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Abstract. We show that a temperate distribution belongs to an ordered intersection or union of admissible Banach or Frechét spaces if and only if the corresponding global wave-front set of union or intersection type is empty. We also discuss the situation where intersections and unions of sequences of spaces with two indeces are involved. A main situation where the present theory applies is given by sequences of weighted, general modulation spaces.

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1. Introduction

Wave-front sets of global type are a standard tool to investigate the regularity, local and at infinity, of solutions $f \in \mathscr{S}' = \mathscr{S}'(\mathbf{R}^d)$ to equations

$$Tf = g, \tag{1.1}$$

for a (pseudo)differential operator T and some given distribution $g \in \mathscr{S}'$.

In particular, it is often necessary to deal with situations where T is not globally hypoelliptic, and the involved distributions f and g belong to appropriate Banach or Fréchet spaces of temperate distributions $\mathcal{B}$ and $\mathcal{C}$, respectively.

In this paper we complete the analysis carried on in [4] (see also [6]), by giving the detailed proofs of some results involving global wave-front sets associated with sequences of appropriate function spaces, compatible with the class of pseudo-differential operators we use here, namely, the SG-operators. The SG-calculus of pseudo-differential operators was introduced in the '70s, independently by C. Parenti [17] and H.O. Cordes, see e.g. [2] (for a different approach, see R. Melrose [16]). This calculus allows to treat global problems associated, e.g., with any linear partial differential operator with constant coefficients, the Klein-Gordon's equation, the Schrödinger equations for different atoms, and classes of Dirac-type operators on $\mathbf{R}^d$. The definition and some basic facts about the SG-operators can be found in Section 2.

The general assumptions on $\mathcal{B}$ and $\mathcal{C}$ in (1.1) are described in detail in Section 1 of [4]. For example, $\mathcal{B}$ and $\mathcal{C}$ can be modulation spaces, a family of Banach spaces of functions and tempered distributions, introduced by H.G. Feichtinger, see e.g. [8], and developed further and generalized by H.G. Feichtinger and K.H. Gröchenig [7], see also [9, 11]. For the convenience of the reader, we included the essential definitions in Section 2, following the approach in [9]. We remark that the family of modulation spaces is broad, in the sense that it contains the Sobolev spaces H^2_s and the Sobolev-Kato spaces $H^2_{s,t}$, see Remark 2.4 below.

In Section 3 we recall the definition of the global wave-front set $\mathrm{WF}_{\mathcal{B}}(f)$ of the distribution f, with respect to the Banach or Fréchet space $\mathcal{B}$, given in [4]. This object, loosely speaking, gives informations about the local regularity (smoothness) and the behavior at infinity (decay and oscillation properties within certain cones) of f. It is a remarkable fact that, whenever the space $\mathcal{B}$ is SG-admissible (see Section 2 below for the precise definition), then

$$\mathrm{WF}_{\mathcal{B}}(f) = \emptyset \quad \Longleftrightarrow \quad f \in \mathcal{B}. \tag{1.2}$$

This fact, together with the mapping properties which hold for these global wave-front sets under the action of SG-pseudo-differential operators (namely, the so-called microlocality and microellipticity properties), can be used to obtain rather precise relations between the regularity properties of f and g in (1.1).

Note that, if $\mathcal{B}$ equals $\mathscr{S}$ or $H^2_{s,t}$, then $\mathrm{WF}_{\mathcal{B}}(f)$ agrees with the wave-front sets of f with respect to $\mathscr{S}$ and $H^2_{s,t}$, respectively, given in [5] and [16]. Consequently, we recover also all the properties that hold for wave-front sets of Sobolev type introduced by Hörmander [14], and classical wave-front sets with respect to smoothness (cf. Sections 8.1 and 8.2 in [13]), as well as for wave-front sets of Banach function types in [3] (cf. also [18, 19]), and wave-front sets with respect to $\mathscr{S}$ and $H^2_{s_1,s_2}$ in [5, 16].

In order to get even more detailed information on the links between the regularity properties of f and g in (1.1), in [4] we introduced global wave-front sets with respect to *sequences* of SG-admissible spaces, whose definitions we recall in Section 4. Microlocality and microellipticity under the action of SG-pseudo-differential operators hold also for this refined type of wave-front set. Here we deal with the relations between the union/intersection type wave-front sets and the unions/intersections of the wave-front sets of the spaces belonging to the involved sequences. The main focus is the extension of (1.2) to this more general case, which in Section 4 is proved to hold as well. In other words, the property of a distributions to belong to an intersection, union, union of intersections, or intersection of unions of SG-admissible spaces is still equivalent to the emptiness of the corresponding type of wave-front set, under rather mild assumptions on the involved sequences of spaces.

An example where our theory applies is the case where $\mathcal{B} = \mathcal{C}$ equals $\mathscr{S}(\mathbf{R}^d)$, $Q_0(\mathbf{R}^d)$ or $Q(\mathbf{R}^d)$, where

$$Q_0(\mathbf{R}^d) \equiv \{ f \in C^\infty(\mathbf{R}^d) : |\partial^\alpha f(x)| \lesssim \langle x \rangle^N \text{for some } N \text{ and every } \alpha \in \mathbf{Z}^d \}$$

and

$$Q(\mathbf{R}^d) \equiv \{ f \in C^\infty(\mathbf{R}^d) : |\partial^\alpha f(x)| \lesssim \langle x \rangle^{N_\alpha}, \text{ where } N_\alpha \text{ depends on } \alpha \in \mathbf{Z}^d \},$$

where $A(x) \lesssim B(x)$ means that there exists $C > 0$, independent of x, such that, for any value of x on which A and B are defined, $A(x) \le C \cdot B(x)$.

The proofs of the results mentioned in Sections 2 and 3 can be found in [4].

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2. Preliminaries

We start by recalling some basic definition and concept which will be needed throughout the paper. The material in this section comes mainly from [4].

2.1. Weight functions

Let ω and v be positive measurable functions on $\mathbf{R}^d$. Then ω is called v-moderate if

$$\omega(x + y) \lesssim \omega(x)v(y). \tag{2.1}$$

If v in (2.1) can be chosen as a polynomial, then ω is called a function or weight of *polynomial type*. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomial type functions on $\mathbf{R}^d$. If $\omega(x,\xi) \in \mathscr{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable or the ξ-variable, then we sometimes write $\omega(\xi)$, respectively $\omega(x)$, instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathscr{P}(\mathbf{R}^{2d})$ or in $\mathscr{P}(\mathbf{R}^d)$ depending on the situation. We say that v is submultiplicative if (2.1) holds for $\omega = v$. For convenience we assume that all submultiplicative weights are even, and we always let v and v_j stand for submultiplicative weights, if nothing else is stated.

Without loss of generality we may assume that every $\omega \in \mathscr{P}(\mathbf{R}^d)$ is smooth and satisfies the ellipticity condition $\partial^\alpha \omega/\omega \in L^\infty$. In fact, by Lemma 1.2 in [20] it follows that for each $\omega \in \mathscr{P}(\mathbf{R}^d)$, there is a smooth and elliptic $\omega_0 \in \mathscr{P}(\mathbf{R}^d)$ which is equivalent to ω in the sense

$$\omega \asymp \omega_0, \tag{2.2}$$

where $A \asymp B$ means $A \lesssim B \lesssim A$.

The weights involved in the sequel have to satisfy additional conditions. More precisely let $r, \rho \ge 0$. Then $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is the set of all $\omega(x,\xi)$ in $\mathscr{P}(\mathbf{R}^{2d}) \bigcap C^\infty(\mathbf{R}^{2d})$ such that

$$\langle x \rangle^{r|\alpha|} \langle \xi \rangle^{\rho|\beta|} \frac{\partial_x^\alpha \partial_\xi^\beta \omega(x,\xi)}{\omega(x,\xi)} \in L^\infty(\mathbf{R}^{2d}),$$

for every multi-indices α and β. Note that $\mathscr{P}_{r,\rho}$ is different here compared to [3], and that there are elements in $\mathscr{P}(\mathbf{R}^{2d})$ which have no equivalent elements in $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. On the other hand, if $s,t\in\mathbf{R}$ and $r,\rho\in[0,1]$, then $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ contains all weights of the form $\omega(x,\xi)=\langle x\rangle^t\langle\xi\rangle^s$, which are one of the most common type of weights in the applications.

