the necessary condition that

$$\int_0^T \sigma(s)^2 \int_{\mathbb{R}^d} |[\mathcal{F}\Lambda(t,s,x)](\xi)|^2 \mu(d\xi) ds < \infty,$$

which is actually weaker than (2.6), in the sense that there is no supremum over η, and corresponds to the one given in [20, Example 9].

In the next Theorem 2.3 we deal with the pathwise integral $\int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y)dyds$ that appears in (1.2), and state sufficient conditions to give it a rigorous meaning. We assume that the spatial Fourier transform of the coefficient $\gamma(s)$ is a measure with finite total variation, denoted by χ_s.

Theorem 2.3. *For $(t,s,x) \in \Delta_T \times \mathbb{R}^d$, let $(t,s,x) \mapsto \Lambda(t,s,x)$ be a deterministic function taking values in $\mathcal{S}'(\mathbb{R}^d)_\infty$ and let $\gamma \in L^2([0,T], C_b(\mathbb{R}^d))$. Assume that*

- ***(A3)*** *the function $(t,s,x,\xi) \mapsto [\mathcal{F}\Lambda(t,s,x)](\xi)$ is measurable, the function $s \mapsto \mathcal{F}\gamma(s) = \chi_s \in L^2([0,T], \mathcal{M}_b(\mathbb{R}^d))$, and, moreover,*

$$\int_0^T \Big(\sup_{\eta\in\mathbb{R}^d} |[\mathcal{F}\Lambda(t,s,x)](\eta)|^2 \Big) |\chi_s|^2_{\text{tv}}\, ds < \infty; \tag{2.8}$$

- ***(A4)*** *let Λ and γ be as in **(A3)**, and also satisfy*

$$\lim_{h\downarrow 0} \int_0^T \Big(\sup_{\eta\in\mathbb{R}^d} \sup_{r\in(s,s+h)} |\mathcal{F}(\Lambda(t,s,x) - \Lambda(t,r,x))(\eta)|^2 \Big) |\chi_s|^2_{\text{tv}}\, ds = 0.$$

Then, the pathwise integral is well-defined and

$$\Big(\int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y)dyds \Big)^2 \leq C \int_0^t \Bigg(\sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Lambda(t,s,x)(\eta)|^2 \Bigg) |\chi_s|^2_{\text{tv}}\, ds. \tag{2.9}$$

As in Remark 2.2, the assumptions **(A3)** and **(A4)** can be weakened when the coefficient γ does not depend on the spatial argument. Note, moreover, that the two conditions **(A3)** and **(A4)** coincide with **(A1)** and **(A2)**, respectively, if $\mu = \delta_0$.

Summing up:

Theorem 2.4. *With the notation of Theorems 2.1 and 2.3, let us assume $\Lambda(t,s,x)$ to be an $\mathcal{S}'(\mathbb{R}^d)_\infty$-valued deterministic function on $\Delta_T \times \mathbb{R}^d$ and $\sigma, \gamma \in L^2([0,T], C_b(\mathbb{R}^d))$ such that assumptions **(A1)**, **(A2)**, **(A3)** **(A4)** hold true. Assume, moreover, that*

- ***(A5)*** *for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $v_0(t,x)$ is finite.*

Then, the random-field solution of the SPDE (1.1) given by (1.2) is well-defined.

Theorem 2.4 gives sufficient conditions for the existence of a well-defined (mild) random-field solution (1.2). In Section 4 we will deal with partial differential operatos L satisfying (1.4), weakly hyperbolic and with characteristics of constant multiplicity. We are going to construct the fundamental solution operator, find the corresponding kernel Λ, and, proving that it fulfills the assumptions of Theorem 2.4, achieve that (1.2) makes sense. For a proof of Theorem 2.4, see [7, Theorem 2.6].

3. Microlocal analysis and fundamental solution to first order hyperbolic systems with polynomially bounded coefficients

We recall here the basic definitions and facts about the so-called *SG*-calculus of pseudodifferential and Fourier integral operators, through standard material appeared, e.g., in [4] and elsewhere (sometimes with slightly different notational choices). Some additional details on these topics, which we include for the convenience of the reader, can be found in the Appendix.

The class $S^{m,\mu} = S^{m,\mu}(\mathbb{R}^d)$ of *SG* symbols of order $(m,\mu) \in \mathbb{R}^2$ is given by all the functions $a(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with the property that, for any multiindices $\alpha, \beta \in \mathbb{N}_0^d$, there exist constants $C_{\alpha\beta} > 0$ such that the conditions

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle x\rangle^{m-|\alpha|} \langle \xi\rangle^{\mu-|\beta|}, \qquad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{3.1}$$

hold. These classes were first introduced in the '70s by H.O. Cordes [13] and C. Parenti [32], see also R. Melrose [27]. For $m,\mu \in \mathbb{R}$, $\ell \in \mathbb{N}_0$, $a \in S^{m,\mu}$, the quantities

$$\|a\|_\ell^{m,\mu} = \max_{|\alpha+\beta|\le \ell} \sup_{x,\xi\in\mathbb{R}^d} \langle x\rangle^{-m+|\alpha|} \langle \xi\rangle^{-\mu+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \tag{3.2}$$

are a family of seminorms, defining the Fréchet topology of $S^{m,\mu}$.

The corresponding classes of pseudodifferential operators $\operatorname{Op}(S^{m,\mu}) = \operatorname{Op}(S^{m,\mu}(\mathbb{R}^d))$ are given by

$$(\operatorname{Op}(a)u)(x) = (a(.,D)u)(x) = (2\pi)^{-d} \int e^{ix\xi} a(x,\xi) \hat{u}(\xi) d\xi, \quad a \in S^{m,\mu}(\mathbb{R}^d), u \in \mathcal{S}(\mathbb{R}^d), \tag{3.3}$$

extended by duality to $\mathcal{S}'(\mathbb{R}^d)$. The operators in (3.3) form a graded algebra with respect to composition, i.e.,

$$\operatorname{Op}(S^{m_1,\mu_1}) \circ \operatorname{Op}(S^{m_2,\mu_2}) \subseteq \operatorname{Op}(S^{m_1+m_2,\mu_1+\mu_2}).$$

The symbol $c \in S^{m_1+m_2,\mu_1+\mu_2}$ of the composed operator $\operatorname{Op}(a) \circ \operatorname{Op}(b)$, $a \in S^{m_1,\mu_1}$, $b \in S^{m_2,\mu_2}$, admits the asymptotic expansion

$$c(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi), \tag{3.4}$$

which implies that the symbol c equals $a \cdot b$ modulo $S^{m_1+m_2-1,\mu_1+\mu_2-1}$.

The residual elements of the calculus are operators with symbols in

$$S^{-\infty,-\infty} = S^{-\infty,-\infty}(\mathbb{R}^d) = \bigcap_{(m,\mu)\in\mathbb{R}^2} S^{m,\mu}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^{2d}),$$

that is, those having kernel in $\mathcal{S}(\mathbb{R}^{2d})$, continuously mapping $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$. For any $a \in S^{m,\mu}$, $(m,\mu) \in \mathbb{R}^2$, $\operatorname{Op}(a)$ is a linear continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to itself, extending to a linear continuous operator from $\mathcal{S}'(\mathbb{R}^d)$ to itself, and from $H^{z,\zeta}(\mathbb{R}^d)$ to $H^{z-m,\zeta-\mu}(\mathbb{R}^d)$, where $H^{z,\zeta}(\mathbb{R}^d)$, $(z,\zeta) \in \mathbb{R}^2$, denotes the Sobolev-Kato (or *weighted Sobolev*) space

$$H^{z,\zeta}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^n) \colon \|u\|_{z,\zeta} = \|\langle\cdot\rangle^z \langle D\rangle^\zeta u\|_{L^2} < \infty\}, \tag{3.5}$$

with the naturally induced Hilbert norm. When $z \ge z'$ and $\zeta \ge \zeta'$, the continuous embedding $H^{z,\zeta} \hookrightarrow H^{z',\zeta'}$ holds true. It is compact when $z > z'$ and $\zeta > \zeta'$. Since $H^{z,\zeta} = \langle\cdot\rangle^z H^{0,\zeta} = \langle\cdot\rangle^z H^\zeta$, with H^ζ the usual Sobolev space of order $\zeta \in \mathbb{R}$, we find $\zeta > k + \dfrac{d}{2} \Rightarrow H^{z,\zeta} \hookrightarrow C^k$, $k \in \mathbb{N}_0$.

One actually finds

$$\bigcap_{z,\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d) = H^{\infty,\infty}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d), \qquad \bigcup_{z,\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d) = H^{-\infty,-\infty}(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d), \tag{3.6}$$

as well as, for the space of *rapidly decreasing distributions*, see [5, 36],

$$\mathcal{S}'(\mathbb{R}^d)_\infty = \bigcap_{z\in\mathbb{R}} \bigcup_{\zeta\in\mathbb{R}} H^{z,\zeta}(\mathbb{R}^d). \tag{3.7}$$

Cordes introduced the class $O(m,\mu)$ of the *operators of order* (m,μ) as follows, see, e.g., [13].

Definition 3.1. A linear continuous operator $A\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ belongs to the class $\mathcal{O}(m,\mu)$, $(m,\mu)\in\mathbb{R}^2$, of the operators of order (m,μ) if, for any $(z,\zeta)\in\mathbb{R}^2$, it extends to a linear continuous operator $A_{z,\zeta}\colon H^{z,\zeta}(\mathbb{R}^d)\to H^{z-m,\zeta-\mu}(\mathbb{R}^d)$. We also define

$$\mathcal{O}(\infty,\infty) = \bigcup_{(m,\mu)\in\mathbb{R}^2} \mathcal{O}(m,\mu), \quad \mathcal{O}(-\infty,-\infty) = \bigcap_{(m,\mu)\in\mathbb{R}^2} \mathcal{O}(m,\mu).$$

Remark 3.2. (1) Trivially, any $A\in\mathcal{O}(m,\mu)$ admits a linear continuous extension $A_{\infty,\infty}\colon \mathcal{S}'(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$. In fact, in view of (3.6), it is enough to set $A_{\infty,\infty}|_{H^{z,\zeta}(\mathbb{R}^d)} = A_{z,\zeta}$.
(2) Theorem A.1 implies $\mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))\subset\mathcal{O}(m,\mu)$, $(m,\mu)\in\mathbb{R}^2$.
(3) $\mathcal{O}(\infty,\infty)$ and $\mathcal{O}(0,0)$ are algebras under operator multiplication, $\mathcal{O}(-\infty,-\infty)$ is an ideal of both $\mathcal{O}(\infty,\infty)$ and $\mathcal{O}(0,0)$, and $\mathcal{O}(m_1,\mu_1)\circ\mathcal{O}(m_2,\mu_2)\subset\mathcal{O}(m_1+m_2,\mu_1+\mu_2)$.

We now introduce the class of SG-phase functions.

Definition 3.3 (SG-phase function)**.** A real valued function $\varphi\in C^\infty(\mathbb{R}^{2d})$ belongs to the class $\mathfrak{P}$ of SG-phase functions if it satisfies the following conditions:
(1) $\varphi\in S^{1,1}(\mathbb{R}^d)$;
(2) $\langle\varphi'_x(x,\xi)\rangle \asymp \langle\xi\rangle$ as $|(x,\xi)|\to\infty$;
(3) $\langle\varphi'_\xi(x,\xi)\rangle \asymp \langle x\rangle$ as $|(x,\xi)|\to\infty$.

