Lower and upper bounds for a semilinear boundary value problem.

(Article begins on next page)

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LOWER AND UPPER BOUNDS
OF A SEMILINEAR BOUNDARY VALUE PROBLEM.

Domenico Delbosco,   Gabriella Viola

Abstract. In the first part of this paper we indicate the best numerical values of the constants entering in the a priori estimates for the solution to nonlinear two point boundary value homogeneous Dirichlet problem. The numerical values are depending only on the first positive zero of some equations connected to the non linearity f. In the special case, where f is a function power with exponent $p > 1$ the constants are depending on the esponent p.In the second part we indicate the numerical values of the constants in connection with a priori bound for a Dirichlet problem for a semilinear elliptic equation with nonlinearity a function power with exponent $p > 1$.

AMS CLASSIFICATION :

1. Introduction.

D.G. Figueireido, P.L. Lions and R. Nussbaum proved the existence of an a priori bound for a semilinear Dirichlet boundary value problem on a bounded domain with smooth boundary. In such work no indication is given for the value of the constant entering in the estimation. The goal of this paper is to indicate in the cases where the dimension N $\geq$ 2 the numerical values of the constants in connection with nonlinearity a function power with exponent $p > 1$ a the domain is ball and in the case $N = 1$ with a more general nonlinearity. In the first part of this paper we give numerical values for the constants in wiew of the a priori estimates for the solution to nonlinear two point boundary value Dirichlet problem. The numerical values are depending only on the first positive zero of some equations connected to the non linearity f. In the special case, where f is a function power with exponent $p > 1$ the constants are depending on the esponent p.In the second part we indicate the numerical values for the constants in connection with a priori bound for a

Dirichlet problem for a semilinear elliptic equation con non linearity a function power with exponent $p > 1$.

2.One dimensional case. Main results.

At first we consider the one dimensional case and we get a lower and an upper bound for any solution with constants depending only on the exponent p.

Theorem 1. *Let $p > 1$. If u be a solution of the following nonlinear two point problem*

$$u'' + u^p = 0$$

$$(1) \quad u(x) > 0 \quad x \in (-1, 1)$$

$$u(-1) = u(1) = 0$$

the u satisfies the following inequalities

$$[\frac{1}{2}(p+1)]^{\frac{1}{p-1}} < \|u\|_\infty < [2(p+1)]^{\frac{1}{p-1}}$$

where $\|u\|_\infty = \sup_{x \in [-1,1]} |u(x)|$.

Proof.

The uniqueness of the solution follows, for example, from a result of Ni and Nussbaum [2].

Since $u(x) > 0$ in $(-1, 1)$, then $u'' = -u^p < 0$: hence u is a concave mapping. Uniqueness of solution to problem (1) implies that $u(-x) = u(x)$, $u'(0) = 0$ and $\|u\|_\infty = u(0)$.

Since u is concave, we have

$$u(0) < |u'(1)|.$$

Multiplying by u' equation (1) and integrating on $[-1, 0]$ we obtain

$$\int_{-1}^0 u'' u' dx + \int_{-1}^0 u^p u' dx = 0$$

and

$$\frac{1}{2}[u'^2]_{-1}^0 + \frac{1}{p+1}[u^{p+1}]_{-1}^0 = 0.$$

Hence we get

$$(1.2) \qquad -\frac{1}{2}|u'(-1)|^2 + \frac{1}{p+1}u^{p+1}(0) = 0.$$

Taking into account (1.1) and (1.2) we deduce the lower bound

$$\|u\|_\infty = u(0) > [\frac{1}{2}(p+1)]^{\frac{1}{p+1}}.$$

To obtain an upper bound for $\|u\|_\infty$, we note that u' is a positive concave mapping on [-1, 0] and that u' is a negative concave mapping on [0, 1]. At first since $u'(0) = 0$ and $u''(x) < 0$ for all $x \in [-1, 0]$ then it follows that

$$u'(x) > 0 \quad x \in [-1, 0].$$

Similarly since $u'(0) = 0$ and $u''(x) < 0$ for all $x \in [0, 1]$, it follows that

$$u'(x) < 0 \quad x \in [0, 1].$$

Now we get

$$(u')'' = (u'')' = (-u^p)' = -pu^{p-1}u' > 0, \quad x \in [0, 1].$$

More precisely we have
$$u'(x) < u'(1)x$$
for all $x \in [0, 1]$.

