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Published version:

DOI:10.1016/j.cam.2018.06.054

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Abstract

In this paper we use spline quasi-interpolating projectors on a bounded interval for the numerical solution of nonlinear integral equations. In particular, we propose a spline quasi-interpolating projection method with high order of convergence and a spline quasi-interpolating collocation method, both in case of smooth kernels and in case of Green's function type ones. We explicitly construct the approximate solutions and we get results related to the convergence orders. Finally, we provide numerical tests, that confirm the theoretical results.
Keywords: Spline quasi-interpolation, Spline projector, Nonlinear integral equation
2008 MSC: 65R20, 65J15, 65D07

1 Introduction

Recently, the use of the spline quasi-interpolation has been proved to work well for the approximation of the solution of linear Fredholm integral equations (see e.g. [1, 2, 3, 10, 12, 13]). In particular, in [1] a degenerate kernel method based on (left and right) partial approximation of the kernel by a quartic spline quasi-interpolant is provided. In [2], the authors propose and analyse a collocation method and a modified Kulkarni's scheme based on quasi-interpolating spline operators, which are not projectors. In [10], the numerical solution of the integral equation is constructed by approximating the kernel by using two types of bivariate spline quasi-interpolants. In [13] quadratic and cubic quasi-interpolating projectors are proposed and analysed in Galerkin, Kantorovich, Sloan and Kulkarni schemes. Finally, in [3, 12], quasi-interpolating operators have been presented for the numerical solution of 2D and surface integral equations, respectively.

Moreover, in [9], the authors use the Nyström method based on a quadrature formula associated with non-uniform spline quasi-interpolation for solving nonlinear integral equations of Hammerstein type.

This paper deals with nonlinear integral equations of the form

$$x - K(x) = f, \tag{1.1}$$

where K is the Urysohn integral operator

$$K(x)(s) := \int_0^1 k(s,t,x(t))dt, \qquad s \in I := [0,1], \qquad x \in X := C[0,1]. \tag{1.2}$$

The kernel $k(s,t,u)$ is a real valued function defined on $[0,1] \times [0,1] \times \mathbb{R}$ and we assume that, for $f \in C[0,1]$, (1.1) has a unique solution φ.

We recall that classical methods to solve nonlinear integral equations [6] are the projection ones, as the Galerkin, collocation methods and, recently, a modified projection method, proposed by Kulkarni, based on a sequence $\{\pi_n\}$ of orthogonal projectors or interpolatory projectors, onto finite dimensional subspaces X_n approximating X. X_n is usually the space of piecewise polynomials of degree d at most continuous (see e.g.

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[8, 14, 15, 16] and references therein). Other methods to numerically solve (1.1) are the Nyström ones (see e.g [4, 5, 7] and references therein).

In this paper we propose methods based on spline quasi-interpolation projectors for the solution of (1.1). In particular, in Section 2, we introduce spline quasi-interpolating projectors (abbr. QIPs) of a general degree d and class C^{d-1}, proving their properties and we present some special ones. Then, in Section 3, we consider integral equations (1.1) with smooth kernels and we propose two methods based on the above spline quasi-interpolating projectors. The first method is of Kulkarni's type and has a high convergence order and the second one is a quasi-interpolating spline collocation method. We explicitly construct the approximate solutions and we study their convergence orders. Finally, in Section 4 we consider and analyse the above methods for the solution of nonlinear integral equations with a Green's function type kernel. Both in Section 3 and in Section 4, we provide numerical tests, confirming the theoretical results and illustrating the approximation properties of the applied methods, with particular reference to the convergence order and the fact that the approximate solution is of class C^{d-1}, if the kernel of K satisfies certain smoothness requirements, contrary to classical projection methods, based on piecewise polynomials of degree d, that provide a solution at most continuous.

2 Spline quasi-interpolating projectors

Let $X_n := \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$ be the space of splines of degree d and class C^{d-1} on the uniform knot sequence $\mathcal{T}_n := \{t_i = ih, 0 \le i \le n\}$, with $h = 1/n$.

For $x \in C[0,1]$, let π_n be a QIP on $\mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$ defined by

$$\pi_n x := \sum_{i=1}^{N} \lambda_i(x) B_i, \tag{2.1}$$

where $N :=\dim(\mathcal{S}_d^{d-1}(I, \mathcal{T}_n)) = n + d$, the B_i's are the B-splines with support $[t_{i-d-1}, t_i]$, on the usual extended knot sequence $\mathcal{T}_n^e := \mathcal{T}_n \cup \{t_{-d} = \ldots = t_0 = 0; 1 = t_n = \ldots = t_{n+d}\}$, and are a basis for $\mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$. The coefficients in (2.1) are given by local continuous linear functionals λ_i. They have the form

$$\lambda_i(x) := \sum_{j=2(i-d-1)}^{2i} \sigma_{i,j} x(\xi_j), \tag{2.2}$$

where $\xi_{2i} := t_i$, for $0 \le i \le n$, $\xi_{2i-1} := s_i := \frac{1}{2}(t_{i-1} + t_i)$, for $1 \le i \le n$, and the $\sigma_{i,j}$'s are chosen such that $\pi_n x = x$, for all $x \in \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$. We remark that the quasi-interpolation nodes ξ_j involved in (2.2) are inside the support of B_i.

Since the λ_i are continuous linear functionals, the operator π_n is bounded, i.e. $\|\pi_n\| := \sup \frac{\|\pi_n x\|}{\|x\|} < \infty$, $x \in C[0,1]$. Therefore, as π_n is a projector, classical results in approximation theory provide

$$\|x - \pi_n x\|_\infty \le C \operatorname{dist}(x, \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)),$$

where $C := 1 + \|\pi_n\|_\infty$. Using the Jackson type theorem for splines [11], we can conclude that, for $x \in C^j[0,1]$, there exists a constant $\overline{C}_j$, depending on C and j, such that,

$$\|x - \pi_n x\|_\infty \le \overline{C}_j h^j \omega(x^{(j)}, h), \quad \text{with} \quad 0 \le j \le d, \tag{2.3}$$

where ω is the modulus of continuity of $x^{(j)}$. Moreover, if x has the derivative of order $d+1$ continuous, we obtain

$$\|x - \pi_n x\|_\infty = O(h^{d+1}). \tag{2.4}$$

The QIP π_n can also be written in the quasi-Lagrange form

$$\pi_n x = \sum_{i=0}^{2n} x_i L_i, \tag{2.5}$$

with $x_i := x(\xi_i)$, for $0 \le i \le 2n$ and the L_i's are compactly supported functions, called fundamental functions, obtained as linear combination of B-splines, according to (2.2).

The following theorems present some interesting properties of the projectors π_n, in case of even degree d.