2.2. Modulation spaces

Let $\phi\in\mathscr{S}(\mathbf{R}^d)$. Then the *short-time Fourier transform* of $f\in\mathscr{S}(\mathbf{R}^d)$ with respect to (the window function) ϕ is defined by

$$V_\phi f(x,\xi)=(2\pi)^{-d/2}\int_{\mathbf{R}^d} f(y)\overline{\phi(y-x)}e^{-i\langle y,\xi\rangle}\,dy. \tag{2.3}$$

More generallly, the short-time Fourier transform of $f\in\mathscr{S}'(\mathbf{R}^d)$ with respect to $\phi\in\mathscr{S}'(\mathbf{R}^d)$ is defined by

$$(V_\phi f)=\mathscr{F}_2F,\quad \text{where}\quad F(x,y)=(f\otimes\overline{\phi})(y,y-x), \tag*{(2.3)$'$}$$

where $\mathscr{F}_2F$ is the partial Fourier transform of $F(x,y)\in\mathscr{S}'(\mathbf{R}^{2d})$ with respect to the y-variable. We refer to [10, 11] for more facts about the short-time Fourier transform. We now recall the notion of translation invariant BF-space on $\mathbf{R}^d$.

Definition 2.1. Let $\mathscr{B}$ be a Banach space which is continuously embedded in $L^1_{\mathrm{loc}}(\mathbf{R}^d)$, and let $v\in\mathscr{P}(\mathbf{R}^d)$ be submultiplicative. Then $\mathscr{B}$ is called a *translation invariant BF-space on* $\mathbf{R}^d$ (with respect to v), if there is a constant C such that the following conditions are fulfilled:

1. $\mathscr{S}(\mathbf{R}^d)\subseteq\mathscr{B}\subseteq\mathscr{S}'(\mathbf{R}^d)$ (continuous embeddings);
2. if $x\in\mathbf{R}^d$ and $f\in\mathscr{B}$, then $f(\cdot-x)\in\mathscr{B}$, and
$$\|f(\cdot-x)\|_{\mathscr{B}}\le Cv(x)\|f\|_{\mathscr{B}}; \tag{2.4}$$
3. if $f,g\in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ satisfy $g\in\mathscr{B}$ and $|f|\le|g|$ almost everywhere, then $f\in\mathscr{B}$ and
$$\|f\|_{\mathscr{B}}\le C\|g\|_{\mathscr{B}}.$$

The following definition of modulation spaces is due to Feichtinger [9].

Definition 2.2. Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$ with respect to $v\in\mathscr{P}(\mathbf{R}^{2d})$, $\phi\in\mathscr{S}(\mathbf{R}^d)\backslash 0$ and let $\omega\in\mathscr{P}(\mathbf{R}^{2d})$ be such that ω is v-moderate. The *modulation space* $M(\omega,\mathscr{B})$ consists of all $f\in\mathscr{S}'(\mathbf{R}^d)$ such that $V_\phi f\cdot\omega\in\mathscr{B}$. We note that $M(\omega,\mathscr{B})$ is a Banach space with the norm

$$\|f\|_{M(\omega,\mathscr{B})}\equiv\|V_\phi f\omega\|_{\mathscr{B}} \tag{2.5}$$

(cf. [7]).

Remark 2.3. Assume that $p,q\in[1,\infty]$, and let $L^{p,q}_1(\mathbf{R}^{2d})$ and $L^{p,q}_2(\mathbf{R}^{2d})$ be the sets of all $F\in L^1_{\mathrm{loc}}(\mathbf{R}^{2d})$ such that

$$\|F\|_{L^{p,q}_1}\equiv\Big(\int\Big(\int|F(x,\xi)|^p\,dx\Big)^{q/p}\,d\xi\Big)^{1/q}<\infty$$

and

$$\|F\|_{L_2^{p,q}} \equiv \Big(\int \Big(\int |F(x,\xi)|^q \, d\xi \Big)^{p/q} \, dx \Big)^{1/p} < \infty .$$

Then $M(\omega, L_1^{p,q}(\mathbf{R}^{2d}))$ is equal to the *classical* modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and $M(\omega, L_2^{p,q}(\mathbf{R}^{2d}))$ is equal to the space $W^{p,q}_{(\omega)}(\mathbf{R}^d)$, related to Wiener-amalgam spaces (cf. [8, 9, 7, 11]). We will set $M^p_{(\omega)} = M^{p,p}_{(\omega)} = W^{p,p}_{(\omega)}$. Furthermore, if $\omega = 1$, then we write $M^{p,q}$, M^p and $W^{p,q}$ instead of $M^{p,q}_{(\omega)}$, $M^p_{(\omega)}$ and $W^{p,q}_{(\omega)}$ respectively.

In what follows we let σ_s and $\sigma_{s,t}$ be the weights

$$\sigma_s(x,\xi) = \langle x,\xi\rangle^s \quad \text{and} \quad \sigma_{s,t}(x,\xi) = \langle x\rangle^t \langle \xi\rangle^s, \quad x,\xi \in \mathbf{R}^d. \tag{2.6}$$

Remark 2.4. Several important spaces agree with certain modulation spaces. In fact, let $s,t \in \mathbf{R}$. Then $M^2_{(\sigma_{s,t})}(\mathbf{R}^d)$ is equal to the weighted Sobolev space (or Sobolev-Kato space) $H^2_{s,t}(\mathbf{R}^d)$ in [5, 16], the set of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $\langle x\rangle^t \langle D\rangle^s f \in L^2$. In particular, $M^2_{(\sigma_{s,0})}$ and $M^2_{(\sigma_{0,t})}$ are equal to H^2_s and L^2_t, respectively. Furthermore, $M^2_{(\sigma_s)}(\mathbf{R}^d)$ is equal to the Shubin-Sobolev space of order s. (Cf. e. g. [15]).

2.3. Pseudo-differential operators and SG-symbol classes

Next we recall some facts in Chapter XVIII in [14] concerning pseudo-differential operators. Let $a \in \mathscr{S}(\mathbf{R}^{2d})$, and $t \in \mathbf{R}$ be fixed. Then the pseudo-differential operator $\operatorname{Op}_t(a)$ is the linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$ defined by the formula

$$(\operatorname{Op}_t(a)f)(x) = (2\pi)^{-d} \iint a((1-t)x + ty, \xi) f(y) e^{i\langle x-y,\xi\rangle} \, dy d\xi. \tag{2.7}$$

For general $a \in \mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_t(a)$ is defined as the continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

$$K_{t,a}(x,y) = (2\pi)^{-d/2} (\mathscr{F}_2^{-1} a)((1-t)x + ty, x-y). \tag{2.8}$$

If $t = 0$, then $\operatorname{Op}_t(a)$ is the Kohn-Nirenberg representation $\operatorname{Op}(a) = a(x,D)$, and if $t = 1/2$, then $\operatorname{Op}_t(a)$ is the Weyl quantization.

We now recall the definition of the generalized SG-symbol classes. Let $m,\mu,r,\rho \in \mathbf{R}$ be fixed. Then $\mathrm{SG}^{m,\mu}_{r,\rho}(\mathbf{R}^{2d})$ is the set of all $a \in C^\infty(\mathbf{R}^{2d})$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \lesssim \langle x\rangle^{m-r|\alpha|} \langle \xi\rangle^{\mu-\rho|\beta|},$$

for all multi-indices α and β. Usually we assume that $r,\rho \ge 0$ and $\rho + r > 0$.

