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The Approximation Theorem for the $\Lambda\mu$-Calculus

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We consider a notion of approximation for terms of de Groote-Saurin $\Lambda\mu$-calculus. Then we introduce an intersection type assignment system for that calculus which is invariant under subject conversion. The type assignment system also induces a filter model, which is an extensional $\Lambda\mu$-model in the sense of Nakazawa and Katsumata. We then establish the approximation theorem, stating that a type can be assigned to a term in the system if and only if it can be assigned to same of its approximations.

Introduction

The $\Lambda\mu$-calculus is an extension of Parigot's $\lambda\mu$-calculus (Parigot 1992), proposed by de Groote (de Groote 1994a; de Groote 1994b) and developed by Saurin (Saurin 2005; Saurin 2008b; Saurin 2010a). The interest of $\Lambda\mu$ lies in the fact that it preserves the separability property, namely the Böhm Theorem of the λ-calculus with $\beta\eta$-conversion (Saurin 2005), which is not the case of $\lambda\mu$ (David *et al.* 2001). Indeed many basic concepts and properties from ordinary λ-calculus in the classic book (Barendregt 1984) extend to $\Lambda\mu$: confluence of the reduction relation even in presence of the η-rule (Py 1998); Böhm-out technique and separability (Saurin 2005); standardisation, head-normal forms and solvability, Böhm trees (Saurin 2012).

The Approximation Theorem is a central result in the study of sensible λ-theories, relating them to the structure of Scott's D_∞ λ-models. This was investigated by Wadsworth, Hyland and Levy leading to the construction of models based on Böhm trees in (Barendregt 1984). The theorem states that the denotation $[\![M]\!]$ of any term M in a domain theoretic model is the directed sup of the denotations $[\![A]\!]$ of its approximations A, where A is a partially defined term (including a constant Ω for the undefined parts) recording the stable part of M reducts.

The invention of intersection type systems by Coppo and Dezani has further widened the knowledge of the λ-calculus, and greatly simplified the proof of approximation theorems of various models and λ-theories, via the concept of filter model introduced in (Barendregt *et al.* 1983) with Barendregt. With intersection types the approximation theorem rephrases into the claim that M has a type σ if and only if there is an approximation A of M that can be typed by σ (Dezani *et al.* 2001).

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In (van Bakel *et al.* 2011), joint work with van Bakel and Barbanera, we obtained an intersection type assignment system by presenting Streicher and Reus's model of $\lambda\mu$ (Streicher *et al.* 1998) as a filter model. In the same paper it was remarked that the type system could be adapted to type all $\Lambda\mu$-terms, but this was not pursued any further because Streicher and Reus's model, on which the system has been based, does not validate Parigot's renaming axiom in the case of the larger set of $\Lambda\mu$-terms, where it has been called the (β_S)-rule by Saurin.

Recently Nakazawa and Katsumata (Nakazawa *et al.* 2012) have proposed a variant of Streicher and Reus's construction, which is a model of $\Lambda\mu$. Both Streicher-Reus and Nakazawa-Katsumata models are D_∞ λ-models with more structure, accommodating the concept of continuation or, as it has been called by Saurin after Parigot's suggestion, stream.

The construction of D_∞ models of the extensional $\Lambda\mu$–theories makes interesting the study of the approximation property, and the pairing of the new models with Saurin's results on separability and Böhm trees for the $\Lambda\mu$-calculus. Here we pursue this goal by means of an intersection type system adapted to $\Lambda\mu$ where, as in (van Bakel *et al.* 2011), we use two sorts of types namely term types (arrow types) and stream types (product types), plus intersection and ω.

In this paper, after some basic facts about the $\Lambda\mu$ theory (section 1), we study Saurin's notion of approximate normal form for $\Lambda\mu$, together with a pre-congruence relation such that the set $\mathcal{A}(M)$ of approximants of a (closed) $\Lambda\mu$-term M is an ideal (section 2). We then introduce our intersection type pre-order and type assignment system and prove that it is invariant under subject conversion (section 3). It comes out that the set of filters determined by the type pre-order is an "extensional $\Lambda\mu$-model" in the sense of (Nakazawa *et al.* 2012), and that term interpretation coincides with the set of types that can be assigned to terms (section 4). Finally we establish the main result of the paper, namely that a term-type δ can be assigned to a $\Lambda\mu$-term M in some bases Γ, Δ (for term and stream variables respectively) if and only if δ can be assigned to some approximation A of M in the same Γ, Δ. By the filter model construction and known results on "domain logic" (Abramsky 1991), this implies the approximation theorem for extensional $\Lambda\mu$-models in the category of ω-algebraic lattices (section 5).

1. $\Lambda\mu$-calculus

The distinctive feature of the $\Lambda\mu$-calculus w.r.t. Parigiot's $\lambda\mu$ consists into abolishing the distinction between named terms $[\alpha]M$ and unnamed or ordinary terms. In particular the restrictions that in $\mu\alpha.M$ the subexpression M is named and that in $\lambda x.N$ term N is unnamed are dropped. In the grammar below we have adopted Saurin's notation $(M)\alpha$ for Parigot's and de Groote's $[\alpha]M$. This makes explicit the intuition that α represents a potentially infinite stream of terms to which M is applied.

Definition 1.1 (Term Syntax).

$$M, N ::= x \mid \lambda x.M \mid (M)N \mid \mu\alpha.M \mid (M)\alpha$$

Var_T and Var_S are denumerable sets of *term variables* and *stream variables* respectively. $\Sigma_{\Lambda\mu}$ is the set of terms generated by the grammar in Definition 1.1 and $\Sigma^c_{\Lambda\mu}$ is the subset of closed terms. Bound and free variables, written $fv(M)$, are defined as usual, with both λ and μ as binders. We identify terms up to renaming of bound variables and assume Barendregt's convention that free and bound variables have distinct names in the same expression.

Definition 1.2 (Structural Substitution). For $M, N \in \Sigma_{\Lambda\mu}$ and $\alpha \in \mathsf{Var}_S$ define $M[\alpha \Leftarrow N]$ as the replacement of any subterm $(P)\alpha$ of M with $\alpha \in fv(M)$, by the subterm $(P)N\alpha$. In particular:

$$((M)\alpha)[\alpha \Leftarrow N] = (M[\alpha \Leftarrow N])N\alpha.$$

$M[\alpha \Leftarrow N]$ is de Groote's notation for Parigot's $M[(P)\alpha := (P)N\alpha]$, which we avoid because of the use of the (bounded) metavariable P that might be confusing.

Definition 1.3 (Axioms).

$$\begin{array}{llcll}
(\beta_T) & (\lambda x.M)N & = & M[x := N] & \\
(\beta_S) & (\mu\alpha.M)\beta & = & M[\alpha := \beta] & \\
(\eta_T) & \lambda x.(M)x & = & M & \text{if } x \notin fv(M) \\
(\eta_S) & \mu\alpha.(M)\alpha & = & M & \text{if } \alpha \notin fv(M) \\
(\mu) & (\mu\alpha.M)N & = & \mu\alpha.M[\alpha \Leftarrow N] &
\end{array}$$

We write $\vdash M = N$ if this equality is derivable from the axioms above in the formal extension to $\Sigma_{\Lambda\mu}$ of the $\beta\eta$-theory. We also use $M =_{ax} N$ to indicate that $M = N$ is an instance of axiom *ax*. When $\vdash$ is omitted or there is no subscript, $M = N$ is just syntactical equality, up to renaming of bound variables.

By orienting the axioms from left to right one obtains a non-confluent notion of reduction (Saurin 2008b):

$$\mu\alpha.x \;{}_\eta\!\longleftarrow \lambda y.(\mu\alpha.x)y \longrightarrow_{fst} \lambda y.\mu\alpha.(x[\alpha \Leftarrow y]) = \lambda y.\mu\alpha.x.$$

This is fixed by replacing the left-to-right version of the (μ)-axiom by rule (*fst*) below:

Definition 1.4 (Reduction). The reduction relation $\longrightarrow$ over $\Sigma_{\Lambda\mu}$ is the compatible closure of the rewriting rules obtained by orienting from left to right the axioms $(\beta_T), (\beta_S), (\eta_T)$ and (η_S), and adding the rule:

$$(\mathit{fst}) \quad \mu\alpha.M \quad \longrightarrow \quad \lambda x.\mu\alpha.M[\alpha \Leftarrow x] \quad \text{if } x \notin fv(M).$$

Lemma 1.5. The equality axiomatised in Def. 1.3 is the conversion relation induced by the reduction $\longrightarrow$.

Proof. Immediate. See also (Saurin 2008b). □

The following is Theorem 2.16 in (Py 1998) and Theorem 3.1 in (Saurin 2010c).

Theorem 1.6 (Confluence of $\longrightarrow$ w.r.t. $\Sigma^c_{\Lambda\mu}$). For $M, M_1, M_2 \in \Sigma^c_{\Lambda\mu}$:

$$M \longrightarrow^* M_1, M_2 \Rightarrow \exists M_3 \in \Sigma^c_{\Lambda\mu}.\ M_1, M_2 \longrightarrow^* M_3.$$

Corollary 1.7. For $M, N \in \Sigma^c_{\Lambda\mu}$:

$$\vdash M = N \iff \exists L \in \Sigma^c_{\Lambda\mu}.\ M \longrightarrow^* L \ {}^*\!\!\longleftarrow N.$$

The restriction to $\Sigma^c_{\Lambda\mu}$ in 1.6 and 1.7 is essential, as e.g. we have (Py 1998; Saurin 2010c):

$$x = x[\alpha := \beta] \ {}_{\beta_S}\!\!\longleftarrow (\mu\alpha.x)\beta \longrightarrow_{fst} (\lambda y.\mu\alpha.x[\alpha \Leftarrow y])\beta = (\lambda y.\mu\alpha.x)\beta.$$

2. Head Normal Forms and Approximants

In "sensible" theories of the ordinary λ-calculus a term M has computational meaning if and only if it reduces to a head normal form (Barendregt 1984). By replacing β-redexes in M by a constant Ω and equating $\Omega N = \Omega = \lambda x.\Omega$ one obtains a context which remains unchanged in any reduction out of M, because the only parts that can be affected by reduction are those subterms replaced by Ω. If one orders $\lambda\Omega$-terms (which are obtained by adding the constant Ω to the grammar of λ-terms) by the compatible closure of the inequality $\Omega \preceq M$, one gets the notion of approximate normal form in the λ-calculus.

In this section we follow (Saurin 2010a) adapting this concept to $\Lambda\mu$, but for the fact that we adapt to the present context Levy's idea of taking ideals of approximate normal forms in place of Böhm trees. To avoid technical intricacies, we do not consider as approximant any element of the larger set of $\Lambda\mu$-terms extended with the constant Ω, nor the reduction $\longrightarrow_\perp$ as e.g. in (Barendregt 1984) for ordinary λ-calculus, to calculate approximate normal forms. Rather we directly associate approximate normal forms to terms.