Theorem 1. *Let π_n be a QIP on $\mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$ of kind (2.1) and let the degree d be even. If the functionals λ_i, $i = d+1, \dots, n$, are such that the values $\sigma_{i,j}$, in (2.2), associated with quasi-interpolation nodes symmetric with respect to the center of the support of B_i, are equal, then*

$$\int_{t_{i-1}}^{t_i} (\pi_n m_{d+1}(t) - m_{d+1}(t))dt = 0, \qquad i = d+1, \dots, n-d, \tag{2.6}$$

where $m_{d+1}(t) = t^{d+1}$.

Proof. Let $\mathbb{P}_d$ be the space of polynomials of degree at most d. Setting $m_{d+1}(t) := (t - s_i)^{d+1} + p_d(t) = p_{d+1}(t) + p_d(t)$, where $p_d \in \mathbb{P}_d$, as $\pi_n p_d = p_d$, we can write

$$\int_{t_{i-1}}^{t_i} (\pi_n m_{d+1}(t) - m_{d+1}(t))dt = \int_{t_{i-1}}^{t_i} (\pi_n p_{d+1}(t) - p_{d+1}(t))dt.$$

Now, as $\int_{t_{i-1}}^{t_i} p_{d+1}(t)dt = 0$, it is sufficient to prove that $\int_{t_{i-1}}^{t_i} \pi_n p_{d+1}(t) = 0$.

Taking into account the locality of the B-splines, the symmetry of the data points with respect to s_i, the fact that p_{d+1} satisfies $p_{d+1}(s_i + w) = -p_{d+1}(s_i - w)$ and the assumptions on the coefficient $\lambda_i(x)$, $i = d+1, \dots, n$, we can deduce $\int_{t_{i-1}}^{t_i} \pi_n p_{d+1}(t) = 0$.

Therefore, we get (2.6). □

Theorem 2. *If the hypotheses of Theorem 1 hold, for any function $g \in W^{1,1}$ (i.e. with $\|g'\|_1$ bounded) and any function x such that $\left\|x^{(d+2)}\right\|_\infty$ is bounded, there results*

$$\left| \int_0^1 g(t)(\pi_n x(t) - x(t))dt \right| = O(h^{d+2}). \tag{2.7}$$

Proof. We have that

$$\begin{aligned} \left| \int_0^1 g(t)(\pi_n x(t) - x(t))dt \right| &= \left| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} g(t)(\pi_n x(t) - x(t))dt \right| \\ &\le \underbrace{\left| \sum_{j=d+1}^{n-d} \int_{t_{j-1}}^{t_j} g(t)\left(\pi_n x(t) - x(t)\right)dt \right|}_{(I)} + \underbrace{\left| \sum_{j=1}^{d} \int_{t_{j-1}}^{t_j} g(t)\left(\pi_n x(t) - x(t)\right)dt \right|}_{(II)} \\ &\quad + \underbrace{\left| \sum_{j=n-d+1}^{n} \int_{t_{j-1}}^{t_j} g(t)\left(\pi_n x(t) - x(t)\right)dt \right|}_{(III)} \end{aligned}$$

For (II) and (III), taking into account (2.4), we have

$$\begin{aligned} (II) + (III) \quad &\le \sum_{j=1}^{d} \int_{t_{j-1}}^{t_j} |g(t)|\,|\pi_n x(t) - x(t)|\,dt + \sum_{j=n-d+1}^{n} \int_{t_{j-1}}^{t_j} |g(t)|\,|\pi_n x(t) - x(t)|\,dt \\ &\le 2dh\,\|g\|_\infty \,\|\pi_n x - x\|_\infty = O(h^{d+2}). \end{aligned} \tag{2.8}$$

In order to bound the term (I), firstly we consider

$$\int_{t_d}^{t_{n-d}} g(t)(\pi_n m_{d+1}(t) - m_{d+1}(t))dt. \tag{2.9}$$

Setting $\gamma_j := \frac{1}{h}\int_{t_{j-1}}^{t_j} g(s)ds$, for all $j = d+1 \dots n-d$, and defining the piecewise constant function γ by $\gamma(t) = \gamma_j$ for $t \in (t_{j-1}, t_j)$, then

$$\|g-\gamma\|_{1,I_d} = \int_{t_d}^{t_{n-d}} |g(t)-\gamma(t)|dt = \sum_{j=d+1}^{n-d}\int_{t_{j-1}}^{t_j} |g(t)-\gamma_j|dt, \tag{2.10}$$

with $I_d := [t_d, t_{n-d}]$. Since

$$|g(t)-\gamma_j| \le \frac{1}{h}\int_{t_{j-1}}^{t_j} |g(t)-g(s)|ds = \frac{1}{h}\int_{t_{j-1}}^{t_j}\int_s^t |g'(v)|dvds \le \int_{t_{j-1}}^{t_j} |g'(v)|dv, \tag{2.11}$$

from (2.10) and (2.11)

$$\|g-\gamma\|_{1,I_d} \le \sum_{j=d+1}^{n-d}\int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{t_j} |g'(v)|dvdt = h\sum_{j=d+1}^{n-d}\int_{t_{j-1}}^{t_j} [g'(v)|dv = h\,\|g'\|_{1,I_d}.$$

Therefore, we consider (2.9) and we get

$$\begin{aligned}
&\left|\int_{t_d}^{t_{n-d}} g(t)(\pi_n m_{d+1}(t) - m_{d+1}(t))dt\right| \\
&\le \left|\int_{t_d}^{t_{n-d}} (g(t)-\gamma(t))(\pi_n m_{d+1}(t) - m_{d+1}(t))dt\right| + \left|\int_{t_d}^{t_{n-d}} \gamma(t)(\pi_n m_{d+1}(t) - m_{d+1}(t))dt\right| \\
&\le \int_{t_d}^{t_{n-d}} |g(t)-\gamma(t)|\,|\pi_n m_{d+1}(t) - m_{d+1}(t)|\,dt + \left|\sum_{j=d+1}^{n-d}\gamma_j \underbrace{\int_{t_{j-1}}^{t_j} (\pi_n m_{d+1}(t) - m_{d+1}(t))dt}_{=0 \text{ by Theorem 1}}\right| \\
&\le \|\pi_n m_{d+1} - m_{d+1}\|_{\infty,I_d}\,\|g-\gamma\|_{1,I_d} = O(h^{d+2})
\end{aligned} \tag{2.12}$$

in view of (2.4).