More generally, assume that $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. Then $\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$ consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \lesssim \omega(x,\xi) \langle x\rangle^{-r|\alpha|} \langle \xi\rangle^{-\rho|\beta|}, \qquad x,\xi \in \mathbf{R}^d, \tag{2.9}$$

for all multi-indices α and β. We note that

$$\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d}) = S(\omega, g_{r,\rho}), \tag{2.10}$$

when $g = g_{r,\rho}$ is the Riemannian metric on $\mathbf{R}^{2d}$, defined by the formula

$$\big(g_{r,\rho}\big)_{(y,\eta)}(x,\xi) = \langle y\rangle^{-2r}|x|^2 + \langle \eta\rangle^{-2\rho}|\xi|^2 \tag{2.11}$$

(cf. Section 18.4–18.6 in [14]). Furthermore, $\mathrm{SG}^{(\omega)}_{r,\rho} = \mathrm{SG}^{m,\mu}_{r,\rho}$ when $\omega(x,\xi) = \langle x\rangle^m\langle \xi\rangle^\mu$.

It is a well-known fact that SG-operators give rise to linear continuous mappings from $\mathscr{S}(\mathbf{R}^d)$ to itself, extendable as linear continuous mappings from $\mathscr{S}'(\mathbf{R}^d)$ to itself. They also act continuously between modulation spaces. Indeed, see [4], if $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, then $\mathrm{Op}_t(a)$ is continuous from $M(\omega,\mathscr{B})$ to $M(\omega/\omega_0,\mathscr{B})$. Moreover, there exist $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ such that for every choice of $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and every translation invariant BF-space $\mathscr{B}$ on $\mathbf{R}^{2d}$, the mappings

$$\mathrm{Op}_t(a)\,:\,\mathscr{S}(\mathbf{R}^d)\to\mathscr{S}(\mathbf{R}^d),\quad \mathrm{Op}_t(a)\,:\,\mathscr{S}'(\mathbf{R}^d)\to\mathscr{S}'(\mathbf{R}^d)$$

$$\text{and}\quad \mathrm{Op}_t(a)\,:\,M(\omega,\mathscr{B})\to M(\omega/\omega_0,\mathscr{B}).$$

are continuous bijections with inverses $\mathrm{Op}_t(b)$.

Definition 2.5. Let $r,\rho\in[0,1]$, $t\in\mathbf{R}$, $\mathcal{B}$ be a topological vector space of distributions on $\mathbf{R}^d$ such that

$$\mathscr{S}(\mathbf{R}^d)\subseteq\mathcal{B}\subseteq\mathscr{S}'(\mathbf{R}^d)$$

with continuous embeddings. Then $\mathcal{B}$ is called SG-*admissible (with respect to r, ρ and d)* when $\mathrm{Op}_t(a)$ maps $\mathcal{B}$ continuously into itself, for every $a\in\mathrm{SG}^{0,0}_{r,\rho}$.

If $\mathcal{B}$ and $\mathcal{C}$ are SG-admissible with respect to r, ρ and d, and $\omega_0\in\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, then the pair $(\mathcal{B},\mathcal{C})$ is called SG-*ordered (with respect to ω_0)*, when the mappings

$$\mathrm{Op}_t(a)\,:\,\mathcal{B}\to\mathcal{C}\quad\text{and}\quad\mathrm{Op}_t(b)\,:\,\mathcal{C}\to\mathcal{B}$$

are continuous for every $a\in\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ and $b\in\mathrm{SG}^{(1/\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$.

Remark 2.6. Let t, r, ρ, ω and ω_0 be as in Definition 2.5, and let $\mathcal{B}$ be SG-admissible with respect to r, ρ and d. Then there is a unique SG-admissible $\mathcal{C}$ such that $(\mathcal{B},\mathcal{C})$ is an SG-ordered pair with respect to ω_0. In fact, let a be as above. Then $\mathcal{C}$ is the image of $\mathcal{B}$ under $\mathrm{Op}_t(a)$.

In particular, $\mathscr{S}(\mathbf{R}^d)$, $\mathscr{S}'(\mathbf{R}^d)$ and $M(\omega,\mathscr{B})$ are SG-admissible, and

$$(\mathscr{S}(\mathbf{R}^d),\mathscr{S}(\mathbf{R}^d)),\quad(\mathscr{S}'(\mathbf{R}^d),\mathscr{S}'(\mathbf{R}^d))\quad\text{and}\quad(M(\omega,\mathscr{B}),M(\omega/\omega_0,\mathscr{B}))$$

are SG-ordered with respect to ω_0.

If $a\in\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, then

$$|a(x,\xi)|\lesssim\omega_0(x,\xi).$$

On the other hand, a is invertible, in the sense that $1/a$ is a symbol in $\mathrm{SG}^{(1/\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, if and only if

$$\omega_0(x,\xi)\lesssim|a(x,\xi)|. \tag{2.12}$$

A slightly relaxed condition appears when (2.12) hold for all points (x,ξ), outside a compact set $K\subseteq\mathbf{R}^{2d}$. In this case we say that a is *elliptic* (with respect to ω_0).

Definition 2.7. Let $r, \rho \geq 0$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ and let $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$.

1. a is called *locally* or *type*-1 *invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \backslash 0)$, if there exist a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and a positive constant R such that (2.12) holds for $x \in X$, $\xi \in \Gamma$ and $|\xi| \geq R$.
2. a is called *Fourier-locally* or *type*-2 *invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in (\mathbf{R}^d \setminus 0) \times \mathbf{R}^d$, if there exist an open conical neighbourhood Γ of x_0, a neighbourhood X of ξ_0 and a positive constant R such that (2.12) holds for $x \in \Gamma$, $|x| \geq R$ and $\xi \in X$.
3. a is called *oscillating* or *type*-3 *invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in (\mathbf{R}^d \setminus 0) \times (\mathbf{R}^d \setminus 0)$, if there exist open conical neighbourhoods Γ_1 of x_0 and Γ_2 of ξ_0, and a positive constant R such that (2.12) holds for $x \in \Gamma_1$, $|x| \geq R$, $\xi \in \Gamma_2$ and $|\xi| \geq R$.

If $m \in \{1, 2, 3\}$ and a is *not* type-m invertible with respect to ω_0 at (x_0, ξ_0), then (x_0, ξ_0) is called *type-m characteristic* for a with respect to ω_0. The set of type-m characteristic points for a with respect to ω_0 is denoted by $\mathrm{Char}^m_{(\omega_0)}(a)$.

The *(global) set of characteristic points* (the characteristic set), for a symbol $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ with respect to ω_0, is

$$\mathrm{Char}(a) = \mathrm{Char}_{(\omega_0)}(a) = \mathrm{Char}^1_{(\omega_0)}(a) \bigcup \mathrm{Char}^2_{(\omega_0)}(a) \bigcup \mathrm{Char}^3_{(\omega_0)}(a).$$

Remark 2.8. In the case $\omega_0 = 1$ we exclude the phrase "with respect to ω_0" in Definition 2.7. For example, $a \in \mathrm{SG}^{0,0}_{r,\rho}(\mathbf{R}^{2d})$ is *type*-1 *invertible* at $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \backslash 0)$ if $(x_0, \xi_0) \notin \mathrm{Char}^1_{(\omega_0)}(a)$ with $\omega_0 = 1$. This means that there exist a neighbourhood X of x_0, an open conical neighbourhood Γ of ξ_0 and $R > 0$ such that (2.12) holds for $\omega_0 = 1$, $x \in X$ and $\xi \in \Gamma$ satisfies $|\xi| \geq R$.

In the next definition we introduce different classes of cutoff functions (see also Definition 1.9 in [3]).

Definition 2.9. Let $X \subseteq \mathbf{R}^d$ be open, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone, $x_0 \in X$ and let $\xi_0 \in \Gamma$.

1. A smooth function φ on $\mathbf{R}^d$ is called a *cutoff (function)* with respect to x_0 and X, if $0 \leq \varphi \leq 1$, $\varphi \in C_0^\infty(X)$ and $\varphi = 1$ in an open neighbourhood of x_0. The set of cutoffs with respect to x_0 and X is denoted by $\mathscr{C}_{x_0}(X)$ or $\mathscr{C}_{x_0}$.
2. A smooth function ψ on $\mathbf{R}^d$ is called a *directional cutoff (function)* with respect to ξ_0 and Γ, if there is a constant $R > 0$ and open conical neighbourhood $\Gamma_1 \subseteq \Gamma$ of ξ_0 such that the following is true:
   - $0 \leq \psi \leq 1$ and $\operatorname{supp} \psi \subseteq \Gamma$;
   - $\psi(t\xi) = \psi(\xi)$ when $t \geq 1$ and $|\xi| \geq R$;
   - $\psi(\xi) = 1$ when $\xi \in \Gamma_1$ and $|\xi| \geq R$.

