Multiple homoclinic solutions for a one-dimensional Schrodinger equation

DOI:10.3934/dcdss.2016040

Multiple homoclinic solutions for a one-dimensional Schrödinger equation*

Walter Dambrosio†   Duccio Papini‡

Abstract

In this paper we study the problem of the existence of homoclinic solutions to a Schrödinger equation of the form

$$x'' - V(t)x + x^3 = 0,$$

where is a stepwise potential. The technique of proof is based on a topological method, relying on the properties of the transformation of continuous planar paths (the S.A.P. method), together with the application of the classical Conley-Ważewski's method.

AMS-Subject Classification. 34C25; 34C28; 34C37.
Keywords. Schrödinger equation; homoclinic solutions.

1 Introduction

Let us consider the equation

$$x'' - V(t)x + x^3 = 0, \tag{1.1}$$

where $V : \mathbb{R} \to \mathbb{R}$ is a L^∞-function; we assume that there exist $0 < V_1 < V_2$ and $t_0 < t_1 < \ldots < t_{2K} < t_{2K+1}$ such that

$$\begin{aligned} & V_1 \leq V(t) \leq V_2, \quad \text{for a.e. } t \in \mathbb{R}, \\ & V(t) = V_1, \quad \forall\, t \in [t_{2j}, t_{2j+1}], \quad j = 0, \ldots, K, \\ & V(t) = V_2, \quad \forall\, t \in [t_{2j+1}, t_{2j+2}], \quad j = 0, \ldots, K-1. \end{aligned} \tag{1.2}$$

*Under the auspices of GNAMPA-INdAM: the authors are supported by the GNAMPA-INdAM project"Equazioni Differenziali Ordinarie sulla retta reale". The first author is supported by the P.R.I.N. Project "Variational and perturbative aspects of nonlinear differential problems".

†Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy. e-mail: `walter.dambrosio@unito.it`

‡Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze 206, 33100 Udine, Italy. e-mail: `duccio.papini@uniud.it`

The equation (1.1) is the very classical one-dimensional Schrödinger equation, which comes from the study of stationary waves of a nonlinear Schrödinger equation in $\mathbb{R}^N$; there is a huge literature on this equation and several different results have been proved (see for instance, among the others, [1], [5], [8], [20]). Very recently, some results on the lines of our approach have been proved in [24]; indeed, in the quoted paper the authors study the existence of chaotic dynamics for the equation (1.1), using the S.A.P. method.

We will be concerned with the existence of solutions of (1.1) homoclinic to the equilibrium point $(0,0)$ (in the phase-plane) and having some prescribed behavior - in terms of the number of zeros of the derivative - in the intervals $[t_i, t_{i+1}]$, $i = 0, \dots, 2K$. Many papers can be found in the literature on this topic; we only mention [10, 15, 17] and the very recent [2, 6, 9]. In particular, the results in [2, 6] are exactly in the same spirit of the one of the present paper, but they are concerned with a different form of equations. For related results on this direction see also [3].

We will able to prove the following result:

Theorem 1.1. *There exists $S^* > 0$ and for any integer $M \geq 1$, there exists $T^* = T^*(M) > 0$, such that if*

$$\begin{aligned} &t_1 - t_0 \geq S^*, \\ &t_{j+1} - t_j \geq T^*, \quad j = 1, \dots, 2K-1, \\ &t_{2K+1} - t_{2K} \geq S^*, \end{aligned} \tag{1.3}$$

there exist M^{2K-1} geometrically distinct globally defined positive solutions u of (1.1) *such that*

$$\lim_{t \to \pm\infty} (u(t), u'(t)) = (0,0).$$

We point out that the same result can be proved also for a more general equation of the form

$$x'' - V(t)x + f(x) = 0,$$

where $f(x) = xh(x)$, for every $x \in \mathbb{R}$, and $h \in C^1(\mathbb{R};\mathbb{R})$ is such that $h(0) = 0$, $h'(x) > 0$ for every $x > 0$ and

$$\lim_{x \to +\infty} h(x) = +\infty$$

(cfr. [24]).

The proof of Theorem 1.1 is based on the combination of the Conley-Ważewski's method (see [4, 23]) and of the so-called S.A.P. method (see [16, 17, 19]); the technique is a slight variant of the ones already used in [2, 6]. More precisely, we first show the existence of the stable and unstable manifolds of the equilibrium point $(0, 0)$; then we are able to connect these two sets by suitable orbits of (2.1) (coming from the application of the S.A.P. method), solving a kind of generalized Sturm-Liouville problem for (1.1). The crucial point in the present situation is that we need an approximation procedure in order to replace the stable and unstable manifolds (which are in general merely continua) by images of continuous paths.

We observe that the problem of connecting continuous paths can be found already in the paper [22]; more recently, it has been considered in [7, 12, 13, 15, 21].

The plan of the paper is the following. In Section 2, we prove the existence of stable and unstable manifolds to the equilibrium $(0, 0)$; in Section 3 we study the dynamics of the equation on bounded intervals, while in Section 4 we prove our result.

Throughout the paper by a *path* or *curve* γ we mean a continuous map $\gamma : [0, 1] \to \mathbb{R}^2$. We will make a systematic abuse of notation by using the same symbol γ to stand for the curve and for its image $\gamma([0, 1])$.

2 Continua of asymptotic solutions

In this section we prove the existence of stable and unstable manifolds to the equilibria of the nonautonomous planar system

$$\begin{cases} x' = y \\ y' = V(t)x - x^3, \end{cases} \tag{2.1}$$

with $V : \mathbb{R} \to \mathbb{R}$ a locally integrable function. We use the notation $z(t; p, t_0)$ for the (unique) solution to (2.1) satisfying the condition $z(t_0) = p$. Moreover, we define

$$E_\mu(x, y) = \frac{1}{2}y^2 - \frac{1}{2}\mu x^2 + \frac{1}{4}x^4, \quad \forall\ (x, y) \in \mathbb{R}^2, \quad \mu > 0;$$

let us observe that E_μ is the energy associated to the autonomous system

$$\begin{cases} x' = y \\ y' = \mu x - x^3, \end{cases}$$

Finally, for every $V_1 < V_2$, $V_1, V_2 \in \mathbb{R}$, let

$$\begin{aligned} \mathcal{T}_- &= \{(x, y) \in \mathbb{R}^2 :\ E_{V_2}(x, y) \le 0 \le E_{V_1}(x, y),\ x \in [0, \sqrt{2V_1}],\ y \ge 0\} \\ \mathcal{T}_+ &= \{(x, y) \in \mathbb{R}^2 :\ E_{V_2}(x, y) \le 0 \le E_{V_1}(x, y),\ x \in [0, \sqrt{2V_1}],\ y \le 0\}, \end{aligned} \tag{2.2}$$

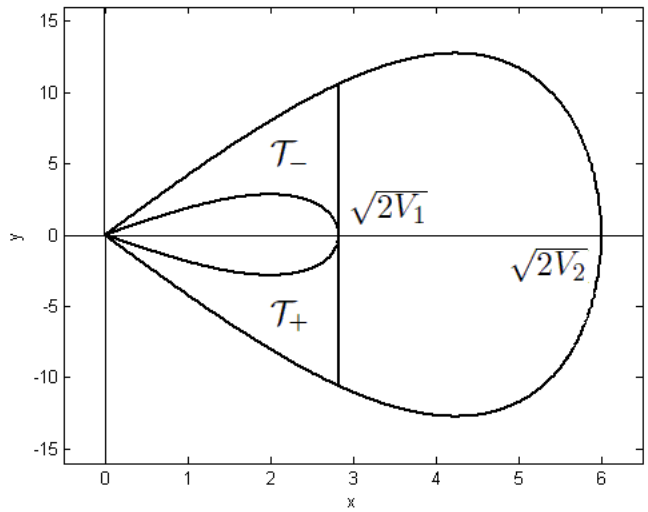

Figure 1: A picture of the two sets $\mathcal{T}_-$ and $\mathcal{T}_+$ when $V_1 = 4$ and $V_2 = 18$.

