Takeoff vs. Stagnation in Endogenous Recombinant Growth Models

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Abstract

This paper concludes the study of transition paths in the continuous-time recombinant endogenous growth model by providing numerical methods to estimate the threshold initial value of capital (a Skiba-type point) above which the economy takes off toward sustained growth in the long run, while it is doomed to stagnation otherwise. The model is based on the setting first introduced by Tsur and Zemel and then further specified by Privileggi, in which knowledge evolves according to the Weitzman recombinant process. We pursue a direct approach based on the comparison of welfare estimations along optimal consumption trajectories either diverging to sustained growth or converging to a steady state. To this purpose, we develop and test three algorithms capable of numerically simulating the initial Skiba-value of capital, each corresponding to initial stock of knowledge values belonging to three different ranges, thus covering all possible scenarios.

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Key words: Knowledge Production, Endogenous Recombinant Growth, Transition Dynamics, Turnpike, Skiba Point, Hamilton-Jacobi-Bellman equation.

1 Introduction

This paper provides a further contribution to the study of the two-sector continuous-time endogenous growth model introduced by Tsur and Zemel [24] in which knowledge evolves according to the Weitzman [25] recombinant process. Given any feasible initial stock of knowledge, we provide three numeric algorithms capable of approximating the corresponding critical initial capital value above which the economy "takes off" toward an asymptotic balanced growth path (ABGP), while it is led toward a steady point – *i.e.*, to stagnation in the long run – whenever the initial capital lies below such threshold. To this purpose we elaborate on the functional forms introduced by Privileggi [19], which are suitable to 'detrend' the model and thus obtain a closed form for the ODE defining the optimal policy that, in turn, can be approximated with a sufficient degree of accuracy by means of a projection method discussed in [20].

Weitzman [25] departs from the mainstream endogenous growth literature[1] flourished after the original works of either Romer [21], [22] – based on technology spillovers/externalities –

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[1] For recent comprehensive surveys see [4], [1] and, more oriented toward the creative-destruction Shumpeterian-style models, [3].

or Grossman and Helpman [8] and Aghion and Howitt [2] – building on the Shumpeterian tradition of the creative-destruction process involved in innovation activities – by focussing on two peculiar elements that drive knowledge generation: the process of creation of new ideas and the resources needed to turn these ideas into "productive" knowledge. The evolution of ideas is assumed to follow a *recombinant mechanism*: existing ideas are combined (matched) to generate new ideas. The number of possible matchings is a combinatorial function of the number of existing ideas that spontaneously would give rise to an unrealistic over-exponential growth. As a matter of fact, such explosive dynamic is contained by the fact that turning potentially fruitful ideas into useful knowledge requires physical resources, whose optimal allocation by a social planner has been first analyzed by Tsur and Zemel [24] in a continuous-time setting.

We consider a specification of the model in [24] where the probability of a successful matching among existing seed ideas has a hyperbolic form, a composite final good is produced in a competitive sector by means of a Cobb-Douglas function using the stock of knowledge and physical capital as input factors, and the representative household has a CIES utility function. A social planner efficiently maximizes the discounted utility of a representative consumer over an infinite time horizon by directly financing new knowledge production through a tax levied on the households; at each instant new knowledge is produced by an independent R&D sector under the supervision of the social planner. Hence, we are pursuing a first-best, social planner-type equilibrium approach, setting aside all issues regarding incentives to innovate, spillovers, externalities, etc., and the related scale effects involved by knowledge production.[2]

This hyperbolic-Cobb-Douglas-CIES specification of the model allows for a closed-form ODE defining the optimal transition dynamics (see [19]) along a characteristic curve in the knowledge-capital state space that will be labeled as (transitory) *turnpike* when the conditions for sustained long-run growth provided by [24] are met. The solution of such ODE is numerically approximated through an appropriate projection method (see [20]) and can be used to compute, by means of a finite-difference, Runge-Kutta method, the optimal time-path trajectories of the stock of knowledge, capital, output and consumption, as well as their transition growth rates, along the turnpike. However, whenever the initial capital is different than its unique value on the turnpike, different types of transition paths appear; they can either reach a point on the turnpike in a finite time period and then continue along the turnpike itself toward sustained growth, or can converge to a steady state which is a point on another characteristic curve in the knowledge-capital state space that will be called the *stagnation line*.

The aim of this paper is to thoroughly investigate the latter type of (initial) transitions. Tsur and Zemel [24] showed that, for each given initial stock of knowledge, there corresponds a unique critical value for the initial capital such that for any value above this threshold the economy will first follow a path toward the turnpike and then, along a path evolving along the turnpike itself, toward sustained growth along a ABGP. Conversely, when the initial capital is below such threshold, the process generating new knowledge does not take off and the economy eventually dies in stagnation by converging asymptotically to steady values for both knowledge and capital on the stagnation line. The properties of this threshold value are akin to those first discussed by Skiba [23]; hence we shall refer to this point as the *Skiba-point*.

We first develop a numerical method (Algorithm 1) that computes the Skiba-point *on the turnpike*, labeled as $\left(A_m, k_m^{sk}\right)$, by equating the welfare when sustained growth is triggered with the welfare associated to a path leading toward stagnation, starting from the same point $\left(A_m, k_m^{sk}\right)$. Next, we consider initial values of the stock of knowledge, A_0, which lie on the left of A_m and build a more complex Bisection method (Algorithm 2), again with the goal of matching the welfare when taking off toward the ABGP with the welfare of the economy converging to

[2] For a general and exhaustive discussion on all these issues see [13], [14] and [15].

a steady state, to find the Skiba-point when the economy starts on an initial capital, k_0, lying *above the turnpike* value corresponding to A_0. Finally, we focus on initial values A_0 lying to the right of A_m; in this case, to estimate the Skiba-point we propose another Bisection method (Algorithm 3) with the aim of equating the welfare generated by the trajectory that starts from an initial capital, k_0, *below the turnpike*, climbs up toward the turnpike, reaches it in a finite time period, and then keeps following it thereafter toward the ABGP, with that produced by the trajectory converging to a steady state starting from the same initial point (A_0, k_0).