For any $a\in S^{m,\mu}$, $(m,\mu)\in\mathbb{R}^2$, $\varphi\in\mathfrak{P}$, the SG FIOs are defined, for $u\in\mathcal{S}(\mathbb{R}^n)$, as

$$(\mathrm{Op}_\varphi(a)u)(x) = (2\pi)^{-d}\int e^{\mathrm{i}\varphi(x,\xi)}a(x,\xi)\widehat{u}(\xi)\,d\xi, \tag{3.8}$$

and

$$(\mathrm{Op}^*_\varphi(a)u)(x) = (2\pi)^{-d}\iint e^{\mathrm{i}(x\cdot\xi-\varphi(y,\xi))}\overline{a(y,\xi)}u(y)\,dyd\xi. \tag{3.9}$$

Here the operators $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(a)$ are sometimes called SG FIOs of type I and type II, respectively, with symbol a and (SG-)phase function φ. Note that a type II operator satisfies $\mathrm{Op}^*_\varphi(a)=\mathrm{Op}_\varphi(a)^*$, that is, it is the formal L^2-adjoint of the type I operator $\mathrm{Op}_\varphi(a)$.

The analysis of SG FIOs started in [14], where composition results with the classes of SG pseudodifferential operators, and of SG FIOs of type I and type II with regular phase functions, have been proved. Also the basic continuity properties in $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ of operators in the class have been proved there, as well as a version of the Asada-Fujiwara $L^2(\mathbb{R}^d)$-continuity, for operators $\mathrm{Op}_\varphi(a)$ with symbol $a\in S^{0,0}$ and regular SG-phase function $\varphi\in\mathfrak{P}_\delta$, see Definition 3.5. The following theorem summarizes composition results between SG pseudodifferential operators and SG FIOs of type I that we are going to use in the present paper, see [14] for proofs and composition results with SG FIOs of type II.

Theorem 3.4. *Let $\varphi\in\mathfrak{P}$ and assume $b\in S^{m_1,\mu_1}(\mathbb{R}^d)$, $a\in S^{m_2,\mu_2}(\mathbb{R}^d)$, $(m_j,\mu_j)\in\mathbb{R}^2$, $j=1,2$. Then,*

$$\mathrm{Op}(b)\circ\mathrm{Op}_\varphi(a) = \mathrm{Op}_\varphi(c_1+r_1) = \mathrm{Op}_\varphi(c_1) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d)),$$

$$\mathrm{Op}_\varphi(a)\circ\mathrm{Op}(b) = \mathrm{Op}_\varphi(c_2+r_2) = \mathrm{Op}_\varphi(c_2) \mod \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d)),$$

for some $c_j\in S^{m_1+m_2,\mu_1+\mu_2}(\mathbb{R}^d)$, $r_j\in S^{-\infty,-\infty}(\mathbb{R}^d)$, $j=1,2$.

To consider the composition of SG FIOs of type I and type II some more hypotheses are needed, leading to the definition of the classes $\mathfrak{P}_\delta$ and $\mathfrak{P}_\delta(\lambda)$ of regular SG-phase functions.

Definition 3.5 (Regular SG-phase function)**.** Let $\lambda\in[0,1)$ and $\delta>0$. A function $\varphi\in\mathfrak{P}$ belongs to the class $\mathfrak{P}_\delta(\lambda)$ if it satisfies the following conditions:
(1) $|\det(\varphi''_{x\xi})(x,\xi)|\geq\delta$, $\forall(x,\xi)$;
(2) the function $J(x,\xi):=\varphi(x,\xi)-x\cdot\xi$ is such that

$$\sup_{\substack{x,\xi\in\mathbb{R}^d\\|\alpha+\beta|\leq 2}} \frac{|D^\alpha_\xi D^\beta_x J(x,\xi)|}{\langle x\rangle^{1-|\beta|}\langle\xi\rangle^{1-|\alpha|}} \leq \lambda. \tag{3.10}$$

If only condition (1) holds, we write $\varphi\in\mathfrak{P}_\delta$.

The result of a composition of SG FIOs of type I and type II with the same regular SG-phase functions is a SG pseudodifferential operator, see again [14]. The continuity properties of regular SG FIOs on the Sobolev-Kato spaces can be expressed as follows, using the operators of order $(m,\mu)\in\mathbb{R}^2$ introduced above.

Theorem 3.6. *Let φ be a regular SG phase function and $a\in S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu)\in\mathbb{R}^2$. Then, $\mathrm{Op}_\varphi(a)\in\mathcal{O}(m,\mu)$.*

Applications of the SG FIOs theory to SG-hyperbolic Cauchy problems were initially given in [15, 18]. Many authors have, since then, expanded the SG FIOs theory and its applications to the solution of hyperbolic problems in various directions. To mention a few, see, e.g., M. Ruzhansky, M. Sugimoto [34], E. Cordero, F. Nicola, L Rodino [12], and the references quoted there and in [4].

In [4], the results in Theorem A.3 have been applied to study classes of SG-hyperbolic Cauchy problems, constructing their fundamental solution $\{E(t,s)\}_{0\le s\le t\le T}$. The existence of the fundamental solution provides, via Duhamel's formula, existence and uniqueness of the solution to the system, for any given Cauchy data in the weighted Sobolev spaces $H^{z,\zeta}(\mathbb{R}^d)$, $(z,\zeta)\in\mathbb{R}^2$. A remarkable feature, typical for these classes of hyperbolic problems, is the *well-posedness with loss/gain of decay at infinity*, observed in [2, 3, 18].

In the present paper we will focus on the class of equations of the form (1.1) and operators L which are *weakly SG-hyperbolic with (roots of) constant multiplicities*, that is, $\mathcal{L}_m$ satisfies (1.4) and the real-valued, characteristic roots can be divided into n groups ($1\le n\le m$) of distinct and separated roots, in the sense that, possibly after a reordering of the τ_j, $j=1,\dots,m$, there exist $l_1,\dots l_n\in\mathbb{N}$ with $l_1+\ldots+l_n=m$ and n sets

$$G_1=\{\tau_1=\cdots=\tau_{l_1}\},\quad G_2=\{\tau_{l_1+1}=\cdots=\tau_{l_1+l_2}\},\quad\ldots\quad G_n=\{\tau_{m-l_n+1}=\cdots=\tau_m\},$$

satisfying, for a constant $C>0$,

$$\tau_j\in G_p,\tau_k\in G_q,\ p\ne q,\ 1\le p,q\le n\Rightarrow|\tau_j(t,x,\xi)-\tau_k(t,x,\xi)|\ge C\langle x\rangle\langle\xi\rangle,\quad\forall(t,x,\xi)\in[0,T]\times\mathbb{R}^{2d}. \tag{3.11}$$

The number $l=\max_{j=1,\dots,n}l_j$ is the *maximum multiplicity of the roots of $\mathcal{L}_m$*. Notice that, in the case $n=1$, we have only one group of m coinciding roots, that is, $\mathcal{L}_m$ admits a single real root of multiplicity m, while for $n=m$ we recover the strictly hyperbolic case (that is, $l=1$, meaning that all the characteristic roots are distict).

Another interesting case is the one when (1.1) and L are *SG-hyperbolic with involutive roots*, that is, $\mathcal{L}_m$ satisfies (1.4) with real-valued characteristic roots such that

$$\begin{aligned}[D_t-\mathrm{Op}(\tau_j(t)),D_t-\mathrm{Op}(\tau_k(t))]=\;&\mathrm{Op}(a_{jk}(t))\,(D_t-\mathrm{Op}(\tau_j(t))\\ +\;&\mathrm{Op}(b_{jk}(t))\,(D_t-\mathrm{Op}(\tau_k(t)))+\mathrm{Op}(c_{jk}(t)),\end{aligned} \tag{3.12}$$

for some $a_{jk},b_{jk},c_{jk}\in C^\infty([0,T],S^{0,0}(\mathbb{R}^d))$, $j,k=1,\dots,m$.

Remark 3.7. Recall that roots of constant multiplicities are always involutive. In fact, with the notation of (3.11) above, since $\tau_j(t)-\tau_k(t)$ is SG-elliptic for any $\tau_j\in G_p$, $\tau_k\in G_q$, $p\ne q$, $1\le p,q\le n$, see the Appendix, for every $r\in S^{1,1}(\mathbb{R}^d)$ we have that $r\cdot(\tau_j(t)-\tau_k(t))^{-1}\in S^{0,0}(\mathbb{R}^d)$. Then, denoting by $\{g,h\}:=\sum_{j=1}^d\left(\partial_{x_j}g\partial_{\xi_j}h-\partial_{\xi_j}g\partial_{x_j}h\right)$ the Poisson bracket of $g(x,\xi)$ and $h(x,\xi)$, using the pseudodifferential calculus we find, for suitable $a_{jk}(t),c_{jk}(t),\widetilde{c}_{jk}(t)\in S^{0,0}(\mathbb{R}^d)$,

$$\begin{aligned}[D_t-\mathrm{Op}(\tau_j(t)),D_t-\mathrm{Op}(\tau_k(t))]&=D_t^2+\mathrm{iOp}(\partial_t\tau_k(t))-\mathrm{Op}(\tau_k(t))D_t-\mathrm{Op}(\tau_j(t))D_t+\mathrm{Op}(\tau_j(t))\circ\mathrm{Op}(\tau_k(t))\\
&\quad-\left(D_t^2+\mathrm{iOp}(\partial_t\tau_j(t))-\mathrm{Op}(\tau_j(t))D_t-\mathrm{Op}(\tau_k(t))D_t+\mathrm{Op}(\tau_k(t))\circ\mathrm{Op}(\tau_j(t)\right)\\
&=\mathrm{iOp}(\partial_t\tau_k(t)-\partial_t\tau_j(t))+[\mathrm{Op}(\tau_j(t)),\mathrm{Op}(\tau_k(t))]\\
&=\mathrm{Op}(\underbrace{\mathrm{i}\partial_t(\tau_k(t)-\tau_j(t))-\mathrm{i}\{\tau_j(t),\tau_k(t)\}}_{=\widetilde{a}_{jk}(t)\in S^{1,1}(\mathbb{R}^d)})+\mathrm{Op}(\widetilde{c}_{jk}(t))\\
&=\mathrm{Op}\left(\underbrace{\frac{\widetilde{a}_{jk}(t)}{\tau_k(t)-\tau_j(t)}}_{=a_{jk}(t)\in S^{0,0}(\mathbb{R}^d)}(\tau_k(t)-\tau_j(t))\right)+\mathrm{Op}(\widetilde{c}_{jk}(t))\\
&=\mathrm{Op}(a_{jk}(t))\mathrm{Op}(\tau_k(t)-\tau_j(t))+\mathrm{Op}(c_{jk}(t))\\
&=\mathrm{Op}(a_{jk}(t))(D_t-\mathrm{Op}(\tau_j(t)))+\mathrm{Op}(-a_{jk}(t))(D_t-\mathrm{Op}(\tau_k(t)))+\mathrm{Op}(c_{jk}(t)),\end{aligned}$$

as claimed. The converse statement is not true in general, as shown in Example 1.2.

Definition 3.8. We will say that the operator L in (1.3) is *weakly (SG-)hyperbolic with constant multiplicities*, or *weakly (SG-)hyperbolic with involutive roots*, respectively, if such properties are satisfied by the roots of $\mathcal{L}_m$, as explained above.

By the hyperbolicity hypotheses (3.11) or (3.12), as it will be shown below, to obtain the term v_0 and the kernel Λ, associated with the operator in (1.1), it is enough to know the fundamental solution of certain first order systems. Namely, let us consider the Cauchy problem

$$\begin{cases}(D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\kappa_0(t)))W(t) = Y(t), & t \in [0,T],\\ W(s) = W_0, & s \in [0,T],\end{cases} \tag{3.13}$$

where the $(\nu \times \nu)$-system is hyperbolic with diagonal principal part, that is:

- the matrix κ_1 satisfies $\kappa_1 \in C^\infty([0,T], S^{1,1})$, it is real-valued and diagonal, and each entry on the principal diagonal coincides with the value of one of the roots $\tau_j \in C^\infty([0,T]; S^{1,1})$, possibly repeated a number of times, depending on their multiplicities;
- the matrix κ_0 satisfies $\kappa_0 \in C^\infty([0,T], S^{0,0})$.