which are drown in figure 1.
We are able to prove the following result:

Theorem 2.1. *The following statements hold true.*

1. Assume that

$$0 < V_1 \leq V(t) \leq V_2, \quad \text{for a.e. } t \in (-\infty, t_0],$$

for some $t_0 \in \mathbb{R}$, $V_1, V_2 \in \mathbb{R}$. *Then there exists a continuum* $\Gamma_{-\infty} \subset \mathbb{R}^2$, *with* $\Gamma_{-\infty} \subset \mathcal{T}_-$, $(0,0) \in \Gamma_{-\infty}$, $\Gamma_{-\infty} \cap \{\sqrt{2V_1}\} \times \mathbb{R} \neq \emptyset$ *and such that for any* $p \in \Gamma_{-\infty}$ *it holds that*

$$\lim_{t \to -\infty} z(t; p, t_0) \to (0,0).$$

2. Assume that

$$0 < V_1 \leq V(t) \leq V_2, \quad \text{for a.e. } t \in [t_0, +\infty),$$

for some $t_0 \in \mathbb{R}$, $V_1, V_2 \in \mathbb{R}$. *Then there exists a continuum* $\Gamma_{+\infty} \subset \mathbb{R}^2$, *with* $\Gamma_{+\infty} \subset \mathcal{T}_+$, $(0,0) \in \Gamma_{+\infty}$, $\Gamma_{+\infty} \cap \{\sqrt{2V_1}\} \times \mathbb{R} \neq \emptyset$ *and such that for any* $p \in \Gamma_{+\infty}$, *it holds that*

$$\lim_{t \to +\infty} z(t; p, t_0) \to (0,0).$$

Proof. We observe that Statement 2 is a consequence of Statement 1 and the symmetry enjoyed by the vector field in (2.1) with respect to the axis $y = 0$.
Concerning the proof of Statement 1, let us fix $\epsilon > 0$ such that

$$0 < V_1 - \epsilon$$

and define

$$\mathcal{T}_\epsilon = \{(x, y) \in \mathbb{R}^2 : \; E_{V_2+\epsilon}(x, y) \leq 0 \leq E_{V_1-\epsilon}(x, y), \; x \in [0, \sqrt{2(V_1 - \epsilon)}], \; y \geq 0\}.$$

We first prove that any solution $z(t) = (x(t), y(t))$ which remains in $\mathcal{T}_\epsilon$ for every $t \leq t_0$ has to satisfy

$$\lim_{t \to -\infty} z(t) = (0, 0).$$

Indeed, for such a solution we have that, for $t \leq t_0$,

$$x(t) \in \left(0, \sqrt{2(V_1 - \epsilon)}\right], \qquad x'(t) > 0,$$

and, hence, there exists

$$\lim_{t \to -\infty} x(t) = L < \sqrt{2(V_1 - \epsilon)}.$$

If $L > 0$ then there exists $t_1 \leq t_0$ such that

$$L \leq x(t) \leq x(t_1) < \sqrt{2(V_1 - \epsilon)}, \quad \forall \, t \leq t_1.$$

As a consequence, since $(x(t), x'(t)) \in \mathcal{T}_\epsilon$ for every $t \leq t_1$, this implies that

$$x'(t) \geq c > 0, \quad \forall \, t \leq t_1,$$

with

$$c = \min\left\{\sqrt{(V_1 - \epsilon)L^2 - L^4/2}, \sqrt{(V_1 - \epsilon)x(t_1)^2 - x(t_1)^4/2}\right\}.$$

This contradicts the fact that L is finite.
Finally, since $z(t) \in \mathcal{T}_\epsilon$ for every $t \leq t_0$, the fact that $x(t) \to 0$ for $t \to -\infty$ implies that $x'(t) \to 0$ for $t \to -\infty$.

Now we write system (2.1) as an autonomous system in $\mathbb{R}^3$:

$$\begin{cases} x' = y \\ y' = \mu x - x^3 \\ t' = 1 \end{cases} \tag{2.3}$$

and let $\pi(\cdot; s_0, P)$ be the unique solution to (2.3) starting from $P \in \mathbb{R}^3$ at $s = s_0$. We set

$$W = \mathcal{T}_\epsilon \times (-\infty, t_0], \quad U = (\gamma_- \setminus \{(0,0)\}) \times (-\infty, t_0], \quad V = (\gamma_+ \setminus \{(0,0)\}) \times (-\infty, t_0],$$

where γ_- and γ_+ are the portions of the boundary of $\mathcal{T}_\epsilon$ lying on $E_{V_1-\epsilon} = 0$ and $E_{V-2+\epsilon} = 0$, respectively.

Let us study the behavior of the vector field associated with system (2.3) at any point $P = (x_1, y_1, t_1) \in \partial W$. First, if $(x_1, y_1) = (0,0)$, it is clear that $\pi(s; t_1, P) = (0, 0, s)$ for all s. Next, if $P \in U \cup V$, the vector field points strictly inwards W; therefore, $\pi(s; t_1, P) \notin W$ for all s in a left neighborhood of t_1. Finally, if either

$$(x_1, y_1) \in \partial\mathcal{T}_\epsilon \setminus (\gamma_- \cup \gamma_+ \cup \{(0,0)\})$$

or

$$(x_1, y_1) \in \mathcal{T}_\epsilon \setminus (\gamma_- \cup \gamma_+ \cup \{(0,0)\}) \quad \text{and} \quad t_1 = 0$$

then $\pi(s; t_1, P) \in W$ for all s in a left neighborhood of t_1, since in those points the vector field of (2.3) points strictly outwards W.

Following the terminology introduced in [4], a point $P \in \partial W$ is called an entry point in $W$for the flow π if $\pi(s; 0, P) \notin W$ for all $s \in [-\epsilon, 0)$ and some $\epsilon > 0$; moreover it is called a strict entry point if in addition $\pi(s; 0, \epsilon) \in \overset{\circ}{W}$ for all $s \in (0, \epsilon]$ and some $\epsilon > 0$. The analysis carried above shows that all the entry points in W are strict and that their set is $U \cup V$.

Now, we consider the set $D \subset W$ given by

$$D = \{P = (x_1, y_1, t_1) \in W : \exists s < t_1 \,\text{s.t.}\, \pi(s; t_1, P) \notin W\}$$

and the map

$$\Phi : D \to \partial W$$

such that $\Phi(P)$ is the last entry point in W before P for the solution of (2.3) starting from the point P, namely $\Phi(P) = \pi(s^*; t_1, P)$ where

$$s^* = \sup\{s < t_1 : \pi(s; t_1, P) \notin D\}.$$

Using the fact that all entry points in W are strict, it follows from the results in [4] that Φ is continuous on D; moreover, by the previous discussion, $\Phi(D) = U \cup V$ is not connected and U and V are its connected components.