   The set of directional cutoffs with respect to ξ_0 and Γ is denoted by $\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$ or $\mathscr{C}^{\mathrm{dir}}_{\xi_0}$.

Remark 2.10. Let $X \subseteq \mathbf{R}^d$ be open and $\Gamma, \Gamma_1, \Gamma_2 \subseteq \mathbf{R}^d \backslash 0$ be open cones. Then the following is true.

1. if $x_0 \in X$, $\xi_0 \in \Gamma$, $\varphi \in \mathscr{C}_{x_0}(X)$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$, then $c_1 = \varphi \otimes \psi$ belongs to $\mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$, and is type-1 invertible at (x_0, ξ_0);
2. if $x_0 \in \Gamma$, $\xi_0 \in X$, $\psi \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma)$ and $\varphi \in \mathscr{C}_{\xi_0}(X)$, then $c_2 = \psi \otimes \varphi$ belongs to $\mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$, and is type-2 invertible at (x_0, ξ_0);
3. if $x_0 \in \Gamma_1$, $\xi_0 \in \Gamma_2$, $\psi_1 \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma_1)$ and $\psi_2 \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma_2)$, then $c_3 = \psi_1 \otimes \psi_2$ belongs to $\mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$, and is type-3 invertible at (x_0, ξ_0).

The next proposition shows that $\mathrm{Op}_t(a)$ for $t \in \mathbf{R}$ satisfies convenient invertibility properties of the form

$$\mathrm{Op}_t(a)\,\mathrm{Op}_t(b) = \mathrm{Op}_t(c) + \mathrm{Op}_t(h), \tag{2.13}$$

outside the set of characteristic points for a symbol a. Here $\mathrm{Op}_t(b)$, $\mathrm{Op}_t(c)$ and $\mathrm{Op}_t(h)$ have the roles of "local inverse", "local identity" and smoothing operators respectively. From these statements it also follows that our set of characteristic points in Definition 2.7 are related to those in [5, 14].

We let $\mathbb{I}_m$ and Ω_m, $m = 1, 2, 3$, be the sets

$$\mathbb{I}_1 \equiv [0,1] \times (0,1], \qquad \mathbb{I}_2 \equiv (0,1] \times [0,1], \qquad \mathbb{I}_3 \equiv (0,1] \times (0,1] = \mathbb{I}_1 \cap \mathbb{I}_2,$$

and

$$\begin{aligned} &\Omega_1 = \mathbf{R}^d \times (\mathbf{R}^d \setminus 0), \qquad \Omega_2 = (\mathbf{R}^d \setminus 0) \times \mathbf{R}^d, \\ &\Omega_3 = (\mathbf{R}^d \setminus 0) \times (\mathbf{R}^d \setminus 0), \end{aligned} \tag{2.14}$$

which will be useful in the formulation of our results.

Proposition 2.11. *Let $m \in \{1, 2, 3\}$, $(r, \rho) \in \mathbb{I}_m$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ and let $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$. Also let Ω_m be as in* (2.14)*, $(x_0, \xi_0) \in \Omega_m$, and let (r_0, ρ_0) be equal to $(r, 0)$, $(0, \rho)$ and (r, ρ) when m is equal to* 1*,* 2 *and* 3*, respectively. Then the following conditions are equivalent:*

1. $(x_0, \xi_0) \notin \mathrm{Char}^m_{(\omega_0)}(a)$*;*
2. *there is an element $c \in \mathrm{SG}^{0,0}_{r,\rho}$ which is type-m invertible at (x_0, ξ_0), and an element $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ such that $ab = c$;*
3. (2.13) *holds for some $c \in \mathrm{SG}^{0,0}_{r,\rho}$ which is type-m invertible at (x_0, ξ_0), and some elements $h \in \mathrm{SG}^{-r_0,-\rho_0}_{r,\rho}$ and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$;*
4. (2.13) *holds for some $c_m \in \mathrm{SG}^{0,0}_{r,\rho}$ in Remark* 2.10 *which is type-m invertible at (x_0, ξ_0), and some elements h and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$, where $h \in \mathscr{S}$ when $m \in \{1, 3\}$ and $h \in \mathrm{SG}^{-\infty,0}$ when $m = 2$.*

   Furthermore, if $t = 0$, then the supports of b and h can be chosen to be contained in $X \times \mathbf{R}^d$ when $m = 1$, in $\Gamma \times \mathbf{R}^d$ when $m = 2$, and in $\Gamma_1 \times \mathbf{R}^d$ when $m = 3$.

3. Global wave-front sets and SG-pseudo-differential operators

In this section we recall the definition given in [4] of global wave-front sets for temperate distributions with respect to Banach or Fréchet spaces and state some of their properties. We first introduce the complements of the wave-front sets. More precisely, let Ω_m, $m \in \{1,2,3\}$, be given by (2.14), $\mathcal{B}$ be a Banach or Fréchet space such that $\mathscr{S}(\mathbf{R}^d) \subseteq \mathcal{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the point $(x_0, \xi_0) \in \Omega_m$ is called *type-m* regular for f with respect to $\mathcal{B}$, if

$$\mathrm{Op}(c_m) f \in \mathcal{B}, \tag{3.1}$$

for some c_m in Remark 2.10. The set of all type-m regular points for f with respect to $\mathcal{B}$, is denoted by $\Theta^m_{\mathcal{B}}(f)$.

Definition 3.1. Let $m \in \{1,2,3\}$, Ω_m be as in (2.14), and let $\mathcal{B}$ be a Banach or Fréchet space such that $\mathscr{S}(\mathbf{R}^d) \subseteq \mathcal{B} \subset \mathscr{S}'(\mathbf{R}^d)$.

1. the *type-m wave-front set* of $f \in \mathscr{S}'(\mathbf{R}^d)$ with respect to $\mathcal{B}$ is the complement of $\Theta^m_{\mathcal{B}}(f)$ in Ω_m, and is denoted by $\mathrm{WF}^m_{\mathcal{B}}(f)$;
2. the *global wave-front set* $\mathrm{WF}_{\mathcal{B}}(f) \subseteq (\mathbf{R}^d \times \mathbf{R}^d) \backslash 0$ is the set

$$\mathrm{WF}_{\mathcal{B}}(f) \equiv \mathrm{WF}^1_{\mathcal{B}}(f) \bigcup \mathrm{WF}^2_{\mathcal{B}}(f) \bigcup \mathrm{WF}^3_{\mathcal{B}}(f).$$

The sets $\mathrm{WF}^1_{\mathcal{B}}(f)$, $\mathrm{WF}^2_{\mathcal{B}}(f)$ and $\mathrm{WF}^3_{\mathcal{B}}(f)$ in Definition 3.1, are also called the *local*, *Fourier-local* and *oscillating* wave-front set of f with respect to $\mathcal{B}$.

From now on we assume that $\mathcal{B}$ in Definition 3.1 is SG-admissible, and recall that Sobolev-Kato spaces and, more generally, modulation spaces, and $\mathscr{S}(\mathbf{R}^d)$ are SG-admissible. (Cf. Definition 2.5, and Remarks 2.4 and 2.6.)

The next result describes the relation between "regularity with respect to $\mathcal{B}$" of temperate distributions and global wave-front sets, which is the aspect of the theory we are focused on in this paper.