All optimal trajectories are estimated through a mix of *projection methods* and *Runge-Kutta* type algorithms. First a projection method – based either on *OLS* or on *Orthogonal Collocation* and with a *residual function* defined by means of *Chebyshev polynomials* (see, *e.g.*, Chapter 11 in [16], Chapter 6 in [12], or Paragraph 5.5.2 in [18]) – is applied to the ODE defining the optimal policy. The approximation thus obtained can then be used in a Runge-Kutta method to generate all transition time-path trajectories. Welfare estimates along the turnpike or toward stagnation are performed through direct computation of the value function by means of the *Hamilton-Jacobi-Bellman equation* in which the derivative of the value function is calculated through the celebrated Benveniste and Scheinkman [5] result as the derivative of the instantaneous utility at the initial optimal consumption value. While this technique is immediately available for the dynamics converging to a steady state because in this case the model boils down to a standard concave Ramsey-type model, for the dynamics along the turnpike, eventually leading to steady growth, we must rely on the Hamilton-Jacobi verification principle to establish that the Hamilton-Jacobi-Bellman equation actually delivers the true value function (Proposition 4), as in this case the model turns out to be not concave in early-times. Along initial trajectories outside the turnpike eventually reaching it after a finite time period, welfare is estimated trough *Gauss-Legendre quadrature* routines, themselves built on the simulations of the consumption time-path trajectories previously calculated.

All simulations produce a rich variety of early-transition dynamic patterns, which are interesting per se. Our main findings are summarized in a plot in the knowledge-capital space reporting a number of estimated Skiba-points (Figure 8); the figure suggests that all Skiba-points lie on a decreasing curve plunging to zero as the initial knowledge approaches the intersection point between the turnpike and the stagnation line. However, the performances of Algorithm 3 rapidly degenerate as the initial knowledge level approaches this intersection point.

Because the set of Skiba-points turns out to be a curve, our results contribute to the literature, started by Haunschmied et al. [11], focussed on numerically computing the DNS-curve (so named in honour of the pioneering works of Skiba [23] and Dechert and Nishimura [7] who first introduced the notion of Skiba-point) separating the basins of attraction of different locally stable steady states (or cycling orbits) in continuous-time economic models.[3]

Section 2 reports some well known preliminary results that will be used throughout the paper. Section 3 recalls the main facts related to endogenous recombinant growth according to Weitzman [25] and Tsur and Zemel [24], and introduces the specification of Privileggi [19], [20]. Section 4 focusses on the Skiba-point $\left(A_m, k_m^{sk}\right)$ on the turnpike, characterizing the optimal dynamics along the turnpike and toward stagnation, and describes our welfare estimation techniques for this case. Sections 5 and 6 characterize the early transition dynamics outside the turnpike, leading to the elaboration of two algorithms for Skiba-point estimations above and below the turnpike respectively. In Section 7 all the algorithms are then used to approximate all types of Skiba-points for a specific example. Finally, Section 8 concludes, while the Appendix contains the proof of our main theoretical result (Proposition 4).

[3] See also Haunschmied et al. [10] and Caulkins et al. [6] and the references quoted therein.

2 Preliminaries

Here a few well known results that will be used throughout the paper for welfare evaluation purposes are reported without proof. Given a technology set $T \subseteq \mathbb{R}^{2n}$, consider the standard continuous-time problem

$$V(x_0) = \sup \int_0^{\infty} e^{-\rho t} U[x(t), \dot{x}(t)]\, dt \tag{1}$$
$$\text{subject to } [x(t), \dot{x}(t)] \in T \text{ for all } t \text{ and } x(0) = x_0,$$

where $\rho > 0$ is the discount rate and $U(\cdot, \cdot)$ is the instantaneous felicity.

Lemma 1 (Principle of optimality) *Suppose that $(x^*(t; x_0), \dot{x}^*(t; x_0))$ is a solution of (1) originating at $x(0) = x_0$. Then, for all $t_0 \geq 0$*

$$V(x_0) = \int_0^{t_0} e^{-\rho t} U[x^*(t; x_0), \dot{x}^*(t; x_0)]\, dt + e^{-\rho t_0} V[x^*(t_0; x_0)]. \tag{2}$$

We denote by $T(x)$ the x-section of the set T, *i.e.*, $T(x) = \{(x, \dot{x}) \in \mathbb{R}^{2n} : (x, \dot{x}) \in T\}$.

Theorem 1 (Hamilton-Jacobi verification principle) *Assume that:*

(i) *$w : \mathbb{R}^n \to \mathbb{R}$ is of class C^1 and satisfies the Hamilton-Jacobi-Bellman equation, i.e.,*

$$\rho w(x) = \max_{\dot{x} \in T(x)} [U(x, \dot{x}) + \nabla w(x) \cdot \dot{x}]; \tag{3}$$

(ii) *for every initial condition x_0 there is a feasible $\dot{x}^*(t; x_0)$ such that the max is attained in (3), i.e.,*

$$\rho w[x^*(t; x_0)] = U[x^*(t; x_0), \dot{x}^*(t; x_0)] + \nabla w[x^*(t_0; x_0)] \cdot \dot{x}^*(t; x_0) \tag{4}$$

for all $t \geq 0$, a.e.;

(iii) $\lim_{t \to \infty} e^{-\rho t} w[x(t; x_0)] = 0$ *for every feasible path $x(t; x_0)$.*

Then $w(x)$ is the value function of (1), i.e., $V(x) = w(x)$, and $(x^*(t; x_0), \dot{x}^*(t; x_0))$ is a solution of (1).