Therefore we get

$$u(0) = -\int_0^1 u'(t)dt > -u'(1)\int_0^1 tdt = -\frac{1}{2}u'(1)$$

i.e.

$$(1.3) \qquad u(0) > \frac{1}{2}|u'(1)|.$$

Now using (1.3) and (1.2) we obtain

$$u(0) > \frac{1}{2}(\frac{2}{p+1})^{\frac{1}{2}}u^{\frac{p+1}{2}}$$

hence

$$u(0)^{\frac{p+1}{2}} < 2(\frac{p+1}{2})^{\frac{1}{2}} = [2(p+1)]^{\frac{1}{2}}$$

and finally
$$\|u\|_\infty = u(0) < [2(p+1)]^{\frac{1}{p+1}}.$$

We want to generalize the previous result to a class of mapping including the power function $f(t) = t^p$, with $p > 1$.

Theorem 2. *Let $f : R \to R$ be a differentiable function satisfying the following hypothesis :*
  i) $f(0) = 0$; $f(t) > 0$ for $t > 0$
  ii) $tf'(t) > f(t) > 0$ for $t > 0$.
  The unique positive solution of the following nonlinear two point problem

$$(1) \quad u'' + f(u) = 0$$

with Dirichlet boundary condition

$$u(-1) = u(1) = 0$$

satisfies the following inequalities

$$m < \|u\|_\infty < M$$

where m denotes the first positive solution of the equation $t^2 - F(t) = 0$, M is the first positive solution of equation $F(t) - 2t^2 = 0$, where $F(t) = \int_0^t f(s)ds$.

We recall that in [1] (Remarks 3.1, pag.30) there a proof of the existence of an upper bound for the positive under the following more general hypothesis that f is a continuous locally Lipschitzian mapping.

Moreover we remark that uniqueness of positive solution follows from a result contained in [2].More precisely condition ii) assures uniqueness.

Proof of Theorem 2.

We remark that u' is a positive concave mappimg on [-1, 0]. Indeed we have
$$(u')'' = (u'')' = (-f(u))' = -f'u' < 0.$$

On the same line of Theorem 1 we get

$$u(0) = \int_{-1}^0 u'(t)dt > u'(1) \int_{-1}^0 (-t)dt = \frac{1}{2}u'(1)$$

i.e.
$$(1.3) \qquad u(0) > \frac{1}{2}u'(1).$$

Multiplying by u' equation (1) and integrating on $[-1,0]$ we obtain

$$\int_{-1}^{0} u''u'\,dx + \int_{-1}^{0} f(u)u'\,dx = 0$$

and

$$\frac{1}{2}[u'^2]_{-1}^{0} + [F(u)]_{-1}^{0} = 0.$$

Thus one has

$$-\frac{1}{2}|u'(-1)|^2 + F(u(0)) = 0.$$

hence

$$(2.1) \qquad |u'(-1)| = [2F(u(0))]^{\frac{1}{2}}.$$

Now taking into account of (1.3) and of (2.1) we have

$$2u(0)^2 > F(u(0)).$$

Setting $t = u(0)$, we have to solve the following inequality

$$F(t) - 2t^2 < 0.$$

Let M be the first positive zero of equation

$$F(t) - 2t^2 = 0$$

Note that $F(t) - 2t^2 < 0$ when $0 < t < M$. Finally we obtain the upper bound

$$\|u\|_\infty = u(0) = t < M.$$

The lower bound for $\|u\|$ can be obtained from

$$u(0) < |u'(-1)|$$

and

$$|u'(-1)| = [2F(u(0))]^{\frac{1}{2}}$$

Thus we have to solve the inequality

$$u(0)^2 - 2F(u(0)) < 0$$

On the same line of the first part of this proof we set $t = u(0)$ and we consider the equation

$$t^2 - 2F(t) = 0$$

Now we have that $t^2 - 2F(t) > 0$ for $t > 0$ and near 0.

If we denote m the first positive solution of previous equation, then we get that $t^2 - 2F(t) < 0$ for $t > m$.

Therefore we have the lower bound

$$\|u\|_\infty = u(0) = t > m.$$

1 D.G. Figueiredo, P.L. Lions and R.D. Nussbaum, A priori estimates and existence of positive solutions of semilinear elliptic equations, J. Math. Pures et Appl. vol.61 (1982),41-63.

2 W. M. Ni and R.D. Nussbaum, Uniqueness and Nonuniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$., Comm.Pure Appl. Math. vol. 28 (1985),67-108.