Theorem 3.2. *Let $\mathcal{B}$ be* SG*-admissible, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then*

$$f \in \mathcal{B} \quad \Longleftrightarrow \quad \mathrm{WF}_{\mathcal{B}}(f) = \emptyset.$$

For the sake of completeness, we recall that microlocality and microellipticity hold for our global wave-front sets and pseudo-differential operators in $\mathrm{Op}(\mathrm{SG}^{(\omega_0)}_{r,\rho})$, see [4]. This implies that operators which are elliptic with respect to $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$ when $0 < r, \rho \le 1$ preserve the global wave-front set of temperate distributions. We recall that a and $\mathrm{Op}(a)$ are called SG-*elliptic* with respect to $\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ or ω_0, if there is a compact set $K \subset \mathbf{R}^{2d}$ such that (2.12) holds when $(x,\xi) \notin K$. By (2.9) it follows that

$$|D^\alpha_x D^\beta_\xi a(x,\xi)| \lesssim |a(x,\xi)| \langle x \rangle^{-r|\alpha|} \langle \xi \rangle^{-\rho|\beta|}, \qquad (x,\xi) \in \mathbf{R}^{2d} \setminus K,$$

for every multi-index α, when a is SG-elliptic (see, e.g., [14, 1]). The following result is an immediate corollary of microlocality and microellipticity for operators in $\mathrm{Op}(\mathrm{SG}^{(\omega_0)}_{r,\rho})$:

Theorem 3.3. *Let* $m \in \{1,2,3\}$*,* $(r,\rho) \in \mathbb{I}_m$*,* $t \in \mathbf{R}$*,* $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$*,* $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ *be* SG*-elliptic with respect to* ω_0 *and let* $f \in \mathscr{S}'(\mathbf{R}^d)$*. Moreover, let* $(\mathcal{B},\mathcal{C})$ *be a* SG*-ordered pair with respect to* ω_0*. Then*

$$\mathrm{WF}^m_{\mathcal{C}}(\mathrm{Op}_t(a)f) = \mathrm{WF}^m_{\mathcal{B}}(f).$$

4. Wave-front sets with respect to sequences of spaces

In this section we recall the definition of wave-front sets based on sequences of admissible spaces, see [4], and prove the result corresponding to Theorem 3.2 in this more general situation. In the first part we consider sequences of spaces which are parameterized with one index. Thereafter we discuss further extensions where we consider sequences of spaces which are parameterized with two indices. Here we also recall wave-front sets which are related to "classical wave-front sets", in the sense that they are wave-front sets with respect to classical spaces of smooth functions. In particular, a refinement of the wave-front set of Schwartz-type treated in [5] can also be obtained as a wave-front set based on sequences of admissible spaces, see [4]. An example is discussed at the end of the section.

4.1. Wave-front sets with respect to sequences with one index parameter

Again we start by introducing the complements of the wave-front sets. More precisely, let J be an index set of integers, Ω_m, $m \in \{1,2,3\}$, be given by (2.14), $(\mathcal{B}_j) = (\mathcal{B}_j)_{j\in J}$, be a sequence of Banach or Fréchet spaces such that $\mathscr{S}(\mathbf{R}^d) \subseteq \mathcal{B}_j \subseteq \mathscr{S}'(\mathbf{R}^d)$, for every j, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the point $(x_0,\xi_0) \in \Omega_m$ is called *type*-$(m,\cup)$ regular (*type*-$(m,\cap)$ regular) for f with respect to $(\mathcal{B}_j)$, if

$$\mathrm{Op}(c_m)f \in \bigcap_j \mathcal{B}_j \qquad \left(\mathrm{Op}(c_m)f \in \bigcup_j \mathcal{B}_j\right), \tag{4.1}$$

for some c_m in Remark 2.10 The set of all type-$m,\cup$ regular points (type-$m,\cap$ regular points) for f with respect to $(\mathcal{B}_j)$, is denoted by $\Theta^{m,\cup}_{(\mathcal{B}_j)}(f)$ $(\Theta^{m,\cap}_{(\mathcal{B}_j)}(f))$.

It is also desirable that right-hand sides of (4.1) should be a vector space, which is guaranteed by imposing that $(\mathcal{B}_j)$ should be *ordered*, i. e. $\mathcal{B}_j$ should be increasing or decreasing with respect to $j \in J$.

Definition 4.1. Let J be an index set of integers, $m \in \{1,2,3\}$, Ω_m be as in (2.14), and let $(\mathcal{B}_j)_{j\in J}$ be a sequence of Banach or Fréchet space such that $\mathscr{S}(\mathbf{R}^d) \subseteq \mathcal{B}_j \subset \mathscr{S}'(\mathbf{R}^d)$, for every j.

1. the *type*-$(m,\cup)$ *wave-front set* (*type*-$(m,\cap)$ *wave-front set*) of $f \in \mathscr{S}'(\mathbf{R}^d)$ with respect to $(\mathcal{B}_j)$ is the complement of $\Theta^{m,\cup}_{(\mathcal{B}_j)}(f)$ $(\Theta^{m,\cap}_{(\mathcal{B}_j)}(f))$ in Ω_m, and is denoted by $\mathrm{WF}^{m,\cup}_{(\mathcal{B}_j)}(f)$ $(\mathrm{WF}^{m,\cap}_{(\mathcal{B}_j)}(f))$;

2. the *global wave-front sets* $\mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f) \subseteq (\mathbf{R}^d \times \mathbf{R}^d)\backslash 0$ and $\mathrm{WF}^{\cap}_{(\mathcal{B}_j)}(f) \subseteq (\mathbf{R}^d \times \mathbf{R}^d)\backslash 0$, of $\cup$ and $\cap$ types, respectively, are the sets

$$\mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f) \equiv \mathrm{WF}^{1,\cup}_{(\mathcal{B}_j)}(f) \bigcup \mathrm{WF}^{2,\cup}_{(\mathcal{B}_j)}(f) \bigcup \mathrm{WF}^{3,\cup}_{(\mathcal{B}_j)}(f),$$

$$\mathrm{WF}^{\cap}_{(\mathcal{B}_j)}(f) \equiv \mathrm{WF}^{1,\cap}_{(\mathcal{B}_j)}(f) \bigcup \mathrm{WF}^{2,\cap}_{(\mathcal{B}_j)}(f) \bigcup \mathrm{WF}^{3,\cap}_{(\mathcal{B}_j)}(f).$$

Example. We can consider wave-front sets with respect to sequences of the form

$$(\mathcal{B}_j) \equiv (\mathcal{B}_j)_{j\in J}, \quad \text{with} \quad \mathcal{B}_j = M(\omega_j, \mathscr{B}_j), \tag{4.2}$$

where $\omega_j \in \mathscr{P}(\mathbf{R}^{2d})$, $\mathscr{B}_j$ is a translation invariant BF-space on $\mathbf{R}^d$, and j belongs to some index set J.

Remark 4.2. Let $p_j, q_j \in [1, \infty]$, $\mathscr{B}_j = L^{p_j,q_j}_1(\mathbf{R}^{2d})$, $\omega_j(x,\xi) = \langle x,\xi\rangle^{-j}$ and let $\mathcal{B}_j$ be as in (4.2) for $j \in J = \mathbf{N}_0$. Then it follows that $\mathrm{WF}^{m,\cup}_{(\mathcal{B}_j)}(f)$, $m = 1,2,3$, in Definition 4.1 are equal to the wave-front sets $\mathrm{WF}^{\psi}(f)$, $\mathrm{WF}^{e}(f)$ and $\mathrm{WF}^{\psi e}(f)$ in [5], respectively. In particular, it follows that $\mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f)$ is equal to the global wave-front set $\mathrm{WF}_{\mathscr{S}}(f)$, which in [5] is denoted by $\mathrm{WF}_{\mathcal{S}}(f)$.

Remark 4.3. Evidently, if $\mathcal{B}_j = \mathcal{B}$ for every $j \in J$, then

$$\mathrm{WF}^{m,\cup}_{(\mathcal{B}_j)}(f) = \mathrm{WF}^{m,\cap}_{(\mathcal{B}_j)}(f) = \mathrm{WF}^{m}_{\mathcal{B}}(f), \quad m = 1,2,3.$$

Proposition 4.4. *Let $m \in \{1,2,3\}$, $\mathcal{B}_j$ be the same as in Definition* 4.1*, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then*

$$\bigcup \mathrm{WF}^{m}_{\mathcal{B}_j}(f) \subseteq \mathrm{WF}^{m,\cup}_{(\mathcal{B}_j)}(f), \quad \bigcup \mathrm{WF}_{\mathcal{B}_j}(f) \subseteq \mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f)$$

and

$$\bigcap \mathrm{WF}^{m}_{\mathcal{B}_j}(f) = \mathrm{WF}^{m,\cap}_{(\mathcal{B}_j)}(f), \quad \bigcap \mathrm{WF}_{\mathcal{B}_j}(f) = \mathrm{WF}^{\cap}_{(\mathcal{B}_j)}(f).$$

Proof. It suffices to prove the first and third relation. Let Ω_m be as in (2.14), and let $X_0 = (x_0, \xi_0) \in \Omega_m$.