Definition 2.1 (Head Normal Forms). $M \in \Sigma_{\Lambda\mu}$ is a *head normal form*, $M \in \mathrm{HNF}$, if

$$M = \lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \dots \mu\alpha_n\lambda\vec{x}_n.(y)\vec{M}_0\beta_1\vec{M}_1 \dots \beta_m\vec{M}_m.$$

M has a head normal form if $\vdash M = H$ for some $H \in \mathrm{HNF}$.

We say that a term of the shape $(y)\vec{M}_0\beta_1\vec{M}_1 \dots \beta_m\vec{M}_m$ is *$\lambda\mu$-free*.

Lemma 2.2.

$$H \in \mathrm{HNF} \ \&\ H \longrightarrow^* N \Rightarrow N \in \mathrm{HNF}.$$

Proof. Let $H = \lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \dots \mu\alpha_n\lambda\vec{x}_n.(y)\vec{M}_0\beta_1\vec{M}_1 \dots \beta_m\vec{M}_m \longrightarrow N$. If this is either an η_T or an η_S reduction, the thesis is immediate. Otherwise either the contracted redex is a subterm of the $M_{i,j}$ or it is

$$\mu\alpha_h\lambda\vec{x}_h \dots \mu\alpha_n\lambda\vec{x}_n.(y)\vec{M}_0\beta_1\vec{M}_1 \dots \beta_m\vec{M}_m$$

and according to rule (*fst*) its contractum in N is

$$\lambda z.\mu\alpha_h\lambda\vec{x}_h \dots \mu\alpha_n\lambda\vec{x}_n.((y)\vec{M}_0\beta_1\vec{M}_1 \dots \beta_m\vec{M}_m)[\alpha_h \Leftarrow z]$$

for some fresh $z \in \mathsf{Var}_T$. Since the application of structural substitution to a $\lambda\mu$-free term results into a $\lambda\mu$-free term, in both cases $N \in \mathrm{HNF}$ as well. $\square$

Definition 2.3 (Approximate Normal Forms).

$$A ::= \Omega \mid \lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \dots \mu\alpha_n\lambda\vec{x}_n.(y)\vec{A}_0\beta_1\vec{M}_1 \dots \beta_m\vec{A}_m$$

Let $\mathcal{A}$ be the set of approximate normal forms.

Definition 2.4 (Pre-Congruence over $\mathcal{A}$). Let $\preceq$ be the least pre-order over $\mathcal{A}$ and $\simeq = \preceq \cap \succeq$, which are (pre-)congruences and such that:

1 $\Omega \preceq A$,
2 $\lambda x.(A)x \simeq A$, if $x \notin fv(A)$ and A is $\lambda\mu$-free,
3 $\mu\alpha.(A)\alpha \simeq A$, if $\alpha \notin fv(A)$ and A is $\lambda\mu$-free,
4 $\mu\alpha.A \preceq \lambda x.\mu\alpha.A[\alpha \Leftarrow x]$, if $A \neq \Omega$ and $x \notin fv(A)$.

The requirement that A is $\lambda\mu$-free in clauses (2) and (3) above is actually redundant, because if A is not $\lambda\mu$-free then $\lambda x.(A)x, \mu\alpha.(A)\alpha \notin \mathcal{A}$. Similarly in clause (4) if $A = \Omega$ then $\mu\alpha.A \notin \mathcal{A}$.

Note that if $x, \alpha \notin fv(A)$ and A is $\lambda\mu$-free then:

$$A \simeq \mu\alpha.(A)\alpha \preceq \lambda x.\mu\alpha.((A)\alpha)[\alpha \Leftarrow x] = \lambda x.\mu\alpha.(A)x\alpha \simeq \lambda x.(A)x \simeq A.$$

Definition 2.5 (Pre-Redex). A term R is a *pre-redex* if it has one of the shapes $(\lambda x.M)N$, $(\lambda x.M)\alpha$, $(\mu\alpha.M)N$ or $(\mu\alpha.M)\beta$.

Pre-redexes are from (Saurin 2008a) Def. 3.17. The pre-redexes $(\lambda x.M)N$ and $(\mu\alpha.M)\beta$ are actual redexes. The pre-redex $(\mu\alpha.M)N$ includes the (fst)-redex $\mu\alpha.M$, which after contraction gives rise to the (β_T)-redex $(\lambda x.\mu\alpha.M[\alpha \Leftarrow x])N$. The pre-redex $(\lambda x.M)\alpha$ is only a potential redex, which becomes an actual redex after a (fst)-reduction only if it occurs as a subterm of some term in which α is bound.

Definition 2.6 (Approximants). Let $\phi : \Sigma_{\Lambda\mu} \to \mathcal{A}$ be the map:

1 $\phi(\lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \dots \mu\alpha_n\lambda\vec{x}_n.(R)\vec{M}_0\beta_1\vec{M}_1 \dots \beta_m\vec{M}_m) = \Omega$, if R is a pre-redex,
2 $\phi(\lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \dots \mu\alpha_n\lambda\vec{x}_n.(y)\vec{M}_0\beta_1\vec{M}_1 \dots \beta_m\vec{M}_m) =$

$$\lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \dots \mu\alpha_n\lambda\vec{x}_n.(y)\phi(\vec{M}_0)\beta_1\phi(\vec{M}_1) \dots \beta_m\phi(\vec{M}_m),$$

where $\phi(\vec{M}_i)$ is the componentwise application of ϕ to the terms of the vector $\vec{M}_i$.

For $M \in \Sigma_{\Lambda\mu}$ we define the sets:

$$\begin{aligned} \mathcal{A}'(M) &= \{\phi(N) \in \mathcal{A} \mid M \longrightarrow^* N\}, \\ \mathcal{A}(M) &= \{A \in \mathcal{A} \mid \exists N.\ M \longrightarrow^* N \ \&\ A \preceq \phi(N)\}. \end{aligned}$$

By the very definition we have that $\phi(M) \neq \Omega$ if and only if $M \in$ HNF. The set $\mathcal{A}(M)$ is the downward closure of $\mathcal{A}'(M)$ w.r.t. $\preceq$. If $A \in \mathcal{A}(M)$ then we say that A is an *approximant* of M, and if $A = \phi(M)$ then it is called the *immediate approximant* of M.

Lemma 2.7. The map ϕ is well defined and total, in the sense that any $M \in \Sigma_{\Lambda\mu}$ has either a pre-redex R in head position, or it is a head normal form.

Proof. By a straightforward induction over the structure of M. □

Lemma 2.8.

1 $\phi(\lambda x.M) \neq \Omega$ and $\phi(\mu\alpha.M) \neq \Omega$ iff $\phi(M) \neq \Omega$,
2 $\phi((M)N) \neq \Omega$ and $\phi((M)\alpha) \neq \Omega$ iff $\phi(M) \neq \Omega$ & M is $\lambda\mu$-free,
3 $\phi(M) \neq \Omega \Rightarrow \phi(\lambda x.M) = \lambda x.\phi(M)$ & $\phi(\mu\alpha.M) = \mu\alpha.\phi(M)$,
4 M is $\lambda\mu$-free iff $\phi((M)N) = (\phi(M))\phi(N)$ & $\phi((M)\alpha) = (\phi(M))\alpha$.

Proof. Immediate by definition. □

Lemma 2.9. If M is either a (β_T), (β_S), (η_T) or (η_S)-redex and N is its contractum, then $\phi(M) \preceq \phi(N)$.

Proof. If M is either a (β_T) or a (β_S)-redex then $\phi(M) = \Omega \preceq \phi(N)$. If M is an (η_T)-redex then $M = \lambda x.(M')x$ with $x \notin fv(M')$. If $\phi(M) \neq \Omega$, then $\phi((M')x) \neq \Omega$ by Lemma 2.8, that is $\phi(M') \neq \Omega$ and M' is $\lambda\mu$-free by the same lemma. Hence by (3) and (4) of Lemma 2.8:

$$\phi(M) = \lambda x.\phi((M')x) = \lambda x.((\phi(M'))x \simeq \phi(M') = \phi(N).$$

By a similar argument we establish the thesis when M is an (η_S)-redex. □

Lemma 2.10.

1 $\phi(M[\alpha \Leftarrow N]) = \phi(M)[\alpha \Leftarrow \phi(N)]$,
2 $\phi(\mu\alpha.M) \preceq \phi(\lambda x.\mu\alpha.M[\alpha \Leftarrow x])$, for $x \notin fv(M)$.

Proof. Part (1) is proved by induction over M. The interesting case is when $M = (M')\alpha$, that is $M[\alpha \Leftarrow N] = (M'[\alpha \Leftarrow N])N\alpha$. Now observe that

$$\phi((M'[\alpha \Leftarrow N])N\alpha) = \Omega \Longleftrightarrow \phi((M'[\alpha \Leftarrow N])N) = \Omega,$$

since otherwise by part (2) of Lemma 2.8 we should have that $(M'[\alpha \Leftarrow N])N$ is $\lambda\mu$-free, which is impossible.

In case that $\phi((M'[\alpha \Leftarrow N])N\alpha) \neq \Omega$ we have that $M'[\alpha \Leftarrow N]$ is $\lambda\mu$-free, so that:

$$\begin{array}{lll} \phi((M'[\alpha \Leftarrow N])N\alpha) = & & \\ \quad = & \phi((M'[\alpha \Leftarrow N]))\phi(N)\alpha & \text{by (4) of Lemma 2.8} \\ \quad = & (\phi(M')[\alpha \Leftarrow \phi(N)])\phi(N)\alpha & \text{by induction} \\ \quad = & \phi((M')\alpha)[\alpha \Leftarrow \phi(N)] & \text{by (4) of Lemma 2.8.} \end{array}$$

Part (2) is trivially true if $\phi(\mu\alpha.M) = \Omega$. If not then, by part (1) of Lemma 2.8, $\phi(M) \neq \Omega$ so that by (3) of Lemma 2.8 we have that $\phi(\mu\alpha.M) = \mu\alpha.\phi(M)$. On the other hand by part (1) of the present lemma we know that $\phi(M[\alpha \Leftarrow x]) = \phi(M)[\alpha \Leftarrow \phi(n)] \neq \Omega$, hence

$$\begin{array}{lll} \phi(\lambda x.\mu\alpha.M[\alpha \Leftarrow x]) = & & \\ \quad = & \lambda x.\mu\alpha.\phi(M[\alpha \Leftarrow x]) & \text{by (3) of Lemma 2.8} \\ \quad = & \lambda x.\mu\alpha.\phi(M)[\alpha \Leftarrow \phi(x)] & \text{by part (1)} \\ \quad = & \lambda x.\mu\alpha.\phi(M)[\alpha \Leftarrow x] & \text{since } \phi(x) = x. \end{array}$$

But $\mu\alpha.\phi(M) \preceq \lambda x.\mu\alpha.\phi(M)[\alpha \Leftarrow x]$ holds by clause (4) of Def. 2.4. □

Lemma 2.11.

$$M \longrightarrow^* N \Rightarrow \phi(M) \preceq \phi(N).$$

Proof. By induction over the length of the reduction $M \longrightarrow^* N$, using the fact that $\preceq$ is a pre-congruence and lemmas 2.9 and 2.10 part (2). $\square$

A subset $I \subseteq X$ over a pre-ordered set X is *directed* if for any finite $Z \subseteq I$, which is bounded above in X, there exists $z \in I$ which is an upper bound of Z. An *ideal* over X is a downward closed directed subset I of X. Since $\emptyset$ is vacuously a finite bounded subset of X, all ideals are non-empty.