Now, we consider the quasi-Lagrange form (2.5) of π_n and the Taylor expansion for x

$$x(\xi_i) = \sum_{k=0}^{d}\frac{x^{(k)}(t)(\xi_i-t)^k}{k!} + \frac{x^{(d+1)}(t)}{(d+1)!}(\xi_i-t)^{d+1} + R_i, \tag{2.13}$$

where

$$R_i = \frac{x^{(d+2)}(v_i)}{(d+2)!}(\xi_i-t)^{d+2}, \quad \text{with } v_i \text{ between } \xi_i \text{ and } t.$$

By considering the binomial expansion and by using the exactness of π_n on $\mathbb{P}_d$, since d is even, from (2.5) and (2.13), we obtain

$$\begin{aligned}
\pi_n x(t) &= \sum_{i=0}^{2n} x(\xi_i)L_i(t) = x(t) + \frac{x^{(d+1)}(t)}{(d+1)!}\sum_{i=0}^{2n}(\xi_i-t)^{d+1}L_i(t) + \sum_{i=0}^{2n} R_iL_i(t) \\
&= x(t) + \frac{x^{(d+1)}(t)}{(d+1)!}\sum_{i=0}^{2n}(\xi_i^{d+1}-t^{d+1})L_i(t) + \sum_{i=0}^{2n} R_iL_i(t) \\
&= x(t) + \frac{x^{(d+1)}(t)}{(d+1)!}(\pi_n m_{d+1}(t) - m_{d+1}(t)) + \sum_{i=0}^{2n} R_iL_i(t).
\end{aligned} \tag{2.14}$$

Assuming $t \in [t_{j-1}, t_j]$, since the fundamental functions L_i are compactly supported, $|R_i| \le \dfrac{\left\|x^{(d+2)}\right\|_\infty}{(d+2)!} C h^{d+2}$, with C a suitable positive constant. Therefore

$$|R| = \left| \sum_{i=0}^{2n} R_i L_i(t) \right| \le \frac{\left\|x^{(d+2)}\right\|_\infty}{(d+2)!} C h^{d+2} \sum_{i=0}^{2n} |L_i(t)| \, .$$

As the Lebesgue function $\sum_{i=0}^{2n} |L_i(t)|$ is bounded independently of n, we see that

$$|R| = O(h^{d+2}). \tag{2.15}$$

Therefore, from (2.14)

$$\pi_n x(t) - x(t) = \frac{x^{(d+1)}(t)}{(d+1)!} (\pi_n m_{d+1}(t) - m_{d+1}(t)) + R. \tag{2.16}$$

Consequently, from (2.16)

$$(I) = \left| \int_{t_d}^{t_{n-d}} g(t)(\pi_n x(t) - x(t)) dt \right| = \left| \int_{t_d}^{t_{n-d}} g(t) \left(\frac{x^{(d+1)}(t)}{(d+1)!} (\pi_n m_{d+1}(t) - m_{d+1}(t)) + R \right) dt \right| .$$

Since the function $g(t) \frac{x^{(d+1)}(t)}{(d+1)!} \in W^{1,1}$, from (2.12) and (2.15), we get

$$(I) = O(h^{d+2}). \tag{2.17}$$

Therefore, from (2.8) and (2.17) we obtain (2.7). □

Here we report three examples of spline QIPs of the form (2.1), that we denote by Q_2, $\bar{Q}_2$ and Q_3.

The operators Q_2 and $\bar{Q}_2$ are defined in the space $\mathcal{S}_2^1(I, \mathcal{T}_n)$ of C^1 quadratic splines ([13], [11, p. 155]) as follows

- $Q_2 x := \sum_{i=1}^{n+2} \lambda_i(x) B_i$, with

$$\lambda_1(x) := x_0, \qquad \lambda_2(x) := 2x_1 - \frac{1}{2}(x_0 + x_2),$$

$$\lambda_{n+1}(x) := 2x_{2n-1} - \frac{1}{2}(x_{2n-2} + x_{2n}), \qquad \lambda_{n+2}(x) := x_{2n},$$

$$\lambda_i(x) := \tfrac{1}{14} x_{2i-6} - \tfrac{2}{7} x_{2i-5} + \tfrac{10}{7} x_{2i-3} - \tfrac{2}{7} x_{2i-1} + \tfrac{1}{14} x_{2i}, \ 3 \le i \le n.$$

- $\bar{Q}_2 x := \sum_{i=1}^{n+2} \lambda_i(x) B_i$, with

$$\lambda_1(x) := x_0, \quad \lambda_i(x) := 2x_{2i-3} - \frac{1}{2}(x_{2i-4} + x_{2i-2}), \ 2 \le i \le n+1, \quad \lambda_{n+2}(x) := x_{2n}.$$

The operator Q_3 is defined on the space $\mathcal{S}_3^2(I, \mathcal{T}_n)$ of C^2 cubic splines [13] as follows:

$$Q_3 x := \sum_{i=1}^{n+3} \lambda_i(x) B_i,$$

with

$$\lambda_1(x) := x_0, \qquad \lambda_2(x) := -\frac{5}{18}x_0 + \frac{20}{9}x_1 - \frac{4}{3}x_2 + \frac{4}{9}x_3 - \frac{1}{18}x_4,$$

$$\lambda_3(x) := \frac{1}{8}x_0 - x_1 + \frac{19}{8}x_2 - \frac{19}{24}x_4 + \frac{1}{3}x_5 - \frac{1}{24}x_6,$$

$$\lambda_{n+1}(x) := \frac{1}{8}x_{2n} - x_{2n-1} + \frac{19}{8}x_{2n-2} - \frac{19}{24}x_{2n-4} + \frac{1}{3}x_{2n-5} - \frac{1}{24}x_{2n-6},$$

$$\lambda_{n+2}(x) := -\frac{5}{18}x_{2n} + \frac{20}{9}x_{2n-1} - \frac{4}{3}x_{2n-2} + \frac{4}{9}x_{2n-3} - \frac{1}{18}x_{2n-4}, \quad \lambda_{n+3}(x) := x_{2n},$$

$$\lambda_i(x) := -\frac{1}{30}x_{2i-8} + \frac{4}{15}x_{2i-7} - \frac{19}{30}x_{2i-6} + \frac{9}{5}x_{2i-4} - \frac{19}{30}x_{2i-2} + \frac{4}{15}x_{2i-1} - \frac{1}{30}x_{2i}, \ 4 \le i \le n.$$

It is easy to verify that Q_2 and $\bar{Q}_2$ satisfy the hypotheses of Theorem 1 and Theorem 2. Moreover, they are superconvergent on the set of evaluation points $\{\xi_i\}_{i=0}^{2n}$, i.e. the following theorem holds.

Theorem 3. *If $\|x^{(4)}\|_\infty$ is bounded, then, for $\pi_n = Q_2, \bar{Q}_2$ it holds*

$$|\pi_n x(\xi_i) - x(\xi_i)| = O(h^4), \qquad 0 \le i \le 2n. \tag{2.18}$$

Proof. The result (2.18) is proved in [13] for Q_2. Following the same logical scheme, we can get (2.18) also for $\bar{Q}_2$. □

3 Projection spline methods for Uryshon integral equation with smooth kernels

We consider the integral equation (1.1) with a smooth kernel. Given a spline QIP operator $\pi_n : C[0,1] \to \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$, defined as in Section 2, we introduce the following projection methods based on it:

1. QIP spline Kulkarni's type method, where, in (1.1), K is approximated by

   $$K_n^k := \pi_n K + K\pi_n - \pi_n K \pi_n. \tag{3.1}$$

   The approximate equation is then

   $$\varphi_n^k - K_n^k(\varphi_n^k) = f. \tag{3.2}$$

2. QIP spline collocation method, where, in (1.1), K is approximated by $K_n^c := \pi_n K \pi_n$ and the right hand side f by $\pi_n f$. The approximate equation is then

   $$\varphi_n^c - \pi_n K(\varphi_n^c) = \pi_n f. \tag{3.3}$$

3.1 Convergence of the methods

In this section we study the convergence of the spline projection methods (3.2) and (3.3).