Let $\gamma : [0,1] \to \Sigma_\epsilon$ be a continuous path such that $\gamma(0) \in \gamma_- \setminus \{(0,0)\}$ and $\gamma(1) \in \gamma_+ \setminus \{(0,0)\}$. Then we have that $\Phi(\gamma(0)) \in U$, $\Phi(\gamma(1)) \in V$. Since $[0,1]$ is connected, there must be $\tau \in (0,1)$ such that $\gamma(\tau) \notin D$. By the topological lemma [21, Corollary 6] there exists a continuum $\Gamma_{-\infty} \subset \mathcal{T}_\epsilon \setminus D$ such that

$$(0,0) \in \Gamma_{-\infty} \quad \text{and} \quad \Gamma_{-\infty} \cap \left(\{\sqrt{2(V_1-\epsilon)}\} \times \mathbb{R} \right) \neq \emptyset.$$

Letting $\epsilon \to 0^+$, it is actually possible to show that $\Gamma_{-\infty} \subset \mathcal{T}_-$ and reaches the vertical line $x = \sqrt{2V_1}$ (see [11, Theorem 6, §47, II, p.171]). $\square$

3 Dynamics on bounded intervals

In this section, we fix two real positive constants $V_1 < V_2$ and we study the dynamics associated to the autonomous system

$$\begin{cases} x' = y \\ y' = V_i x - x^3, \quad i = 1, 2, \end{cases} \tag{3.1}$$

which will be referred to as (V_i).
For $T \in \mathbb{R}$ and $i = 1, 2$, we define the map Ψ_i^T as the Poincaré map associated with system (V_i) from $t = 0$ to $t = T$, i.e.

$$\Psi_i^T(x_0, y_0) = (x(T; x_0, y_0), y(T; x_0, y_0)), \quad \forall\, (x_0, y_0) \in \mathbb{R}^2,$$

where $(x(\cdot; x_0, y_0), y(\cdot; x_0, y_0))$ is the unique solution to (V_i) satisfying the initial condition $(x(0), y(0)) = (x_0, y_0)$. Observe that $(\Psi_i^T)^{-1} = \Psi_i^{-T}$.
It is well known that the orbits of (V_i) with negative energy E_{V_i} are periodic and turn around the equilibrium $(\sqrt{V_i}, 0)$ in clockwise sense. Hence it is useful to introduce a pair of angular coordinates θ_i, for $i = 1, 2$, which are centered at $(\sqrt{V_i}, 0)$ and are measured in clockwise sense starting from the half line $\left[\sqrt{V_i}, +\infty\right) \times \{0\}$. In particular we denote by $\theta(t; (x_0, y_0))$ the angular coordinate of the solution of (V_i) starting at time $t = 0$ from the point $(x_0, y_0) \neq (\sqrt{V_i}, 0)$ with the choice $\theta(0; (x_0, y_0)) \in [-\pi, \pi)$.

Let us now consider the points $(\sqrt{2V_1}, \pm\sqrt{2V_1(V_2 - V_1)})$ which are two vertices of the triangular regions $\mathcal{T}_\pm$ lying where the homoclinic orbit of (V_2) crosses the vertical line $x = \sqrt{2V_1}$. They also lie on the same orbit of (V_1), namely the one with energy $E_{V_1}(x, y) = V_1(V_2 - V_1)$, which crosses the positive x-axis at the abscissa $\hat{x} = \sqrt{V_1 + \sqrt{V_1(4V_2 - 3V_1)}} \in (\sqrt{2V_1}, \sqrt{2V_2})$. We observe that each solution of (V_1) that starts from a point in $\mathcal{T}_-$ crosses the positive x-axis at a point lying between $(\sqrt{2V_1}, 0)$ and $(\hat{x}, 0)$, and the same holds for the solutions of (V_1) arriving at a point in $\mathcal{T}_+$.

We set

$$\mathcal{A}_2 = \{(x, y) \in \mathbb{R}^2 : \ c_1 \leq E_{V_2}(x, y) \leq c_2, \ x > 0\}, \tag{3.2}$$

with c_1, c_2 fixed in the following way:

1. $c_1 < c_2 < 0$;
2. $c_1 > \max\{E_{V_2}(\sqrt{2V_1}, 0), E_{V_2}(\hat{x}, 0)\}$.

We observe that condition 1. guarantees that $\mathcal{A}_2$ is filled up by periodic orbits of system (V_2), that surround the segment $[\sqrt{2V_1}, \hat{x}] \times \{0\}$ by condition 2. We denote by $\partial^e \mathcal{A}_2$ and $\partial^i \mathcal{A}_2$ the outer ($E_{V_2} = c_2$) and inner ($E_{V_2} = c_1$) boundary of $\mathcal{A}_2$, respectively. In particular, $\partial^i \mathcal{A}_2$ and $\partial^e \mathcal{A}_2$ cross the segment $(0, \sqrt{V_2}) \times \{0\}$ on the x-axis at the point $(x'_i, 0)$ and $(x'_e, 0)$, respectively, which both lie inside the region bounded by the homoclinic orbit of (V_1) by condition 2, i.e. they satisfy $E_{V_1}(x'_i, 0), E_{V_1}(x'_e, 0) < 0$.

Now, let

$$\mathcal{A}_1 = \{(x, y) \in \mathbb{R}^2 : \ c_3 \le E_{V_1}(x, y) \le 0, \ x \ge 0\},$$

where $0 > c_3 > \max\{E_{V_1}(x'_e, 0), E_{V_1}(x'_i, 0)\}$; by these choices, the internal boundary $\partial^i \mathcal{A}_1$ of the annulus $\mathcal{A}_1$ (i.e. the level set $E_{V_1} = c_3$) surrounds the segment $[x'_e, x'_i] \times \{0\}$ on the positive x-axis and, hence, the set

$$\mathcal{A}_1 \cap \mathcal{A}_2$$

is non-empty and it has exactly two connected components. We denote by $\mathcal{R}_1$ the connected component contained in $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ and by $\mathcal{R}_2$ the other one (see Figure 2). Moreover, we name explicitly some of the "sides" of $\mathcal{R}_1$ and $\mathcal{R}_2$:

$$\begin{aligned} \mathcal{R}_1^l &= \partial \mathcal{R}_1 \cap \partial^e \mathcal{A}_2 \qquad & \mathcal{R}_1^r &= \partial \mathcal{R}_1 \cap \partial^i \mathcal{A}_2 \\ \mathcal{R}_2^l &= \partial \mathcal{R}_2 \cap \partial^e \mathcal{A}_1 \qquad & \mathcal{R}_2^r &= \partial \mathcal{R}_2 \cap \partial^i \mathcal{A}_1 \end{aligned}$$

where the superscript l, r stands for "left" and "right", respectively. Setting $\mathcal{R}_1^- = \mathcal{R}_1^l \cup \mathcal{R}_1^r$ and $\mathcal{R}_2^- = \mathcal{R}_2^l \cup \mathcal{R}_2^r$, we obtain that each of the pairs

$$\widetilde{\mathcal{R}}_1 = (\mathcal{R}_1, \mathcal{R}_1^-), \quad \widetilde{\mathcal{R}}_2 = (\mathcal{R}_2, \mathcal{R}_2^-)$$

is an *oriented rectangle*, according to [16, 18].

Oriented rectangles are crucial in the following definition (see again [16, 18]).