First let $X_0 \in \Theta^{m,\cup}_{\mathcal{B}_j}(f)$. Then $\mathrm{Op}(c_m)f \in \cap\mathcal{B}_j$, for some c_m as in Remark 2.10, giving that $\mathrm{Op}(c_m)f \in \mathcal{B}_j$ for every j. This implies that $X_0 \in \Theta^{m}_{\mathcal{B}_j}(f)$ for every j, i. e. $X_0 \in \cap\Theta^{m}_{\mathcal{B}_j}(f)$, and the first relation is proved.

The third relation follows from the relations

$$X_0 \in \cup\Theta^{m}_{\mathcal{B}_j}(f) \quad \Longleftrightarrow \quad X_0 \in \Theta^{m}_{\mathcal{B}_j}(f) \text{ for some } j \quad \Longleftrightarrow$$

$$\mathrm{Op}(c_m)f \in \mathcal{B}_j \text{ for some } c_m \text{ as in Remark 2.10, and some } j \quad \Longleftrightarrow$$

$$\mathrm{Op}(c_m)f \in \cup\mathcal{B}_j \text{ for some } c_m \text{ as in Remark 2.10} \quad \Longleftrightarrow \quad X_0 \in \Theta^{m,\cap}_{(\mathcal{B}_j)}(f).$$

The proof is complete. □

We can now prove the result corresponding to Theorem 3.2 for wave-front sets associated with one-parameter sequences of admissible spaces.

Theorem 4.5. *Let $\mathcal{B}_j$ be* SG*-admissible for every j, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then*

$$f \in \bigcap \mathcal{B}_j \quad \Longleftrightarrow \quad \mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f) = \emptyset \quad \Longleftrightarrow \quad \bigcup \mathrm{WF}_{\mathcal{B}_j}(f) = \emptyset,$$

and if in addition $(\mathcal{B}_j)$ is ordered, then

$$f \in \bigcup \mathcal{B}_j \quad \Longleftrightarrow \quad \mathrm{WF}^{\cap}_{(\mathcal{B}_j)}(f) = \emptyset \quad \Longleftrightarrow \quad \bigcap \mathrm{WF}_{\mathcal{B}_j}(f) = \emptyset.$$

Proof. Let (1)–(6) be the statements in the theorem. Then (2) $\Rightarrow$ (3) and (5) $\Leftrightarrow$ (6), by Proposition 4.4.

Next we prove (1) $\Rightarrow$ (2). Let $f \in \cap \mathcal{B}_j$. Then $\mathrm{Op}(c_m)f \in \cap \mathcal{B}_j$ for every c_m in Remark 2.10, and then $\mathrm{WF}^{m,\cup}_{(\mathcal{B}_j)}(f) = \emptyset$, and the implication follows.

Next we prove (3) $\Rightarrow$ (1). We have

$$\cup \, \mathrm{WF}^m_{\mathcal{B}_j}(f) = \emptyset \quad \Longrightarrow \quad \mathrm{WF}^m_{\mathcal{B}_j}(f) = \emptyset \text{ for all } j \quad \Longrightarrow$$

$$f \in \mathcal{B}_j \text{ for all } j \quad \Longrightarrow \quad f \in \cap \mathcal{B}_j,$$

and we have proved that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

We have

$$f \in \cup \mathcal{B}_j \quad \Longrightarrow \quad f \in \mathcal{B}_j \text{ for some } j = j_0 \quad \Longrightarrow$$

$$\mathrm{Op}(c_m)f \in \mathcal{B}_{j_0} \text{ for every } c_m \text{ as in Remark 2.10} \quad \Longrightarrow$$

$$\mathrm{WF}^m_{\mathcal{B}_{j_0}}(f) = \emptyset \quad \Longrightarrow \quad \cap \, \mathrm{WF}^m_{\mathcal{B}_j}(f) = \emptyset,$$

which shows that (4) $\Rightarrow$ (6).

Finally, if $\cap \, \mathrm{WF}^m_{\mathcal{B}_j}(f) = \emptyset$, then $\cap_{k=1}^N \mathrm{WF}^m_{\mathcal{B}_{j_k}}(f) = \emptyset$ for some $j_1, \dots, j_N$, by compactness. Let $\mathcal{B}' = \cup_1^N \mathcal{B}_{j_k}$. Then $\mathrm{WF}^m_{\mathcal{B}'}(f) = \emptyset$, since $(\mathcal{B}_j)$ are ordered. This implies that $f \in \cup_1^N \mathcal{B}_{j_k}$, which in turn implies that $f \in \cup \mathcal{B}_j$. This gives that (6) $\Rightarrow$ (4), and the proof is complete. □

4.2. Wave-front sets with respect to sequences of spaces with two indices parameters

Next we shall consider wave-front sets with respect to sequences of spaces, parameterized with two indices, and start by introducing the complements of the wave-front sets. More precisely, let J be an index set of integers, Ω_m, $m \in \{1,2,3\}$, be given by (2.14), $(\mathcal{B}_{j,k}) = (\mathcal{B}_{j,k})_{j,k \in J}$, be a sequence of Banach or Fréchet spaces such that $\mathscr{S}(\mathbf{R}^d) \subseteq \mathcal{B}_{j,k} \subseteq \mathscr{S}'(\mathbf{R}^d)$, for every j,k, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the point $(x_0, \xi_0) \in \Omega_m$ is called *type*-$(m, \cup\cap)$ regular (*type*-$(m, \cap\cup)$ regular) for f with respect to $(\mathcal{B}_{j,k})$, if

$$\mathrm{Op}(c_m)f \in \bigcap_j \left(\bigcup_k \mathcal{B}_{j,k} \right) \qquad \left(\mathrm{Op}(c_m)f \in \bigcup_j \left(\bigcap_k \mathcal{B}_{j,k} \right) \right), \tag{4.3}$$

for some c_m in Remark 2.10. The set of all type-$m, \cup\cap$ regular points (type-$m, \cap\cup$ regular points) for f with respect to $(\mathcal{B}_{j,k})$, is denoted by $\Theta^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)$ ($\Theta^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f)$).

Also in here it is desirable that right-hand sides of (4.3) should be a vector space. For this reason, the sequence $(\mathcal{B}_{j,k})$ is called *ordered with respect to* j, if $\mathcal{B}_{j,k}$ increases with j for every k fixed, or decreases with j for every k fixed. The definition of ordered sequences with respect to k is defined in analogous way.

Definition 4.6. Let J be an index set, $m \in \{1,2,3\}$, Ω_m be as in (2.14), and let $(\mathcal{B}_{j,k})_{j,k\in J}$ be a sequence of Banach or Fréchet space such that $\mathscr{S}(\mathbf{R}^d) \subseteq \mathcal{B}_{j,k} \subset \mathscr{S}'(\mathbf{R}^d)$, for every j.