Concerning the existence and uniqueness of the approximate solutions φ_n^k and φ_n^c, we can refer to the general results given in [16] and [8], respectively, that also hold for the spline QIPs considered.

To establish the convergence order of the proposed methods, we firstly need the following assumptions.

Let φ be the unique solution of (1.1) and let a and b be two real numbers such that

$$[\min_{s\in[0,1]} \varphi(s), \max_{s\in[0,1]} \varphi(s)] \subset (a,b).$$

Define $\Omega := [0,1] \times [0,1] \times [a,b]$. Let $\alpha \geq 1$. We assume that $k \in C^{\alpha}(\Omega)$, $\frac{\partial k}{\partial x} \in C^{2\alpha}(\Omega)$, $f \in C^{\alpha}[0,1]$. Therefore, K is a compact operator from $C[0,1]$ to $C^{\alpha}[0,1]$ and $\varphi \in C^{\alpha}[0,1]$.

The operator K is Fréchet differentiable and the Fréchet derivative is given by

$$(K'(x)q)(s) = \int_0^1 \frac{\partial k}{\partial u}(s,t,x(t))q(t)dt. \tag{3.4}$$

We get the convergence orders of the spline projection methods (3.2) and (3.3). Moreover, we prove a superconvergence property of (3.2) in case of even d, under some hypotheses on the functionals defining the spline QIP and on the smoothness of the kernel K.

Theorem 4. *For $\alpha \geq 1$, let $k \in C^{\alpha}(\Omega), \frac{\partial k}{\partial u} \in C^{2\alpha}(\Omega)$ and $f \in C^{\alpha}[0,1]$. Let φ be the unique solution of (1.1) and assume that 1 is not an eigenvalue of $K'(\varphi)$. Let $\pi_n : C[0,1] \to \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$ be a spline QIP operator of kind (2.1). Let φ_n^k be the unique solution of (3.2). Then*

$$\|\varphi_n^k - \varphi\|_\infty = O(h^{2\beta}),$$

with $\beta := \min\{\alpha, d+1\}$.

Proof. It is directly obtained from the approximation properties (2.3) of the spline projector π_n and from the general Theorem 2.4 of [16]. □

If the kernel of K is sufficiently smooth, that is $\alpha \geq d+1$, we have also the following result.

Theorem 5. *For $\alpha \geq d+1$, let $k \in C^{\alpha}(\Omega), \frac{\partial k}{\partial u} \in C^{2\alpha}(\Omega)$ and $f \in C^{\alpha}[0,1]$. Let φ be the unique solution of (1.1) and assume that 1 is not an eigenvalue of $K'(\varphi)$. Let $\pi_n : C[0,1] \to \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$ be a spline QIP of kind (2.1). Let φ_n^k be the unique solution of (3.2). Then*

$$\|\varphi_n^k - \varphi\|_\infty = \begin{cases} O(h^{2d+2}), & \text{if } d \text{ is odd} \\ O(h^{2d+3}), & \text{if } d \text{ is even and } \varphi \text{ satisfies the hypothesis of Theorem 2} \end{cases}.$$

Proof. If d is odd, the result is an immediate consequence of Theorem 4 with $\beta = d+1$.

Now we suppose d even. We know that, by assumption, $I - K'(\varphi)$ is invertible. From (3.1), we have

$$(K_n^k)'(\varphi) = \pi_n K'(\varphi) + (I - \pi_n)K'(\pi_n\varphi)\pi_n.$$

Consequently,

$$K'(\varphi) - (K_n^k)'(\varphi) = (I - \pi_n)K'(\varphi)(I - \pi_n) + (I - \pi_n)(K'(\varphi) - K'(\pi_n\varphi))\pi_n.$$

Therefore,

$$\|K'(\varphi) - (K_n^k)'(\varphi)\| \leq \|(I - \pi_n)K'(\varphi)(I - \pi_n)\| + \|(I - \pi_n)(K'(\varphi) - K'(\pi_n\varphi))\pi_n\|.$$

Since π_n converges to the identity operator pointwise on $C[0,1]$ and $K'(\varphi)$ is compact, it follows that $\|(I - \pi_n)K'(\varphi)\| \to 0$, as $n \to \infty$.

For $\delta_0 > 0$, let

$$B(\varphi, \delta_0) = \{\psi \in X : \|\varphi - \psi\|_\infty < \delta_0\}.$$

Since by assumption $\frac{\partial k}{\partial x} \in C^{2\alpha}(\Omega)$, it follows that K' is Lipschitz continuous in a neighborhood $B(\varphi, \delta_0)$ of φ, that means, there exists a constant γ such that

$$\|K'(\varphi) - K'(x)\| \leq \gamma \|\varphi - x\|_\infty, \qquad x \in B(\varphi, \delta_0).$$

Therefore, we get

$$\|K'(\varphi) - K'(\pi_n\varphi)\| \leq \gamma\|\varphi - \pi_n\varphi\|_\infty \to 0, \quad \text{as } n \to \infty.$$

Thus, since the sequence $\{\|\pi_n\|\}$ is uniformly bounded,

$$\|K'(\varphi) - (K_n^k)'(\varphi)\| \to 0 \quad \text{as } n \to \infty.$$

It follows that $I - (K_n^k)'(\varphi)$ is invertible, for n big enough, and

$$\|(I - (K_n^k)'(\varphi))]^{-1}\| \le 2\|(I - K'(\varphi))^{-1}\|.$$

Since the spline operator π_n is a projector, by using Theorem 1.1 in [16], we obtain

$$\|\varphi_n^k - \varphi\|_\infty \le 4\|(I - K'(\varphi))^{-1}\|\|(I - \pi_n)(K(\varphi) - K(\pi_n\varphi))\|_\infty. \tag{3.5}$$

Now we can write

$$(I-\pi_n)(K(\varphi)-K(\pi_n(\varphi))) = \underbrace{-(I - \pi_n)[K(\pi_n\varphi) - K(\varphi) - K'(\varphi)(\pi_n\varphi - \varphi)]}_{(\square)}\underbrace{-(I - \pi_n)K'(\varphi)(\pi_n\varphi - \varphi)}_{(\diamond)}.$$

By applying Lemma 2.1 in [16], that holds for the spline projector π_n, since $\beta = d + 1$, we have

$$\|(\square)\|_\infty = O(h^{3d+3}). \tag{3.6}$$

Moreover, from (2.4), we get

$$\|(\diamond)\|_\infty \le C\|(K'(\varphi)(\pi_n\varphi - \varphi))^{(d+1)}\|_\infty h^{d+1}. \tag{3.7}$$