Definition 3.1. *Let* $\widetilde{\mathcal{R}}_1 = (\mathcal{R}_1, \mathcal{R}_1^-)$, $\widetilde{\mathcal{R}}_2 = (\mathcal{R}_2, \mathcal{R}_2^-)$ *be oriented rectangles and let* $\Psi : \mathcal{D}_\Psi \subset \mathbb{R}^2 \to \mathbb{R}^2$ *be a continuous map.*

We say that $(\mathcal{H}, \Psi)$ *stretches* $\widetilde{\mathcal{R}}_1$ *to* $\widetilde{\mathcal{R}}_2$ *along the paths and write*

$$(\mathcal{H}, \Psi) : \widetilde{\mathcal{R}}_1 \Leftrightarrow\!\!\!\!\to \widetilde{\mathcal{R}}_2$$

if $\mathcal{H} \subset \mathcal{R}_1 \cap \mathcal{D}_\Psi$ *is a compact subset and for every continuous path* $\gamma : [0, 1] \to \mathcal{R}_1$ *such that* $\gamma(0) \in \mathcal{R}_1^l$ *and* $\gamma(1) \in \mathcal{R}_1^r$ *there exists a sub-interval* $[s', s''] \subset [0, 1]$ *such that* $\gamma([s', s'']) \subset \mathcal{H}$, $\Psi(\gamma([s', s''])) \subset \mathcal{R}_2$ *and* $\Psi(\gamma(s')) \in \mathcal{R}_2^l$, $\Psi(\gamma(s'')) \in \mathcal{R}_2^r$ *(or viceversa).*

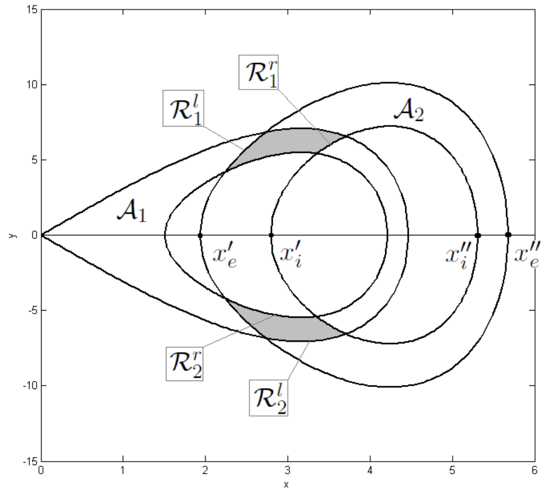

Figure 2: The shadowed regions are the two topological rectangles whose orientation is given by choosing their "left" and "right" sides. Here we have chosen $V_1 = 10$, $V_2 = 18$, $c_1 = -55$, $c_2 = -30$, $c_3 = -10$.

We say that Ψ stretches $\widetilde{\mathcal{R}}_1$ to $\widetilde{\mathcal{R}}_2$ along the paths with crossing number $M \geq 1$ and write

$$\Psi : \widetilde{\mathcal{R}}_1 \Leftrightarrow\!\!\!\!\!\longrightarrow^{M} \widetilde{\mathcal{R}}_2$$

if there exist M pairwise disjoint compact sets $\mathcal{H}_1, \ldots, \mathcal{H}_M \subset \mathcal{R}_1 \cap \mathcal{D}_\Psi$ such that $(\mathcal{H}_i, \Psi) : \widetilde{\mathcal{R}}_1 \Leftrightarrow\!\!\!\!\!\longrightarrow \widetilde{\mathcal{R}}_2$ for $i = 1, \ldots, M$.

As an easy consequence of the definition, we have that the stretching property has a good behavior with respect to compositions of maps.

Proposition 3.2. *Let $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{C}}$ be oriented rectangles and Φ, Ψ be suitable continuous maps such that $(\mathcal{H}, \Psi) : \widetilde{\mathcal{A}} \Leftrightarrow\!\!\!\!\!\longrightarrow \widetilde{\mathcal{B}}$ and $(\mathcal{K}, \Phi) : \widetilde{\mathcal{B}} \Leftrightarrow\!\!\!\!\!\longrightarrow \widetilde{\mathcal{C}}$ for some compact sets $\mathcal{H} \subseteq \mathcal{A}$ and $\mathcal{K} \subseteq \mathcal{B}$. Then the set $\mathcal{H} \cap \Psi^{-1}(\mathcal{K})$ is a non-empty compact set and*

$$(\mathcal{H} \cap \Psi^{-1}(\mathcal{K}), \Phi \circ \Psi) : \widetilde{\mathcal{A}} \Leftrightarrow\!\!\!\!\!\longrightarrow \widetilde{\mathcal{C}}.$$

In particular, if Ψ stretches $\widetilde{\mathcal{A}}$ to $\widetilde{\mathcal{B}}$ with crossing number M and Φ stretches $\widetilde{\mathcal{B}}$ to $\widetilde{\mathcal{C}}$ with crossing number N, then the composition $\Phi \circ \Psi$ stretches $\widetilde{\mathcal{A}}$ to $\widetilde{\mathcal{C}}$ with crossing number $M \times N$.

The following crucial result on the stretching can be proved arguing as in [24, Sect. 3]:

Proposition 3.3. *For any $M \in \mathbb{N}$, there exist $T_*(M) > 0$ such that, for every $T_1 > T_*(M)$ and $T_2 > T_*(M)$, we have*

$$\Psi_2^{T_2} : \widetilde{\mathcal{R}}_1 \Longleftrightarrow^M \widetilde{\mathcal{R}}_2, \qquad \Psi_1^{T_1} : \widetilde{\mathcal{R}}_2 \Longleftrightarrow^M \widetilde{\mathcal{R}}_1.$$

Proof. The proof is essentially contained in [24, Sect. 3]. We only highlight some key points of the argument for $\Psi_2^{T_2} : \widetilde{\mathcal{R}}_1 \Longleftrightarrow^M \widetilde{\mathcal{R}}_2$ which will be useful in the sequel.

We observe that $\Psi_2^{T_2}$ moves the points of $\mathcal{R}_1$ along the periodic orbits of system (V_2) which fill up the annulus $\mathcal{A}_2$ and that the period τ_i of the orbit on $\partial^i\mathcal{A}_2$ is smaller than the period τ_e of the orbit on $\partial^e\mathcal{A}_2$. Hence, if we choose T_2 such that

$$T_2 \geq (M+2)\frac{\tau_i\tau_e}{\tau_e - \tau_i}, \tag{3.3}$$

any path $\gamma : [0,1] \to \mathcal{R}_1$ with $\gamma(0) \in \mathcal{R}_1^l \subset \partial^e\mathcal{A}_2$ and $\gamma(1) \in \mathcal{R}_1^r \subset \partial^i\mathcal{A}_2$, is transformed by $\Psi_2^{T_2}$ into a path winding at least $M+1$ times around $(\sqrt{V_2}, 0)$ in clockwise sense. In particular $\Psi_2^{T_2}(\gamma)$ crosses $\mathcal{R}_2$ at least M times across the components of the boundary in R_2^-. As a consequence it is possible to select M sub-paths $\gamma_1, \ldots, \gamma_M$ of γ such that $\Psi_2^{T_2}(\gamma_i)$ is contained in $\mathcal{R}_2$ and crosses both components of $\mathcal{R}_2^-$ for $i = 1, \ldots, M$. More precisely, we observe that

$$\begin{aligned}
\theta_2(T_2; \gamma(0)) &\leq \theta_2(0; \gamma(0)) + 2\pi \left\lceil \frac{T_2}{\tau_e} \right\rceil \leq 2\pi \left\lceil \frac{T_2}{\tau_e} \right\rceil \\
\theta_2(T_2; \gamma(1)) &\geq \theta_2(0; \gamma(1)) + 2\pi \left\lfloor \frac{T_2}{\tau_i} \right\rfloor \geq 2\pi \left\lfloor \frac{T_2}{\tau_i} \right\rfloor - \pi
\end{aligned}$$

and, therefore, as s ranges in $[0,1]$, the angular function $s \mapsto \theta_2(T_2; \gamma(s))$ spans all the M pairwise disjoint intervals $[2\pi(n_* + n - 1), 2\pi(n_* + n) - \pi]$, where $n_* = \lceil T_2/\tau_e \rceil$ and $n = 1, \ldots, M$. Indeed, the choice (3.3) of T_2 implies that

$$\frac{T_2}{\tau_i} - \frac{T_2}{\tau_e} \geq M + 2$$

and, thus:

$$\left\lfloor \frac{T_2}{\tau_i} \right\rfloor - \left\lceil \frac{T_2}{\tau_e} \right\rceil \geq M.$$