Theorem 2.12. For all $M \in \Sigma^c_{\Lambda\mu}$ the set $\mathcal{A}(M)$ is an ideal over $(\mathcal{A}, \preceq)$.

Proof. The set $\mathcal{A}(M)$ is downward closed by definition, so that it remains to show that $\mathcal{A}(M)$ is directed. First $\phi(M) \in \mathcal{A}(M) \neq \emptyset$. Next suppose that $A_1, A_2 \in \mathcal{A}(M)$, so that there exist $N_1, N_2 \in \Sigma_{\Lambda\mu}$ s.t. $M \longrightarrow^* N_i$ and $A_i \preceq \phi(N_i)$ for $i = 1, 2$. Since reduction cannot introduce new free variables and M is closed, we have that $N_1, N_2 \in \Sigma^c_{\Lambda\mu}$ and, by Theorem 1.6, there exists $N_3 \in \Sigma^c_{\Lambda\mu}$ such that

$$N_1 \longrightarrow^* N_3 \;{}^*\!\longleftarrow N_2.$$

Consequently $\phi(N_3) \in \mathcal{A}(M)$ and $A_1, A_2 \preceq \phi(N_3)$ by Lemma 2.11. $\square$

Corollary 2.13. $M \in \Sigma^c_{\Lambda\mu}$ has a head normal form if and only if $\mathcal{A}(M)$ is a non-trivial ideal.

Proof. By Theorem 2.12 $\mathcal{A}(M)$ is an ideal. On the other hand if $M \longrightarrow^* H \in \mathrm{HNF}$ then $\phi(H) \in \mathcal{A}(M)$ by definition and we know that $\phi(H) \neq \Omega$. Vice versa if $\Omega \neq A \in \mathcal{A}(M)$ then there exists N such that $M \longrightarrow^* N$ and $A \preceq \phi(N)$, so that $\phi(N) \neq \Omega$. This implies that $N \in \mathrm{HNF}$. $\square$

3. Intersection Types for $\Lambda\mu$

Even if there is just one kind of terms in the $\Lambda\mu$ syntax, to type them we have to make assumptions also about stream variables. Consequently we follow (van Bakel *et al.* 2011) by having two kinds of types for terms and term variables, and for stream variables.

Type syntax is motivated by the semantics (see section 4), where terms denote functions from streams to term denotations, and streams are infinite tuples of term denotations. Therefore term types are (intersections of) arrows of the shape $\sigma \to \delta$ where σ is a stream type and δ a term type. Stream types are (intersections of) product types of the shape $\sigma = \delta_1 \times \cdots \times \delta_k \times \omega$, where the δ_i are term types. The ending ω expresses the fact that σ only encodes a finite information about any infinite stream $s = \langle d_1, \ldots, d_k, \ldots\rangle$ such that each d_i satifies the respective δ_i, while ω is the type of the infinite tail.

Definition 3.1 (Intersection Types).

$$\begin{array}{llll} \mathcal{T}_T: & \delta & ::= & \varphi \mid \sigma \to \delta \mid \delta \wedge \delta \mid \omega_T \\ \mathcal{T}_S: & \sigma & ::= & \delta \times \sigma \mid \sigma \wedge \sigma \mid \omega_S \end{array}$$

where φ varies over a denumerable set of atomic types.

$\mathcal{T}_T$ is the set of *term types* and $\mathcal{T}_S$ the set of *stream types*. When clear from the context we shall write just ω in place of ω_T or ω_S.

Definition 3.2 (Subtyping). The relations $\leq_T$ and $\leq_S$ over $\mathcal{T}_T$ and $\mathcal{T}_S$ respectively are the least preorders such that:

1 $\delta \leq_T \delta$; $\sigma \leq_S \sigma$
2 $\delta_1 \leq_T \delta_2 \leq_T \delta_3 \Rightarrow \delta_1 \leq_T \delta_3$; $\sigma_1 \leq_S \sigma_2 \leq_S \sigma_3 \Rightarrow \sigma_1 \leq_S \sigma_3$
3 $\delta \leq_T \omega$; $\sigma \leq_S \omega$
4 $\delta_1 \wedge \delta_2 \leq_T \delta_i$; $\sigma_1 \wedge \sigma_2 \leq_S \sigma_i$, for $i = 1, 2$
5 $\delta \leq_T \delta_1, \delta_2 \Rightarrow \delta \leq_T \delta_1 \wedge \delta_2$; $\sigma \leq_S \sigma_1, \sigma_2 \Rightarrow \sigma \leq_S \sigma_1 \wedge \sigma_2$
6 $\omega \leq_T \omega \to \omega$
7 $\varphi \leq_T \omega \to \varphi \leq_T \varphi$
8 $(\sigma \to \delta_1) \wedge (\sigma \to \delta_2) \leq_T \sigma \to (\delta_1 \wedge \delta_2)$
9 $\sigma_2 \leq_S \sigma_1, \delta_1 \leq_T \delta_2 \Rightarrow \sigma_1 \to \delta_1 \leq_T \sigma_2 \to \delta_2$
10 $\omega \leq_S \omega \times \omega$
11 $(\delta_1 \times \sigma_1) \wedge (\delta_2 \times \sigma_2) \leq_S (\delta_1 \wedge \delta_2) \times (\sigma_1 \wedge \sigma_2)$
12 $\delta_1 \leq_T \delta_2, \sigma_1 \leq_S \sigma_2 \Rightarrow \delta_1 \times \sigma_1 \leq_S \delta_2 \times \sigma_2$

We abbreviate by $\delta_1 \sim_T \delta_2$ the inequalities $\delta_1 \leq_T \delta_2$ and $\delta_2 \leq_T \delta_1$. Similarly for $\sigma_1 \sim_S \sigma_2$. Among the consequences of the above axioms we remark $\omega \sim_T \omega \to \omega$ and $\omega \sim_S \omega \times \omega$, which together imply $\varphi \sim_T \omega \times \omega \to \varphi$. In the inequations (8) and (11) of Def. 3.2 the $\leq$ can be replaced by $\sim$. Finally we shall implicitly use the fact:

$$\omega \sim_T \omega \to \omega \leq_T \sigma \to \omega \leq_T \omega,$$

which implies that $\sigma \to \delta \sim_T \omega$ if and only if $\delta \sim_T \omega$.

Lemma 3.3 (Subtyping Properties).

1 $\forall \sigma \in \mathcal{T}_S\ \exists k \in \mathbb{N}, \delta_1, \ldots, \delta_k \in \mathcal{T}_T.\ \sigma \sim_S \delta_1 \times \cdots \times \delta_k \times \omega$,
2 $\delta_1 \times \cdots \times \delta_h \times \omega \leq_S \delta'_1 \times \cdots \times \delta'_k \times \omega \Longleftrightarrow h \leq k\ \&\ \forall i \leq h.\ \delta_i \leq_T \delta'_i$,
3 If $\delta \not\sim_T \omega$ and I is a finite and non-empty set of indexes then

$$\bigwedge_{i \in I} (\sigma_i \to \delta_i) \leq_T \sigma \to \delta \iff \exists J \subseteq I.\ \sigma \leq_S \bigwedge_{j \in J} \sigma_j\ \&\ \bigwedge_{j \in J} \delta_j \leq_T \delta.$$

Proof. Parts (1) and (2) follow by induction on derivations in the formal presentation of $\leq_T$ and $\leq_S$. Part (3) is standard with intersection types. $\square$

A *basis for term variables* $\Gamma = \{x_1 : \delta_1, \ldots, x_n : \delta_n\}$ is a finite set with $n \in \mathbb{N}$, pairwise distinct $x_i \in \mathsf{Var}_T$ and (not necessarily distinct) $\delta_i \in \mathcal{T}_T$; $dom(\Gamma) = \{x_1, \ldots, x_n\}$ is the *domain* of Γ. A *basis for stream variables* $\Delta = \{\alpha_1 : \sigma_1, \ldots, \alpha_m : \sigma_m\}$ is a finite set with $m \in \mathbb{N}$, pairwise distinct $\alpha_j \in \mathsf{Var}_S$ and (not necessarily distinct) $\sigma_j \in \mathcal{T}_S$; $dom(\Delta) = \{\alpha_1, \ldots, \alpha_m\}$ is the *domain* of Δ. Γ and Δ are called *bases* for short. A *judgement* is an expression $\Gamma \vdash M : \delta \mid \Delta$, where Γ, Δ are bases, $M \in \Sigma_{\Lambda\mu}$ and $\delta \in \mathcal{T}_T$. The writing $\Gamma, x : \delta$ abbreviates $\Gamma \cup \{x : \delta\}$ with $x \notin dom(\Gamma)$. Similarly for $\alpha : \sigma, \Delta$.

Definition 3.4 (Type Assignment System for $\Lambda\mu$).

$$\frac{}{\Gamma, x:\delta \vdash x:\delta \mid \Delta}\,(Ax)$$

$$\frac{\Gamma, x:\delta_1 \vdash M:\sigma\to\delta_2 \mid \Delta}{\Gamma \vdash \lambda x.M:\delta_1\times\sigma\to\delta_2 \mid \Delta}\,(\lambda) \qquad \frac{\Gamma \vdash M:\delta_1\times\sigma\to\delta_2 \mid \Delta \quad \Gamma\vdash N:\delta_1 \mid \Delta}{\Gamma \vdash (M)N:\sigma\to\delta_2 \mid \Delta}\,(App)$$

$$\frac{\Gamma \vdash M:\delta \mid \alpha:\sigma,\Delta}{\Gamma \vdash \mu\alpha.M:\sigma\to\delta \mid \Delta}\,(\mu) \qquad \frac{\Gamma \vdash M:\sigma\to\delta \mid \alpha:\sigma,\Delta}{\Gamma \vdash (M)\alpha:\delta \mid \alpha:\sigma,\Delta}\,(S)$$

$$\frac{}{\Gamma \vdash M:\omega \mid \Delta}\,(\omega)$$

$$\frac{\Gamma \vdash M:\delta_1 \mid \Delta \quad \Gamma \vdash M:\delta_2 \mid \Delta}{\Gamma \vdash M:\delta_1\wedge\delta_2 \mid \Delta}\,(\wedge) \qquad \frac{\Gamma \vdash M:\delta_1 \mid \Delta \quad \delta_1 \le_T \delta_2}{\Gamma \vdash M:\delta_2 \mid \Delta}\,(\le)$$

As we shall see in the next section, terms denote functions from streams to term denotations. The functional behaviour of a term is however different when it is a λ or a μ-abstraction. Rules (λ) and (App) express the fact that ordinary λ-abstraction and application deal with the first component of the stream, which is their (implicit) argument. On the contrary μ-abstraction and application to a stream variable depend on their stream argument as a whole; hence rules (μ) and (S) have the shape of arrow introduction and elimination from simple type discipline. The remaining rules $(\omega), (\wedge)$ and $(\le)$ are familiar since (Barendregt *et al.* 1983).