Since $\frac{\partial k}{\partial u} \in C^{2\alpha}(\Omega)$, it follows that

$$(K'(\varphi)(\pi_n\varphi - \varphi))^{(d+1)}(s) = \int_0^1 \frac{\partial k^{d+2}}{\partial s^{d+1}\partial u}(s, t, \varphi(t))(\pi_n\varphi - \varphi)(t)dt,$$

then, since d is even, by Theorem 2 with $g(t) = \frac{\partial k^{d+2}}{\partial s^{d+1}\partial u}(s, t, \varphi(t))$, we obtain

$$\|(K'(\varphi)(\pi_n\varphi - \varphi))^{(d+1)}\|_\infty = O(h^{d+2}). \tag{3.8}$$

Thus, from (3.7) and (3.8)

$$\|(\diamond)\|_\infty = O(h^{2d+3}). \tag{3.9}$$

Finally, from (3.5), (3.6) and (3.9), we can conclude that $\|\varphi_n^k - \varphi\|_\infty = O(h^{2d+3})$. □

For the QIP spline collocation method (3.3), as we expect from classical literature (see e.g [6, 8]), we have

$$\|\varphi_n^c - \varphi\|_\infty = O(h^\beta). \tag{3.10}$$

If $\beta = d + 1$, from (2.4), we have the following order of convergence:

$$\|\varphi_n^c - \varphi\|_\infty = O(h^{d+1}). \tag{3.11}$$

3.2 Construction of the approximate solutions

Considering the equations (3.2) and (3.3), here we propose the construction of the corresponding approximate solutions.

1. QIP spline Kulkarni's type method.

   From (3.1) and (3.2), taking into account that π_n is a spline QIP, after some algebra we obtain

   $$\pi_n\varphi_n^k - \pi_n K(\varphi_n^k) = \pi_n f, \quad \varphi_n^k = \pi_n\varphi_n^k + (I - \pi_n)(K(\pi_n\varphi_n^k) + f) \tag{3.12}$$

   and

   $$\pi_n\varphi_n^k - \pi_n K(\pi_n\varphi_n^k + (I - \pi_n)(K(\pi_n\varphi_n^k) + f)) = \pi_n f. \tag{3.13}$$

Defining $\psi_n = \pi_n \varphi_n^k$ and

$$F_n(y) = y - \pi_n K(y + (I - \pi_n)(K(y) + f)) - \pi_n f, \quad y \in \mathcal{S}_d^{d-1}(I, \mathcal{T}_n), \tag{3.14}$$

whose Fréchet derivative is given by

$$(F_n)'(y)q = q - \pi_n K'(y + (I - \pi_n)(K(y) + f))(I + (I - \pi_n)K'(y))q, \tag{3.15}$$

the equation (3.13) becomes $\psi_n - \pi_n K(\psi_n + (I - \pi_n)(K((\psi_n) + f)) = \pi_n f$, which is equivalent to $F_n(\psi_n) = 0$ and it is iteratively solved by applying the Newton-Kantorovich method.

Given an initial approximation $\psi_n^{(0)}$, the iterates $\psi_n^{(r)}$, $r = 0, 1, 2, \ldots$, are given by

$$\psi_n^{(r+1)} = \psi_n^{(r)} - ((F_n)'(\psi_n^{(r)}))^{-1} F_n(\psi_n^{(r)}). \tag{3.16}$$

Defining, according to (3.12),

$$\varphi_n^{(r)} := \psi_n^{(r)} + (I - \pi_n)(K(\psi_n^{(r)}) + f), \tag{3.17}$$

then, from (3.14) and (3.15), (3.16) can be written as

$$\begin{aligned} &\psi_n^{(r+1)} - \pi_n K'(\varphi_n^{(r)})\psi_n^{(r+1)} - \pi_n K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})\psi_n^{(r+1)} \\ &= \pi_n(K(\varphi_n^{(r)}) + f) - \pi_n K'(\varphi_n^{(r)})\psi_n^{(r)} - \pi_n K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})\psi_n^{(r)}. \end{aligned} \tag{3.18}$$

Since $\psi_n^{(r)} \in \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$,

$$\psi_n^{(r)} = \sum_{j=1}^{N} x_n^{(r)}(j) B_j, \qquad x_n^{(r)} \in \mathbb{R}^N \tag{3.19}$$

and, from (3.18), after some algebra, we obtain the following linear system of size N:

$$\begin{aligned} &x_n^{(r+1)}(i) - \sum_{j=1}^{N} x_n^{(r+1)}(j)\lambda_i(K'(\varphi_n^{(r)})B_j) - \sum_{j=1}^{N} x_n^{(r+1)}(j)\lambda_i\left(K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})B_j\right) \\ &= \lambda_i\left(K(\varphi_n^{(r)})\right) + \lambda_i(f) - \sum_{j=1}^{N} x_n^{(r)}(j)\lambda_i\left(K'(\varphi_n^{(r)})B_j\right) - \sum_{j=1}^{N} x_n^{(r)}(j)\lambda_i\left(K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})B_j\right), \end{aligned}$$

$i = 1, \ldots, N$, whose matrix form is

$$\left(I - A_n^{(r)} - B_n^{(r)}\right) x_n^{(r+1)} = d_n^{(r)}, \tag{3.20}$$

where, for $i, j = 1, 2, \ldots, N$,

- $A_n^{(r)}(i, j) := \lambda_i\left(K'(\varphi_n^{(r)})B_j\right)$,
- $B_n^{(r)}(i, j) := \lambda_i\left(K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})B_j\right)$,
- $d_n^{(r)}(i) := \lambda_i\left(K(\varphi_n^{(r)})\right) + \lambda_i(f) - (A_n^{(r)}x_n^{(r)})(i) - (B_n^{(r)}x_n^{(r)})(i)$.

After solving the system (3.20), we get the vector $x_n^{(r+1)}$, defining $\psi_n^{(r+1)}$ as in (3.19). Finally, the approximate solution φ_n^k at the $(r+1)$ iteration is $\varphi_n^{(r+1)}$, constructed as in (3.17).

From (3.17), we can remark that the approximate solution $\varphi_n^k \in C^{d-1}[0, 1]$, if the kernel k is sufficiently smooth, contrary to the classical projection methods based on piecewise polynomials of degree d that are at most continuous.