In particular, if we define

$$\begin{aligned}
\mathcal{H}^1 &= \{z \in \mathcal{R}_1 : \Psi_2^{T_2}(z) \in \mathcal{R}_2\} \\
\mathcal{H}^1_n &= \left\{ z \in \mathcal{H}^1 : n_* + n - 1 \leq \frac{\theta_2(T_2; z)}{2\pi} \leq n_* + n - \frac{1}{2} \right\} \quad \text{for } n = 1, \ldots, M,
\end{aligned}$$

it is possible to find points $s_1^n, s_2^n \in [0,1]$, with $n = 1, \dots, M$ and

$$0 \le s_1^1 < s_2^1 < s_1^2 < \cdots < s_2^{M-1} < s_1^M < s_2^M \le 1,$$

such that $\gamma([s_1^n, s_2^n]) \subset \mathcal{H}_n^1$, $\Psi_2^{T_2}(\gamma(s_1^n)) \in \mathcal{R}_2^l \subset \partial^e \mathcal{A}_1$ and $\Psi_2^{T_2}(\gamma(s_2^n)) \in \mathcal{R}_2^r \subset \partial^i \mathcal{A}_1$. This shows that the following stretching properties hold:

$$(\mathcal{H}_n^1, \Psi_2^{T_2}) : \widetilde{\mathcal{R}}_1 \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{R}}_2 \quad \text{for all } n = 1, \dots, M.$$

We remark that each solution of (V_2) starting at time $t = 0$ from any point $z \in \mathcal{H}_n^1$ ends in $\mathcal{R}_2$ at $t = T_2$ after exactly $2(n_* + n) - 1$ crossings with the x-axis.

In a very similar way, one can show also that, if we denote by σ_i the period of the orbit $\partial^i \mathcal{A}_i$ of (V_1) and choose $T_1 \ge M\sigma_i$, then also the following stretching properties are true:

$$(\mathcal{H}_n^2, \Psi_1^{T_1}) : \widetilde{\mathcal{R}}_2 \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{R}}_1 \quad \text{for all } n = 1, \dots, M,$$

with the positions:

$$\mathcal{H}^2 = \{z \in \mathcal{R}_2 : \Psi_1^{T_1}(z) \in \mathcal{R}_1\}$$
$$\mathcal{H}_n^2 = \left\{ z \in \mathcal{H}^2 : n - \frac{1}{2} \le \frac{\theta_1(T_1; z)}{2\pi} \le n \right\} \quad \text{for } n = 1, \dots, M,$$

and with analogous considerations about the nodal behavior of the solution of (V_1) starting from points of $\mathcal{H}_n^2$. Actually in this case the argument is slightly simpler thanks to the fact that the external boundary $\partial^e \mathcal{A}_1$ of $\mathcal{A}_1$ is the homoclinic orbit of (V_1) and not a periodic one.

In conclusion, the lemma is proved with the choice

$$T_*(M) = \max\left\{ (M+2)\frac{\tau_i \tau_e}{\tau_e - \tau_i}, M\sigma_i \right\}. \qquad \square$$

4 Proof of the result

We fix $M \ge 1$ and $\{(n_j)\}_{j=1,\dots,2K-1}$ such that $n_j \in \{1, \dots, M\}$ for all $j = 1, \dots, 2K-1$. We then consider $T^* = T^*(M)$ as given in Proposition 3.3 and a weight function V as in (1.2)-(1.3), with S^* given below.

We first apply Proposition 2.1 on the intervals $(-\infty, t_0]$ and $[t_{2K}, +\infty)$ in order to find the continua $\Gamma_{\pm\infty}$. We then select $P_\pm = (\sqrt{2V_1}, y_\pm) \in \Gamma_{\pm\infty}$ and let

$$S^* = \max\left\{ \frac{1}{\sqrt{2}} \int_{\sqrt{2V_1}}^{x_-} \frac{dx}{\sqrt{F_{V_1}(x_-) - F_{V_1}(x)}}, \frac{1}{\sqrt{2}} \int_{\sqrt{2V_1}}^{x_+} \frac{dx}{\sqrt{F_{V_1}(x_+) - F_{V_1}(x)}} \right\} \tag{4.1}$$

where, for every $\mu > 0$, F_μ is the primitive of

$$f_\mu(x) = -\mu x + x^3$$

such that $F_\mu(0) = 0$, and $x_\pm > 0$ are such that

$$(x_\pm, 0) \in \{(x, y) \in \mathbb{R}^2 : \ E_{V_1}(x, y) = E_{V_1}(P_\pm)\},$$

as before. In other words, S^* is the maximum between the times needed to run from P_- to $(x_-, 0)$ and from $(x_+, 0)$ to P_+ along the orbits of (V_1).[1]

In order to prove Theorem 1.1 we will show that the following Sturm-Liouville-type boundary value problem:

$$\begin{cases} x' = y \\ y' = V(t)x - x^3 & \text{for } t \in [t_1, t_{2K+1}] \\ (x(t_0), y(t_0)) \in \Gamma_{-\infty} \\ (x(t_{2K+1}), y(t_{2K+1})) \in \Gamma_{+\infty} \end{cases} \tag{4.2}$$

has at least a solution whose oscillatory behavior in each interval (t_j, t_{j+1}) is described by the fixed number $n_j \in \{1, \dots, M\}$, for each $j = 1, \dots, 2K-1$, in a way that will be clearer along the proof and that, however, is closely related to the definition of the sets $\mathcal{H}^i_n$ in Proposition 3.3. In fact, any solution of (4.2) is necessarily a global solution of (1.1) and satisfies $(x(\pm\infty), x'(\pm\infty)) = (0, 0)$ by the very definition of $\Gamma_{\pm\infty}$.

Since our shooting argument involves mainly paths of initial points and the sets $\Gamma_{\pm\infty}$ are just continua, we first replace the two continua in the boundary conditions with (the image of) two suitable paths that lie in an ϵ-neighborhood of $\Gamma_{\pm\infty}$, as in the following claim.

Claim 1. For every $\epsilon > 0$ there exists a continuous path $\gamma^\pm_\epsilon : [0, 1] \to \mathcal{T}_\pm$ such that

$$\gamma^\pm_\epsilon(0) = (0, 0), \quad \gamma^\pm_\epsilon(1) = P_\pm$$

and

$$\operatorname{dist}(\gamma^\pm_\epsilon(s), \Gamma_\pm) < \epsilon, \quad \forall\ s \in [0, 1].$$

[1]The constant S^* thus defined is not very explicit since it depends on the knowledge of points of the continua $\Gamma_{\pm\infty}$. A more explicit, albeit rougher, constant S^* could be obtained in the following way. Each solution of (V_1), starting from a point $(\sqrt{2V_1}, y_0)$ on the "right side" $\{\sqrt{2V_1}\} \times [0, \sqrt{2V_1(V_2 - V_1)}]$ of the region $\mathcal{T}_-$ that contains $\Gamma_{-\infty}$ (see (2.2)), hits the positive x-axis at the abscissa $\hat{x}(y_0) \in (\sqrt{2V_1}, \sqrt{2V_2})$ which can be explicitly evaluated together with the time needed to reach the point $(\hat{x}(y_0), 0)$. By letting

$$S^* = \max\left\{ \frac{1}{\sqrt{2}} \int_{\sqrt{2V_1}}^{\hat{x}(y_0)} \frac{dx}{\sqrt{F_{V_1}(\hat{x}(y_0)) - F_{V_1}(x)}} : 0 \le y_0 \le \sqrt{2V_1(V_2 - V_1)} \right\}$$

we obtain a larger bound that doesn't depend on the explicit knowledge of $\Gamma_{\pm\infty}$.