We write $\Gamma \vdash M:\delta \mid \Delta$ ambiguously for the judgement itself and for the claim that it is derivable in the system of Def. 3.4. Below we write $\Gamma(x) = \delta$ if $x:\delta \in \Gamma$, $\Gamma(x) = \omega$ else. Similarly for $\Delta(\alpha)$. By this we identify bases with functions from variables to $\mathcal{T}_T \cup \mathcal{T}_S$ with values of the appropriate sort, which are equal to ω but for finitely many variables.

We set $\Gamma \le \Gamma'$ if and only if $\Gamma(x) \le_T \Gamma'(x)$ for all $x \in \mathsf{Var}_T$; $\Delta \le \Delta'$ is similarly defined w.r.t. $\le_S$. Also we define $\Gamma_1\wedge\Gamma_2$ as the least basis such that $(\Gamma_1\wedge\Gamma_2)(x) = \Gamma_1(x)\wedge\Gamma_2(x)$, and similarly for $\Delta_1\wedge\Delta_2$.

Lemma 3.5 (Generation Lemma).

1 $\Gamma \vdash x:\delta \mid \Delta \Longleftrightarrow \exists\,\delta' \in \mathcal{T}_T.\ \Gamma(x) = \delta'\ \&\ \delta' \le_T \delta$,

2 $\Gamma \vdash \lambda x.M:\delta \mid \Delta \Longleftrightarrow \exists\, I, \sigma_i \in \mathcal{T}_S, \delta_i, \delta_i' \in \mathcal{T}_T.$

$$\forall i \in I.\ \Gamma, x:\delta_i \vdash M:\sigma_i\to\delta_i' \mid \Delta\ \&\ \textstyle\bigwedge_{i\in I}(\delta_i\times\sigma_i\to\delta_i') \le_T \delta,$$

3 $\Gamma \vdash (M)N:\delta \mid \Delta \Longleftrightarrow \exists\,\delta' \in \mathcal{T}_T, \sigma \in \mathcal{T}_S.\ \Gamma \vdash M:\delta'\times\sigma \mid \Delta\ \&\ \Gamma \vdash N:\delta' \mid \Delta$,

4 $\Gamma \vdash \mu\alpha.M:\delta \mid \Delta \Longleftrightarrow \exists\,\delta' \in \mathcal{T}_T, \sigma \in \mathcal{T}_S.\ \Gamma \vdash M:\delta' \mid \alpha:\sigma,\Delta\ \&\ \sigma\to\delta' \le_T \delta$,

5 $\Gamma \vdash (M)\alpha:\delta \mid \Delta \Longleftrightarrow$

$$\exists\delta' \in \mathcal{T}_T, \sigma \in \mathcal{T}_S.\ \Delta(\alpha) = \sigma\ \&\ \Gamma \vdash M:\sigma\to\delta' \mid \Delta\ \&\ \sigma\to\delta' \le_T \delta.$$

Proof. Standard. □

Lemma 3.6 (Strengthening).

1 $\Gamma, x:\delta \vdash M:\delta' \mid \Delta\ \&\ \delta'' \le_T \delta \Rightarrow \Gamma, x:\delta'' \vdash M:\delta' \mid \Delta$,

2 $\Gamma \vdash M : \delta \mid \alpha : \sigma, \Delta \ \&\ \sigma' \leq_S \sigma \Rightarrow \Gamma \vdash M : \delta \mid \alpha : \sigma', \Delta$.

Proof. Part (1) is proved as the in case of intersection type systems with subtyping for the ordinary λ-calculus. The proof of part (2) is slightly more complex. Let

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma' \vdash M' : \sigma \to \delta' \mid \alpha : \sigma, \Delta'\end{array}}{\Gamma' \vdash (M')\alpha : \delta' \mid \alpha : \sigma, \Delta'}\,(S)$$

be a subderivation of the derivation of $\Gamma \vdash M : \delta \mid \alpha : \sigma, \Delta$ involving the application of the subterm M' of M to the stream variable α, if any such subderivation exists. Since $\sigma' \leq_S \sigma$ implies $\sigma \to \delta' \leq_T \sigma' \to \delta'$ by the contravariance of the arrow in its first argument, we can replace this subderivation by:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma' \vdash M' : \sigma \to \delta' \mid \alpha : \sigma', \Delta'\end{array} \quad \sigma \to \delta' \leq_T \sigma' \to \delta'}{\Gamma' \vdash M' : \sigma' \to \delta' \mid \alpha : \sigma', \Delta'}\,(\leq)}{\Gamma' \vdash (M')\alpha : \delta' \mid \alpha : \sigma', \Delta'}\,(S)$$

Then an inductive argument over the derivation of $\Gamma \vdash M : \delta \mid \alpha : \sigma, \Delta$ completes the proof. □

Remark 3.7. If $x \notin dom(\Gamma)$ then $\Gamma(x) = \omega$ by convention. Therefore if $\Gamma \subseteq \Gamma'$ as sets then $\Gamma' \leq \Gamma$ as functions. Similarly $\Delta \subseteq \Delta'$ implies $\Delta' \leq \Delta$. Now Lemma 3.6 reads as the statement that the following *strengthening rule* is admissible:

$$\dfrac{\Gamma' \leq \Gamma \quad \Gamma \vdash M : \delta \mid \Delta \quad \Delta' \leq \Delta}{\Gamma' \vdash M : \delta \mid \Delta'}$$

From the above observation it follows that the *weakening rule*:

$$\dfrac{\Gamma \subseteq \Gamma' \quad \Gamma \vdash M : \delta \mid \Delta \quad \Delta \subseteq \Delta'}{\Gamma' \vdash M : \delta \mid \Delta'}$$

is a particular case of strengthening, and hence it is admissible as well.

Lemma 3.8 (Substitution).

1 $\Gamma \vdash M[x := N] : \delta_1 \mid \Delta \iff \exists \delta_2 \in \mathcal{T}_T.\ \Gamma, x : \delta_2 \vdash M : \delta_1 \mid \Delta \ \&\ \Gamma \vdash N : \delta_2 \mid \Delta$
2 $\Gamma \vdash M[\alpha \Leftarrow N] : \delta_1 \mid \alpha : \sigma, \Delta \iff$

$$\exists \delta_2 \in \mathcal{T}_T.\ \Gamma \vdash M : \delta_1 \mid \alpha : \delta_2 \times \sigma, \Delta \ \&\ \Gamma \vdash N : \delta_2 \mid \Delta$$

Proof. The proof of part (1) is the standard one for type assignment systems to ordinary λ-terms. Part (2) is by induction over the derivation of $\Gamma \vdash M[\alpha \Leftarrow N] : \delta_1 \mid \alpha : \sigma, \Delta$ ($\Rightarrow$) and of $\Gamma \vdash M : \delta_1 \mid \alpha : \delta_2 \times \sigma, \Delta$ ($\Leftarrow$). Consider e.g. the case $M[\alpha \Leftarrow N] = ((M')\alpha)[\alpha \Leftarrow N] = (M'[\alpha \Leftarrow N])N\alpha$ and the derivation ends by:

$$\dfrac{\Gamma \vdash (M'[\alpha \Leftarrow N])N : \sigma \to \delta \mid \alpha : \sigma, \Delta}{\Gamma \vdash (M'[\alpha \Leftarrow N])N\alpha : \delta \mid \alpha : \sigma, \Delta}\,(S)$$

for some δ'. By (3) of 3.5, $\Gamma \vdash M'[\alpha \Leftarrow N] : \delta' \times \sigma \to \delta \mid \alpha : \sigma, \Delta$ and $\Gamma \vdash N : \delta' \mid \alpha : \sigma, \Delta$. By induction hypothesis ($\Rightarrow$) there exists δ'' s.t.

$$\Gamma \vdash M' : \delta' \times \sigma \to \delta \mid \alpha : \delta'' \times \sigma, \Delta \qquad (*)$$

and $\Gamma \vdash N : \delta'' \mid \Delta$. Note that $\alpha \notin dom(\Delta)$ so that $\alpha \notin fv(N)$, hence the assumption $\alpha : \sigma$ is never used in the derivation of $\Gamma \vdash N : \delta' \mid \alpha : \sigma, \Delta$, that is $\Gamma \vdash N : \delta' \mid \Delta$. Then we derive $\Gamma \vdash N : \delta' \wedge \delta'' \mid \Delta$ by rule ($\wedge$).

On the other hand $\delta' \wedge \delta'' \leq_T \delta', \delta''$ implies $\delta' \times \sigma \to \delta \leq_T (\delta' \wedge \delta'') \times \sigma \to \delta$ and $(\delta' \wedge \delta'') \times \sigma \leq_S \delta'' \times \sigma$. Therefore from the derivation of $(*)$, by rule ($\leq$) and by (2) of Lemma 3.6, we obtain $\Gamma \vdash M' : (\delta' \wedge \delta'') \times \sigma \to \delta \mid \alpha : (\delta' \wedge \delta'') \times \sigma, \Delta$ as desired. □

Theorem 3.9 (Subject Conversion).

$$\vdash M = N \;\&\; \Gamma \vdash M : \delta \mid \Delta \;\Rightarrow\; \Gamma \vdash N : \delta \mid \Delta$$

Proof. It suffices to check the axioms in Def. 1.3 and their symmetric equations. We treat some relevant cases only, as the others are similar. Let us consider the axiom (μ): $(\mu\alpha.M)N = \mu\alpha.M[\alpha \Leftarrow N]$. Supposing that $(\mu\alpha.M)N$ has been given an arrow type, we have:

$$\begin{array}{lll}
 & \Gamma \vdash (\mu\alpha.M)N : \sigma \to \delta \mid \Delta & \\
\Rightarrow & \exists \delta'.\, \Gamma \vdash \mu\alpha.M : \delta' \times \sigma \to \delta \mid \Delta \;\&\; \Gamma \vdash N : \delta' \mid \Delta & \text{by (3) of 3.5} \\
\Rightarrow & \Gamma \vdash M : \delta \mid \alpha : \delta' \times \sigma, \Delta & \text{by (4) of 3.5} \\
\Rightarrow & \vdash M[\alpha \Leftarrow N] : \delta \mid \alpha : \sigma, \Delta & \text{by (2, }\Leftarrow\text{) of 3.8} \\
\Rightarrow & \Gamma \vdash \mu\alpha.M[\alpha \Leftarrow N] : \sigma \to \delta \mid \Delta & \text{by rule } (\mu).
\end{array}$$

The symmetric of axiom (μ) is proved similarly, by using Lemma 3.5 and (2, $\Rightarrow$) of 3.8.