1. the *type-*$(m,\cup\cap)$ *wave-front set* (*type-*$(m,\cap\cup)$ *wave-front set*) of $f \in \mathscr{S}'(\mathbf{R}^d)$ with respect to $(\mathcal{B}_{j,k})$ is the complement of $\Theta^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)$ $(\Theta^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f))$ in Ω_m, and is denoted by $\mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)$ $(\mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f))$;
2. the *global wave-front sets* $\mathrm{WF}^{\cup\cap}_{(\mathcal{B}_{j,k})}(f) \subseteq (\mathbf{R}^d\times\mathbf{R}^d)\backslash 0$ and $\mathrm{WF}^{\cap\cup}_{(\mathcal{B}_{j,k})}(f) \subseteq (\mathbf{R}^d\times\mathbf{R}^d)\backslash 0$, of $\cup\cap$ and $\cap\cup$ types, respectively, are the sets

$$\mathrm{WF}^{\cup\cap}_{(\mathcal{B}_{j,k})}(f) \equiv \mathrm{WF}^{1,\cup\cap}_{(\mathcal{B}_{j,k})}(f) \bigcup \mathrm{WF}^{2,\cup\cap}_{(\mathcal{B}_{j,k})}(f) \bigcup \mathrm{WF}^{3,\cup\cap}_{(\mathcal{B}_{j,k})}(f),$$

$$\mathrm{WF}^{\cap\cup}_{(\mathcal{B}_{j,k})}(f) \equiv \mathrm{WF}^{1,\cap\cup}_{(\mathcal{B}_{j,k})}(f) \bigcup \mathrm{WF}^{2,\cap\cup}_{(\mathcal{B}_{j,k})}(f) \bigcup \mathrm{WF}^{3,\cap\cup}_{(\mathcal{B}_{j,k})}(f).$$

Remark 4.7. In analogy with Remark 4.3 we notice that if $\mathcal{B}_{j,k} = \mathcal{B}_j$ is independent of $k \in J$, then

$$\mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f) = \mathrm{WF}^{m,\cup}_{(\mathcal{B}_j)}(f), \quad \mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f) = \mathrm{WF}^{m,\cap}_{(\mathcal{B}_j)}(f), \quad m = 1,2,3.$$

Hence, the families of wave-front sets in Definition 4.6 contain the wave-front sets in Definition 4.1.

Remark 4.8. We observe that if $m \in \{1,2,3\}$, $\mathcal{B}_{j,k}$ is SG-admissible for every j,k, Ω_m is given by (2.14) and $f \in \mathscr{S}'(\mathbf{R}^d)$, then $\mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)$ and $\mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f)$ are closed subsets of Ω_m.

From now on we assume that the involved sequence spaces, $(\mathcal{B}_{j,k})$, are ordered with respect to k when wave-front sets of the form $\mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)$ are involved, and ordered with respect to j when wave-front sets of the form $\mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f)$ are involved.

Proposition 4.9. *Let* $m \in \{1,2,3\}$*,* $\mathcal{B}_{j,k}$ *be the same as in Definition* 4.6*, and let* $f \in \mathscr{S}'(\mathbf{R}^d)$*. Then*

$$\bigcup_j \Big(\bigcap_k \mathrm{WF}^m_{\mathcal{B}_{j,k}}(f) \Big) \subseteq \mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f), \quad \bigcup_j \Big(\bigcap_k \mathrm{WF}_{\mathcal{B}_{j,k}}(f) \Big) \subseteq \mathrm{WF}^{\cup\cap}_{(\mathcal{B}_{j,k})}(f),$$

and

$$\bigcap_j \Big(\bigcup_k \mathrm{WF}^m_{\mathcal{B}_{j,k}}(f) \Big) \subseteq \mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f), \quad \bigcap_j \Big(\bigcup_k \mathrm{WF}_{\mathcal{B}_{j,k}}(f) \Big) \subseteq \mathrm{WF}^{\cap\cup}_{(\mathcal{B}_{j,k})}(f).$$

Proof. It suffices to prove the first and third inclusion. Let $X_0 \in \Theta^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)$. Then $\operatorname{Op}(c_m)f \in \cap_j(\cup_k \mathcal{B}_{j,k})$ for some c_m in Remark 2.10, which implies that $\operatorname{Op}(c_m)f \in \cup_k \mathcal{B}_{j,k}$ for every j, i.e. for every j, there is a $k = k(j)$ such that $\operatorname{Op}(c_m)f \in \mathcal{B}_{j,k(j)}$.

This means that $X_0 \in \Theta^m_{j,k(j)}$ for every j, giving that $X_0 \in \cup_k \Theta^m_{\mathcal{B}_{j,k}}(f)$ for every j. Hence $X_0 \in \cap_j(\cup_k \Theta^m_{\mathcal{B}_{j,k}}(f))$, and the first inclusion follows.

Next assume that $X \in \Theta^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f)$. Then $\operatorname{Op}(c_m)f \in \cup_j(\cap_k \mathcal{B}_{j,k})$ for some c_m in Remark 2.10. Hence $\operatorname{Op}(c_m)f \in \cap_k \mathcal{B}_{j,k}$, for some $j = j_0$. This implies that $\operatorname{Op}(c_m)f \in \mathcal{B}_{j_0,k}$ for every k, giving that $X_0 \in \Theta^m_{\mathcal{B}_{j_0,k}}(f)$ for every k. Hence $X_0 \in \cap_k \Theta^m_{\mathcal{B}_{j_0,k}}(f) \subseteq \cup_j(\cap_k \Theta^m_{\mathcal{B}_{j,k}}(f))$, and the third inclusion follows. The proof is complete. □

Theorem 4.10. *Let $\mathcal{B}_{j,k}$ be* SG-*admissible for every j and k, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then*

$$f \in \bigcap_j \Big(\bigcup_k \mathcal{B}_{j,k} \Big) \quad \Longleftrightarrow \quad \mathrm{WF}^{\cup\cap}_{(\mathcal{B}_{j,k})}(f) = \emptyset \quad \Longleftrightarrow \quad \bigcup_j \Big(\bigcap_k \mathrm{WF}_{\mathcal{B}_{j,k}}(f) \Big) = \emptyset,$$

provided $(\mathcal{B}_{j,k})$ is ordered with respect to k, and if instead $(\mathcal{B}_{j,k})$ is ordered with respect to j, then

$$f \in \bigcup_j \Big(\bigcap_k \mathcal{B}_{j,k} \Big) \quad \Longleftrightarrow \quad \mathrm{WF}^{\cap\cup}_{(\mathcal{B}_{j,k})}(f) = \emptyset \quad \Longrightarrow \quad \bigcap_j \Big(\bigcup_k \mathrm{WF}_{\mathcal{B}_{j,k}}(f) \Big) = \emptyset.$$

Remark 4.11. Some steps of the proof are dependent of the parametrix constructions in the framework of Proposition 2.11. We note that if $\mathcal{B}_{jk}$ are as in Proposition 4.9 and $\Theta^m_{\mathcal{B}_{j_0,k}}(f) = \Omega_m$, then for every $X \in \Omega_m$, there are elements $c_m = c_{m,X}$ as in Remark 2.10 such that $\operatorname{Op}(c_{m,X})f \in \cap_j(\cup_k \mathcal{B}_{j,k})$. By compactness, there are $c_{m,X_1}, \dots c_{m,X_N}$ such that if $a = c_1 + \dots + c_N$, then $a \ge 1$ outside a compact set in Ω_m. Furthermore, $\operatorname{Op}(a)f \in \cap_j(\cup_k \mathcal{B}_{j,k})$.

Let b be as in Proposition 2.11 (4). Then $\operatorname{Op}(b)$ maps $\cap_j(\cup_k \mathcal{B}_{j,k})$ into itself. Hence Proposition 2.11 gives

$$f = \operatorname{Op}(b)\operatorname{Op}(a)f \mod \mathscr{S} \subseteq \cap_j(\cup_k \mathcal{B}_{j,k}).$$

Proof of Theorem 4.10. Let (1)–(6) be the statements in the theorem. It is clear that (2) ⇒ (3) and (5) ⇒ (6), in view of Proposition 4.9.

If $f \in \cap_j(\cup \mathcal{B}_{j,k})$, then $\operatorname{Op}(c_m)f \in \cap_j(\cup \mathcal{B}_{j,k})$, for every c_m as in Remark 2.10. Consequently, $\mathrm{WF}^{\cup\cap}_{(\mathcal{B}_{j,k})}(f) = \emptyset$, and we have proved that (1) ⇒ (2). By Remark 4.11, it follows that (2) ⇒ (1), and the equivalence between (1) and (2) follows.

Next assume that (3) holds, i.e. $\cup_j(\cap_k \mathrm{WF}_{\mathcal{B}_{j,k}}(f)) = \emptyset$. Then $\cap_k \mathrm{WF}_{\mathcal{B}_{j,k}}(f) = \emptyset$ for every j. Hence $f \in \cup_k \mathcal{B}_{j,k}$ for every j, by Theorem 4.5. This implies that $f \in \cap_j(\cup \mathcal{B}_{j,k})$, and the equivalences between (1)–(3) follows.