Proof. We give the details of the proof for γ_ϵ^-; the existence of γ_ϵ^+ can be obtained in a similar way, by reversing the direction of time.

For every $\epsilon > 0$ let

$$B_\epsilon = \{(x,y) \in \mathbb{R}^2 : \ \mathrm{dist}((x,y), \Gamma_{-\infty}) < \epsilon\};$$

since B_ϵ is arcwise connected, there exists a path $\gamma_\epsilon^- : [0,1] \to B_\epsilon$ such that $\gamma_\epsilon^-(0) = (0,0)$ and $\gamma_\epsilon^-(1) = P_-$.
We observe that we can choose γ_ϵ^- in such a way that $\gamma_\epsilon^-([0,1]) \subset \mathcal{T}_-$; indeed, if this condition fails, it is sufficient to replace the components u, v of the curve $\gamma_\epsilon^- = (u,v)$ with the components $\tilde{u}, \tilde{v}$ given by

$$\begin{aligned}
\tilde{u}(s) &= \min\left\{0, \max(u(s), \sqrt{2V_1})\right\}, \\
\tilde{v}(s) &= \min\left\{\sqrt{-2f_{V_2}(\tilde{u}(s))}, \max\left\{v(s), \sqrt{-2f_{V_1}(\tilde{u}(s))}\right\}\right\},
\end{aligned}$$

for all $s \in [0,1]$. ☐

For each $\epsilon > 0$ we look for solutions of the approximating boundary value problems

$$\begin{cases} x' = y \\ y' = V(t)x - x^3 & \text{for } t \in [t_1, t_{2K+1}] \\ (x(t_0), y(t_0)) \in \gamma_\epsilon^- \\ (x(t_{2K+1}), y(t_{2K+1})) \in \gamma_\epsilon^+ \end{cases} \tag{4.3}$$

with suitable oscillatory behavior. At the end of the argument we will let $\epsilon \to 0$. Our shooting argument starts by following the forward evolution of the points on γ_ϵ^- from $t = t_0$ to $t = t_1$ and the backward evolution of the points on γ_ϵ^+ from $t = t_{2K+1}$ to $t = t_{2K}$, along the flow generated by the differential equations in (4.3).

Claim 2. There exist $0 < s'_\pm < s''_\pm < 1$ such that

$$\begin{aligned}
&\Psi_1^{t_1 - t_0}(\gamma_\epsilon^-([s'_-, s''_-])) \in \mathcal{A}_2 \cap [0, +\infty) \times [0, +\infty), \\
&\Psi_1^{t_1 - t_0}(\gamma_\epsilon^-(s'_-)) \in \partial^e \mathcal{A}_2, \quad \Psi_1^{t_1 - t_0}(\gamma_\epsilon^-(s''_-)) \in \partial^i \mathcal{A}_2
\end{aligned}$$

and

$$\begin{aligned}
&\Psi_1^{t_{2K} - t_{2K+1}}(\gamma_\epsilon^+([s'_+, s''_+])) \in \mathcal{A}_2 \cap [0, +\infty) \times (-\infty, 0], \\
&\Psi_1^{t_{2K} - t_{2K+1}}(\gamma_\epsilon^+(s'_+)) \in \partial^e \mathcal{A}_2, \quad \Psi_1^{t_{2K} - t_{2K+1}}(\gamma_\epsilon^+(s''_+)) \in \partial^i \mathcal{A}_2.
\end{aligned}$$

Proof. As before, we give the details only for the time interval $[t_0, t_1]$.
Let us consider $z(\cdot;\gamma_\epsilon^-(1), t_0) = (x(\cdot;\gamma_\epsilon^-(1), t_0), y(\cdot;\gamma_\epsilon^-(1), t_0))$ and let

$$s_0 = \inf\{s \in (0,1] \mid \exists\, t \in (t_0, t_1] : y(t;\gamma_\epsilon^-(t_0), t_0) = 0\},$$

which is well defined by the construction of γ_ϵ^- and the fact that $t_1 - t_0 \geq S^*$. Indeed, the condition $t_1 - t_0 \geq S^*$ implies that $z(\cdot;\gamma_\epsilon^-(1), t_0) = z(\cdot; P_-, t_0)$ crosses the positive x-axis at least once in $(t_0, t_1]$ (see (4.1)). Moreover, the crossing point lies in the interior of the inner boundary $\partial^i \mathcal{A}_2$ of the annulus $\mathcal{A}_2$ by the very construction of $\mathcal{A}_2$ (see (3.2)). By the way, we observe that $0 < s_0 \leq 1$. Hence, $\Psi_1^{t_1 - t_0} \circ (\gamma_\epsilon^-)|_{[0,s_0]}$ is a continuous path in $\mathcal{T}_-$ which connects $(0,0)$ with the point $\Psi_1^{t_1-t_0}(\gamma_\epsilon^-(s_0))$ on the x-axis. Therefore, its support crosses $\mathcal{A}_2$ in the upper half plane; this allows to define

$$\begin{aligned} s'_- &= \max\{s \in [0, s_0] : \ \Psi_1^{t_1-t_0}(\gamma_\epsilon^-(s)) \in \partial^e \mathcal{A}_2\} \\ s''_- &= \min\{s \in [s_1, s_0] : \ \Psi_1^{t_1-t_0}(\gamma_\epsilon^-(s)) \in \partial^i \mathcal{A}_2\} \end{aligned}$$

and conclude the proof of the claim. □

Now we follow the forward evolution of the solutions starting at $t = t_0$ from γ_ϵ^- up to the time $t = t_{2K}$. The main idea here is to select a sub-path of γ_ϵ^- that gives rise to solutions with the desired oscillatory properties in the intermediate intervals and that is transformed at time $t = t_{2K}$ into a path whose position inside the annulus $\mathcal{A}_2$ is in some sense good.

By Claim 2, the path $s \mapsto z(t_1;\gamma_\epsilon^-(s), t_0)$ crosses the annulus $\mathcal{A}_2$ in the upper half plane as s ranges in $[s'_-, s''_-]$. Since $t_2 - t_1 \geq T^*(M)$ it is possible to argue as in the proof of Proposition 3.3 and show that the map $\Psi_2^{t_2-t_1}$ transforms the curve $s \mapsto z(t_1;\gamma_\epsilon^-(s), t_0)$ into a path that winds in $\mathcal{A}_2$ and crosses $\mathcal{R}_2$ at least M times as s ranges in $[s'_-, s''_-]$. In fact, there exists a non-trivial subinterval $[s'_2, s''_2] \subset [s'_-, s''_-]$ such that $z(t_2;\gamma_\epsilon^-(s), t_0) \in \mathcal{R}_2$ for all $s \in [s'_2, s''_2]$, $z(t_2;\gamma_\epsilon^-(s'_2), t_0) \in \mathcal{R}_2^l$ and $z(t_2;\gamma_\epsilon^-(s''_2), t_0) \in \mathcal{R}_2^r$. More precisely the interval $[s'_2, s''_2]$ can be chosen among M different ones, according to the number of oscillations that the solution performs in the interval $[t_1, t_2]$. Based on the choice of $n_1 \in \{1, \ldots, M\}$ that we made at the beginning of the proof, the interval $[s'_2, s''_2]$ can be characterized by the following property:

$$n_*^1 + n_1 - 1 \leq \frac{\theta_2(t_2 - t_1;\gamma_\epsilon^-(s))}{2\pi} \leq n_*^1 + n_1 - \frac{1}{2}, \qquad \forall s \in [s'_2, s''_2]$$

where $n_*^1 = \lceil (t_2 - t_1)/\tau_e \rceil$. We omit the details since they are analogous to those employed in the proof of Proposition 3.3. We only remark that all the solutions $z(\cdot;\gamma_\epsilon^-(s), t_0)$, for all $s \in [s'_2, s''_2]$ cross the positive x-axis exactly $2(n_*^1 + n_1) - 1$ times in the interval (t_1, t_2).