Let us consider the symmetric equation of axiom (η_T), namely $M = \lambda x.(M)x$ where $x \notin fv(M)$. Suppose that $\Gamma \vdash M : \varphi \mid \Delta$; since $\varphi \sim_T \omega \times \omega \to \varphi$, and $\Gamma \vdash x : \omega \mid \Delta$, we have $\Gamma \vdash (M)x : \omega \to \varphi \mid \Delta$ and hence $\Gamma \vdash \lambda x.(M)x : \omega \to \omega \to \varphi \mid \Delta$ by rule (λ) and finally $\Gamma \vdash \lambda x.(M)x : \varphi \mid \Delta$ by rule ($\leq$). This argument is the basis case of an inductive proof establishing that $\Gamma \vdash M : \delta \mid \Delta$ implies $\Gamma \vdash \lambda x.(M)x : \delta \mid \Delta$ for all $\delta \in \mathcal{T}_T$. □

4. An Extensional $\Lambda\mu$ Filter Model

The type system introduced in the previous section is motivated by the semantics of $\Lambda\mu$-terms as defined in (Nakazawa *et al.* 2012). An extensional $\Lambda\mu$-model is essentially a D_∞ model endowed with more structure to provide stream denotations. There are two domains: D for denotations and S for streams, which are assumed to satisfy the domain equations $D = [S \to D]$ and $S = D \times S$. The denotation $[\![M]\!]$ is then a map from environments e, interpreting term and stream variables in D and S respectively, and from streams $s \in S$ to denotations $[\![M]\!]\ e\ s \in D$. The space $[S \to D]$ is made of Scott-continuous functions which implies that $[\![M]\!]\ e$ will always use a finite amount of information about s to yield a finite information of its value; in particular $[\![M]\!]\ e\ s$ depends only on a finite number of components of s.

It is known that intersection types denote the compact elements of ω-algebraic lattices, which can be recovered by taking filters of appropriate intersection type pre-orders. Therefore, by considering a solution of the domain equations $D = [S \to D]$ and $S = D \times S$ in the category of ω-algebraic lattices, we show that the type assignment system in Definition 3.4 induces a filter-model, which is an extensional $\Lambda\mu$-model. In the model the denotation of a term M coincides with the filters of the types that can be assigned to M, which is the content and of Theorem 4.8.

Definition 4.1 (Extensional $\Lambda\mu$-model). An *extensional $\Lambda\mu$-model* is a quadruple $(D, S, ::, [\![\cdot]\!])$ where:

1 D and S are non-empty sets satisfying, up to isomorphism, the equations:

$$D = [S \to D] \subseteq D^S, \qquad S = D \times S;$$

2 $(::) : D \times S \to S$ is the inverse of the isomorphism $S = D \times S$;
3 $[\![\cdot]\!] : \Sigma_{\Lambda\mu} \to \mathsf{Env} \to S \to D$, where $\mathsf{Env} = (\mathsf{Var}_T \to D) + (\mathsf{Var}_S \to S)$, is such that:

$$\begin{array}{lcl} [\![x]\!]\, e\, s & = & e(x) \\ [\![\lambda x.M]\!]\, e\, (d :: s) & = & [\![M]\!]\, e[x \mapsto d]\, s \\ [\![(M)N]\!]\, e\, s & = & [\![M]\!]\, e\, (([\![N]\!]\, e) :: s) \\ [\![\mu\alpha.M]\!]\, e\, s & = & [\![M]\!]\, e[\alpha \mapsto s] \\ [\![(M)\alpha]\!]\, e\, s & = & [\![M]\!]\, e\, e(\alpha) \end{array}$$

where $e[x \mapsto d](y) = d$ if $x = y$, $e(y)$ else. $e[\alpha \mapsto s]$ has a similar meaning.

Extensional $\Lambda\mu$-models are a variant of Streicher and Reus's models in (Streicher *et al.* 1998). In both cases one has that terms denote functions of infinite tuples of term denotations; the difference is that the range of these functions is a parametric domain of "results" according to (Streicher *et al.* 1998), and the set of term denotations itself according to (Nakazawa *et al.* 2012).

Nakazawa and Katsumata's construction models all $\Lambda\mu$-axioms, while Streicher and Reus's models do not validate all instances of axiom (β_S), as remarked in (van Bakel *et al.* 2011). The following is Theorem 1 in (Nakazawa *et al.* 2012).

Theorem 4.2 (Soundness). If $(D, S, ::, [\![\cdot]\!])$ is an extensional $\Lambda\mu$-model then for all $M, N \in \Sigma_{\Lambda\mu}$:

$$\vdash M = N \Rightarrow [\![M]\!] = [\![N]\!].$$

A *filter* over an inf-semilattice is a non-empty, upward closed subset which is closed under finite meets. Let $\mathcal{F}_T$ and $\mathcal{F}_S$ be the sets of filters over $(\mathcal{T}_T, \leq_T)$ and over $(\mathcal{T}_S, \leq_S)$ respectively. It is known from the literature that the set of filters over a countable inf-semilattice is an ω-algebraic lattice w.r.t. subset inclusion, whose compact elements are the upward cones of the elements of the original semilattice, also called *principal filters*.

Definition 4.3. For $d \in \mathcal{F}_T$ and $s \in \mathcal{F}_S$ define:

1 $d :: s = \{\delta \times \sigma \in \mathcal{T}_S \mid \delta \in d \,\&\, \sigma \in s\}$,
2 $d \cdot s = \{\delta \in \mathcal{T}_T \mid \exists\, \sigma \in s.\, \sigma \to \delta \in d\}$.

$(\mathcal{F}_T, \subseteq)$ and $(\mathcal{F}_S, \subseteq)$ are taken with the Scott topology. The product $\mathcal{F}_T \times \mathcal{F}_S$ and the function space $[\mathcal{F}_S \to \mathcal{F}_T]$ are in the category of algebraic lattices; in particular $[\mathcal{F}_S \to \mathcal{F}_T]$ is the space of the Scott continuous functions from $\mathcal{F}_S$ to $\mathcal{F}_T$, ordered pointwise.

Lemma 4.4. If $d \in \mathcal{F}_T$ and $s \in \mathcal{F}_S$ then $d :: s \in \mathcal{F}_S$ and $d \cdot s \in \mathcal{F}_T$. Moreover the mappings $::$ and $\cdot$ are continuous in both their arguments.

Proof. Standard. □

Lemma 4.5. The mappings

$$(::) : \mathcal{F}_T \times \mathcal{F}_S \to \mathcal{F}_S \qquad \text{and} \qquad \lambda d \in \mathcal{F}_T\ \lambda s \in \mathcal{F}_S.\, d \cdot s : \mathcal{F}_T \to [\mathcal{F}_S \to \mathcal{F}_T]$$

are isomorphisms of algebraic lattices.

Proof. By unfolding definitions and Lemma 4.4. □

Definition 4.6. For $e \in \mathsf{Env}_{\mathcal{F}} = (\mathsf{Var}_T \to \mathcal{F}_T) + (\mathsf{Var}_S \to \mathcal{F}_S)$ we set:

$$\Gamma, \Delta \models e \iff \forall x \in \mathsf{Var}_T.\ \Gamma(x) \in e(x)\ \&\ \forall \alpha \in \mathsf{Var}_S.\ \Delta(\alpha) \in e(\alpha).$$

Then for all $e \in \mathsf{Env}_{\mathcal{F}}$ and $s \in \mathcal{F}_S$ define the map $[\![\cdot]\!] : \Sigma_{\Lambda\mu} \times \mathsf{Env}_{\mathcal{F}} \times \mathcal{F}_S \to \mathcal{F}_T$ by

$$[\![M]\!]\, e\, s = \{\delta \in \mathcal{T}_T \mid \exists \Gamma, \Delta.\ \Gamma, \Delta \models e\ \&\ \Gamma \vdash M : \delta \mid \Delta\} \cdot s.$$

Lemma 4.7. The mapping $[\![\cdot]\!]$ in Definition 4.6 is well defined.

Proof. By Lemma 4.4, it suffices to show that the set

$$[\![M]\!]\, e = \{\delta \in \mathcal{T}_T \mid \exists \Gamma, \Delta.\ \Gamma, \Delta \models e\ \&\ \Gamma \vdash M : \delta \mid \Delta\}$$

belongs to $\mathcal{F}_T$. By rule (ω) we have that $\omega \in [\![M]\!]\, e \neq \emptyset$ and by rule $(\leq)$ it is upward closed. To see that it is closed under finite meets let $\delta_1, \delta_2 \in [\![M]\!]\, e$, so that there exist $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ such that $\Gamma_i, \Delta_i \models e$ and $\Gamma_i \vdash M : \delta_i \mid \Delta_i$ for $i = 1, 2$. Because of the fact that $e(x)$ and $e(\alpha)$ are filters for any x and α, $\Gamma_i(x) \in e(x)$ implies $(\Gamma_1 \wedge \Gamma_2)(x) = \Gamma_1(x) \wedge \Gamma_2(x) \in e(x)$, and similarly we have that $(\Delta_1 \wedge \Delta_2)(\alpha) \in e(\alpha)$. Hence we have $\Gamma_1 \wedge \Gamma_2, \Delta_1 \wedge \Delta_2 \models e$.

On the other hand, since $(\Gamma_1 \wedge \Gamma_2)(x) \leq_T \Gamma_i(x)$ and $(\Delta_1 \wedge \Delta_2)(\alpha) \leq_S \Delta_i(\alpha)$ for all x and α and $i = 1, 2$, by Lemma 3.6 we have that

$$\Gamma_1 \wedge \Gamma_2 \vdash M : \delta_i \mid \Delta_1 \wedge \Delta_2 \quad (i = 1, 2)$$

from which, by rule $(\wedge)$, we get

$$\Gamma_1 \wedge \Gamma_2 \vdash M : \delta_1 \wedge \delta_2 \mid \Delta_1 \wedge \Delta_2,$$

which establishes the thesis. □

Theorem 4.8 (Filter Model). The structure $(\mathcal{F}_T, \mathcal{F}_S, ::, [\![\cdot]\!])$ is an extensional $\Lambda\mu$-model.

Proof. By lemmas 4.5 and 4.7 it remains to show that the mapping $[\![\cdot]\!]$ satisfies the equations in Def. 4.1. This follows by unravelling definitions and using Lemma 3.5. E.g. we prove $[\![\lambda x.M]\!]\, e\, (d :: s) \subseteq [\![M]\!]\, e[x \mapsto d]\, s$ as follows. If $\delta \in [\![\lambda x.M]\!]\, e\, (d :: s)$ then

$\sigma \to \delta \in [\![\lambda x.M]\!]\, e$ for some $\sigma \in d :: s$. This implies that $\sigma \sim_S \delta' \times \sigma'$ for some $\delta' \in d$ and $\sigma' \in s$, and that $\Gamma \vdash \lambda x.M : \delta' \times \sigma' \to \delta \mid \Delta$ for some $\Gamma, \Delta \models e$. By (2) of 3.5 we have also $\Gamma, x : \delta' \vdash M : \sigma' \to \delta \mid \Delta$. The fact that $\delta' \in d$ implies $\Gamma, x : \delta', \Delta \models e[x \mapsto d]$ and therefore $\sigma' \to \delta \in [\![M]\!]\, e[x \mapsto d]$, and we conclude from $\sigma' \in s$ that $\delta \in [\![M]\!]\, e[x \mapsto d]\, s$.