Theorem 2 (Benveniste and Scheinkman) *Assume that:*

(i) *T is convex and* $\operatorname{int}(T) \neq \varnothing$*;*

(ii) *$U : T \to \mathbb{R}$ is continuously differentiable on* $\operatorname{int}(T)$ *and concave;*

(iii) *an optimal solution $x^*(t; x_0)$ from x_0 (not necessarily unique) exists and $V(x)$ in (1) is defined in some neighborhood of x_0;*

(iv) *the optimal solution is interior in the following sense: there exist $h > 0$, $\varepsilon > 0$ and $M > 0$ such that for $t \in [0, h]$, $\|x^*(t; x_0), \dot{x}^*(t; x_0)\| \leq M$ and if $(z, z') \in \mathbb{R}^{2n}$ satisfies $\|(x^*(t; x_0), \dot{x}^*(t; x_0)) - (z, z')\| \leq \varepsilon$ for some $t \in [0, h]$, then $(z, z') \in T$;*

(v) *$\dot{x}^*(t; x_0)$ is a piecewise continuous function of time t.*

Then $V(x)$ in (1) is of class C^1 at x_0 and

$$\nabla V(x_0) = -\nabla_{\dot{x}} U\left[x_0, \dot{x}^*\left(0^+; x_0\right)\right], \tag{5}$$

where $\nabla_{\dot{x}} U(\cdot,\cdot)$ denotes the vector of partial derivatives of U with respect to its second argument.

A proof of Theorem 2 can be found in [5]. The following corollary provides a converse result of Theorem 1 under the value function differentiability provided by Theorem 2.

Corollary 1 *Under the same assumptions of Theorem 2 the value function $V(x)$ in (1) satisfies the Hamilton-Jacobi-Bellman equation (3), and the maximum is attained at $\dot{x} = \dot{x}^*(0^+; x_0)$, i.e.,*

$$\rho V(x_0) = U\left[x_0, \dot{x}^*\left(0^+; x_0\right)\right] + \nabla V(x_0) \cdot \dot{x}^*\left(0^+; x_0\right). \tag{6}$$

3 The Model

In continuous-time the flow of successful new ideas accruing the existing stock of knowledge is given by

$$\dot{A}(t) = H(t)\,\pi\left[J(t)/H(t)\right], \tag{7}$$

where $A(t)$ is the stock of knowledge at time t (measured as the total number of fruitful ideas), $\dot{A}(t)$ denotes its time-derivative, $H(t)$ is the number of yet unprocessed (*seed*) ideas at instant t which are combined together in order to obtain new *hybrid* seed ideas of which only a fraction turns out to be successful, according to a probability function $\pi(\cdot)$ that itself depends on the ratio between a measure of the physical resources devoted to the R&D recombinant process, $J(t)$, and the available seeds, $H(t)$, at instant t (see [24], [19] and [20]).

A. 1 *The success probability function is independent of time and is given by*[4]

$$\pi(x) = \beta x / (\beta x + 1), \qquad \beta > 0. \tag{8}$$

Parameter β is a measure of the '*degree of efficiency*' of the recombinant process: the larger β the higher the probability of obtaining a new successful idea out of each seeds matching. The continuous-time setting implies that $\dot{A}(t)$ has the same value both while looking forward to the new output – equation (7) – and while looking backward, *i.e.*, to the formation of seed ideas, which is given by

$$H(t) = C'_m\left[A(t)\right]\dot{A}(t), \tag{9}$$

where $C_m(A) = A!/\left[m!\,(A-m)!\right]$ is of the *number of different combinations of m seed ideas* as a function of the stock A and $C'_m(A)$ denotes its derivative. (9) is the continuous-time version of (26) on p. 345 in [25]. We assume that only *pairs of seed ideas* are combined together: $m = 2$. Hence, $C_m(A) = C_2(A) = A(A-1)/2$ and $C'_m(A) = C'_2(A) = A - 1/2$, so that (9) boils down to

$$H(t) = \left[A(t) - 1/2\right]\dot{A}(t). \tag{10}$$

Within this approach both the seed production in (10) and the production of new ideas in (7) are referred (as a limit) to the same time instant, so that (10) can be substituted into (7) to yield the following law of motion for the stock of knowledge:

$$\dot{A}(t) = J(t)/\varphi\left[A(t)\right], \tag{11}$$

[4] $\pi : \mathbb{R}_+ \to [0,1)$ in (8) satisfies Weitzman's assumptions (p. 345 in [25]): $\pi' > 0$, $\pi'' < 0$, $\pi(0) = 0$ and $\pi(\infty) \leq 1$; moreover, $\pi'(0) = \beta < +\infty$.

where, under Assumption A.1 and for $m = 2$,

$$\varphi(A) = C_2'(A)\,\pi^{-1}[1/C_2'(A)] = (1/\beta)[1 + 2/(2A-3)], \tag{12}$$

is the *expected unit cost of knowledge production*, which is defined for[5] $A > 3/2$, is decreasing in A, and $\lim_{A\to\infty}\varphi(A) = 1/\pi'(0) = 1/\beta > 0$.

The social planner chooses the optimal amount J to be employed in production of new knowledge according to (11) in order to maximize the discounted utility of a representative consumer over an infinite time horizon. J is levied as a tax on the representative consumer, and the new "productive" knowledge obtained is immediately and freely passed to the output producing firms. We assume that labor is constant and normalized to one: $L \equiv 1$.

A. 2 *Output is produced according to a Cobb-Douglas technology:*

$$y(t) = \theta[k(t)]^{\alpha}[A(t)]^{1-\alpha}, \qquad \theta > 0,\ 0 < \alpha < 1, \tag{13}$$

depending on aggregate capital, $k(t)$, and knowledge-augmented labor, $A(t)L(t)$, for $L(t) \equiv 1$.

Output producing firms maximize instantaneous profit by renting capital k and hiring labor L from the households, taking as given the capital rental rate, r, the labor wage and the stock of knowledge, A. As these firms operate in a competitive market, it follows from A.2 that:

$$\theta\alpha[k(t)/A(t)]^{\alpha-1} = r(t). \tag{14}$$

We slightly depart from [19] and [20] by setting an upper bound on investment in R&D activities: $J(t) \le y(t)$ for all $t \ge 0$.[6] Hence, capital evolves through time according to

$$\dot{k}(t) = y(t) - J(t) - c(t), \tag{15}$$

where it is assumed that capital does not depreciate.