In analogy with the terminology introduced above, we will say that the system (3.13) is hyperbolic with constant multiplicities, when the elements on the main diagonal of κ_1 satisfy (3.11). Similarly, we will say that the system is hyperbolic with involutive roots when they satisfy (3.12). We will also generally assume $W_0 \in H^{z,\zeta}$, $Y \in C([0,T], H^{z,\zeta})$, $(z,\zeta) \in \mathbb{R}^2$.

The fundamental solution, or *solution operator*, of (3.13) is a family $\{E(t,s)\colon (t,s) \in [0,T_0]^2\}$, $0 < T_0 \le T$, of linear continuous operators in the strong topology of $\mathscr{L}(H^{z,\zeta}, H^{z,\zeta})$, $(z,\zeta) \in \mathbb{R}^2$. In the cases of weak SG-hyperbolicity with constant multiplicities or involutive roots, such family can be explicitly expressed in terms of suitable (matrices of) SG FIOs of type I, modulo smoothing terms, see [1, 15, 18] and Section 4 below. In the case of SG-hyperbolicity with variable multiplicities, the operator family $E(t,s)$ is, in general, a limit of a sequence of a family of (matrices of) SG FIOs of type I. In all cases, it satisfies

$$\begin{cases}(D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\kappa_0(t)))E(t,s) = 0, & (t,s) \in [0,T_0]^2,\\ E(s,s) = I, & s \in [0,T_0].\end{cases} \tag{3.14}$$

Indeed, there are various techniques to switch from a Cauchy problem for an hyperbolic operator L of order $M \ge 1$ to a Cauchy problem for a first order system (3.13), see, e.g., [1, 13, 15, 30]. In the approach we follow here, which is the same used in [18], the key results for this aim is an adapted version of the so-called Mizohata Lemma of Perfect Factorization, see Proposition A.9 and Lemma A.12 in the Appendix[1].

We conclude this section with the following two results, similar to those discussed in [7] for the case of uniformly bounded coefficients, and needed in the construction of the stochastic integral with respect to a martingale measure. The proofs follow from arguments similar to those given in [7].

Proposition 3.9. *Let $A = \mathrm{Op}_\varphi(a)$ be a SG FIO of type I, with symbol $a \in S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu) \in \mathbb{R}^2$, and phase function $\varphi \in \mathfrak{P}$, and let K_A denote its Schwartz kernel. Then, the Fourier transform with respect to the second argument of K_A, $\mathcal{F}_{\cdot\mapsto\eta} K_A(x,\cdot)$, is given by*

$$\mathcal{F}_{\cdot\mapsto\eta} K_A(x,\cdot) = \mathrm{e}^{\mathrm{i}\varphi(x,-\eta)} a(x,-\eta). \tag{3.15}$$

Proof. Here we can argue as in the proof of [7, Proposition 3.11]. □

Lemma 3.10. *Let $A = \mathrm{Op}_\varphi(a)$ be a SG FIO of type I with symbol $a \in S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu) \in \mathbb{R}^2$, and let K_A denote its Schwartz kernel. Then, for every $x \in \mathbb{R}^d$, $K_A(x,\cdot) \in \mathcal{S}'(\mathbb{R}^d)_\infty$. More precisely, we find $K_A \in C^\infty(\mathbb{R}^d, \mathcal{S}'(\mathbb{R}^d)_\infty)$.*

Proof. Given a fixed $x \in \mathbb{R}^d$, by [5, Theorem 3.3], to see that $K_A(x,\cdot) \in \mathcal{S}'(\mathbb{R}^d)_\infty$ it suffices to show that or every $\chi \in \mathcal{D}(\mathbb{R}^d)$, $K_A(x,\cdot) * \chi \in \mathcal{S}(\mathbb{R}^d)$. We already know [36, p. 244/245] that $K_A(x,\cdot) * \chi$ is a C^∞ function of slow growth. Computing now by Proposition 3.9 its Fourier transform (in the distributional sense), we see that

$$\mathcal{F}_{\cdot\mapsto\eta}(K_A(x,\cdot) * \chi)(\eta) = \mathcal{F}_{\cdot\mapsto\eta} K_A(x,\eta)\widehat{\chi}(\eta) = \mathrm{e}^{\mathrm{i}\varphi(x,-\eta)} a(x,-\eta)\widehat{\chi}(\eta) \in \mathcal{S}(\mathbb{R}^d_\eta);$$

so its inverse Fourier transform $K_A(x,\cdot) * \chi \in \mathcal{S}(\mathbb{R}^d)$, too. Finally, the fact that the map

$$x \mapsto K_A(x,y) = \int_{\mathbb{R}^d} e^{i[\varphi(x,\xi) - y\cdot\xi]} a(x,\xi) đ\xi$$

[1] See also [26, 28, 29], for the original version of such results.

belongs to $C^\infty(\mathbb{R}^d, \mathcal{S}'(\mathbb{R}^d)_\infty)$ follows from the general properties of oscillatory integrals, taking into account that φ and a are smooth functions with respect to x. This completes the proof. □

4. Existence of a random-field solution

In this section we prove our main results of existence of a random-field solution of the SPDE (1.1) under suitable assumptions of hyperbolicity for the operator L, see (1.3), (1.4). We work here with a class of operators with more general symbols than the (polynomial) ones appearing in (1.3). Namely, we consider operators of the form

$$L = D_t^m - \sum_{j=1}^{m} A_j(t,x,D_x) D_t^{m-j}, \tag{4.1}$$

where $A_j(t) = \mathrm{Op}(a_j(t))$ are SG pseudo-differential operators with symbols $a_j \in C^\infty([0,T], S^{j,j})$, $1 \le j \le m$. Notice that, of course, (1.3) is a particular case of (4.1). The hyperbolicity condition on L becomes

$$\mathcal{L}_m(t,x,\tau,\xi) = \tau^m - \sum_{j=1}^{m} \tilde{A}_j(t,x,\xi)\tau^{m-j} = \prod_{j=1}^{m} \Big(\tau - \tau_j(t,x,\xi)\Big), \tag{4.2}$$

where $\tilde{A}_j$ stands for the principal part of A_j, with characteristic roots $\tau_j(t,x,\xi) \in \mathbb{R}$, $\tau_j \in C^\infty([0,T]; S^{1,1})$.

We consider the corresponding Cauchy problem

$$\begin{cases} Lu(t,x) = g(t,x) = \gamma(t,x) + \sigma(t,x)\dot{\Xi}(t,x), & (t,x) \in (0,T] \times \mathbb{R}^d, \\ D_t^j u(0,x) = u_j(x), & x \in \mathbb{R}^d,\ 0 \le j \le m-1, \end{cases} \tag{4.3}$$

with the aim of finding conditions on L, on the stochastic noise $\dot{\Xi}$, and on σ, γ, u_j, $j = 0, \dots, m-1$, such that (4.3) admits a random-field solution. The conditions on the stochastic noise will be given on the spectral measure μ corresponding to the correlation measure Γ related to the noise $\dot{\Xi}$. Our main result is Theorem 4.1, where we assume that $\mathbf{(WH)_{CMR}}$ holds true. We will also state a further result, corresponding to the case $\mathbf{(WH)_{IR}}$. For the sake of brevity, we will give the full argument only in the case $\mathbf{(WH)_{CMR}}$, treating the strictly hyperbolic case as its particular case $\ell = 1$. In the case $\mathbf{(WH)_{IR}}$, we will shortly address some aspects, since the full treatment requires a careful and quite long and technical analysis (see [1]). Notice that all the results on SG-hyperbolic differential operators recalled in the previous Section 3 and in the Appendix, in particular, Proposition A.9 and Lemma A.12, still hold true for SG-hyperbolic operators of the form (4.1). We adopt the same terminology and definitions also for this more general operators, with straightforward modifications, where needed.

4.1. **The weakly hyperbolic case with constant multiplicites.** We now give the precise statement and the proof of our main result, which holds under the hypothesis $\mathbf{(WH)_{CMR}}$. As it is very well-known in the usual hyperbolic theory, in the case of weak hyperbolicity the principal term does not provide enough information, by itself, to imply well-posedness of the Cauchy problem. In other words, lower order terms are also relevant in this case, and one needs to impose additional conditions on them. We will then assume, in this and the next subsection, that L satisfies the SG-Levi condition

$$h_{jk}^{(p)} \in C^\infty([0,T], S^{0,0}(\mathbb{R}^d)), \quad p, j = 1, \dots, n, k = 1, \dots, l_j, \tag{4.4}$$

see Corollary A.10. Let us observe that, actually, (4.4) needs to be fulfilled only for a single value of $p = 1, \dots, n$. Moreover, (4.4) is always true if L is a strictly hyperbolic operators, that is, when $l = \max_{j=1,\dots,n} l_j = 1$. If L satisfies (4.4) we will also say that L is of Levi type.

Theorem 4.1. *Let us consider the Cauchy problem (4.3) for an SPDE associated with an operator of the form (4.1), under the hyperbolicity hypothesis (4.2). Moreover, assume that L is weakly SG-hyperbolic with constant multiplicities, that is, $\mathcal{L}_m$ satisfies (1.4) and the characteristic roots τ_j, $j = 1, \dots, m$, can be divided into n groups, $1 \le n \le m$, of distinct and separated roots, in the sense that, possibly after a reordering of the τ_j, $j = 1, \dots, m$, there exist $l_1, \dots l_n \in \mathbb{N}$ with $l_1 + \dots + l_n = m$ and n sets*

$$G_1 = \{\tau_1 = \dots = \tau_{l_1}\}, \quad G_2 = \{\tau_{l_1+1} = \dots = \tau_{l_1+l_2}\}, \quad \dots \quad G_n = \{\tau_{m-l_n+1} = \dots = \tau_m\},$$

satisfying (3.11) for some constant $C > 0$. Assume also that L is of Levy type, that is, with the notation of Corollary A.10, it satisfies (4.4). Assume also, for the initial conditions, that $u_j \in H^{z+m-j-1, \zeta+m-j-1}(\mathbb{R}^d)$, $0 \le j \le m-1$, with $z \in \mathbb{R}$ and $\zeta > d/2$. Furthermore, assume for the spectral measure that

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{m-l}} \mu(d\xi) < \infty, \tag{4.5}$$

where $l = \max_{j=1,\dots,n} l_j$ is the maximum multiplicity of the roots of L_m. Finally, assume that $\gamma \in C([0,T]; H^{z,\zeta}(\mathbb{R}^d))$, $\sigma \in C([0,T], H^{0,\zeta})$, $s \mapsto \mathcal{F}\sigma(s) = \nu_s \in L^2([0,T], \mathcal{M}_b(\mathbb{R}^d))$.

Then, for some time horizon $0 < T_0 \leq T$, there exists a random-field solution u of (4.3). Moreover, $\mathbb{E}[u] \in C([0,T_0], H^{z+m-l,\zeta+m-l}(\mathbb{R}^d))$.

Remark 4.2. In the (more restrictive) case $\gamma \in \bigcap_{j\geq 0} C^j([0,T], H^{z-j,\zeta-j}(\mathbb{R}^d))$, our method of proof provides a random-field solution u such that $\mathbb{E}[u] \in \bigcap_{j\geq 0} C^j([0,T_0], H^{z+m-l-j,\zeta+m-l-j}(\mathbb{R}^d))$. Some details on this point will be given in the proof of Theorem 4.1.

Remark 4.3. Let us notice that condition (4.5) on the spectral measure for the strict hyperbolic case, that is, $l = 1 \Leftrightarrow n = m$, is exactly the one obtained for strictly hyperbolic equations with uniformly bounded coefficients in [7]. The class of the stochastic noises which are admissible, if we want to obtain a random-field solution of the Cauchy problem for an SPDE through our method, is described by (4.5) for all weakly *SG*-hyperbolic operators L with constant multiplicities. Condition (4.5) can be understood as a *compatibility condition* between the noise and the equation: as the order of the equation increases, we can allow for rougher stochastic noises Ξ.