Now, about the dynamics in the intermediate interval $[t_2, t_{2K-1}]$, we observe that the map $p \mapsto z(t_{2K}; p, t_2)$ is the alternate composition, for suitable values of j, of the maps $\Psi_2^{t_{2j+2}-t_{2j+1}}$ and $\Psi_1^{t_{2j+1}-t_{2j}}$ for which the stretching properties of Proposition 3.3 with crossing number M hold thanks to assumption (1.3). However, the compact sets $\mathcal{H}_n^i$ with respect to which the S.A.P. property holds are different for each of those maps since they depend explicitly on the length of the time intervals $t_{j+1} - t_j$. To make things slightly more complicated, in the case $i = 1$ (i.e. j odd) one has also to be careful with the minimum number of turns n^* which also depends on the lengths $t_{j+1} - t_j$. In order to take into account all these facts, we modify our notation in the following way: the superscript j in $\mathcal{H}^j$ and $\mathcal{H}_n^j$ is allowed to range in $2, \ldots, 2K-2$ and, from now on, it refers to the stretching property in the interval $[t_j, t_{j+1}]$ for the appropriate map, depending on the value that V has in that interval according to (1.2). More precisely, by Proposition 3.3 we then have

$$(\mathcal{H}_n^{2j+1}, \Psi_2^{t_{2j+2}-t_{2j+1}}) : \widetilde{\mathcal{R}}_1 \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{R}}_2,$$

for $n = 1, \ldots, M$, and $j = 1, \ldots, K-2$, where

$$\begin{aligned}
\mathcal{H}^{2j+1} &= \{z \in \mathcal{R}_1 : \Psi_2^{t_{2j+2}-t_{2j+1}}(z) \in \mathcal{R}_2\}, \\
\mathcal{H}_n^{2j+1} &= \left\{ z \in \mathcal{H}^{2j+1} : n_*^{2j+1} + n - 1 \leq \frac{\theta_2(t_{2j+2} - t_{2j+1}; z)}{2\pi} \leq n_*^{2j+1} + n - \frac{1}{2} \right\}, \\
n_*^{2j+1} &= \left\lceil \frac{t_{2j+2} - t_{2j+1}}{\tau_e} \right\rceil,
\end{aligned}$$

and

$$(\mathcal{H}_n^{2j}, \Psi_1^{t_{2j+1}-t_{2j}}) : \widetilde{\mathcal{R}}_2 \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{R}}_1 \quad \text{for all } n = 1, \ldots, M,$$

for $n = 1, \ldots, M$, and $j = 1, \ldots, K-1$, where:

$$\begin{aligned}
\mathcal{H}^{2j} &= \{z \in \mathcal{R}_2 : \Psi_1^{t_{2j+1}-t_{2j}}(z) \in \mathcal{R}_1\}, \\
\mathcal{H}_n^{2j} &= \left\{ z \in \mathcal{H}^{2j} : n - \frac{1}{2} \leq \frac{\theta_1(t_{2j+1} - t_{2j}; z)}{2\pi} \leq n \right\}.
\end{aligned}$$

Thanks to this notation, to the assumption (1.3) and to Proposition 3.3, we have the following chain of successive stretching maps:

$$\begin{aligned}
(\mathcal{H}_{n_2}^2, \Psi_1^{t_3-t_2}) &: \widetilde{\mathcal{R}}_2 \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{R}}_1 \\
(\mathcal{H}_{n_3}^3, \Psi_2^{t_4-t_3}) &: \widetilde{\mathcal{R}}_1 \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{R}}_2 \\
&\vdots \\
(\mathcal{H}_{n_{2K-3}}^{2K-3}, \Psi_2^{t_{2K-2}-t_{2K-3}}) &: \widetilde{\mathcal{R}}_1 \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{R}}_2 \\
(\mathcal{H}_{n_{2K-2}}^{2K-2}, \Psi_1^{t_{2K-1}-t_{2K-2}}) &: \widetilde{\mathcal{R}}_2 \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathcal{R}}_1
\end{aligned}$$

and, therefore, by Proposition 3.2 there exists a non-trivial subinterval $[s'_{2K-1}, s''_{2K-1}] \subset [s'_2, s''_2]$ such that the path $s \mapsto z(t_{2K-1}; \gamma_\epsilon^-(s), t_0)$ crosses $\mathcal{R}_1$, with:

$$\begin{aligned}
&z(t_{2K-1}; \gamma_\epsilon^-(s'_{2K-1}), t_0) \in \mathcal{R}_1^l, \\
&z(t_{2K-1}; \gamma_\epsilon^-(s''_{2K-1}), t_0) \in \mathcal{R}_1^r, \\
&z(t_j; \gamma_\epsilon^-(s), t_0) \in \mathcal{H}_{n_j}^j \quad \forall s \in [s'_{2K-1}, s''_{2K-1}] \text{ and } \forall j = 2, \ldots, 2K-2.
\end{aligned}$$

The last condition, in particular, grants that all the solutions starting from $\gamma_\epsilon^-(s)$ at time $t = t_0$ for all $s \in [s'_{2K-1}, s''_{2K-1}]$ have the oscillatory behavior prescribed by the integer n_j in the interval $[t_j, t_{j+1}]$, for $j = 1, \ldots, 2K-2$ (the behavior for $j = 1$ is already granted by the choice of s'_2, s''_2).

Since finally $t_{2K} - t_{2K-1} \geq T^*(M)$, also the path $s \mapsto z(t_{2K-1}; \gamma_\epsilon^-(s), t_0)$ is transformed by $\Psi_2^{t_{2K}-t_{2K-1}}$ into a path that winds in $\mathcal{A}_2$ at least M times. More precisely it is possible to find a non-trivial subinterval $[s'_{2K}, s''_{2K}] \subset [s'_{2K-1}, s''_{2K-1}]$ such that

$$\begin{aligned}
&n_*^{2K-1} + n_{2K-1} - 1 \leq \frac{\theta_2(t_{2K} - t_{2K-1}; z(t_{2K-1}; \gamma_\epsilon^-(s)))}{2\pi} \leq n_*^{2K-1} + n_{2K-1} - \frac{1}{2}, \\
&\theta_2(t_{2K} - t_{2K-1}; z(t_{2K-1}; \gamma_\epsilon^-(s'_{2K}))) = 2\pi(n_*^{2K-1} + n_{2K-1}) - 2\pi, \\
&\theta_2(t_{2K} - t_{2K-1}; z(t_{2K-1}; \gamma_\epsilon^-(s''_{2K}))) = 2\pi(n_*^{2K-1} + n_{2K-1}) - \pi.
\end{aligned}$$

In other words, as s ranges in $[s'_{2K}, s''_{2K}]$, the solutions $z(\cdot; \gamma_\epsilon^-(s), t_0)$ cross the positive x axis exactly $2(n_*^{2K-1} + n_{2K-1}) - 1$ times in the time interval (t_{2K-1}, t_{2K}), while the curve $s \mapsto z(t_{2K}; \gamma_\epsilon^-(s), t_0)$ describes an arc that lies in the lower half of the annulus $\mathcal{A}_2$ and joins the segments $[x'_e, x'_i] \times \{0\}$ and $[x''_i, x''_e] \times \{0\}$, which are the two components of the intersection of the annulus $\mathcal{A}_2$ with the positive x-axis (here $x'_e < x'_i < \sqrt{V_2} < x''_i < x''_e$).