A. 3 *All households enjoy an instantaneous CIES utility,*

$$u(c) = \left(c^{1-\sigma} - 1\right)/(1-\sigma), \qquad \sigma \ge 1, \tag{16}$$

and have a common discount rate, $\rho > 0$.

Thus, the welfare maximization problem faced by the social planner is

$$V(A_0, k_0) = \max_{[c(t),J(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t}\frac{[c(t)]^{1-\sigma} - 1}{1-\sigma}dt \tag{17}$$

subject to the dynamic constraints (11) and (15), with the additional constraints $J(t) \le y(t)$, $c(t) \le k(t) + y(t)$, and usual non-negativity constraints, given the initial stock of physical capital, $k_0 > 0$, and knowledge, $A_0 > 3/2$. Suppressing the time argument, the current-value Hamiltonian associated to (17) is

$$H(A, k, J, c, \lambda, \delta) = \left(c^{1-\sigma} - 1\right)/(1-\sigma) + \lambda\left(\theta k^{\alpha}A^{1-\alpha} - J - c\right) + \delta J/\varphi(A), \tag{18}$$

[5] Note that the RHS in (12) is negative for $1/2 < A < 3/2$, while for $A \le 1$ the interpretation of the Weitzman's process based on the combination of (more than 1) ideas becomes meaningless.

[6] As in the original framework of [25] and [24], the social planner cannot spend more than the aggregate output in new knowledge production.

where λ and δ are the costate variables associated with k and A respectively and $\varphi(A)$ is defined by (12). Necessary conditions are:[7]

$$
\begin{gathered}
u'(c) = \lambda \\
J = \begin{cases} 0 & \text{if } \delta/\varphi(A) < \lambda \\ \tilde{J} & \text{if } \delta/\varphi(A) = \lambda \\ \theta k^{\alpha} A^{1-\alpha} & \text{if } \delta/\varphi(A) > \lambda \end{cases} \qquad (19) \\
\dot{\lambda} = \rho\lambda - \lambda\theta\,(k/A)^{\alpha-1} \\
\dot{\delta} = \rho\delta - \lambda\theta\,(1-\alpha)\,(k/A)^{\alpha} + \delta J \varphi'(A)\,/\,[\varphi(A)]^{2} \\
\lim_{t\to\infty} H(t)\, e^{-\rho t} = 0,
\end{gathered}
$$

where $\tilde{J}$ in (19) is defined by (23) below.

Remark 1 *While the costates λ and δ are continuous functions of time,*[8] *conditions (19) imply a discontinuous optimal R&D financing (a* 'bang-bang' *solution) due to linearity of the Hamiltonian (18) in the variable J. On the other hand, the necessary condition $c^{-\sigma} = \lambda$ and continuity of λ in time implies that* the optimal trajectory of consumption must be a continuous function of time.

The solution of (17) in this regulated economy is described by means of the following three characteristic curves in the space (A, k).

1. The locus on which the marginal product of capital equals that of knowledge per unit cost, which, under Assumptions A.1 and A.2, using (12) can be rewritten as a function of the only variable A:

$$\tilde{k}(A) = [\alpha/(1-\alpha)]\,\varphi(A)\,A = \{\alpha/[\beta(1-\alpha)]\}\,[1 + 2/(2A-3)]\,A. \qquad (20)$$

   We call $\tilde{k}(A)$ in (20) the *(transitory) turnpike*.

2. The function $\tilde{k}(A)$ in (20) for large A becomes affine, defining the curve

$$\tilde{k}_{\infty}(A) = \{\alpha/[\beta(1-\alpha)]\}\,(A+1). \qquad (21)$$

   We call $\tilde{k}_{\infty}(A)$ in (21) the *asymptotic turnpike*. $\tilde{k}(A)$ lies above $\tilde{k}_{\infty}(A)$, that is, $\tilde{k}(A) > \tilde{k}_{\infty}(A)$ for all $A < \infty$, and approaches $\tilde{k}_{\infty}(A)$ as $A \to \infty$.

3. Finally, on the locus $\theta\alpha\,(k/A)^{\alpha-1} = \rho$ the marginal product of capital equals the individual discount rate, which, by (14), implies $r = \rho$. It can be written as a linear function of A:

$$\hat{k}(A) = (\theta\alpha/\rho)^{1/(1-\alpha)}\,A. \qquad (22)$$

   We call $\hat{k}(A)$ in (22) the *stagnation line*, as it defines the set of all possible steady pairs (k, A) on which the economy might eventually end up in stagnation.

[7] See conditions (27) – (31) in [24] or conditions (15) – (19) in [19].
[8] More precisely, according to [9] they are *continuous and piecewise continuously differentiable.*

Differentiating $\tilde{k}(A)$ in (20) with respect to time and substituting into the dynamic constraints (11) and (15) yields

$$\tilde{J} = (\tilde{y} - \tilde{c})\,\varphi(A) \,/ \left[\tilde{k}'(A) + \varphi(A)\right], \tag{23}$$

where $\tilde{y} = \theta\left[\tilde{k}(A)\right]^{\alpha} A^{1-\alpha}$, and $\tilde{c}$ denotes optimal consumption when the economy is constrained to grow along the turnpike $\tilde{k}(A)$. Condition (23) relates the optimal investment in R&D, $\tilde{J}$, as a function of the other control variable, $\tilde{c}$, along the transitory turnpike; that is, in view of (19), when $\delta/\varphi(A) = \lambda$ holds.