2. QIP spline collocation method.

   Defining $F_n(y) = y - \pi_n K(y) - \pi_n f$, where $y \in \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$, we solve $F_n(\varphi_n^c) = 0$ iteratively by using the Newton-Kantorovich method. Given an initial approximation $\varphi_n^{(0)}$, the iterates $\varphi_n^{(r)}$, $r = 0, 1, 2, \ldots$, are

$$\varphi_n^{(r+1)} - \pi_n K'(\varphi_n^{(r)})\varphi_n^{(r+1)} = \pi_n(K(\varphi_n^{(r)}) + f) - \pi_n K'(\varphi_n^{(r)})\varphi_n^{(r)}. \tag{3.21}$$

   Setting

$$\varphi_n^{(r)} := \sum_{j=1}^{N} x_n^{(r)}(j)B_j, \quad x_n^{(r)} \in \mathbb{R}^N, \tag{3.22}$$

   (3.21) is equivalent to the following system of linear equations of size N

$$\left(I - C_n^{(r)}\right) x_n^{(r+1)} = w_n^{(r)}, \tag{3.23}$$

   where, for $i, j = 1, 2, \ldots, N$,

   - $C_n^{(r)}(i,j) := \lambda_i \left(K'(\varphi_n^{(r)})B_j\right)$,
   - $w_n^{(r)}(i) := \lambda_i \left(K(\varphi_n^{(r)})\right) + \lambda_i(f) - (C_n^{(r)} x_n^{(r)})(i)$.

   After solving the system (3.23), we get the vector $x_n^{(r+1)}$, defining $\varphi_n^{(r+1)}$ as in (3.22). Finally, the approximate solution φ_n^c at the $(r+1)$ iteration is $\varphi_n^{(r+1)}$. From (3.22), we notice that the solution $\varphi_n^{(c)}$ is a spline of degree d and class C^{d-1}.

From a computational point of view, the QIP spline Kulkarni's type method is more complicated than the QIP spline collocation one that is easier to implement. However, as we have proved in Section 3.1, the first one has a higher convergence order.

Remark 1. *Our method (3.1)-(3.2), based on quasi-interpolatory projectors in the space of splines of degree d and smoothness C^{d-1}, is comparable with those recently proposed in [5] and [16], based on interpolatory projectors in the space of piecewise polynomials of degree d at most continuous. In general, for sufficiently smooth kernels, the convergence order is $2d+2$ for the above three methods. However, the schemes in [5] and [16] provide a numerical solution that is at most continuous for any degree d, while the approximate solution provided by our scheme has the advantage of having smoothness that depends on the degree d, indeed it is C^{d-1}. Moreover, we remark that our method has convergence order $2d+3$ if d is even and those proposed in [5] and [16] can achieve convergence order $3d+3$ using interpolatory projectors at the set of Gauss points.*

3.3 Numerical results

In this section we present some test equations of kind (1.1), also considered in [4, 5, 16]. We solve them both by the QIP spline Kulkarni's type method and the QIP spline collocation one, based on the spline projectors $Q_2, \bar{Q}_2$ and Q_3.

The integrals appearing in (3.20) and (3.23) are evaluated numerically by using a classical composite Gauss-Legendre quadrature formula with high accuracy.

For all the tests, for increasing values of n, we compute:

i) the maximum absolute error

$$E_\infty^\mu := \max_{v \in G} |\varphi(v) - \varphi_n^\mu(v)|,$$

   where G is a set of 1500 equally spaced points in $[0,1]$, $\mu = k$ in case of QIP spline Kulkarni's type methods and $\mu = c$ in case of QIP spline collocation ones, based on the projectors Q_2, $\bar{Q}_2$, Q_3. We also compute the corresponding numerical convergence order O_∞^μ, obtained by the logarithm to base 2 of the ratio between two consecutive errors;

ii) the maximum absolute error at the quasi-interpolation nodes

$$ES^\mu := \max_{0\le i\le 2n} |\varphi(\xi_i) - \varphi_n^\mu(\xi_i)|,$$

with $\mu = k$ in case of QIP spline Kulkarni's type methods and $\mu = c$ in case of QIP spline collocation ones, based on Q_2, $\bar{Q}_2$. The corresponding numerical convergence order O^μ is also computed.

The numerical tests confirm the theoretical properties proved in Section 3.1. Moreover, the comparisons of our results with those reported in [4, 5, 16] show that the statements of Remark 1 hold. Finally, we remark that methods based on quasi-interpolating spline projectors provide smooth approximate solutions of class C^1 when $\pi_n = Q_2$, $\bar{Q}_2$ and of class C^2 when $\pi_n = Q_3$.

Test 1

Consider the following Hammerstein integral operator K with a degenerate kernel, defined as follows

$$K(x)(s) = \int_0^1 p(s)q(t)x^2(t)dt, \qquad s \in [0,1],$$

where $p(s) = \cos(11\pi s)$, $q(t) = \sin(11\pi t)$.

Then K is compact and the integral equation $\varphi - K(\varphi) = f$ has a unique solution for $f \in C[0,1]$.

We choose $f(s) = \left(1 - \frac{2}{33\pi}\right)\cos(11\pi s)$, $s \in [0,1]$, so that $\varphi(s) = \cos(11\pi s)$, $s \in [0,1]$.

By using computational procedures constructed in the Matlab environment, we obtain the results reported in Table 1, that confirm the theoretical ones stated in Theorem 5 for the QIP spline Kulkarni's type method and in (3.11) for the QIP spline collocation one. In particular, we remark that in case $d = 2$, there is a superconvergence phenomenon, i.e. the order of the QIP spline Kulkarni's type method is 7, as proved in Theorem 5. Moreover, we can notice the superconvergence on the set of evaluation points, as stated in Theorem 3. Finally, we remark that the approximate solutions are smooth. In particular they are of class C^1 when $\pi_n = Q_2$, $\bar{Q}_2$ and of class C^2 when $\pi_n = Q_3$.

Test 2

Consider the following Hammerstein integral equation

$$\varphi(s) - \int_0^1 e^{s-2t}\varphi^3(t)dt = e^{s+1}, \qquad 0 \le s \le 1,$$

with the exact solution $\varphi(s) = e^s$.

We obtain the results reported in Table 2, that confirm the theoretical ones stated in Theorem 5 for the QIP spline Kulkarni's type method and in (3.11) for the QIP spline collocation one.

Test 3

Consider the following Urysohn integral equation

$$\varphi(s) - \int_0^1 \frac{dt}{s + t + \varphi(t)} = f(s), \qquad 0 \le s \le 1,$$

where f is chosen so that $\varphi(t) = \frac{1}{t+c}$, $c > 0$, is a solution.

We consider $c = 1$, $c = 0.1$ and we remark that the exact solution is ill behaved in the case $c = 0.1$.