The opposite inclusion, as well as all the other cases are similar. $\square$

5. The Approximation Theorem

The approximation property of the interpretation mapping $[\![\cdot]\!]$ w.r.t. a model D states that: $[\![M]\!]\, e = \bigsqcup\{\, [\![A]\!]\, e \mid A \in \mathcal{A}(M)\}$ for all environment e, where $[\![\cdot]\!]$ extends to approximate normal forms by setting $[\![\Omega]\!]\, e = \bot$, namely the bottom of D.

In case of the filter model generated by the intersection type assignment system in section 4, it is equivalent to the statement that $\Gamma \vdash M : \delta \mid \Delta$ if and only if $\Gamma \vdash A : \delta \mid \Delta$ for some $A \in \mathcal{A}(M)$. The if part is a rather easy consequence of the fact that any type that can be assigned to Ω is equivalent to ω, so that if $\Gamma \vdash A : \delta \mid \Delta$ when $A = \phi(M)$, then A is M but for some subterms which have been replaced by Ω: then the derivation of $\Gamma \vdash M : \delta \mid \Delta$ is obtained from the type derivation of $\Gamma \vdash A : \delta \mid \Delta$ typing with ω exactly those subterms. The general case $A \in \mathcal{A}(M)$ is then proved by means of subject conversion and the monotonicity property of typing w.r.t. $\preceq$.

As usual the difficult part of the proof is the only if one. To adapt Tait's computability argument the concept of (syntactical) stream in Definition 5.4 and the subsequent remark are crucial. Indeed a term represents a function of streams, which are represented not by stream variables only, rather by contexts of the shape $[\;]M_1 \ldots M_k\alpha$, which we call a *stream* by overloading terminology.

In this section the type assignment system is extended to approximate normal forms in $\mathcal{A}$ by allowing more subjects in the typing judgements, but without adding any new type nor typing rule.

Lemma 5.1.

1 $\Gamma \vdash \Omega : \delta \mid \Delta \iff \delta \sim_T \omega$,
2 $\Gamma \vdash A : \delta \mid \Delta \;\&\; A \preceq A' \Rightarrow \Gamma \vdash A' : \delta \mid \Delta$, for $A, A' \in \mathcal{A}$.

Proof. (1): by induction over the derivation of $\Gamma \vdash \Omega : \delta \mid \Delta$, by observing that the only possible rules by which the derivation can end are (ω), $(\wedge)$ and $(\leq)$.

(2): by (1) we have that if $A = \Omega$ then $\delta \sim_T \omega$, and the thesis is obvious. Otherwise $A = \lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \ldots \mu\alpha_n\lambda\vec{x}_n.(y)\vec{A}_0\beta_1\vec{M}_1 \ldots \beta_m\vec{A}_m$, and either

$$A' = \lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \ldots \mu\alpha_n\lambda\vec{x}_n.(y)\vec{A}'_0\beta_1\vec{A}'_1 \ldots \beta_m\vec{A}'_m$$

with $\vec{A}_i \preceq \vec{A}'_i$ (componentwise) for all i, or $A = \lambda x.(A')x$ ($x \notin fv(A')$), or $A = \mu\alpha.(A')\alpha$ ($\alpha \notin fv(A')$) or $A = \mu\alpha.A''$ and $A' = \lambda x.\mu\alpha.A''[\alpha \Leftarrow x]$ ($x \notin fv(A'')$).

In the first case we use induction hypothesis and Lemma 3.5. In the last three cases we observe that $A \longrightarrow_{\eta_T,\eta_S,fst} A'$, namely by extending the relation $\longrightarrow$ to $\mathcal{A}$ in those cases in which $\mathcal{A}$ is closed under reduction and Ω is never in head-position. In these cases the thesis follows by (the obvious extension of) Theorem 3.9. $\square$

Lemma 5.2.

$$\Gamma \vdash \phi(M) : \delta \mid \Delta \Rightarrow \Gamma \vdash M : \delta \mid \Delta.$$

Proof. By induction on the shape of $\phi(M)$. If $\phi(M) = \Omega$ then the thesis follows by (1) of 5.1. Otherwise $\phi(M) = \lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \dots \mu\alpha_n\lambda\vec{x}_n.(y)\vec{A}_0\beta_1\vec{M}_1 \dots \beta_m\vec{A}_m$ and $M = \lambda\vec{x}_0\mu\alpha_1\lambda\vec{x}_1 \dots \mu\alpha_n\lambda\vec{x}_n.(y)\vec{M}_0\beta_1\vec{M}_1 \dots \beta_m\vec{M}_m$ where $\vec{A}_i = \phi(\vec{M}_i)$ for all i. Then the thesis follows by induction using Lemma 3.5. $\square$

Lemma 5.3.

$$\exists\, A \in \mathcal{A}(M).\ \Gamma \vdash A : \delta \mid \Delta \Rightarrow \Gamma \vdash M : \delta \mid \Delta.$$

Proof. If $A \in \mathcal{A}(M)$ then $A \preceq \phi(N)$ and $M \longrightarrow^* N$ for some N. Now:

$$\begin{array}{rcll} \Gamma \vdash A : \delta \mid \Delta & \Rightarrow & \Gamma \vdash \phi(N) : \delta \mid \Delta & \text{by (2) of Lemma 5.1} \\ & \Rightarrow & \Gamma \vdash N : \delta \mid \Delta & \text{by Lemma 5.2} \\ & \Rightarrow & \Gamma \vdash M : \delta \mid \Delta & \text{by Theorem 3.9.} \end{array}$$

$\square$

Definition 5.4 (Streams). A *stream* $S \in \mathsf{Strm}$ is an applicative context of the shape:

$$S = [\ ]N_1 \dots N_k\beta, \quad k \in \mathbb{N},\ N_1, \dots, N_k \in \Sigma_{\Lambda\mu}.$$

Streams are from (Saurin 2005; Saurin 2008b) and are simply called “contexts” in (Nakazawa *et al.* 2012). When $k = 0$, the stream $[\ ]N_1 \dots N_k\beta$ is just $[\ ]\beta$. Observe that:

$$\begin{array}{rcll} (\mu\alpha.M)N_1 \dots N_k\beta & \longrightarrow^*_{fst,\beta_T t} & (\mu\alpha.M[\alpha \Leftarrow N_1 \dots N_k])\beta & \\ & \longrightarrow_{\beta_S} & M[\alpha \Leftarrow N_1 \dots N_k][\alpha := \beta] & \\ & = & M[\alpha := \beta][\beta \Leftarrow N_1 \dots N_k] & \text{as } \alpha \notin \bigcup_{i=1}^k fv(N_i). \end{array}$$

This justifies the following notations, for $S = [\ ]N_1 \dots N_k\beta = [\ ]\vec{N}\beta \in \mathsf{Strm}$:

$$\begin{array}{rcl} (M)S & = & (M)\vec{N}\beta \\ M[\alpha \Leftarrow S] & = & M[\alpha := \beta][\beta \Leftarrow \vec{N}] \\ M :: S & = & [\ ]MN_1 \dots N_k\beta \end{array}$$

by slightly overloading the notation of the semantic operator :: (although with a related meaning). The following is a refinement of the computability interpretation in (van Bakel *et al.* 2013).

Definition 5.5 (Computability Interpretation). For any bases Γ, Δ and $\delta \in \mathcal{T}_T$, let us define the set:

$$[\delta]_{\Gamma,\Delta} = \{ M \in \Sigma_{\Lambda\mu} \mid \exists A \in \mathcal{A}(M).\ \Gamma \vdash A : \delta \mid \Delta \}.$$

Then for $\delta \in \mathcal{T}_T$ and $\sigma \in \mathcal{T}_S$ we define the sets $[\![\delta]\!]_{\Gamma,\Delta} \subseteq \Sigma_{\Lambda\mu}$ and $[\![\sigma]\!]_{\Gamma,\Delta} \subseteq \mathsf{Strm}$ inductively as follows:

1 $[\![\varphi]\!]_{\Gamma,\Delta} = [\varphi]_{\Gamma,\Delta}$, for all atomic type φ,
2 $[\![\omega_T]\!]_{\Gamma,\Delta} = \Sigma_{\Lambda\mu}$, $[\![\omega_S]\!]_{\Gamma,\Delta} = \mathsf{Strm}$,

3 $M \in [\![\sigma \to \delta]\!]_{\Gamma,\Delta} \iff \forall \Gamma', \Delta', S \in [\![\sigma]\!]_{\Gamma',\Delta'}.\ (M)S \in [\![\delta]\!]_{\Gamma\wedge\Gamma',\Delta\wedge\Delta'}$,
4 $[\ ]\beta \in [\![\sigma]\!]_{\Gamma,\Delta} \iff \Delta(\beta) \leq_S \sigma$,
5 $M :: S \in [\![\delta \times \sigma]\!]_{\Gamma\wedge\Gamma',\Delta\wedge\Delta'} \iff M \in [\![\delta]\!]_{\Gamma,\Delta}\ \&\ S \in [\![\sigma]\!]_{\Gamma',\Delta'}$,
6 $[\![\delta_1 \wedge \delta_2]\!]_{\Gamma,\Delta} = [\![\delta_1]\!]_{\Gamma,\Delta} \cap [\![\delta_2]\!]_{\Gamma,\Delta}$ and $[\![\sigma_1 \wedge \sigma_2]\!]_{\Gamma,\Delta} = [\![\sigma_1]\!]_{\Gamma,\Delta} \cap [\![\sigma_2]\!]_{\Gamma,\Delta}$.

Lemma 5.6.

1 $\delta \leq_T \delta' \Rightarrow [\delta]_{\Gamma,\Delta} \subseteq [\delta']_{\Gamma,\Delta}\ \&\ [\![\delta]\!]_{\Gamma,\Delta} \subseteq [\![\delta']\!]_{\Gamma,\Delta}$,
2 $\sigma \leq_S \sigma' \Rightarrow [\![\sigma]\!]_{\Gamma,\Delta} \subseteq [\![\sigma']\!]_{\Gamma,\Delta}$.