Proposition 1

i) *A necessary condition for the economy to sustain long-run growth is*

$$\rho < \theta\alpha\left[\beta(1-\alpha)/\alpha\right]^{1-\alpha}, \tag{24}$$

where the RHS, $\theta\alpha\left[\beta(1-\alpha)/\alpha\right]^{1-\alpha}$, defines the long-run capital rental rate.

ii) *Under (24), for any given initial knowledge stock $A_0 > 3/2$ there is a unique corresponding* threshold capital stock $k^{sk}(A_0) \geq 0$, *to which we refer as the* Skiba-point, *such that whenever $k_0 \geq k^{sk}(A_0)$ the economy first reaches the turnpike $\tilde{k}(A)$ in a finite time period, and then continues to grow along it until the asymptotic turnpike $\tilde{k}_{\infty}(A)$ is approached as $A \to \infty$. Along $\tilde{k}_{\infty}(A)$ the economy follows a ABGP characterized by the following common constant growth rate of output, knowledge, capital and consumption:*

$$\gamma = \left\{\theta\alpha\left[\beta(1-\alpha)/\alpha\right]^{1-\alpha} - \rho\right\}/\sigma. \tag{25}$$

Moreover, there exist an instant $t_0 \geq 0$ such that $J(t) > 0$ for all $t > t_0$, while, as $t \to \infty$, $J(t) < y(t)$ holds and the income shares devoted to investments in knowledge and capital are constant and given by $s_{\infty} = (1-\alpha)\gamma/\left\{\theta\alpha\left[\beta(1-\alpha)/\alpha\right]^{1-\alpha}\right\}$ and $s_{\infty}^{k} = \alpha\gamma/\left\{\theta\alpha\left[\beta(1-\alpha)/\alpha\right]^{1-\alpha}\right\}$ respectively.

iii) *Conversely, whenever either (24) fails, i.e., if $\rho \geq \theta\alpha\left[\beta(1-\alpha)/\alpha\right]^{1-\alpha}$, or $k_0 < k^{sk}(A_0)$ after a finite instant $t_s \geq 0$ the optimal investment in R&D activities becomes zero and the stock of knowledge remains constant: $J(t) \equiv 0$ and $A(t) \equiv A(t_s)$ for all $t > t_s$. In this scenario – perhaps after a time interval of "full investment" in R&D, during which $J(t) = y(t)$, if $\delta(t)/\varphi[A(t)] > \lambda(t)$ holds in (19) for $0 \leq t \leq t_s$ – eventually optimal capital and consumption follow the usual Ramsey-type saddle-stable time-path trajectory monotonically converging to a steady state on the stagnation line $\hat{k}(A)$ defined in (22), with steady value for the physical capital equal to $\hat{k}[A(t_s)]$, corresponding to zero growth.*

For a proof see [24]. Figure 1 shows all three characteristic curves for the parameters' values considered in Section 7 satisfying condition (24).

4 The Unique Skiba-Point on the Turnpike

Proposition 1 (ii) implies that, under (24) and if $k_0 \geq k^{sk}(A_0)$, the turnpike $\tilde{k}(A)$ is 'trapping', *i.e.*, the economy keeps growing along it after it is entered so to reach the asymptotic turnpike $\tilde{k}_{\infty}(A)$ for $t \to \infty$ and follow the ABGP thereafter. In order to estimate the Skiba-point $k^{sk}(A_0)$ we focus on this scenario, which forecasts two types of transitory behavior:

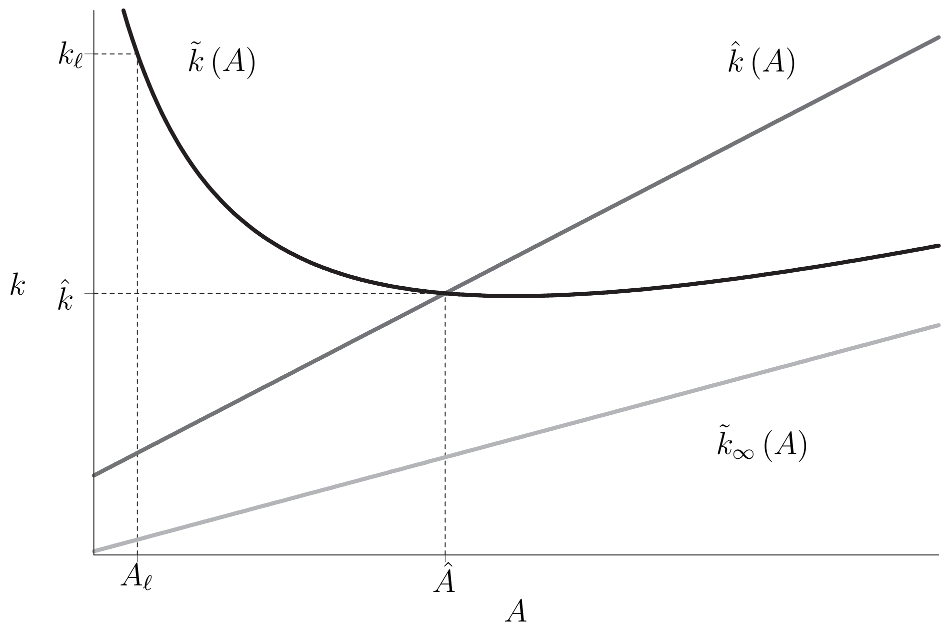

FIGURE 1: the transitory turnpike $\tilde{k}(A)$ (in black), the stagnation line $\hat{k}(A)$ (in dark grey) and the asymptotic turnpike $\tilde{k}_\infty(A)$ (in light grey) of our economy for the parameters' values used in Section 7; $\left(\hat{A}, \hat{k}\right)$ is the intersection point between the transitory turnpike and the stagnation line.

1. the path driving the system toward the turnpike starting from outside it, and
2. the path characterizing the optimal path along $\tilde{k}(A)$ after it has been entered.

Special attention will be devoted on the former, as the latter has been already thoroughly analyzed in [19] and [20].