Remark 4.4. Notice that, if the correlation measure Γ is absolutely continous, then condition (4.5) with $l = 1$ is equivalent to

$$\int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^{m-1}} \mu(d\xi) < \infty, \tag{4.6}$$

see [33]. Condition (4.6) with $m = 2$ on the spectral measure is the one needed for the existence and uniqueness of a random-field solution to a second order SPDE well-known in literature, namely, the stochastic wave equation.

The proof of Theorem 4.1 consists of the following 4 steps:

(1) factorization of the operator L;
(2) reduction of (4.3) to an equivalent first order system of the form (3.13), with the matrices κ_1 and κ_0 satisfying the assumptions described in Section 3 above;
(3) construction of the fundamental solution to (3.13), and then (formally) of the solution u to (4.3);
(4) proof of the fact that v_0 and the stochastic and deterministic integrals, appearing in the (formal) expression (1.2) of u, are well-defined.

For steps (1), (2) and (3) we can rely on Proposition A.9, Corollary A.10 and Lemma A.12, and on the procedure explained in [18]. We recall below the main aspects of this microlocal approach, for the convenience of the reader.

Remark 4.5. We point out that, in the proof of Theorem 4.1, we will show that conditions **(A1)**, **(A2)**, **(A5)** from Section 2 hold true, to achieve, respectively, that the stochastic integral and v_0 in (1.2) are well-defined. In view of the special structure of Λ (kernel of a *SG* FIO), we will not need to verify **(A3)** and **(A4)**, to see that the deterministic integral in (1.2) is well-defined. Indeed, this follows by the general theory of *SG* hyperbolic equations, under the assumptions on γ given in the statement of Theorem 4.1 (which are weaker, with respect to **(A3)** and **(A4)**).

Let us denote by θ_j, $j = 1,\dots,n$, the distinct values of the roots τ_k, $k = 1,\dots,m$, and by ϖ_p, $p = 1,\dots,n$, the reorderings of the n-tuple $(1,\dots,n)$ defined in (A.10). The equivalence of the Cauchy problem for the equation $Lu(t) = g(t)$ and a 1×1 system (3.13) is then trivial for $m = 1$. For $m \geq 2$, we will now introduce a (mn)-dimensional vector of unknowns

$$W = (W_1^{(1)},\dots,W_m^{(1)},W_1^{(2)},\dots,W_m^{(2)},\dots,W_1^{(n)},\dots W_m^{(n)})^t,$$

and construct a corresponding linear first order hyperbolic system, with diagonal principal part and constant multiplicities, equivalent to (4.3). Let us set, for convenience, with the notation introduced in Corollary A.10,

$$l^{(p,k)} = \begin{cases} 0, & k=0, \\ \displaystyle\sum_{1\leq j\leq k} l_{\varpi_p(j)}, & 1\leq k\leq n-1, \text{ if } n\geq 2, \\ m, & k=n, \end{cases} \qquad L^{(p,k)} = \begin{cases} I, & k=0, \\ L^{(p)}_{\varpi_p(k)}\cdots L^{(p)}_{\varpi_p(1)}, & 1\leq k\leq n-1, \text{ if } n\geq 2, \end{cases}$$

$p = 1, \dots, n$, and define

$$W^{(p)}_{l^{(p,k)}+j+1}(t) = (D_t - \mathrm{Op}(\theta_{\varpi_p(k+1)}(t)))^j L^{(p,k)} u(t), \quad p = 1, \dots, n,\ k = 0, \dots, n-1,\ j = 0, \dots, l_{\varpi_p(k+1)} - 1. \tag{4.7}$$

Using Lemma A.12 and the symbolic calculus, it is possible to express the t-derivatives of u in terms of the components of W from (4.7). In fact,

Lemma 4.6. *Under the hypotheses of Lemma A.12, for all $k = 1, \dots, m-1$, $p = 1, \dots, n$, it is possible to find symbols $w^{(p)}_{kj} \in C^\infty([0,T], S^{j,j}(\mathbb{R}^d))$, $j = 1, \dots, k$, such that, with the (nm)-dimensional vector W defined in (4.7),*

$$D^k_t u(t) = \sum_{j=1}^{k} \mathrm{Op}(w^{(p)}_{kj}(t)) W^{(p)}_{k-j+1}(t) + W^{(p)}_{k+1}(t). \tag{4.8}$$

Remark 4.7. Trivially, by the definition (4.7), (4.8) extends to $k = 0$ in the form $u(t) = W^{(p)}_1(t)$, $p = 1, \dots, n$.

The system (3.13) is now obtained by (A.11), (A.12), (4.4), and (4.7), in blocks labeled by $p = 1, \dots, n$, of the type

$$\begin{cases} \dots, \\ (D_t - \mathrm{Op}(\theta_{\varpi_p(1)}(t)))W^{(p)}_{j+1}(t) = W^{(p)}_{j+2}(t), \quad j = 0, \dots, l_{\varpi_p(1)} - 2, \text{ if } l_{\varpi_p(1)} \geq 2, \\ (D_t - \mathrm{Op}(\theta_{\varpi_p(1)}(t)))W^{(p)}_{l^{(p,1)}}(t) = -\displaystyle\sum_{k=1}^{l_{\varpi_p(1)}} \mathrm{Op}(h^{(p)}_{\varpi_p(1)k}(t))W^{(p)}_{l^{(p,1)}-k+1}(t) + W^{(p)}_{l^{(p,1)}+1}(t), \\ (D_t - \mathrm{Op}(\theta_{\varpi_p(2)}(t)))W^{(p)}_{l^{(p,1)}+j+1}(t) = W^{(p)}_{l^{(p,1)}+j+2}(t), \quad j = 0, \dots, l_{\varpi_p(2)} - 2, \text{ if } l_{\varpi_p(2)} \geq 2, n \geq 2, \\ (D_t - \mathrm{Op}(\theta_{\varpi_p(2)}(t)))W^{(p)}_{l^{(p,2)}}(t) = -\displaystyle\sum_{k=1}^{l_{\varpi_p(2)}} \mathrm{Op}(h^{(p)}_{\varpi_p(2)k}(t))W^{(p)}_{l^{(p,2)}-k+1}(t) + W^{(p)}_{l^{(p,2)}+1}(t), \text{ if } n \geq 2, \\ \dots, \\ (D_t - \mathrm{Op}(\tau_{\varpi_p(n)}(t)))W^{(p)}_m(t) = -\displaystyle\sum_{k=1}^{l_{\varpi_p(n)}} \mathrm{Op}(h^{(p)}_{\varpi_p(n)k}(t))W^{(p)}_{m-k+1}(t) \\ \qquad - \displaystyle\sum_{j=1}^{m-1} \left(\sum_{q=1}^{m-j} \mathrm{Op}(r^{(p)}_j(t)) \circ \mathrm{Op}(w^{(p)}_{m-j,q}(t))W^{(p)}_{m-j-q+1}(t) + \mathrm{Op}(r^{(p)}_j(t))W^{(p)}_{m-j+1}(t) \right) \\ \qquad - \mathrm{Op}(r^{(p)}_m(t))W^{(p)}_1(t) + g(t), \\ \dots \end{cases} \tag{4.9}$$

The initial data W_0 is obtained by $W_0 = \mathrm{Op}(b)U_0$, with a $(mn \times m)$-dimensional block-matrix symbol b such that

$$b = \begin{pmatrix} b^{(1)} \\ \hline \dots \\ \hline b^{(n)} \end{pmatrix}, \quad b^{(p)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ b^{(p)}_{10} & 1 & 0 & 0 & \dots \\ b^{(p)}_{20} & b^{(p)}_{21} & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, p = 1, \dots, n, \tag{4.10}$$

the $(m \times m)$-dimensional matrices $b^{(p)}$ satisfying

- if $m \geq 2$, $b^{(p)}_{jk} \in S^{j-k,j-k}$, $j > k$, $j = 1, \dots, m-1$, $k = 0, \dots, j-1$,
- $b^{(p)}_{jj} = 1 \in S^{0,0}$, $j = 0, \dots, m-1$,
- if $m \geq 2$, $b^{(p)}_{jk} = 0$, $j < k$, $j = 0, \dots, m-2$, $k = j+1, \dots, m-1$,

$p = 1, \dots, n$, and

$$U_0 = (u_0, \dots, u_{m-1})^t. \tag{4.11}$$

Remark 4.8. Consider, for instance, the case $n = 1$, that is, $\mathcal{L}_m$ admits a unique real root $\theta_1 = \tau_1$ of maximum multiplicity $l = l_1 = m$. Then, there is a single "reordering" $\varpi_1 = (1)$, the vector W has m components, $W = (W_1^{(1)}, \dots, W_m^{(1)})$, and (4.9) consists of a single block of m equations. Namely, in view of Corollary A.10, assuming $n \geq 2$ and dropping everywhere the $^{(1)}$ label, (4.7) reads, in this case,

$$
\begin{aligned}
W_1(t) &= u(t),\\
W_2(t) &= (D_t - \mathrm{Op}(\tau_1(t)))u(t) = (D_t - \mathrm{Op}(\tau_1(t)))W_1(t),\\
&\dots,\\
W_m(t) &= (D_t - \mathrm{Op}(\tau_1(t)))^{m-1}u(t) = (D_t - \mathrm{Op}(\tau_1(t)))W_{m-1}(t),
\end{aligned}
$$

while $Lu(t) = g(t)$ is then equivalent to

$$
\begin{aligned}
&(D_t - \mathrm{Op}(\tau_1(t)))^m u(t) + \sum_{k=1}^{m} \mathrm{Op}(h_{1k}(t))(D_t - \mathrm{Op}(\tau_1(t)))^{m-k}u(t) + \sum_{j=1}^{m} \mathrm{Op}(r_j(t))D_t^{m-j}u(t) = g(t)\\
&\Leftrightarrow\\
&(D_t - \mathrm{Op}(\tau_1(t)))W_m(t) = -\sum_{k=1}^{m} \mathrm{Op}(h_{1k}(t))W_{m-k+1}(t)\\
&\qquad - \sum_{j=1}^{m-1}\left(\sum_{q=1}^{m-j} \mathrm{Op}(r_j(t)) \circ \mathrm{Op}(w_{m-j,q}(t))W_{m-j-q+1}(t) + \mathrm{Op}(r_j(t))W_{m-j+1}(t)\right)\\
&\qquad - \mathrm{Op}(r_m(t))W_1(t) + g(t),
\end{aligned}
$$

that is,

$$
\begin{cases}
(D_t - \mathrm{Op}(\tau_1(t)))W_1(t) = W_2(t)\\
\dots\\
(D_t - \mathrm{Op}(\tau_1(t)))W_{m-1}(t) = W_m(t)\\
(D_t - \mathrm{Op}(\tau_1(t)))W_m(t) = -\sum_{k=1}^{m} \mathrm{Op}(h_{1k}(t))W_{m-k+1}(t)\\
\qquad - \sum_{j=1}^{m-1}\left(\sum_{q=1}^{m-j} \mathrm{Op}(r_j(t)) \circ \mathrm{Op}(w_{m-j,q}(t))W_{m-j-q+1}(t) + \mathrm{Op}(r_j(t))W_{m-j+1}(t)\right)\\
\qquad - \mathrm{Op}(r_m(t))W_1(t) + g(t),
\end{cases}
$$

which has the form (3.13) with $Y(t) = (\underbrace{0, \dots, 0}_{m-1 \text{ times}}, g(t))^t$, as claimed, since $\kappa_1(t) = \mathrm{diag}(\tau_1(t), \dots, \tau_1(t))$, while the coefficients of the components of W in the right-hand sides of the equations are all symbols of order $(0, 0)$, in view of the inclusion $S^{-\infty,-\infty} \subset S^{0,0}$.