Proof. That $[\delta]_{\Gamma,\Delta} \subseteq [\delta']_{\Gamma,\Delta}$ is an obvious consequence of rule $(\leq)$ of the type system. That $\delta \leq_T \delta' \Rightarrow [\![\delta]\!]_{\Gamma,\Delta} \subseteq [\![\delta']\!]_{\Gamma,\Delta}$ and $\sigma \leq_S \sigma' \Rightarrow [\![\sigma]\!]_{\Gamma,\Delta} \subseteq [\![\sigma']\!]_{\Gamma,\Delta}$ are shown simultaneously, by checking the inequations in Def. 3.2 and by definition unfolding of $[\![\delta]\!]_{\Gamma,\Delta}, [\![\delta']\!]_{\Gamma,\Delta}, [\![\sigma]\!]_{\Gamma,\Delta}$ and $[\![\sigma']\!]_{\Gamma,\Delta}$. □

Corollary 5.7.

$$\Gamma \leq \Gamma'\ \&\ \Delta \leq \Delta' \Rightarrow [\![\delta]\!]_{\Gamma',\Delta'} \subseteq [\![\delta]\!]_{\Gamma,\Delta}\ \&\ [\![\sigma]\!]_{\Gamma',\Delta'} \subseteq [\![\sigma]\!]_{\Gamma,\Delta}.$$

Proof. Routine, using Lemma 5.6 and the admissibility of the strengthening rule, Lemma 3.6. □

Lemma 5.8. If R is a redex and R' its contractum, then for any $S \in \mathsf{Strm}$:

$$(R')S \in [\![\delta]\!]_{\Gamma,\Delta} \Rightarrow (R)S \in [\![\delta]\!]_{\Gamma,\Delta}.$$

Proof. By induction over δ. In the base case $\delta = \varphi$ the hypothesis is equivalent to $(R')S \in [\varphi]_{\Gamma,\Delta}$, that is $\Gamma \vdash A : \delta \mid \Delta$ for some $A \in \mathcal{A}((R')S)$. Since $(R)S \longrightarrow (R')S$ we have that $\mathcal{A}((R')S) \subseteq \mathcal{A}((R)S)$ by Def. 2.6, and therefore $(R)S \in [\varphi]_{\Gamma,\Delta} = [\![\varphi]\!]_{\Gamma,\Delta}$.
The remaining cases are immediate consequence of the inductive hypothesis. □

Lemma 5.9.

1 $(M)x \in [\sigma \to \delta]_{\Gamma,\Delta}\ \&\ \Gamma(x) \leq_T \delta' \Rightarrow M \in [\delta' \times \sigma \to \delta]_{\Gamma,\Delta}$
2 $(M)\alpha \in [\delta]_{\Gamma,\Delta}\ \&\ \Delta(\alpha) \leq_S \sigma\ \&\ \delta \not\sim_T \omega \Rightarrow M \in [\sigma \to \delta]_{\Gamma,\Delta}$

Proof. (1): by assumption there exists $A \in \mathcal{A}((M)x)$ such that $\Gamma \vdash A : \sigma \to \delta \mid \Delta$. If $A = \Omega$ then $A \preceq \phi((M)x)$ and $\sigma \to \delta \sim_T \omega$ by (1) of Lemma 5.1, and $\omega \sim_T \omega \to \omega \sim_T \omega \times \omega \to \omega$, we have that $\Gamma \vdash A : \omega \times \omega \to \omega \mid \Delta$ and $A \preceq \phi(M)$.

Now let us assume that $A \neq \Omega$. Then for some P we know that $(M)x \longrightarrow^* P$ and $A \preceq \phi(P) \neq \Omega$. If $P = (P')x$ with $M \longrightarrow^* P'$ then P' is $\lambda\mu$-free and $A = (A')x$ with $A' \preceq \phi(P')$. By (3) of Lemma 3.5 there exists some δ' s.t. $\Gamma(x) \leq_T \delta'$ and $\Gamma \vdash A' : \delta' \times \sigma \to \delta \mid \Delta$.

If instead $P \neq (P')x$ for any P' then two cases may occur. The first case is when:

$$(M)x \longrightarrow^* (\lambda y.M')x \longrightarrow_{\beta_T} M'[y := x] \longrightarrow^* P,$$

where $M \longrightarrow^* \lambda y.M'$. Since $\Gamma(x) \leq_T \delta'$ and we can assume that $y \notin \mathit{dom}(\Gamma)$, from $\Gamma \vdash A : \sigma \to \delta \mid \Delta$ we infer $\Gamma, y : \delta' \vdash A[x := y] : \sigma \to \delta \mid \Delta$ by (1) of Lemma 3.8, so that by rule (λ) we get $\Gamma \vdash \lambda y.A[x := y] : \delta' \times \sigma \to \delta \mid \Delta$, where $\lambda y.A[x := y] \in \mathcal{A}$ since

$A \neq \Omega$ and hence $A[x := y] \neq \Omega$. Now if $M'[y := x] \longrightarrow^* P$ then $M' \longrightarrow^* P[x := y]$ and therefore $\lambda y.M' \longrightarrow^* \lambda y.P[x := y]$.

On the other hand, if $A \preceq \phi(P) \neq \Omega$ then $A[x := y] \preceq \phi(P[x := y]) \neq \Omega$, hence $\lambda y.A[x := y] \preceq \lambda y.\phi(P[x := y]) = \phi(\lambda y.P[x := y])$ by (3) of Lemma 2.8. But $M \longrightarrow^* \lambda y.M' \longrightarrow^* \lambda y.P[x := y]$, so that $\lambda y.A[x := y] \in \mathcal{A}(M)$.

It remains the case when: $(M)x \longrightarrow^* (\mu\alpha.M')x \longrightarrow^* P$ and $(\mu\alpha.M')x$ is the last term of the shape $(P')x$ in the reduction to P. Since $\phi(P) \neq \Omega$ while $\phi((\mu\alpha.M')x) = \Omega$, it must be the case that:

$$(M)x \longrightarrow^* (\mu\alpha.M')x \longrightarrow_{fst} (\lambda y.\mu\alpha.M'[\alpha \Leftarrow y])x \longrightarrow^* P,$$

which is an instance of the first case above.

To see (2) we reason as for part (1), since e.g. we have to treat the case:

$$(M)\alpha \longrightarrow^* (\mu\beta.M')\alpha \longrightarrow_{\beta_S} M'[\beta := \alpha] \longrightarrow^* P.$$

There is however a third possibility, namely that:

$$(M)\alpha \longrightarrow^* (\lambda y.M')\alpha \longrightarrow^* P.$$

But then $P = (\lambda y.P')\alpha$ for some P' s.t. $M' \longrightarrow^* P'$, which implies that $\phi(P) = \Omega$, a case that has been treated at the beginning of the proof of (1). $\square$

Lemma 5.10.

1 M $\lambda\mu$-free & $M \in [\delta]_{\Gamma,\Delta} \Rightarrow M \in [\![\delta]\!]_{\Gamma,\Delta}$,

2 $M \in [\![\delta]\!]_{\Gamma,\Delta} \Rightarrow M \in [\delta]_{\Gamma,\Delta}$.

Proof. (1) and (2) are proved by simultaneous induction over δ. The base case $\delta = \varphi$ is obvious, as well as the case $\delta = \omega$. The case $\delta = \delta_1 \wedge \delta_2$ follows by induction. The only relevant case is when $\delta = \sigma \to \delta' \not\sim_T \omega$.

Part (1): let $S = [\ ]\vec{N}\beta \in [\![\sigma]\!]_{\Gamma,\Delta}$ be arbitrary, then the thesis is proved if we can show that $(M)S \in [\![\delta']\!]_{\Gamma,\Delta}$. If M is $\lambda\mu$-free then $(M)S$ is such. On the other hand we know that $\Gamma \vdash A : \sigma \to \delta' \mid \Delta$ for some $A \preceq \phi(M')$ with $M \longrightarrow^* M'$ (hence also M' is $\lambda\mu$-free), and it must be the case that $A \neq \Omega$ by (1) of Lemma 5.1, since $\sigma \to \delta' \not\sim_T \omega$. If $\vec{N} = N_1, \ldots, N_k$ we have that $\phi((M')S) = (\phi(M'))\phi(N_1)\cdots\phi(N_k)\beta$ since M' is $\lambda\mu$-free so that $\phi(M') \neq \Omega$; on the other hand $[\ ]\vec{N}\beta \in [\![\sigma]\!]_{\Gamma,\Delta}$ implies that $N_i \in [\![\delta_i]\!]_{\Gamma,\Delta}$ for $i = 1, \ldots, k$ where $\sigma \sim_S \delta_1 \times \cdots \times \delta_k \times \sigma'$ and $\Delta(\beta) \leq_S \sigma'$ by Lemma 3.3 (1). By induction hypothesis (2) there exist $A_1, \ldots, A_k$ such that $A_i \preceq \phi(N_i)$ and $\Gamma \vdash A_i : \delta_i \mid \Delta$ for all i. From $\Omega \neq A \preceq \phi(M')$ and the fact that M' is $\lambda\mu$-free, it follows that A itself is $\lambda\mu$-free and therefore we have that $(A)A_1\cdots A_k\beta \in \mathcal{A}((M')S) \subseteq \mathcal{A}((M)S)$, and $\Gamma \vdash (A)A_1\cdots A_k\beta : \delta' \mid \Delta$ that is $(M)S \in [\delta']_{\Gamma,\Delta}$. From this the desired $(M)S \in [\![\delta']\!]_{\Gamma,\Delta}$ follows by induction hypothesis (1).

Part (2): suppose that $M \in [\![\sigma \to \delta']\!]_{\Gamma,\Delta}$. Let $\sigma = \delta_1 \times \cdots \times \delta_k \times \omega$ and take $\Gamma' = \Gamma \wedge \{x_1 : \delta_1, \ldots, x_k : \delta_k\}$. Then by induction hypothesis (1) $x_i \in [\![\delta_i]\!]_{\Gamma',\Delta}$ for $i = 1, \ldots, k$, so that $S = [\ ]\vec{x}\beta \in [\![\sigma]\!]_{\Gamma',\Delta}$ since trivially $\Delta(\beta) \leq_S \omega$. By Def. 5.5 it follows that $(M)S \in [\![\delta']\!]_{\Gamma',\Delta}$ which implies, by induction hypothesis (2), that $(M)S \in [\delta']_{\Gamma',\Delta}$. From this we get the desired $M \in [\sigma \to \delta']_{\Gamma',\Delta}$ by repeated applications of Lemma 5.9.

$\square$

For $\vec{\alpha} = \alpha_1, \ldots, \alpha_m$ and $\vec{S} = S_1, \ldots, S_m$ vectors of stream variables and streams respectively, we abbreviate:

$$M[\vec{\alpha} \Leftarrow \vec{S}] = M[\alpha_1 \Leftarrow S_1] \ldots [\alpha_m \Leftarrow S_m].$$

Lemma 5.11. Let $\Gamma = \{x_1 : \delta_1, \ldots, x_n : \delta_n\}$ and $\Delta = \{\alpha_1 : \sigma_1, \ldots, \alpha_m : \sigma_m\}$. Suppose that

$$N_i \in [\![\delta_i]\!]_{\Gamma_i, \Delta_i} \quad i = 1, \ldots, n, \;\; \& \;\; S_j \in [\![\sigma_j]\!]_{\Gamma'_j, \Delta'_j} \quad j = 1, \ldots, m.$$

where $dom(\Gamma_i) \cap dom(\Gamma) = \emptyset$ and $dom(\Delta_j) \cap dom(\Delta) = \emptyset$ for all i, j. Then

$$\Gamma \vdash M : \delta \mid \Delta \Rightarrow \widehat{M} \in [\![\delta]\!]_{\widehat{\Gamma}, \widehat{\Delta}}$$

where $\widehat{\Gamma} = (\bigwedge_{i=1}^n \Gamma_i) \wedge (\bigwedge_{j=1}^m \Gamma'_j)$, $\widehat{\Delta} = (\bigwedge_{i=1}^n \Delta_i) \wedge (\bigwedge_{j=1}^m \Delta'_j)$ and

$$\widehat{M} = M[\vec{x} := \vec{N}][\vec{\alpha} \Leftarrow \vec{S}].$$

Remark 5.12. Observe that $dom(\widehat{\Gamma}) \cap dom(\Gamma) = \emptyset$ which is in accordance with the fact that we can freely assume that $\vec{x} \cap fv(\vec{N}) = \emptyset$ so that $\vec{x} \cap fv(M[\vec{x} := \vec{N}]) = \emptyset$. Also it is the case that $dom(\widehat{\Delta}) \cap dom(\Delta) = \emptyset$ and we assume that $\vec{\alpha} \cap fv(\vec{S}) = \emptyset$ so that $\vec{\alpha} \cap fv(\widehat{M}) = \emptyset$.