First, we need to narrow the range of our analysis according to the following preliminary result. From (21) and (22) it is immediately seen that the growth condition (24) states that the slope of the asymptotic turnpike $\tilde{k}_\infty(A)$ must be less than the slope of the stagnation line $\hat{k}(A)$; because the transitory turnpike $\tilde{k}(A)$ lies above $\tilde{k}_\infty(A)$ for all finite A and (20) implies $\lim_{A\to 3/2^+} \tilde{k}(A) = +\infty$, there is a value $\hat{A}$ at which the turnpike $\tilde{k}(A)$ intersects the stagnation line $\hat{k}(A)$, that is, such that $\tilde{k}\left(\hat{A}\right) = \hat{k}\left(\hat{A}\right)$. Such value is unique and is obtained by coupling (20) and (22):

$$\hat{A} = \alpha / \left[\beta(1-\alpha)(\theta\alpha/\rho)^{\frac{1}{1-\alpha}} - \alpha\right] + 3/2, \tag{26}$$

which is well defined whenever the necessary condition for growth (24) is satisfied. Figure 1 illustrates this property for the parameters' values considered in Section 7.

Proposition 2 *Under growth condition (24) for all initial stock of knowledge levels $A_0 \geq \hat{A}$ the economy is bound to sustain growth in the long run independently of the initial stock of capital $k_0 > 0$; that is, whenever $A_0 \geq \hat{A}$, $k^{sk}(A_0) = 0$.*

Proof. It is an immediate consequence of Property 4b on p. 3472 in [24], stating that a steady state cannot lie above the turnpike $\tilde{k}(A)$. ■

Hence, given $\hat{A}$ defined in (26), we confine our attention to levels $3/2 < A_0 < \hat{A}$ for the initial stock of knowledge. From (20) we know that, for each A_0, there is a unique value of physical capital corresponding to A_0, $\tilde{k}(A_0)$, which lies on the turnpike. Clearly, given A_0, the Skiba-point $k^{sk}(A_0)$ can either lie above or below the number $\tilde{k}(A_0)$. However, there is also the very peculiar case in which $k^{sk}(A_0) = \tilde{k}(A_0)$. As the two cases $k^{sk}(A_0) > \tilde{k}(A_0)$ and $k^{sk}(A_0) < \tilde{k}(A_0)$ exhibit quite diverse types of transition dynamics, the scenario $k^{sk}(A_0) = \tilde{k}(A_0)$ determines a boundary value separating these two cases. We start by investigating this boundary regime; specifically, we look for the initial knowledge stock level A_0 that, when the economy starts with an initial capital endowment $k_0 = \tilde{k}(A_0)$, the welfare generated by growing along $\tilde{k}(A)$ and then converging toward a ABGP along $k_\infty(A)$ equals the welfare obtained by converging toward the steady state $\left(A_0, \hat{k}(A_0)\right)$ on the stagnation line along a monotonic Ramsey-type saddle-stable time-path trajectory.

Proposition 3 *If a knowledge value $3/2 < \bar{A} \leq \hat{A}$ exists such that the optimal dynamics converge to a steady state on the stagnation line $\hat{k}(A)$ when the economy starts from $\left(\bar{A}, \tilde{k}(\bar{A})\right)$ on the turnpike, then for all $3/2 < A_0 \leq \bar{A}$ it is optimal to converge to a steady state on the stagnation line when the economy starts from $\left(A_0, \tilde{k}(A_0)\right)$ on the turnpike.*

Proof. It follows immediately from Property 6 on p. 3473 in [24], establishing that the singular policy (23) along the turnpike is trapping. ■

Proposition 3 implies that, if it exists, there is a unique *minimal* knowledge level $3/2 < A_m \leq \hat{A}$ such that when the economy initiates from $\left(A_m, \tilde{k}(A_m)\right)$ on the turnpike, it is bound to proceed along the turnpike toward sustained long-run growth. Indeed, such A_m value corresponds to the *unique Skiba-point lying on the turnpike*, *i.e.*, the unique knowledge level satisfying $k^{sk}(A_m) = \tilde{k}(A_m)$ we are looking for. Hence, A_m is the initial level of the stock of knowledge that equates the welfare obtained by investing $\tilde{J}(t) > 0$ as in (23) for all $t \geq 0$ and thus following the optimal trajectory $\left(A(t), \tilde{k}[A(t)]\right)$ on the turnpike starting from the point $\left(A_m, \tilde{k}(A_m)\right)$, with the welfare yield, according to $\delta/\varphi(A) < \lambda$ in (19), by a zero-R&D investment policy, $J(t) \equiv 0$ for all $t \geq 0$, keeping constant the stock of knowledge at its initial level A_m and leading the economy toward the steady state $\left(A_m, \hat{k}(A_m)\right)$ on the stagnation line. To estimate A_m we thus evaluate the welfare delivered both by trajectories evolving along the turnpike $\tilde{k}(A)$ following the policy (23) and trajectories evolving toward the steady state $\left(A_0, \hat{k}(A_0)\right)$ on the stagnation line through a zero-R&D investment policy – as $J(t) \equiv 0$ implies $A(t) \equiv A_0$ for all $t \geq 0$.

4.1 Optimal Dynamics Along the Turnpike

Optimal trajectories along the turnpike are solutions of the following social planner problem in the only two variables A (state) and c (control), and one dynamic constraint:

$$\tilde{V}(A_0) = \max_{[c]} \int_0^\infty e^{-\rho t} \frac{c^{1-\sigma} - 1}{1-\sigma} dt \tag{27}$$

$$\text{subject to} \left\{ \begin{array}{l} \dot{A} = \left\{ \theta \left[\tilde{k}(A)\right]^\alpha A^{1-\alpha} - c \right\} / \left[\tilde{k}'(A) + \varphi(A)\right] \\ A(0) = A_0, \end{array} \right. \tag{28}$$

where the time argument has been dropped for simplicity, $\tilde{k}(A)$ is defined in (20), $\tilde{k}'(A) = \partial\tilde{k}(A)/\partial A$, and $\varphi(A)$ is given by (12). Necessary conditions on the current-value Hamiltonian for problem (27) yield the following system of ODEs defining the optimal dynamics for A and c along the turnpike:

$$\left\{\begin{array}{l} \dot{A}=\left\{\theta A\left[\tilde{k}\left(A\right)/A\right]^{\alpha}-c\right\}/\left[\tilde{k}'\left(A\right)+\varphi\left(A\right)\right] \\ \dot{c}=c\left\{\theta\alpha\left[\tilde{k}\left(A\right)/A\right]^{\alpha-1}-\rho\right\}/\sigma. \end{array}\right. \tag{29}$$