Then, (4.9) is a hyperbolic first order linear system, with diagonal principal part and constant multiplicities, and right-hand side

$$
Y(t) = (\underbrace{G(t), \dots, G(t)}_{n \text{ times}})^t, \quad G(t) = (\underbrace{0, \dots, 0}_{m-1 \text{ times}}, g(t))^t.
$$

By an extension of the results in [15, 18], we can give an explicit form to the fundamental solution $E(t,s)$ in Theorem A.8, in terms of (smooth families of) SG FIOs of type I, modulo smoothing remainders. With the results of Theorem A.13 at hand, we solve, by means of the so-called *geometrical optics* (or FIOs) method, the system

$$
\begin{cases}
(D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\widetilde{\kappa}_0(t)))\widetilde{E}(t,s) = 0, & t \in [0, T_0],\\
\widetilde{E}(s,s) = I, & s \in [0, T_0).
\end{cases}
\tag{4.12}
$$

Notice that the *approximate solution operator* $\widetilde{A}(t,s)$, $(t,s) \in \Delta_{T_0}$, in terms of SG FIOs solves the corresponding operator problem up to smoothing remainders. Namely, the FIOs family $\widetilde{A}(t,s)$ solves the system

$$
\begin{cases}
(D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\widetilde{\kappa}_0(t)))\widetilde{A}(t,s) = \widetilde{R}_1(t,s), & (t,s) \in \Delta_{T_0},\\
\widetilde{A}(s,s) = I + \widetilde{R}_2(s), & s \in [0, T_0),
\end{cases}
\tag{4.13}
$$

where $\widetilde{R}_1$ and $\widetilde{R}_2$ are suitable smooth families of operators in $\mathcal{O}(-\infty,-\infty)$, coming from the solution method, see [13, 14, 15, 18, 26] for more details. It turns out that $\widetilde{A}(t,s)$ belongs to $\mathcal{O}(0,0)$ for any $(t,s)\in\Delta_{T_0}$. Explicitely,

$$\widetilde{A}(t,s) = \mathrm{diag}(\widetilde{A}^{(1)}(t,s),\dots,\widetilde{A}^{(n)}(t,s)),$$
$$\widetilde{A}^{(p)}(t,s) = \mathrm{diag}(\mathrm{Op}_{\varphi_{\varpi_p(1)}(t,s)}(a_1^{(p)}(t,s)),\dots,\mathrm{Op}_{\varphi_{\varpi_p(m)}(t,s)}(a_m^{(p)}(t,s))), p=1,\dots,n,$$

with phase functions $\varphi_j\in C^\infty(\Delta_{T_0},\mathfrak{P}_\delta(\lambda))$, $\lambda=\lambda(T_0)$ suitably small, solutions of the eikonal equations (A.3) with τ_j in place of $\varkappa$, and symbols $a_j^{(p)}\in C^\infty(\Delta_{T_0},S^{0,0})$, $p=1,\dots,n$, $j=1,\dots,m$, taking values in the linear space of $(m\times m)$-dimensional matrices (indeed, the system can be diagonalized block by block, see [15]). Solving the equations in (4.12) modulo smoothing terms is enough for our aims, as we will see below. Indeed, we have the following result.

Proposition 4.9. *Under the hypotheses of Theorem 4.1, let $A(t,s)=\mathrm{Op}(\omega(t))\circ\widetilde{A}(t,s)\circ\mathrm{Op}(\omega_{-1})(s)$, with $\widetilde{A}(t,s)$ solution of (4.13), $(t,s)\in\Delta_{T_0}$, and $\mathrm{Op}(\omega_{-1})(s)$ parametrix of the perfect diagonalizer $\mathrm{Op}(\omega(s))$, $s\in[0,T]$. Then, the solution $E(t,s)$ of (3.14) and the operator family $A(t,s)$ satisfy $E-A\in C^\infty(\Delta_{T_0},\mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d)))$.*

Proof. The argument is a variant of a similar one in [24], see also [17]. Define $H(t,s)=E(s,t)\circ A(t,s)$, $(t,s)\in\Delta_{T_0}$. Then, by Definition 3.1, Remark 3.2, Theorems A.2 and A.6, and (A.6), with $R_0\in C^\infty([0,T],\mathrm{Op}(S^{-\infty,-\infty}))$ from (A.14),

$$\begin{aligned}
\partial_t H(t,s) &= \ i(D_tE)(s,t)\circ A(t,s)+iE(s,t)\circ(D_tA)(t,s)\\
&= -iE(s,t)\circ(\mathrm{Op}(\kappa_1(t))+\mathrm{Op}(\kappa_0(t)))\circ A(t,s)+iE(s,t)\circ(D_tA)(t,s)\\
&= \ iE(s,t)\circ(D_t-\mathrm{Op}(\kappa_1(t))-\mathrm{Op}(\kappa_0(t)))\circ A(t,s)\\
&= \ iE(s,t)\circ(D_t-\mathrm{Op}(\kappa_1(t))-\mathrm{Op}(\kappa_0(t)))\circ\mathrm{Op}(\omega(t))\circ\widetilde{A}(t,s)\circ\mathrm{Op}(\omega_{-1}(t))\\
&= \ iE(s,t)\circ\mathrm{Op}(\omega(t))\circ[(D_t-\mathrm{Op}(\kappa_1(t))-\mathrm{Op}(\widetilde{\kappa}_0(t)))\widetilde{A}(t,s)]\circ\mathrm{Op}(\omega_{-1}(t))\\
&\quad +iE(s,t)\circ R_0(t)\circ\widetilde{A}(t,s)\circ\mathrm{Op}(\omega_{-1}(t))\\
&= \ iE(s,t)\circ[R_0(t)\circ\widetilde{A}(t,s)+\mathrm{Op}(\omega(t))\circ\widetilde{R}_1(t,s)]\circ\mathrm{Op}(\omega_{-1}(s)))\in\mathrm{Op}(S^{-\infty,-\infty}).
\end{aligned}$$

Integrating with respect to t on $[s,t]$, recalling (4.13) and again Remark 3.2, Theorem A.2 and Theorem A.6, (3), it follows, for some $R_3\in C^\infty([0,T_0],\mathrm{Op}(S^{-\infty,-\infty}))$, $R_4\in C^\infty(\Delta_{T_0},\mathrm{Op}(S^{-\infty,-\infty}))$,

$$\begin{aligned}
H(t,s)-H(s,s) &= E(s,t)\circ A(t,s)-E(s,s)\circ A(s,s)\\
&= E(s,t)\circ A(t,s)-\mathrm{Op}(\omega(s))\circ(I+\widetilde{R}_2(s))\circ\mathrm{Op}(\omega_{-1}(s))\\
&= E(s,t)\circ A(t,s)-I-R_3(t)-\mathrm{Op}(\omega(s))\circ\widetilde{R}_2(s)\circ\mathrm{Op}(\omega_{-1}(s))\in\mathrm{Op}(S^{-\infty,-\infty})\\
&\Rightarrow E(s,t)\circ A(t,s)=I+R_4(s,t).
\end{aligned}$$

Applying $E(t,s)$ to both sides of this last equality we get that $E-A\in C^\infty(\Delta_{T_0},\mathrm{Op}(S^{-\infty,-\infty}))$, as claimed. □

Remark 4.10. Proposition 4.9 means that the Schwartz kernels of E and A differ by a family of elements of $\mathcal{S}(\mathbb{R}^{2d})$, smoothly depending on $(t,s)\in\Delta_{T_0}$.

Using Proposition 4.9, by repeated applications of Theorem 3.4, we finally obtain

$$E(t,s)=E_0(t,s)+R(t,s),\quad (t,s)\in\Delta_{T_0},\tag{4.14}$$

where

- E_0 is a $[(mn)\times(mn)]$-dimensional matrix of operators in $\mathcal{O}(0,0)$ given by
$$E_0(t,s)=\left(\sum_{p=1}^n \mathrm{Op}_{\varphi_p(t,s)}(e_{pjk}(t,s))\right)_{j,k=0,\dots,mn-1},$$
with the regular phase-functions $\varphi_p(t,s)$, solutions of the eikonal equations associated with τ_p, and symbols $e_{pjk}(t,s)\in S^{0,0}$, $j,k=0,\dots,mn-1$, $p=1,\dots,n$, smoothly depending on $(t,s)\in\Delta_{T_0}$;
- R is a $[(mn)\times(mn)]$-dimensional matrix of elements in $C^\infty(\Delta_{T_0},\mathrm{Op}(S^{-\infty,-\infty}))$, operators with kernel in $\mathcal{S}(\mathbb{R}^{2d})$, smoothly depending on $(t,s)\in\Delta_{T_0}$, that is,
$$R=(\mathrm{Op}(r_{jk}(t,s)))_{j,k=0,\dots,mn-1},$$

with symbols $r_{jk} \in C^\infty(\Delta_{T_0}, S^{-\infty,-\infty})$, $j,k = 0,\dots,mn-1$, collecting the remainders of the compositions in $\mathrm{Op}(\omega) \circ \widetilde{A} \circ \mathrm{Op}(\omega_{-1})$ and the difference $E - A$.

The next Lemma 4.11 from [18], see also [9, 10] and [29], is the key result to achieve, from (4.14) and the expressions of E_0 and R, the correct regularity of u.

Lemma 4.11. *There exists a $(m \times mn)$-dimensional matrix $\Upsilon_n \in C^\infty([0,T_0], S^{0,0}(\mathbb{R}^d))$ such that the k-th row consists of symbols of order $(l-m+k, l-m+k)$, $k = 0,\dots,m-1$, and*

$$\begin{pmatrix} u(t) \\ \dots \\ D_t^{m-1}u(t) \end{pmatrix} = \mathrm{Op}(\Upsilon_n(t))W(t), \quad t \in [0,T_0].$$

Assuming for a moment that $g \in C([0,T], H^{z,\zeta})$, $(z,\zeta) \in \mathbb{R}^2$, an application of Theorem A.6 and Lemma 4.11, together with (4.11) and the $[(mn)\times(mn)]$-dimensional operator matrices $E_0(t,s)$ and $R(t,s)$ from (4.14), initially gives

$$\begin{pmatrix} u(t) \\ \dots \\ D_t^{m-1}u(t) \end{pmatrix} = [\mathrm{Op}(\Upsilon_n(t)) \circ (E_0(t,0) + R(t,0)) \circ \mathrm{Op}(b)]U_0 + i\int_0^t [\mathrm{Op}(\Upsilon_n(t)) \circ (E_0(t,s) + R(t,s))]Y(s)ds, t \in [0,T_0].$$

Then, taking into account that the only non-vanishing entries of Y coincide with g, computations with matrices, the structure of the entries of Υ_n and b, and further applications of Theorem 3.4 give

$$\begin{aligned} u(t) &= \sum_{j=0}^{m-1}\left[\sum_{p=1}^{n} \mathrm{Op}_{\varphi_p(t,0)}(z^0_{pj}(t)) + \mathrm{Op}(r^0_j(t))\right]u_j + i\int_0^t \left[\sum_{p=1}^{n} \mathrm{Op}_{\varphi_p(t,s)}(z^1_p(t,s)) + \mathrm{Op}(r^1(t,s))\right]g(s)ds, \\ &= v_0(t) + \int_0^t\int_{\mathbb{R}^d} \Lambda(t,s,.,y)g(s,y)\,dyds, \end{aligned} \tag{4.15}$$

where

- the phase functions φ_p are solution to the eikonal equations (A.3), with θ_p in place of $\varkappa$, $p = 1,\dots,n$;
- $z^0_{pj} \in C^\infty([0,T_0], S^{l-1-j,l-1-j})$, $p = 1,\dots,n$, $r^0_j \in C^\infty([0,T_0], S^{-\infty,-\infty})$, $j = 0,\dots,m-1$, so that $v_0 \in \bigcap_{j\geq 0} C^j([0,T_0], H^{z+m-l-j,\zeta+m-l-j})$;
- $\Lambda \in C^\infty(\Delta_{T_0}, \mathcal{S}')$ is, for any $(t,s) \in \Delta_{T_0}$, the Schwartz kernel of the operator

$$Z_{l-m}(t,s) = i\left[\sum_{p=1}^{n} \mathrm{Op}_{\varphi_p(t,s)}(z^1_p(t,s)) + \mathrm{Op}(r^1(t,s))\right], \tag{4.16}$$

with $z^1_p \in C^\infty(\Delta_{T_0}, S^{l-m,l-m})$, $p = 1,\dots,m$, $r^1 \in C^\infty(\Delta_{T_0}, S^{-\infty,-\infty})$, so that also

$$\int_0^t\int_{\mathbb{R}^d} \Lambda(t,s,.,y)g(s,y)\,dyds \in \bigcap_{j\geq 0} C^j([0,T_0], H^{z+m-l-j,\zeta+m-l-j}).$$