Now, the lower half $\mathcal{A}_2 \cap \{(x,y) : y \leq 0\}$ of $\mathcal{A}_2$ is also a topological rectangle in which the segments $[x'_e, x'_i] \times \{0\}$ and $[x''_i, x''_e] \times \{0\}$ form one couple of opposite sides and the arcs $\partial^e \mathcal{A}_2 \cap \{(x,y) : y \leq 0\}$ and $\partial^i \mathcal{A}_2 \cap \{(x,y) : y \leq 0\}$ form the other couple (see figure 2). The discussion carried on above shows that both the paths $s \mapsto z(t_{2K}; \gamma_\epsilon^-(s), t_0)$, for $s \in [s'_{2K}, s''_{2K}]$, and $s \mapsto z(t_{2K}; \gamma_\epsilon^+(s), t_{2K+1})$, for $s \in [s'_+, s''_+]$ (see Claim 2), lie in that topological rectangle and join different couples of opposite sides. Therefore, those two paths must intersect by [14, Lemma 3]. More precisely, there exists $s_\epsilon \in [s'_{2K}, s''_{2K}]$ such that $z(t_{2K+1}; \gamma_\epsilon^-(s_\epsilon), t_0) \in \gamma_\epsilon^+$ which means that $z(\cdot; \gamma_\epsilon^-(s_\epsilon), t_0)$ is a solution of (4.3) which in addition has the oscillatory behavior prescribed by n_j in each interval (t_j, t_{j+1}) for all $j = 1, \ldots, 2K-1$, as described along the proof.

Now we conclude the proof by passing to the limit as $\epsilon \to 0$. Along a suitable sequence $\epsilon_m \to 0$ as $m \to +\infty$, the sequence $z_m := \gamma_{\epsilon_m}^-(s_{\epsilon_m})$ converges to a point $z_0 = (x_0, y_0) \in \Gamma_{-\infty}$ and, by the continuous dependence on initial data, we have that:

1. $z(\cdot; z_m, t_0)$ converges to $z(\cdot; z_0, t_0)$ uniformly on $[t_0, t_{2K+1}]$;

2. in particular $z(t_{2K+1}; z_m, t_0) \to z(t_{2K+1}; z_0, t_0) \in \Gamma_{+\infty}$ and, thus, $z(\cdot; z_0, t_0)$ is a solution of (4.2);

3. $z(\cdot; z_0, t_0)$ still has the same oscillatory behavior prescribed by the numbers n_j in (t_j, t_{j+1}) as the approximate solutions $z(\cdot; z_m, t_0)$ have, since the inequalities satisfied by the angular coordinates are weak and the angular coordinates are continuous.

We also observe that the first component of the solution z is positive, by construction. Therefore Theorem 1.1 is proved.

References

[1] A. Ambrosetti, M. Badiale and S. Cingolani, *Semiclassical states of nonlinear Schrodinger equations*, Arch. Rational Mech. Anal. **140** (1997), 285300.

[2] A. Boscaggin, W. Dambrosio and D. Papini, *Asymptotic and chaotic solutions of a singularly perturbed Nagumo-type equation*, Nonlinearity **28** (2015), 3465-3485.

[3] A. Capietto, W. Dambrosio and D. Papini, *Superlinear indefinite equations on the real line and chaotic dynamics*, J. Differential Equations **181** (2002), 419–438.

[4] C. Conley, *An application of Ważewski's method to a non-linear boundary value problem which arises in population genetics*, J. Math. Biol. **2** (1975), 241–249.

[5] M. del Pino, P. Felmer and O. Miyagaki, *Existence of positive bound states of nonlinear Schrodinger equations with saddle-like potential*, Nonlinear Anal. **34** (1998), 979-989.

[6] E. Ellero and F. Zanolin, *Homoclinic and heteroclinic solutions for a class of second-order non-autonomous ordinary differential equations: multiplicity results for stepwise potentials*, Bound. Value Probl. **2013**:167 (2013), 23 pp.

[7] E. Ellero and F. Zanolin, *Connected branches of initial points for asymptotic solutions, with applications*, in preparation.

[8] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrodinger equation with a bounded potential*, J. Funct. Anal. **69** (1986), 397-408.

[9] A. Gavioli, *Monotone heteroclinic solutions to non-autonomous equations via phase plane analysis*, NoDEA Nonlinear Differential Equations Appl. **18** (2011), 79–100.

[10] P.J. Holmes and C.A. Stuart, *Homoclinic orbits for eventually autonomous planar flows*, Z. Angew. Math. Phys. **43** (1992), 598–625.

[11] K. Kuratowski, Topology, Vol. 2, Academic Press, New York, 1968.

[12] A. Margheri, C. Rebelo and F. Zanolin, *Connected branches of initial points for asymptotic BVPs, with application to heteroclinic and homoclinic solutions*, Adv. Nonlinear Stud. **9** (2009), 95–135.

[13] J. Mawhin, D. Papini and F. Zanolin, *Boundary blow-up for differential equations with indefinite weight*, J. Differential Equations **188** (2003), 33-51.

[14] J.S. Muldowney and D. Willett, *An elementary proof of the existence of solutions to second order nonlinear boundary value problems*, SIAM J. Math. Anal. **5** (1974), 701–707.

[15] D. Papini and F. Zanolin, *A topological approach to superlinear indefinite boundary value problems*, Topol. Methods Nonlinear Anal. **15** (2000), 203–233.

[16] D. Papini and F. Zanolin, *Fixed points, periodic points, and coin-tossing sequences for mappings defined on two-dimensional cells*, Fixed Point Theory Appl. 2004, 113–134.

[17] D. Papini and F. Zanolin, *On the periodic boundary value problem and chaotic-like dynamics for nonlinear Hill's equations*, Adv. Nonlinear Stud. **4** (2004), 71–91.

[18] A. Pascoletti, M. Pireddu and F. Zanolin, *Multiple periodic solutions and complex dynamics for second order ODEs via linked twist maps*, The 8th Colloquium on the Qualitative Theory of Differential Equations, No. 14, 32 pp., Proc. Colloq. Qual. Theory Differ. Equ., 8, Electron. J. Qual. Theory Differ. Equ., Szeged, 2008.

[19] M. Pireddu and F. Zanolin, *Fixed points for dissipative-repulsive systems and topological dynamics of mappings defined on N-dimensional cells*, Adv. Nonlinear Stud. **5** (2005), 411–440.

[20] P. Rabinowitz, *On a class of nonlinear Schrodinger equations*, Z. Angew. Math. Phys. **43** (1992), 270-291.

[21] C. Rebelo and F. Zanolin, *On the existence and multiplicity of branches of nodal solutions for a class of parameter-dependent Sturm-Liouville problems via the shooting map*, Differential Integral Equations **13** (2000), 1473–1502.

[22] M. Struwe, *Multiple solutions of anticoercive boundary value problems for a class of ordinary differential equations of second order*, J. Differential Equations **37** (1980), 285–295.

[23] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math. **20**, (1947) 279–313 (1948).

[24] C. Zanini and F. Zanolin, *Complex dynamics in one-dimensional nonlinear Schrödinger equations with stepwise potential*, preprint (2014).