Proof. By induction over the derivation $\mathcal{D}$ of $\Gamma \vdash M : \delta \mid \Delta$. If $\mathcal{D}$ ends by (Ax) or (ω) the thesis is obvious. If $\mathcal{D}$ ends by $(\wedge)$ the thesis follows by the induction hypothesis. If $\mathcal{D}$ ends by $(\leq)$ the thesis follows by the induction hypothesis and Lemma 5.6. It remains to consider the cases in which $\mathcal{D}$ ends by (λ), (App), (μ) and (S). Below we abbreviate $M'' = M'[\vec{x} := \vec{N}][\vec{\alpha} \Leftarrow \vec{S}]$ and $N'' = N'[\vec{x} := \vec{N}][\vec{\alpha} \Leftarrow \vec{S}]$.

(λ): then $M = \lambda y.M'$ and the conclusion $\Gamma \vdash \lambda y.M' : \delta' \times \sigma \to \delta'' \mid \Delta$ has been derived from the premise $\Gamma, y : \delta' \vdash M' : \sigma \to \delta'' \mid \Delta$. For any Γ', Δ' and arbitrary $P \in [\![\delta']\!]_{\Gamma', \Delta'}$ we have:

$$\begin{array}{lll} & \widehat{M'} = M''[y := P] \in [\![\sigma \to \delta'']\!]_{\widehat{\Gamma}, \widehat{\Delta}} & \text{by ind. hyp.} \\ \Rightarrow & \forall S \in [\![\sigma]\!]_{\Gamma'', \Delta''}.\ (M''[y := P])S \in [\![\delta'']\!]_{\widehat{\Gamma} \wedge \Gamma'', \widehat{\Delta} \wedge \Delta''} & \text{by Def. 5.5} \\ \Rightarrow & \forall S \in [\![\sigma]\!]_{\Gamma'', \Delta''}.\ (\lambda y.M'')(P :: S) \in [\![\delta'']\!]_{\widehat{\Gamma} \wedge \Gamma'', \widehat{\Delta} \wedge \Delta''} & \text{by Lemma 5.8} \\ \Rightarrow & \widehat{\lambda y.M'} = \lambda y.M'' \in [\![\delta' \times \sigma \to \delta'']\!]_{\widehat{\Gamma}, \widehat{\Delta}} & \text{by Def. 5.5} \end{array}$$

since $P \in [\![\delta']\!]_{\Gamma', \Delta'}$ and $S \in [\![\sigma]\!]_{\Gamma'', \Delta''}$ imply $(P :: S) \in [\![\delta' \times \sigma]\!]_{\Gamma' \wedge \Gamma'', \Delta' \wedge \Delta''}$, and $\widehat{\Gamma} \leq \Gamma' \wedge \Gamma''$ and $\widehat{\Delta} \leq \Delta' \wedge \Delta''$.

(App): then $M = (M')N'$ and the conclusion $\Gamma \vdash (M')N' : \sigma \to \delta'' \mid \Delta$ is derived from the premises $\Gamma \vdash M' : \delta' \times \sigma \to \delta'' \mid \Delta$ and $\Gamma \vdash N' : \delta' \mid \Delta$. Then $\widehat{M} = (M'')N''$ (with M'' and N'' defined above) and, for arbitrary Γ', Δ':

$$\begin{array}{ll} N'' \in [\![\delta']\!]_{\widehat{\Gamma}, \widehat{\Delta}} \;\&\; \forall S \in [\![\delta' \times \sigma]\!]_{\Gamma', \Delta'}.\ (M'')S \in [\![\delta']\!]_{\widehat{\Gamma} \wedge \Gamma', \widehat{\Delta} \wedge \Delta'} & \text{by ind. hyp.} \\ \Rightarrow \forall S' \in [\![\sigma]\!]_{\Gamma', \Delta'}.\ (M'')N''S' = (M'')(N'' :: S') \in [\![\delta'']\!]_{\widehat{\Gamma} \wedge \Gamma', \widehat{\Delta} \wedge \Delta'} & \text{by Def. 5.5} \\ \Rightarrow (M'')N'' \in [\![\sigma \to \delta'']\!]_{\widehat{\Gamma}, \widehat{\Delta}} & \text{by Def. 5.5} \end{array}$$

since $N'' \in [\![\delta']\!]_{\widehat{\Gamma}, \widehat{\Delta}}$ and $S' \in [\![\sigma]\!]_{\Gamma', \Delta'}$ imply $(N'' :: S') \in [\![\delta' \times \sigma]\!]_{\widehat{\Gamma} \wedge \Gamma', \widehat{\Delta} \wedge \Delta'}$.

(μ): then $M = \mu\alpha.M'$ and the conclusion $\Gamma \vdash \mu\alpha.M' : \sigma \to \delta' \mid \Delta$ has been derived from the premise $\Gamma \vdash M' : \delta' \mid \alpha : \sigma, \Delta$. For any Γ', Δ', let $S \in [\![\sigma]\!]_{\Gamma',\Delta'}$ be arbitrary, then:

$$\begin{array}{lll} & \widehat{M'} = M''[\alpha \Leftarrow S] \in [\![\delta]\!]_{\widehat{\Gamma}\wedge\Gamma',\widehat{\Delta}\wedge\Delta'} & \text{by ind. hyp.} \\ \Rightarrow & (\mu\alpha.M'')S \in [\![\delta]\!]_{\widehat{\Gamma}\wedge\Gamma',\widehat{\Delta}\wedge\Delta'} & \text{by Lemma 5.8} \\ \Rightarrow & \widehat{\mu\alpha.M'} = \mu\alpha.M'' \in [\![\sigma \to \delta]\!]_{\widehat{\Gamma},\widehat{\Delta}} & \text{by Def. 5.5} \end{array}$$

(S): then $M = (M')\alpha$ and the conclusion $\Gamma \vdash (M')\alpha : \delta' \mid \alpha : \sigma, \Delta'$ (where $\Delta = \alpha : \sigma, \Delta'$ so that $\alpha = \alpha_j$ for some j) has been obtained from the premise $\Gamma \vdash M' : \sigma \to \delta' \mid \alpha : \sigma, \Delta'$. For any Γ', Δ', let $S = [\ ]\vec{P}\beta \in [\![\sigma]\!]_{\Gamma',\Delta'}$ be arbitrary, then we have:

$$\begin{array}{lll} & \widehat{M'} = M''[\alpha \Leftarrow S] \in [\![\sigma \to \delta']\!]_{\widehat{\Gamma}\wedge\Gamma',\widehat{\Delta}\wedge\Delta'} & \text{by ind. hyp.} \\ \Rightarrow & \widehat{(M')\alpha} = ((M'')\alpha)[\alpha \Leftarrow S] = & \\ & \quad = (M''[\alpha \Leftarrow S])\vec{P}\beta = (M''[\alpha \Leftarrow S])S \in [\![\delta']\!]_{\widehat{\Gamma}\wedge\Gamma',\widehat{\Delta}\wedge\Delta'} & \text{by Def. 5.5} \end{array}$$

where $\widehat{\Gamma}\wedge\Gamma' = \widehat{\Gamma}$ since $\widehat{\Gamma} \leq \Gamma'$, and similarly $\widehat{\Delta}\wedge\Delta = \widehat{\Delta}$.

□

Theorem 5.13 (Approximation Theorem). For all $M \in \Sigma^c_{\Lambda\mu}$ and $\delta \in \mathcal{T}_T$:

$$\Gamma \vdash M : \delta \mid \Delta \iff \exists\, A \in \mathcal{A}(M).\ \Gamma \vdash A : \delta \mid \Delta.$$

Proof. The implication ($\Leftarrow$) is Lemma 5.3. To see ($\Rightarrow$) assume $\Gamma \vdash M : \delta \mid \Delta$ with $\Gamma = \{x_1 : \delta_1, \ldots, y_n : \delta_n\}$ and $\Delta = \{\alpha_1 : \sigma_1, \ldots, \alpha_m : \sigma_m\}$. Now $y_i \in [\delta_i]_{\{y_i:\delta_i\},\emptyset}$ for all $i = 1, \ldots, n$ because $\phi(y_i) = y_i$; then by (1) of Lemma 5.10 $y_i \in [\![\delta_i]\!]_{\{y_i:\delta_i\},\Delta}$ being $\lambda\mu$-free.

On the other hand $[\ ]\beta_j \in [\![\sigma_j]\!]_{\emptyset,\{\beta_j:\sigma_j\}}$, for $j = 1, \ldots, m$, by Def. 5.5. Therefore, by Lemma 5.11, $\Gamma \vdash M : \delta \mid \Delta$ implies $\widehat{M} \in [\![\delta]\!]_{\widehat{\Gamma},\widehat{\Delta}}$ where $\widehat{\Gamma} = \{y_1 : \delta_1, \ldots, y_n : \delta_n\}$, $\widehat{\Delta} = \{\beta_1 : \sigma_1, \ldots, \beta_m : \sigma_m\}$ and $\widehat{M} = M[\vec{x} := \vec{y}][\vec{\alpha} \Leftarrow \vec{[\ ]\beta}] = M[\vec{x} := \vec{y}][\vec{\alpha} := \vec{\beta}]$. Since Def. 5.5 does not depend on the choice of variable names, it follows that $M \in [\![\delta]\!]_{\Gamma,\Delta}$, and we conclude that $M \in [\delta]_{\Gamma,\Delta}$ by (2) of Lemma 5.10.

□

Conclusions

We have provided an intersection type assignment system for $\Lambda\mu$, extending the system in (van Bakel *et al.* 2011), that satisfies all the relevant properties of its homologous intersection type systems for the λ-calculus. It can be argued that the system is a conservative extension of that one in (Barendregt *et al.* 1983). The delicate though fundamental property stated in the approximation theorem also is preserved, and its proof naturally extends techniques and arguments from the λ-calculus.

We think that the use of type theoretic techniques can be useful also to overcome technical difficulties in the study of the $\Lambda\mu$-calculus, in particular when treating open terms in the proof of statements about closed terms only. Possible developments are in the study of delimited control in the line of (Herbelin *et al.* 2008; Saurin 2010b).

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