Table 1: Maximum absolute errors for Test 1 in case of QIP spline Kulkarni's type (k) and QIP spline collocation (c) methods.

n	E^k_∞	O^k_∞	ES^k	O^k	E^c_∞	O^c_∞	ES^c	O^c
Methods based on Q_2								
40	1.08(-06)		6.97(-07)		7.74(-03)		4.98(-03)	
80	4.08(-09)	8.1	2.26(-09)	8.2	6.77(-04)	3.5	3.76(-04)	3.7
160	2.13(-11)	7.6	6.31(-12)	8.5	8.17(-05)	3.0	2.43(-05)	4.0
320	1.42(-13)	7.2	2.14(-14)	8.2	1.01(-05)	3.0	1.53(-06)	4.0
640	-	-	-	-	1.26(-06)	3.0	9.57(-08)	4.0
Methods based on $\bar{Q}_2$								
40	1.50(-06)		1.41(-06)		9.12(-03)		8.58(-03)	
80	1.12(-08)	7.1	7.79(-09)	7.5	7.97(-04)	3.5	5.54(-04)	4.0
160	8.07(-11)	7.1	3.28(-11)	7.9	8.58(-05)	3.2	3.49(-05)	4.0
320	6.13(-13)	7.0	1.32(-13)	8.0	1.02(-05)	3.0	2.19(-06)	4.0
640	5.33(-15)	6.8	-	-	1.27(-06)	3.0	1.37(-07)	4.0
Methods based on Q_3								
40	2.38(-08)				1.53(-03)			
80	9.40(-11)	8			9.27(-05)	4.0		
160	1.12(-13)	9.7			5.58(-06)	4.1		
320	-	-			3.43(-07)	4.0		
640	-	-			1.34(-08)	4.7		

Firstly, we consider $c = 1$ and we obtain the results presented in Table 3.

When we consider $c = 0.1$ by using the same procedures we get the results in the Table 4.

Also in Test 3, the theoretical results stated in Theorem 5 for the QIP spline Kulkarni's type method and in (3.11) for the QIP spline collocation one are confirmed. In case of degree d even, we can remark the superconvergence phenomenon of the QIP spline Kulkarni's type method and the superconvergence on the set of evaluation points, for both QIP spline methods.

4 Projection spline methods for Uryshon integral equation with Green's function type kernels

In this section, we consider integral equations (1.1) with Green's function type kernels in the definition of the integral operator K in (1.2). In particular, we assume k belonging to $\mathcal{G}_2(\alpha, \gamma)$ [8], i.e., given two integers α and γ with $\alpha \geq \gamma$, $\alpha \geq 0$, $\gamma \geq -1$, the kernel k has the following properties:

1. the partial derivative $l(s,t,u) := \dfrac{\partial k(s,t,u)}{\partial u}$ exists for all $(s,t,u) \in \Psi := [0,1] \times [0,1] \times \mathbb{R}$;
2. there are functions $l_i \in C^\alpha(\Psi_i)$, $i = 1, 2$, with

$$l(s,t,u) = \begin{cases} l_1(s,t,u), & (s,t,u) \in \Psi_1, \quad s \neq t, \\ l_2(s,t,u), & (s,t,u) \in \Psi_2, \end{cases}$$

Table 2: Maximum absolute errors for Test 2 in case of QIP spline Kulkarni's type (k) and QIP spline collocation (c) methods.

n	E_∞^k	O_∞^k	ES^k	O^k	E_∞^c	O_∞^c	ES^c	O^c
Methods based on Q_2								
4	2.00(-08)		4.54(-09)		3.35(-04)		7.61(-05)	
8	1.13(-10)	7.5	1.49(-11)	8.3	4.28(-05)	3.0	5.67(-06)	3.7
16	6.83(-13)	7.4	4.10(-14)	8.5	5.36(-06)	3.0	3.87(-07)	3.9
32	1.51(-14)	5.5	-	-	6.68(-07)	3.0	2.53(-08)	3.9
64	-	-	-	-	8.27(-08)	3.0	1.62(-09)	4.0
Methods based on $\bar{Q}_2$								
4	5.35(-08)		1.16(-08)		3.63(-04)		7.88(-05)	
8	4.88(-10)	6.8	5.35(-11)	7.8	4.47(-05)	3.0	4.90(-06)	4.0
16	4.03(-12)	6.9	2.46(-13)	7.8	5.47(-06)	3.0	3.25(-07)	3.9
32	4.80(-14)	6.4	-	-	6.75(-07)	3.0	2.11(-08)	3.9
64	-	-	-	-	8.38(-08)	3.0	1.34(-09)	4.0
Methods based on Q_3								
4	2.37(-11)				2.17(-05)			
8	5.73(-14)	8.7			1.51(-06)	3.8		
16	-	-			9.93(-08)	3.9		
32	-	-			6.36(-09)	4.0		
64	-	-			4.02(-10)	4.0		

and $\Psi_1 := \{(s,t,u) : 0 \le t \le s \le 1,\ u \in \mathbb{R}\}$, $\Psi_2 := \{(s,t,u) : 0 \le s \le t \le 1,\ u \in \mathbb{R}\}$;

3. if $\gamma \ge 0$ then $l \in C^\gamma(\Psi)$. If $\gamma = -1$, then l may have a discontinuity of the first kind along the line $s = t$;

4. there are two functions $k_i \in C^\alpha(\Psi_i)$, $i = 1, 2$, such that

$$k(s,t,u) = \begin{cases} k_1(s,t,u), & (s,t,u) \in \Psi_1, \quad s \ne t, \\ k_2(s,t,u), & (s,t,u) \in \Psi_2, \end{cases}$$

5. $\dfrac{\partial^2 k_i}{\partial^2 u} \in C(\Psi_i)$, $i = 1, 2$.

Thanks to such requirements, the operator K is Fréchet differentiable and the Fréchet derivative is given by (3.4).

In order to obtain the approximate solution for (1.1), given a spline QIP operator $\pi_n : C[0,1] \to \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$, we apply the QIP spline Kulkarni's type method (3.2) and the QIP spline collocation one (3.3), described in Section 3. For the explicit construction of the approximate solutions we can refer to Section 3.2.

Concerning the convergence of the methods, in case of the QIP spline collocation method, as we expect (see e.g [6, 8]), we have the order of convergence stated in (3.10), (3.11) and in case of the QIP Kulkarni's type method we can prove the following result.

Theorem 6. *Let the kernel k of the integral operator K in (1.2) be of class $\mathcal{G}_2(2\alpha, \gamma)$, with $\alpha \ge 1$ and $f \in C^\alpha[0,1]$. Let φ be the unique solution of (1.1) and assume that 1 is*

Table 3: Maximum absolute errors for Test 3, with $c = 1$, in case of QIP spline Kulkarni's type (k) and QIP spline collocation (c) methods.