By Proposition A.2, $\Lambda(t,s)$ differs by an element of $C^\infty(\Delta_{T_0}, \mathcal{S})$ from the kernel of

$$\widetilde{Z}_{l-m}(t,s) = i\sum_{p=1}^{n} \mathrm{Op}_{\varphi_p(t,s)}(z^1_p(t,s)). \tag{4.17}$$

Proof of Theorem 4.1. Let us insert $g(t,x) = \gamma(t,x) + \sigma(t,x)\dot{\Xi}(t,x)$ in (4.15), so that, formally,

$$\begin{aligned} u(t,x) &= v_0(t,x) + \int_0^t\int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y)\,dyds + \int_0^t\int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)\dot{\Xi}(s,y)\,dyds \\ &= v_0(t,x) + v_1(t,x) + v_2(t,x). \end{aligned} \tag{4.18}$$

We find $v_0 \in \bigcap_{j\geq 0} C^j([0,T_0], H^{z+m-l-j,\zeta+m-l-j}) \subset C([0,T_0], H^{z+m-l,\zeta+m-l})$, which is a continuous function in $(t,x) \in [0,T_0]\times\mathbb{R}^d$. This implies that condition **(A5)** holds true. Since $Z_{l-m}(t,s) \in \mathcal{O}(l-m, l-m)$ and $\gamma \in C([0,T], H^{z,\zeta})$, we also find $v_1 \in C([0,T_0], H^{z+m-l,\zeta+m-l})$, which is a well-defined, continuous function in $(t,x) \in [0,T_0]\times\mathbb{R}^d$.

We can rewrite v_2 in (4.18) as

$$v_2(t,x) = \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\sigma(s,y)M(ds,dy),$$

where M is the martingale measure associated with the stochastic noise Ξ, as defined in Section 2. By Proposition 3.9, (4.16) and (4.17), we then find

$$|\mathcal{F}_{y\mapsto\eta}\Lambda(t,s,x,\cdot)(\eta)|^2 = \left|\sum_{p=1}^n e^{i\varphi_p(t,s,x,-\eta)} z_p^1(t,s,x,-\eta) \bmod \mathcal{S}(\mathbb{R}^{2d})\right|^2 \le C_{t,s}\langle x\rangle^{2(l-m)}\langle\eta\rangle^{2(l-m)}. \tag{4.19}$$

where $C_{t,s}$ can be chosen to be continuous in s and t, since Λ differs by an element of $C^\infty(\Delta_{T_0},\mathcal{S})$ from the kernel of (4.17), which is, in turn, in $C(\Delta_{T_0}, S^{l-m,l-m})$. Using (4.19), we get that condition **(A1)**, with $\Lambda(t,s)$ being the Schwartz kernel of $Z_{l-m}(t,s)$, is satisfied if

$$\int_0^t \left(\sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{R}^d} |[\mathcal{F}_{y\mapsto\eta}\Lambda(t,s,x,\cdot)](\eta+\xi)|^2\mu(d\eta)\right)|\nu_s|^2_{\text{tv}}\,ds \lesssim \langle x\rangle^{-2(m-l)}\int_0^t |\nu_s|^2_{\text{tv}}\,ds \sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{\langle\eta+\xi\rangle^{2(m-l)}}\mu(d\eta) < \infty.$$

In view of the assumptions on σ, we conclude that **(A1)** holds true as long as (4.5) does.

To check the continuity condition **(A2)**, with Λ being the Schwartz kernel of Z_{l-m}, which is regular with respect to s and t, it will suffice to show that

$$\sup_{r\in(s,s+h)} |\mathcal{F}(\Lambda(t,s,x)-\Lambda(t,r,x))(\xi+\eta)|^2 \le \frac{C^2_{t,s,h}}{\langle x\rangle^{2(m-l)}\langle\xi+\eta\rangle^{2(m-l)}}, \tag{4.20}$$

with $C_{t,s,h}\to 0$ as $h\to 0$ and $C_{t,s,h}\le C_{T_0}$ for every $h\in[0,t-s]$, $(t,s)\in\Delta_{T_0}$. Indeed, if (4.20) holds true, then:

$$\begin{aligned}
&\lim_{h\to 0}\int_0^t \left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\sup_{r\in(s,s+h)}|\mathcal{F}(\Lambda(t,s,x)-\Lambda(t,r,x))(\xi+\eta)|^2\mu(d\xi)\right)|\nu_s|^2_{\text{tv}}\,ds\\
&\le \lim_{h\to 0}\int_0^t C^2_{t,s,h}\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\langle\xi+\eta\rangle^{-2(m-l)}\mu(d\xi)\right)|\nu_s|^2_{\text{tv}}\,ds\\
&= \left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\langle\xi+\eta\rangle^{-2(m-l)}\mu(d\xi)\right)\lim_{h\to 0}\int_0^t C^2_{t,s,h}|\nu_s|^2_{\text{tv}}\,ds\\
&= 0,
\end{aligned}$$

via Lebesgue's Dominated Convergence Theorem, in view of assumption (4.5), the fact that $|\nu_s|^2_{\text{tv}}\in L^1[0,T]$, and $C_{t,s,h}\le C_{T_0}$.

Then, it only remains to show that (4.20) holds true. This follows from the uniform continuity of $s\mapsto \mathcal{F}\Lambda(t,s,\cdot)(*)$, (3.2) and (4.19). Indeed, the function $s\mapsto\langle\cdot\rangle^{m-l}\langle *\rangle^{m-l}\mathcal{F}\Lambda(t,s,\cdot)(*)$ is, by (4.19), uniformly continuous on $[0,t]$ with values in the Fréchet space $S^{0,0}(\mathbb{R}^{2d})$, endowed with the norm

$$\|a-b\| = \sum_{\ell=0}^\infty \frac{1}{2^\ell}\frac{\|a-b\|^{0,0}_\ell}{1+\|a-b\|^{0,0}_\ell},$$

see Section 3. It follows, for its modulus of continuity,

$$\omega_{t,s}(h) = \sup_{r\in(s,s+h)}\|\langle\cdot\rangle^{m-l}\langle *\rangle^{m-l}(\mathcal{F}\Lambda(t,s,\cdot)(*)-\mathcal{F}\Lambda(t,r,\cdot)(*))\|\to 0$$

as $h \to 0$. By (3.2) with $\ell = 0$ we get

$$
\begin{aligned}
&\sup_{r\in(s,s+h)} |\langle x\rangle^{m-l}\langle \xi+\eta\rangle^{m-l}(\mathcal{F}\Lambda(t,s,x)(\xi+\eta)-\mathcal{F}\Lambda(t,r,x)(\xi+\eta))|\\
&\quad\le \sup_{r\in(s,s+h)} |\langle\cdot\rangle^{m-l}\langle *\rangle^{m-l}(\mathcal{F}\Lambda(t,s,\cdot)(*)-\mathcal{F}\Lambda(t,r,\cdot)(*))|_0^{0,0}\langle x\rangle^0\langle\xi+\eta\rangle^0\\
&\quad= \sup_{r\in(s,s+h)} \Big(\frac{\|\langle\cdot\rangle^{m-l}\langle *\rangle^{m-l}(\mathcal{F}\Lambda(t,s,\cdot)(*)-\mathcal{F}\Lambda(t,r,\cdot)(*))\|_0^{0,0}}{1+\|\langle\cdot\rangle^{m-l}\langle *\rangle^{m-l}(\mathcal{F}\Lambda(t,s,\cdot)(*)-\mathcal{F}\Lambda(t,r,\cdot)(*))\|_0^{0,0}}\times\\
&\qquad\times(1+\|\langle\cdot\rangle^{m-l}\langle *\rangle^{m-l}(\mathcal{F}\Lambda(t,s,\cdot)(*)-\mathcal{F}\Lambda(t,r,\cdot)(*))\|_0^{0,0})\Big)\\
&\quad\le \omega_{t,s}(h)(1+\sup_{r\in(s,s+h)}\|\langle\cdot\rangle^{m-l}\langle *\rangle^{m-l}(\mathcal{F}\Lambda(t,s,\cdot)(*)-\mathcal{F}\Lambda(t,r,\cdot)(*))\|_0^{0,0})\\
&\quad\le \omega_{t,s}(h)(1+2C_{T_0}),
\end{aligned}
\tag{4.21}
$$

where

$$C_{T_0} := \max_{0\le s\le t\le T_0} C_{t,s} < \infty,$$

by (4.19) and the fact that $(t,s)\mapsto C_{t,s}$ is continuous on Δ_{T_0}. Therefore, the term in the last line of (4.21) goes to zero as $h\to 0$. Choosing the constant $C_{t,s,h} = \omega_{t,s}(h)(1+2C_{T_0})$ we get (4.20).

The argument above shows that v_2 is well-defined, as a stochastic integral with respect to the martingale measure canonically associated with Ξ. The regularity claim $\mathbb{E}[u]\in C([0,T_0],H^{z+m-l,\zeta+m-l}(\mathbb{R}^d))$ follows from the regularity properties of Z_{l-m}, of γ and of the Cauchy data, taking expectation on both sides of (4.18), and recalling the fact that the expected value $\mathbb{E}[v_2]$ of the stochastic integral is zero. In fact, Ξ is supposed to be a Gaussian process with mean zero. It follows that the regularity of $\mathbb{E}[u]$ is the same as the one of the solution of the associated deterministic Cauchy problem.

With respect to Remark 4.2, notice that, if $\gamma\in\bigcap_{j\ge0}C^j([0,T],H^{z-j,\zeta-j})$, then we have also $v_1\in\bigcap_{j\ge0}C^j([0,T_0],$ $H^{z+m-l-j,\zeta+m-l-j}(\mathbb{R}^d))\subset C([0,T_0],H^{z+m-l,\zeta+m-l})$. This, together with $v_0\in\bigcap_{j\ge0}C^j([0,T_0],H^{z+m-l-j,\zeta+m-l-j})$ and the fact that $\mathbb{E}[v_2]=0$, gives that $\mathbb{E}[u]\in\bigcap_{j\ge0}C^j([0,T_0],H^{z+m-l-j,\zeta+m-l-j}(\mathbb{R}^d))$. The proof is complete. □

Remark 4.12. Condition (4.5) for the strictly hyperbolic case $l=1$ has already been seen in [19], when dealing with higher-order beam equations. Since beam equations are evolution equations of anisotropic type with real characteristics (in the sense of Petrowski, i.e. in an "anisotropic sense", where τ has the same weight as ξ^2), and deterministic anisotropic equations with real characteristics can be studied with techniques similar to the ones presented in this paper, we conjecture condition (4.5) to be a sufficient condition on the spectral measure to obtain a random-field solution also to stochastic anisotropic evolution equations with real characteristics.

Remark 4.13. One could say that the random-field solution u of (4.3) found in Theorem 4.1 "is unique" in the following sense. First, when $\sigma\equiv 0$, it reduces to the unique solution of the associated deterministic Cauchy problem. Moreover, by linearity, if u_1 and u_2 are two solutions of the linear Cauchy problem (4.3), $u=u_1-u_2$ satisfies the deterministic equation $Lu=0$ with trivial initial conditions, and such Cauchy problem admits in $\mathcal{S}'$ only the trivial solution. The latter follows immediately by the $\mathcal{S}'$ well-posedness (with loss of smoothness and decay) of the Cauchy problem for the homogeneous deterministic linear equation $Lu=0$, proved in [15, 18].