Next we prove (4) ⇒ (5). Therefore, let $f \in \cup_j(\cap \mathcal{B}_{j,k})$. Then $\operatorname{Op}(c_m)f \in \cup_j(\cap \mathcal{B}_{j,k})$ for every c_m in Remark 2.10. Hence $\Theta^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f) = \Omega_m$, and therefore $\mathrm{WF}^{\cap\cup}_{(\mathcal{B}_{j,k})}(f) = \emptyset$. This proves the inclusion.

Finally we prove that (5) $\Rightarrow$ (4). Assume that $\mathrm{WF}^{\cap\cup}_{(\mathcal{B}_{j,k})}(f)=\emptyset$. Then for every $X\in\Omega_m$, there is an $c_m=c_{m,X}$ as in Remark 2.10 such that $\mathrm{Op}(c_m)f\in\cup_j(\cap\mathcal{B}_{j,k})$, giving that $\mathrm{Op}(c_m)\in\cap_k\mathcal{B}_{j,k}$ for some $j=j_0$. This implies that $\mathrm{Op}(c_m)f\in\mathcal{B}_{j_0,k}$, for every k. By similar arguments as in Remark 4.11, it follows that $f\in\mathcal{B}_{j_0,k}$ for every k, i.e. $f\in\cap_k\mathcal{B}_{j_0,k}\subseteq\cup_j(\cap\mathcal{B}_{j,k})$, and the assertion follows. The proof is complete. □

4.3. An example.

In a similar way as in Remark 4.2, we may construct wave-front sets with respect to the spaces $Q_0(\mathbf{R}^d)$ and $Q(\mathbf{R}^d)$ (see [4] for the definition of these spaces). In fact, let

$$p_{j,k},q_{j,k}\in[1,\infty],\quad \mathscr{B}_{j,k}=L_1^{p_{j,k},q_{j,k}}(\mathbf{R}^{2d}),\quad \omega_{j,k}(x,\xi)=\langle x\rangle^{-j}\langle\xi\rangle^{k},$$

$$\mathcal{B}_{j,k}=M(\omega_{j,k},\mathscr{B}_{j,k}),\quad \mathcal{C}_{j,k}=\mathcal{B}_{k,j}\quad\text{when}\quad j,k\in J=\mathbf{N}_0.$$

By similar arguments as in [12, Remark 2.18] it follows that

$$Q_0(\mathbf{R}^d)=\bigcup_j\Big(\bigcap_k\mathcal{B}_{j,k}\Big),\quad Q(\mathbf{R}^d)=\bigcap_j\Big(\bigcup_k\mathcal{C}_{j,k}\Big).$$

Now we define the components of the wave-front sets with respect to Q_0 and Q as

$$\mathrm{WF}^m_{Q_0}(f)=\mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f),\quad \mathrm{WF}^m_{Q}(f)=\mathrm{WF}^{m,\cup\cap}_{(\mathcal{C}_{j,k})}(f),\quad m=1,2,3,$$

when $f\in\mathscr{S}'(\mathbf{R}^d)$. By Theorem 4.10 it follows that (1.2) holds when $\mathcal{B}=Q_0(\mathbf{R}^d)$ or $\mathcal{B}=Q(\mathbf{R}^d)$. We also note that

$$\mathrm{WF}^1_{\mathscr{S}}(f)=\mathrm{WF}^1_{Q_0}(f)=\mathrm{WF}^1_{Q}(f)=\mathrm{WF}^1_{C^\infty}(f)$$

agrees with the classical wave-front set of f (see [4] and Section 8.1 in [14]).

The next result deals with the regularity of the solutions f to $Tf=g$, in terms if the regularity of the datum g, under appropriate hypotheses on the operator T. It is a straightforward consequence of the theory developed above and of the propagation results proved in [4].

Theorem 4.12. *Let $T_1=1-\Delta$ be the harmonic oscillator and $T_2=\partial_t-a(t,x,D)$ be a generalized heat operator, with the symbol $a(t,.,..)\in\mathrm{SG}^{0,0}_{r,\rho}(\mathbf{R}^{2d})$ chosen in such a way that T_2 is elliptic with respect to the weight $\omega_0=1$. Then, both T_1 and T_2 map continuously $\mathscr{S}(\mathbf{R}^d)$ to itself, Q to itself, and Q_0 to itself.*

References

[1] P. Boggiatto, E. Buzano, L. Rodino *Global Hypoellipticity and Spectral Theory*, Mathematical Research, 92, Akademie Verlag, Berlin, 1996.

[2] H. O. Cordes *The Technique of Pseudodifferential Operators*, Cambridge Univ. Press, 1995.

[3] S. Coriasco, K. Johansson, J. Toft *Local wave-front sets of Banach and Fréchet types, and pseudo-differential operators*, Monatsh. Math., 169 (3-4) (2013) 285-316.

[4] S. Coriasco, K. Johansson, J. Toft *Global wave-front sets of Banach, Fréchet and Modulation space types, and pseudo-differential operators*, J. Differential Equations (2013), DOI 10.1016/j.jde.2013.01.014.

[5] S. Coriasco, L. Maniccia *Wave front set at infinity and hyperbolic linear operators with multiple characteristics*, Ann. Global Anal. Geom., **24**, (2003), 375–400.

[6] S. Coriasco, R. Schulz *The global wave front set of tempered oscillatory integrals with inhomogeneous phase functions*, Preprint, arXiv:1207.6813v1, submitted (2012).

[7] H. G. Feichtinger and K. H. Gröchenig *Banach spaces related to integrable group representations and their atomic decompositions, I*, J. Funct. Anal., **86** (1989), 307–340.

[8] H. G. Feichtinger *Modulation spaces on locally compact abelian groups. Technical report*, University of Vienna, Vienna, 1983; also in: M. Krishna, R. Radha, S. Thangavelu (Eds) Wavelets and their applications, Allied Publishers Private Limited, NewDehli Mumbai Kolkata Chennai Hagpur Ahmedabad Bangalore Hyderbad Lucknow, 2003, pp. 99–140.

[9] H. G. Feichtinger *Modulation spaces: Looking back and ahead*, Sampl. Theory Signal Image Process. **5** (2006), 109–140.

[10] G. B. Folland *Harmonic analysis in phase space*, Princeton U. P., Princeton, 1989.

[11] K. H. Gröchenig *Foundations of Time-Frequency Analysis*, Birkhäuser, Boston, 2001.

[12] A. Holst, J. Toft, P. Wahlberg *Weyl product algebras and modulation spaces*, J. Funct. Anal., **251** (2007), 463–491.

[13] L. Hörmander *Fourier Integral Operators I.* Acta Math. **127** (1971), 79–183.

[14] L. Hörmander *The Analysis of Linear Partial Differential Operators*, vol I–III, Springer-Verlag, Berlin Heidelberg NewYork Tokyo, 1983, 1985.

[15] F. Luef, Z. Rahbani *On pseudodifferential operators with symbols in generalized Shubin classes and an application to Landay-Weyl operators*, Banach J. Math. Anal. **5** (2011), 59–72.

[16] R. Melrose Spectral and scattering theory for the Laplacian on asymptotically Euclidean spaces, In *Spectral and scattering theory (Sanda, 1992)*, number 161 in Lecture Notes in Pure and Appl. Math., pages 85–130. Dekker, New York, 1994.

[17] C. Parenti *Operatori pseudodifferenziali in* $\mathbf{R}^n$ *e applicazioni*, Ann. Mat. Pura Appl., **93** (1972), 359–389.

[18] S. Pilipović, N. Teofanov, J. Toft *Micro-local analysis in Fourier Lebesgue spaces. Part I*, J. Fourier Anal. Appl. **17** (2011), 374–407.

[19] S. Pilipović, N. Teofanov, J. Toft *Micro-local analysis in Fourier Lebesgue and modulation spaces. Part II*, J. Pseudo-Differ. Oper. Appl., **1** (2010), 341–376.

[20] J. Toft *Continuity properties for modulation spaces with applications to pseudo-differential calculus, II*, Ann. Global Anal. Geom., **26** (2004), 73–106.

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