n	E^k_∞	O^k_∞	ES^k	O^k	E^c_∞	O^c_∞	ES^c	O^c
Methods based on Q_2								
4	8.48(-08)		5.06(-08)		6.85(-04)		3.84(-04)	
8	7.84(-10)	6.8	3.50(-10)	7.2	9.54(-05)	2.8	3.84(-05)	3.3
16	5.08(-12)	7.3	1.47(-12)	7.9	1.21(-05)	3.0	3.10(-06)	3.6
32	3.08(-14)	7.4	5.55(-15)	8.0	1.50(-06)	3.0	2.21(-07)	3.8
64	-	-	-	-	1.85(-07)	3.0	1.48(-08)	3.0
Methods based on $\bar{Q}_2$								
4	2.00(-07)		1.33(-07)		7.27(-04)		4.59(-04)	
8	2.75(-09)	6.2	1.42(-09)	6.6	9.96(-05)	2.9	4.67(-05)	3.3
16	2.67(-11)	6.7	9.34(-12)	7.2	1.25(-05)	3.0	3.84(-06)	3.6
32	2.24(-13)	6.9	4.60(-14)	7.7	1.53(-06)	3.0	2.78(-07)	3.8
64	-	-	-	-	1.87(-07)	3.0	1.87(-08)	3.9
Methods based on Q_3								
4	1.58(-09)				9.02(-05)			
8	3.30(-12)	8.9			7.77(-06)	3.5		
16	6.55(-15)	9.0			6.84(-07)	3.5		
32	-	-			5.00(-08)	3.8		
64	-	-			3.36(-09)	3.9		

not an eigenvalue of $K'(\varphi)$. Let $\pi_n : C[0,1] \to \mathcal{S}_d^{d-1}(I, \mathcal{T}_n)$ be a spline QIP operator of kind (2.1). Let φ_n^k be the unique solution of (3.2). Then

$$\|\varphi_n^k - \varphi\|_\infty = O(h^{\beta+\tilde{\beta}}), \tag{4.1}$$

where $\beta := \min\{\alpha, d+1\}$ and $\tilde{\beta} := \min\{\beta, \gamma+2\}$.

Proof. Thanks to the approximation properties of the spline QIPs π_n stated in (2.3) and from [15, Theorem 2.3], we obtain (4.1). □

4.1 Numerical results

In this section we consider the following test equation

$$x(s) - \int_0^1 k(s,t) g(t, x(t)) dt = \int_0^1 k(s,t) z(t) dt, \qquad s \in [0,1], \tag{4.2}$$

where

$$k(s,t) = \begin{cases} (1-s)t & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases}$$

$g(t,u) = \dfrac{1}{1+t+u}$ and $z(t)$ is chosen so that $\varphi(s) = \dfrac{s(1-s)}{s+1}$ is a solution of (4.2). We remark that $\gamma = 0$ and α can be chosen as large as we want.

We solve it both by the QIP spline Kulkarni's type method and the QIP spline collocation one, based on the spline projectors $Q_2, \bar{Q}_2$ and Q_3.

Table 4: Maximum absolute errors for Test 3, with $c = 0.1$, in case of QIP spline Kulkarni's type (k) and QIP spline collocation (c) methods.

n	E_∞^k	O_∞^k	ES^k	O^k	E_∞^c	O_∞^c	ES^c	O^c
Methods based on Q_2								
4	4.50(-07)		1.13(-07)		6.80(-01)		5.51(-01)	
8	3.87(-10)	10.2	2.51(-10)	8.8	2.67(-01)	1.3	2.10(-01)	1.4
16	1.00(-11)	5.3	1.01(-12)	8.0	7.04(-02)	1.9	5.12(-02)	2.0
32	1.21(-13)	6.4	1.07(-14)	6.7	1.29(-02)	2.4	7.99(-03)	2.7
64	-	-			1.86(-03)	2.8	8.65(-04)	3.2
Methods based on $\bar{Q}_2$								
4	2.89(-06)		1.39(-06)		7.44(-01)		6.41(-01)	
8	1.26(-08)	7.8	3.43(-09)	8.7	2.94(-01)	1.3	2.48(-01)	1.4
16	8.00(-11)	7.3	1.22(-11)	8.1	7.77(-02)	1.9	6.14(-02)	2.0
32	5.88(-13)	7.1	4.80(-14)	8.0	1.42(-02)	2.4	9.87(-03)	2.6
64	-	-	-	-	2.02(-03)	2.8	1.11(-03)	3.2
Methods based on Q_3								
4	1.97(-08)				4.17(-01)			
8	1.16(-11)	10.7			1.03(-01)	2.0		
16	1.29(-13)	6.5			1.70(-02)	2.6		
32	-	-			1.96(-03)	3.1		
64	-	-			1.75(-04)	3.5		

The integrals appearing in (3.20) and (3.23) are evaluated numerically with high accuracy, by a classical composite Gauss-Legendre quadrature formula in a suitable way, i.e. paying attention to the line $s = t$ in the evaluation of the kernel, by using quadrature nodes that do not lie on such a line.

For increasing values of n, we compute the maximum absolute error

$$E_\infty^\mu := \max_{v \in G} |\varphi(v) - \varphi_n^\mu(v)|,$$

where G is a set of 1500 equally spaced points in $[0, 1]$, $\mu = k$ in case of QIP spline Kulkarni's type methods and $\mu = c$ in case of QIP spline collocation ones, based on the projectors Q_2, $\bar{Q}_2$ and Q_3. We also compute the corresponding numerical convergence order O_∞^μ.

By using computational procedures constructed in the Matlab environment, we obtain the results reported in Table 5. The numerical tests confirm the theoretical results stated in Theorem 6 for the QIP spline Kulkarni's type method and in (3.11) for the QIP spline collocation method.

5 Conclusions

In this paper we have proposed spline projection methods for the numerical solution of nonlinear integral equations, both in case of smooth kernels and in case of Green's function type ones. In particular, we have considered quasi-interpolating spline projectors on a bounded interval for defining a projection method with high order of convergence and a collocation method of classical type. We have analysed the construction of the

Table 5: Maximum absolute errors in case of QIP spline Kulkarni's type (k) and QIP spline collocation (c) methods.

n	E^k_∞	O^k_∞	E^c_∞	O^c_∞
Methods based on Q_2				
4	2.53(-06)		1.41(-03)	
8	9.44(-08)	4.7	1.94(-04)	2.9
16	2.93(-09)	5.0	2.44(-05)	3.0
32	8.53(-11)	5.1	3.02(-06)	3.0
64	2.50(-12)	5.1	3.71(-07)	3.0
Methods based on $\bar{Q}_2$				
4	2.94(-06)		1.54(-03)	
8	1.09(-07)	4.8	2.09(-04)	2.9
16	3.28(-09)	5.0	2.58(-05)	3.0
32	9.25(-11)	5.1	3.12(-06)	3.0
64	2.63(-12)	5.1	3.78(-07)	3.0
Methods based on Q_3				
4	1.29(-07)		1.85(-04)	
8	6.49(-09)	4.3	1.56(-05)	3.6
16	1.83(-10)	5.1	1.36(-06)	3.5
32	3.82(-12)	5.6	1.00(-07)	3.8
64	6.86(-14)	5.8	6.73(-09)	3.9

approximate solutions and we have studied their order of convergence. Then, we have presented some numerical examples, confirming the approximation properties of the proposed methods.

An interesting work in progress is the use of spline quasi-interpolating operators which are not projectors for the numerical solution of nonlinear integral equations, as considered in [2] for the linear case.

Acknowledgements

The authors thank the University of Torino for its support to their research. Moreover, they wish to thank the anonymous referees for their comments which helped them to improve the original manuscript.

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