4.2. **The weakly hyperbolic case with involutive roots.** We conclude the paper shortly discussing our second result, under the hypothesis **(WH)$_{\mathbf{IR}}$**. With these even weaker hyperbolicity assumption we can still switch from (4.3) to an equivalent first order system (3.13), but at the price, as usual, of some further requirement on the lower order terms of the operator L. Namely, we ask, that L admits a factorization (A.7) with symbols h_{jk}, $j=1,\dots,m$, $k=1,\dots,l_j$, such that $h_{jk}\in C^\infty([0,T],S^{0,0})$. Notice that this is automatically true in the case of strict hyperbolicity, and that only the request on the order of the symbols h_{jk} has to be fulfilled in the case of hyperbolicity with constant multiplicities. We say, in the present case, that L satisfies the *strong Levi condition*, or, equivalently, that it is of strong Levi type.

Theorem 4.14. *Let us consider the Cauchy problem (4.3) for an SPDE associated with an operator of the form (4.1), $m\ge 2$, under the hyperbolicity hypothesis (4.2). Assume that L is SG-hyperbolic with involutive roots, that is, all the roots of the principal part L_m of L are real-valued and form an involutive system, in the sense of (3.12). Moreover, assume* *that L is of strong Levi type. Assume also, for the initial conditions, that* $u_j \in H^{z+m-j-1,\zeta+m-j-1}(\mathbb{R}^d)$, $0 \le j \le m-1$, *with* $z \in \mathbb{R}$ *and* $\zeta > d/2$. *Furthermore, assume for the spectral measure that*

$$\int_{\mathbb{R}^d} \mu(d\xi) < \infty. \tag{4.22}$$

Finally, assume that $\gamma \in C([0,T]; H^{z,\zeta}(\mathbb{R}^d))$, $\sigma \in C([0,T], H^{0,\zeta}(\mathbb{R}^d))$, *and* $s \mapsto \mathcal{F}\sigma(s) = \nu_s \in L^2([0,T], \mathcal{M}_b(\mathbb{R}^d))$.

Then, for some time horizon $0 < T_0 \le T$, *there exists a unique random-field solution* u *of* (4.3). *Moreover,* $\mathbb{E}[u] \in C([0,T_0], H^{z,\zeta}(\mathbb{R}^d))$.

By the procedure explained in [15], see also [30], the Cauchy problem (4.3) is turned into an equivalent first order system (3.13) with real-valued, diagonal principal part. However, due to the failure of the ellipticity of the differences $\tau_j(t,x,\xi) - \tau_k(t,x,\xi)$, even in the sense of the constant multiplicities, here we have no possibility to decouple (into blocks) the equations through a perfect diagonalizer $\operatorname{Op}(\omega)$ as in Theorem A.13, and directly proceed, as in the case of scalar equations of order 1, by means of Fourier integral operators.

By Theorem A.8 from [4], we know that the fundamental solution of (3.13) can be obtained as limit of a sequence of matrices of Fourier operators. In [1] we have extended to the *SG* case a result by Taniguchi [37], which allows to simplify such general limit procedure, and obtain E, in the case of involutive roots, again as a finite sum of *SG* FIOs, modulo smoothing terms.

Another difference is that, in this case, we have no improvement in the decay and smoothness order loss, as it is instead provided by the matrix-valued operators $\operatorname{Op}(\Upsilon_n)$, $n \le m$, in the cases of constant multiplicities. So, the symbols z^1_p appearing in the expected kernels of the approximate solution operator in the sense of (4.17) will be of order $(0,0)$. This explains the more restrictive condition (4.22), which allows again to go through an argument similar to those in the proofs of Theorem 4.1.

The full proof of Theorem 4.14 requires a number of technical details, to incorporate in the *SG* theory the analog of the result by Taniguchi mentioned above. In order to keep the present exposition within a reasonable size, and not to heavily divert from the main objects of interests treated here, the proof of Theorem 4.14 is given in [1].

Appendix A.

We collect in this Appendix, for the convenience of the reader, some additional results concerning the *SG*-calculus and its applications to hyperbolic problems, which we mentioned along the main text. This material appeared, sometimes in slightly different form, in [4] and the references quoted therein.

The continuity property of the elements of $\operatorname{Op}(S^{m,\mu})$ on the scale of spaces $H^{z,\zeta}(\mathbb{R}^d)$, $(m,\mu), (z,\zeta) \in \mathbb{R}^2$, is precisely expressed in the next Theorem A.1 (see [13] and the references quoted therein for the result on more general classes of *SG*-symbols).

Theorem A.1. *Let* $a \in S^{m,\mu}(\mathbb{R}^d)$, $(m,\mu) \in \mathbb{R}^2$. *Then, for any* $(z,\zeta) \in \mathbb{R}^2$, $\operatorname{Op}(a) \in \mathcal{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-m,\zeta-\mu}(\mathbb{R}^d))$, *and there exists a constant* $C > 0$, *depending only on* d, m, μ, z, ζ, *such that*

$$\|\operatorname{Op}(a)\|_{\mathcal{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-m,\zeta-\mu}(\mathbb{R}^d))} \le C\|a\|^{m,\mu}_{[\frac{d}{2}]+1}, \tag{A.1}$$

where $[t]$ *denotes the integer part of* $t \in \mathbb{R}$.

The following characterization of the class $O(-\infty,-\infty)$ is often useful, see [13].

Theorem A.2. *The class* $O(-\infty,-\infty)$ *coincides with* $\operatorname{Op}(S^{-\infty,-\infty}(\mathbb{R}^d))$ *and with the class of smoothing operators, that is, the set of all the linear continuous operators* $A\colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. *All of them coincide with the class of linear continuous operators* A *admitting a Schwartz kernel* k_A *belonging to* $\mathcal{S}(\mathbb{R}^{2d})$.

An operator $A = \operatorname{Op}(a)$ and its symbol $a \in S^{m,\mu}$ are called *elliptic* (or $S^{m,\mu}$*-elliptic*) if there exists $R \ge 0$ such that

$$C\langle x\rangle^m \langle \xi\rangle^\mu \le |a(x,\xi)|, \qquad |x| + |\xi| \ge R,$$

for some constant $C > 0$. If $R = 0$, a^{-1} is everywhere well-defined and smooth, and $a^{-1} \in S^{-m,-\mu}$. If $R > 0$, then a^{-1} can be extended to the whole of $\mathbb{R}^{2d}$ so that the extension $\widetilde{a}_{-1}$ satisfies $\widetilde{a}_{-1} \in S^{-m,-\mu}$. An elliptic *SG* operator $A \in \operatorname{Op}(S^{m,\mu})$ admits a parametrix $A_{-1} \in \operatorname{Op}(S^{-m,-\mu})$ such that

$$A_{-1}A = I + R_1, \quad AA_{-1} = I + R_2,$$

for suitable $R_1, R_2 \in \operatorname{Op}(S^{-\infty,-\infty})$, where I denotes the identity operator. In such a case, A turns out to be a Fredholm operator on the scale of functional spaces $H^{z,\zeta}(\mathbb{R}^d)$, $(z,\zeta) \in \mathbb{R}^2$.

The study of the composition of $M \geq 2$ *SG* FIOs of type I $\mathrm{Op}_{\varphi_j}(a_j)$ with regular *SG*-phase functions $\varphi_j \in \mathfrak{P}_\delta(\lambda_j)$ and symbols $a_j \in S^{m_j,\mu_j}(\mathbb{R}^d)$, $j = 1, \dots, M$, has been done in [4]. The result of such composition is still an SG-FIO with a regular SG-phase function φ given by the so-called *multi-product* $\varphi_1 \sharp \cdots \sharp \varphi_M$ of the phase functions φ_j, $j = 1, \dots, M$, and symbol a as in Theorem A.3 here below.

Theorem A.3. *Consider, for $j = 1, 2, \dots, M$, $M \geq 2$, the SG FIOs of type I $\mathrm{Op}_{\varphi_j}(a_j)$ with $a_j \in S^{m_j,\mu_j}(\mathbb{R}^d)$, $(m_j, \mu_j) \in \mathbb{R}^2$, and $\varphi_j \in \mathfrak{P}_\delta(\lambda_j)$ such that $\lambda_1 + \cdots + \lambda_M \leq \lambda \leq \frac{1}{4}$ for some sufficiently small $\lambda > 0$. Then, there exists $a \in S^{m,\mu}(\mathbb{R}^d)$, $m = m_1 + \cdots + m_M$, $\mu = \mu_1 + \cdots + \mu_M$, such that, setting $\phi = \varphi_1 \sharp \cdots \sharp \varphi_M$, we have*

$$\mathrm{Op}_{\varphi_1}(a_1) \circ \cdots \circ \mathrm{Op}_{\varphi_M}(a_M) = \mathrm{Op}_\phi(a).$$

Moreover, for any $\ell \in \mathbb{N}_0$ there exist $\ell' \in \mathbb{N}_0$, $C_\ell > 0$ such that

$$\|a\|_\ell^{m,\mu} \leq C_\ell \prod_{j=1}^{M} \|a_j\|_{\ell'}^{m_j,\mu_j}. \tag{A.2}$$

Theorem A.3 is a corollary of the main Theorem in [4]. There, the *multi-product* of regular *SG*-phase functions is defined and its properties are studied, parametrices and compositions of regular *SG* FIOs with amplitude identically equal to 1 are considered, leading to the general composition $\mathrm{Op}_{\varphi_1}(a_1) \circ \cdots \circ \mathrm{Op}_{\varphi_M}(a_M)$. It is needed for the determination of the fundamental solutions of the hyperbolic operators (1.3), involved in (1.1), in the case of involutive roots with non-constant multiplicities, see [1].

The next one is a key result in the analysis of *SG*-hyperbolic Cauchy problems by means of the corresponding class of Fourier operators. Given a symbol $\varkappa \in C([0, T]; S^{1,1})$, set $\Delta_{T_0} = \{(s, t) \in [0, T_0]^2 \colon 0 \leq s \leq t \leq T_0\}$, $0 < T_0 \leq T$, and consider the eikonal equation

$$\begin{cases} \partial_t \varphi(t, s, x, \xi) = \varkappa(t, x, \varphi'_x(t, s, x, \xi)), & t \in [s, T_0], \\ \varphi(s, s, x, \xi) = x \cdot \xi, & s \in [0, T_0), \end{cases} \tag{A.3}$$

with $0 < T_0 \leq T$. By an extension of the theory developed in [15], it is possible to prove that the following Proposition A.4 holds true.

Proposition A.4. *For any small enough $T_0 \in (0, T]$, equation (A.3) admits a unique solution $\varphi \in C^1(\Delta_{T_0}, S^{1,1}(\mathbb{R}^d))$, satisfying $J \in C^1(\Delta_{T_0}, S^{1,1}(\mathbb{R}^d))$ and*

$$\partial_s \varphi(t, s, x, \xi) = -\varkappa(s, \varphi'_\xi(t, s, x, \xi), \xi), \tag{A.4}$$

for any $(t, s) \in \Delta_{T_0}$. Moreover, for every $\ell \in \mathbb{N}_0$ there exists $\delta > 0$, $c_\ell \geq 1$ and $\widetilde{T}_\ell \in [0, T_0]$ such that $\varphi(t, s, x, \xi) \in \mathfrak{P}_\delta(c_\ell |t - s|)$, with $\|J\|_{2,\ell} \leq c_\ell |t - s|$ for all $(t, s) \in \Delta_{\widetilde{T}_\ell}$.

Remark A.5. Of course, if additional regularity with respect to $t \in [0, T]$ is fulfilled by the symbol $\varkappa$ in the right-hand side of (A.3), this reflects in a corresponding increased regularity of the resulting solution φ with respect to $(t, s) \in \Delta_{T_0}$. Since here we are not dealing with problems concerning the t-regularity of the solution, we assume smooth t-dependence of the coefficients of L. Some of the results below will anyway be formulated in situations of lower regularity with respect to t.