To study the phase diagram associated to (29), using (20) and (12) Privileggi [19] introduces the ratio variables

$$\mu=\tilde{k}\left(A\right)/A=\left[\alpha/\left(1-\alpha\right)\right]\varphi\left(A\right)=\left\{\alpha/\left[\beta\left(1-\alpha\right)\right]\right\}\left[1+2/\left(2A-3\right)\right], \tag{30}$$

$$\chi=c/A, \tag{31}$$

which transform (29) into the following system of ODEs:

$$\left\{\begin{array}{l} \dot{\mu}=\left[1-2\beta\left(1-\alpha\right)\mu/Q\left(\mu\right)\right]\left(\theta\mu^{\alpha}-\chi\right) \\ \dot{\chi}=\left[\left(\theta\alpha\mu^{\alpha-1}-\rho\right)/\sigma-2\alpha\beta\left(1-\alpha\right)\left(\theta\mu^{\alpha}-\chi\right)/Q\left(\mu\right)\right]\chi, \end{array}\right. \tag{32}$$

where

$$Q\left(\mu\right)=-3\beta^{2}\left(1-\alpha\right)^{2}\mu^{2}+2\beta\left(1-\alpha\right)\left(1+2\alpha\right)\mu-\alpha^{2}. \tag{33}$$

Unlike system (29), whose variables A and c diverge in the long-run, μ and χ solving (32) can converge to the steady state (μ^{*},χ^{*}) whose coordinates are defined by

$$\mu^{*}=\alpha/\left[\beta\left(1-\alpha\right)\right] \qquad \text{and} \qquad \chi^{*}=\theta\left\{\alpha/\left[\beta\left(1-\alpha\right)\right]\right\}^{\alpha}\left(1-1/\sigma\right)+\rho/\left[\beta\sigma\left(1-\alpha\right)\right], \tag{34}$$

where μ^{*} is to the constant *long-run capital/knowledge ratio* along the asymptotic turnpike $\tilde{k}_{\infty}\left(A\right)$ [μ^{*} is the slope of $\tilde{k}_{\infty}\left(A\right)$ in (21)] and χ^{*} is the *long-run consumption/knowledge ratio.* (μ^{*},χ^{*}) is *saddle-path stable*, with the stable arm converging to it from north-east whenever the initial values $\left.\left(\mu\left(t\right),\chi\left(t\right)\right)\right|_{t=0}$ are suitably chosen. A detailed discussion on the complete phase diagram, including the other two non attractive steady states for system (32), under the assumption

$$\theta\alpha\left(\mu^{s}\right)^{\alpha-1}<\rho<\theta\alpha\left(\mu^{*}\right)^{\alpha-1}, \tag{35}$$

where μ^{s} will be defined in (42), can be found in [19]; Figure 1 on p. 266 there illustrates such phase diagram (see also Figure 2 below).

According to [17], to solve (32) we *eliminate time* and tackle the ODE given by the ratio between the equations in (32):

$$\chi'\left(\mu\right)=\frac{\left[\left(\alpha\theta\mu^{\alpha-1}-\rho\right)/\sigma\right]Q\left(\mu\right)-2\alpha\beta\left(1-\alpha\right)\left[\theta\mu^{\alpha}-\chi\left(\mu\right)\right]}{\left[Q\left(\mu\right)-2\beta\left(1-\alpha\right)\mu\right]\left[\theta\mu^{\alpha}-\chi\left(\mu\right)\right]}\chi\left(\mu\right), \tag{36}$$

where $Q\left(\mu\right)$ is defined in (33). Following [20], the solution of (36), yielding the optimal policy function $\tilde{\chi}\left(\mu\right)$ along the turnpike, is approximated through a *projection method* based on *OLS* applied to the integral of a residual function built upon an approximation function which is a linear combination of n Chebyshev polynomials.[9] Therefore, our estimate of the policy $\tilde{\chi}\left(\mu\right)$ turns out to be a polynomial of degree n.

[9]The integral of the residual function is itself approximated by means of *Gauss-Chebyshev quadrature* on the relevant interval, while as initial condition for the *Maple 16 nonlinear programming (NLP) solver* with the sequential quadratic programming (sqp) method we use a *Chebyshev regression of order* n (Algorithm 6.2 on p. 223 in [16]) on the line crossing the two steady states (μ^{*},χ^{*}) and (μ^{s},χ^{s}), with coordinates defined in (34) and (42) respectively. All details are reported in [20].

Using (30) and (31), the *optimal consumption policy* for problem (27) corresponding to $\tilde{\chi}(\mu)$ is thus obtained as

$$\tilde{c}(A) = \tilde{\chi}(\mu) A = \tilde{\chi}\left[\frac{\alpha}{\beta(1-\alpha)}\left(1+\frac{2}{2A-3}\right)\right] A. \tag{37}$$

To approximate the time-path trajectory $\tilde{\mu}(t)$, $\tilde{\chi}(\mu)$ is substituted into the first equation of (32) so to obtain a ODE with respect to time which can be numerically solved through the standard *Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant* method available in Maple 16. The time-path trajectory $\tilde{\chi}(t)$ is then computed as $\tilde{\chi}(t) = \tilde{\chi}[\tilde{\mu}(t)]$, while the time-path trajectories of the stock of knowledge and capital, according to (30) and (20), are given by $\tilde{A}(t) = \alpha/[\beta(1-\alpha)\tilde{\mu}(t) - \alpha] + 3/2$ and $\tilde{k}(t) = \tilde{k}\left[\tilde{A}(t)\right]$ respectively. Similarly, the time-path trajectory of output is given by $\tilde{y}(t) = \theta\left[\tilde{k}(t)\right]^{\alpha}\left[\tilde{A}(t)\right]^{1-\alpha}$, while, using (31), the time-path trajectory of the optimal consumption is obtained as $\tilde{c}(t) = \tilde{\chi}(t)\tilde{A}(t)$. Finally, according to (23), the time-path trajectory of optimal investment into new knowledge production is given by $\tilde{J}(t) = [\tilde{y}(t) - \tilde{c}(t)]\varphi\left[\tilde{A}(t)\right] / \left\{\tilde{k}'\left[\tilde{A}(t)\right] + \varphi\left[\tilde{A}(t)\right]\right\}$.