The fundamental solution of a first order *SG*-hyperbolic system with diagonal principal part, $E(t, s)$, has the following properties, which actually hold for the broader class of symmetric first order system of the type (3.13), of which systems with real-valued, diagonal principal part are a special case, see [13], Ch. 6, §3, and [15].

Theorem A.6. *Let the system (3.13) be hyperbolic with diagonal principal part $\kappa_1 \in C^1([0, T], S^{1,1}(\mathbb{R}^d))$, and lower order part $\kappa_0 \in C^1([0, T], S^{0,0}(\mathbb{R}^d))$. Then, for any choice of $W_0 \in H^{z,\zeta}(\mathbb{R}^d)$, $Y \in C([0, T], H^{z,\zeta}(\mathbb{R}^d))$, there exists a unique solution $W \in C([0, T], H^{z,\zeta}(\mathbb{R}^d)) \cap C^1([0, T], H^{z-1,\zeta-1}(\mathbb{R}^d))$ of (3.13), $(z, \zeta) \in \mathbb{R}^2$, given by Duhamel's formula*

$$W(t) = E(t, s)W_0 + i \int_s^t E(t, \vartheta) Y(\vartheta) d\vartheta, \quad t \in [0, T].$$

Moreover, the solution operator $E(t, s)$ has the following properties:

1. *$E(t, s) \colon \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is an operator belonging to $\mathcal{O}(0, 0)$, $(t, s) \in [0, T]^2$; its first order derivatives, $\partial_t E(t, s)$, $\partial_s E(t, s)$, exist in the strong operator convergence of $\mathscr{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-1,\zeta-1}(\mathbb{R}^d))$, $(z, \zeta) \in \mathbb{R}^2$, and belong to $\mathcal{O}(1, 1)$;*
2. *$E(t, s)$ is bounded and strongly continuous from $[0, T]^2_{ts}$ to $\mathscr{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z,\zeta}(\mathbb{R}^d))$, $(z, \zeta) \in \mathbb{R}^2$; $\partial_t E(t, s)$ and $\partial_s E(t, s)$ are bounded and strongly continuous from $[0, T]^2_{ts}$ to $\mathscr{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-1,\zeta-1}(\mathbb{R}^d))$, $(z, \zeta) \in \mathbb{R}^2$;*

(3) *for $t, s, t_0 \in [0, T]$ we have*

$$E(t_0, t_0) = I, \quad E(t, s)E(s, t_0) = E(t, t_0), \quad E(t, s)E(s, t) = I;$$

(4) *$E(t, s)$ satisfies, for $(t, s) \in [0, T]^2$, the differential equations*

$$D_t E(t, s) - (\mathrm{Op}(\kappa_1(t)) + \mathrm{Op}(\kappa_0(t)))E(t, s) = 0, \tag{A.5}$$

$$D_s E(t, s) + E(t, s)(\mathrm{Op}(\kappa_1(s)) + \mathrm{Op}(\kappa_0(s))) = 0; \tag{A.6}$$

(5) *the operator family $E(t, s)$ is uniquely determined by the properties (1)-(3) here above, and one of the differential equations (A.5), (A.6).*

Corollary A.7. (1) *Under the hypotheses of Theorem A.6, $E(t, s)$ is invertible on $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}'(\mathbb{R}^d)$, and $H^{z,\zeta}(\mathbb{R}^d)$, $(z, \zeta) \in \mathbb{R}^2$, with inverse given by $E(s, t)$, $s, t \in [0, T]$.*

(2) *If, additionally, one assumes $\kappa_1 \in C^m([0, T], S^{1,1}(\mathbb{R}^d))$, $\kappa_0 \in C^m([0, T], S^{0,0}(\mathbb{R}^d))$, $m \geq 2$, the partial derivatives $\partial_t^j \partial_s^k E(t, s)$ exist in strong operator convergence of $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, and $\partial_t^j \partial_s^k E(t, s) \in \mathcal{O}(j + k, j + k)$, $j + k \leq m$. Moreover, $\partial_t^j \partial_s^k E(t, s)$ is strongly continuous from $[0, T]^2_{ts}$ to every $\mathscr{L}(H^{z,\zeta}(\mathbb{R}^d), H^{z-j-k,\zeta-j-k}(\mathbb{R}^d))$, $(z, \zeta) \in \mathbb{R}^2$, $j + k \leq m$.*

In [4] we have proved the next Theorem A.8, concerning the structure of $E(t, s)$, in the spirit of the approach followed in [26].

Theorem A.8. *Under the same hypotheses of Theorem A.6, if T_0 is small enough, for every fixed $(t, s) \in \Delta_{T_0}$, $E(t, s)$ is a limit of a sequence of matrices of SG FIOs of type I, with regular phase functions $\varphi_{jk}(t, s)$ belonging to $\mathfrak{P}_\delta(c_h|t - s|)$, $c_h \geq 1$, of class C^1 with respect to $(t, s) \in \Delta_{T_0}$, and amplitudes belonging to $C^1(\Delta_{T_0}, S^{0,0}(\mathbb{R}^d))$.*

The next results are employed to switch from (4.3) to a first order linear system of the form (3.13).

Proposition A.9. *Let L be a hyperbolic operator with constant multiplicities l_j, $j = 1, \dots, n \leq m$. Denote by $\theta_j \in G_j$, $j = 1, \dots, n$, the distinct real roots of $\mathcal{L}_m$ in (1.4). Then, it is possible to factor L as*

$$L = L_n \cdots L_1 + \sum_{j=1}^{m} \mathrm{Op}(r_j(t)) D_t^{m-j} \tag{A.7}$$

with

$$L_j = (D_t - \mathrm{Op}(\theta_j(t)))^{l_j} + \sum_{k=1}^{l_j} \mathrm{Op}(h_{jk}(t)) \, (D_t - \mathrm{Op}(\theta_j(t)))^{l_j - k}, \tag{A.8}$$

$$h_{jk} \in C^\infty([0, T], S^{k-1,k-1}(\mathbb{R}^d)), \quad r_j \in C^\infty([0, T], S^{-\infty,-\infty}(\mathbb{R}^d)), \quad j = 1, \dots, n, k = 1, \dots, l_j. \tag{A.9}$$

The following corollary is an immediate consequence of Proposition A.9, and is proved by means of a reordering of the distinct roots θ_j, $j = 1, \dots, n$.

Corollary A.10. *Let ϖ_j, $j = 1, \dots, n$, denote the reordering of the n-tuple $(1, \dots, n)$, given, for $k = 1, \dots, n$, by*

$$\varpi_j(k) = \begin{cases} j + k - 1 & \text{for } j + k \leq n + 1, \\ j + k - n - 1 & \text{for } j + k > n + 1, \end{cases} \tag{A.10}$$

That is, for $n \geq 2$, $\varpi_1 = (1, \dots, n), \varpi_2 = (2, \dots, n, 1), \dots, \varpi_n = (n, 1, \dots, n - 1)$. Then, under the same hypotheses of Proposition A.9, we have, for any $p = 1, \dots, n$,

$$L = L^{(p)}_{\varpi_p(n)} \dots L^{(p)}_{\varpi_p(1)} + \sum_{j=1}^{m} \mathrm{Op}(r^{(p)}_j(t)) D_t^{m-j} \tag{A.11}$$

with

$$L^{(p)}_j = (D_t - \mathrm{Op}(\theta_j(t)))^{l_j} + \sum_{k=1}^{l_j} \mathrm{Op}(h^{(p)}_{jk}(t)) \, (D_t - \mathrm{Op}(\theta_j(t)))^{l_j - k}, \tag{A.12}$$

$$h^{(p)}_{jk} \in C^\infty([0, T], S^{k-1,k-1}(\mathbb{R}^d)), j = 1, \dots, n, k = 1, \dots, l_j, \quad r^{(p)}_j \in C^\infty([0, T], S^{-\infty,-\infty}(\mathbb{R}^d)), j = 1, \dots, m. \tag{A.13}$$

Remark A.11. Of course, for $n = 1$, we only have the single "reordering" $\varpi_1 = (1)$, $l_1 = l = m$, and

$$L = L_1^{(1)} + \sum_{j=1}^{m} \mathrm{Op}(r_j^{(1)}(t))D_t^{m-j}$$

with

$$L_1^{(1)} = (D_t - \mathrm{Op}(\theta_1(t)))^m + \sum_{k=1}^{m} \mathrm{Op}(h_{1k}^{(1)}(t))\,(D_t - \mathrm{Op}(\theta_1(t)))^{m-k},$$

$$h_{1k}^{(1)} \in C^\infty([0,T], S^{k-1,k-1}(\mathbb{R}^d)), k = 1,\dots,m, \quad r_j^{(1)} \in C^\infty([0,T], S^{-\infty,-\infty}(\mathbb{R}^d)), j = 1,\dots,m$$

With inductive procedures similar to those performed in [9, 10] and [29], respectively, it is possible to prove the following Lemma A.12.

Lemma A.12. *Under the same hypotheses of Proposition A.9, for all $k = 0,\dots,m-1$, it is possible to find symbols $\varsigma_{kpq} \in C^\infty([0,T], S^{k-q+l_p-n,k-q+l_p-n}(\mathbb{R}^d))$, $p = 1,\dots,n$, $q = 0,\dots,l_p-1$, such that, for all $t \in [0,T]$,*

$$\theta^k = \sum_{p=1}^{n} \left[\sum_{q=0}^{l_p-1} \varsigma_{kpq}(t)(\theta - \theta_p(t))^q\right] \cdot \left[\prod_{\substack{1\le j\le n\\ j\neq p}} (\theta - \theta_j(t))^{l_j}\right].$$

In the case of strict hyperbolicity, or, more generally, hyperbolicity with constant multiplicities, we can actually "decouple" the equations in (3.13) into n blocks of smaller dimensions, by means of the so-called *perfect diagonalizer*, an element of $C^\infty([0,T], \mathrm{Op}(S^{0,0}))$. Thus, the solution of (3.13) can be reduced to the solution of n independent smaller systems. The principal part of the coefficient matrix of each one of such decoupled subsystems admits then a single distinct eigenvalue of maximum multiplicity, so that it can be treated, essentially, like a scalar SG hyperbolic equations of first order. Explicitely, see, e.g., [15, 26],

Theorem A.13. *Assume that the system (3.13) is hyperbolic with constant multiplicities ν_j, $j = 1,\dots,N$, $\nu_1+\dots+\nu_n = \nu$, with diagonal principal part $\kappa_1 \in C^\infty([0,T], S^{1,1}(\mathbb{R}^d))$ and $\kappa_0 \in C^\infty([0,T], S^{0,0}(\mathbb{R}^d))$, both of them $(\nu\times\nu)$-dimensional matrices. Then, there exist $(\nu\times\nu)$-dimensional matrices $\omega \in C^\infty([0,T], S^{0,0}(\mathbb{R}^d))$ and $\widetilde{\kappa}_0 \in C^\infty([0,T], S^{0,0}(\mathbb{R}^d))$ such that*

$$\det(\omega) \asymp 1 \Rightarrow \omega^{-1} \in C^\infty([0,T], S^{0,0}(\mathbb{R}^d)), \quad \widetilde{\kappa}_0 = \mathrm{diag}(\widetilde{\kappa}_{01},\dots,\widetilde{\kappa}_{0n}),\ \widetilde{\kappa}_{0j}\ (\nu_j\times\nu_j)\text{-dimensional matrix},$$

and

$$(D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\kappa_0(t)))\mathrm{Op}(\omega(t)) - \mathrm{Op}(\omega(t))(D_t - \mathrm{Op}(\kappa_1(t)) - \mathrm{Op}(\widetilde{\kappa}_0(t))) \in C^\infty([0,T], \mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d)). \tag{A.14}$$

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