4.2 Optimal Dynamics Toward Stagnation

Trajectories initiating on a point $\left(A_0, \tilde{k}(A_0)\right)$ on the turnpike and evolving toward the steady state $\left(A_0, \hat{k}(A_0)\right)$ on the stagnation line according to a constant zero-R&D investment policy, $J(t) \equiv 0$ [corresponding to condition $\delta/\varphi(A) < \lambda$ in (19)], are just standard saddle-path stable trajectories of a typical Ramsey model for a given (constant) level A_0 of knowledge stock. That is, they are solutions of the following social planner problem in the two variables k (state) and c (control), and the usual dynamic constraint:

$$\bar{V}(A_0) = \max_{[c]} \int_0^{\infty} e^{-\rho t}\frac{c^{1-\sigma}-1}{1-\sigma}dt \tag{38}$$
$$\text{subject to } \begin{cases} \dot{k} = \theta k^{\alpha} A_0^{1-\alpha} - c, \\ k(0) = \tilde{k}(A_0), \end{cases}$$

in which the initial condition is the capital value on the turnpike corresponding to A_0, $k(0) = \tilde{k}(A_0)$. Although, as occurs in (27), the value function $\bar{V}$ in (38) depends only on the initial stock of knowledge A_0, the true initial condition of the Ramsey problem here is the initial stock of capital $\tilde{k}(A_0)$ – itself a function of A_0 – because, unlike (27), the only state variable in (38) is physical capital, k, while the stock of knowledge remains constant at level A_0.

We actually solve a problem which is equivalent to (38) but have variables rescaled by the ratios $\mu = k/A_0$ and $\chi = c/A_0$, with A_0 constant. This choice allows for studying the phase diagram of the dynamics of (38) in the same 'detrended' (μ, χ) space that contains the optimal policy $\tilde{\chi}(\mu)$ of model (27) previously built. A constant stock of knowledge $A \equiv A_0$ implies $\dot{A} \equiv 0$, which, in turn, allows to suitably rewrite the necessary conditions for the current-value Hamiltonian associated to (38) in terms of $\mu = k/A_0$ and $\chi = c/A_0$ according to the following system of ODEs describing the detrended optimal dynamics:

$$\begin{cases} \dot{\mu} = \theta\mu^{\alpha} - \chi \\ \dot{\chi} = \chi\left(\theta\alpha\mu^{\alpha-1} - \rho\right)/\sigma. \end{cases} \tag{39}$$

Again we eliminate time by taking their ratio and study the unique ODE characterizing the optimal policy, $\bar{\chi}(\mu)$, in this scenario:

$$\chi'(\mu) = \chi(\mu)\left(\theta\alpha\mu^{\alpha-1} - \rho\right) / \{\sigma[\theta\mu^\alpha - \chi(\mu)]\}. \tag{40}$$

To approximate the solution of (40) we apply a *projection method* based on *Chebyshev Orthogonal Collocation* on n collocation points applied to a residual function built upon an approximation function which is a linear combination of n Chebyshev polynomials.[10] Thus, also here our estimate of the policy $\bar{\chi}(\mu)$ turns out to be a polynomial of degree n, possibly with a different n than that used in the previous Subsection. The *optimal consumption policy* for problem (38), which is a function of the state variable k, is then obtained as

$$\bar{c}(A_0, k) = \bar{\chi}(k/A_0)\, A_0, \tag{41}$$

where also its dependency on the initial stock of knowledge A_0 has been emphasized.

4.3 A Comprehensive Detrended Phase Diagram

Figure 2 reports a unique phase diagram in the detrended (μ, χ) space exhibiting all loci involved,[11] the three relevant steady states, and both optimal policy curves, $\tilde{\chi}(\mu)$ along the turnpike and $\bar{\chi}(\mu)$ toward stagnation (the thick curves in black and dark grey respectively), for the parameters' values considered in Section 7. The saddle-path stable steady state (μ^*, χ^*) of the $\tilde{\chi}(\mu)$ policy with coordinates given by (34) lies on the bottom left, the steady state[12] (μ^s, χ^s) of the $\tilde{\chi}(\mu)$ policy with coordinates

$$\mu^s = \left(1 + 2\alpha + \sqrt{1 + 4\alpha + \alpha^2}\right) / [3\beta(1-\alpha)] \qquad \text{and} \qquad \chi^s = \theta(\mu^s)^\alpha, \tag{42}$$

lies on the top right, while between these two there is the steady state $(\hat{\mu}, \hat{\chi})$ with coordinates

$$\hat{\mu} = (\theta\alpha/\rho)^{1/(1-\alpha)} \qquad \text{and} \qquad \hat{\chi} = \theta(\theta\alpha/\rho)^{\alpha/(1-\alpha)}, \tag{43}$$

which happens to be irrelevant for the $\tilde{\chi}(\mu)$ policy but turns out to be the unique saddle-path stable steady state for the $\bar{\chi}(\mu)$ policy defining the optimal path (the stable arm) for system (39). The latter steady state, $(\hat{\mu}, \hat{\chi})$, with coordinates in (43) corresponds to any steady state $\left(A, \hat{k}(A)\right)$ on the stagnation line defined in (22) to which the economy might eventually

[10]Details can be found in Chapter 11 in [16], in Chapter 6 in [12], or in Paragraph 5.5.2 in [18]. As initial condition for the *Maple 16 'fsolve' routine* used to numerically solve the system of $n+1$ equations setting the residual function equal to zero on each collocation node plus the steady state constraint, $\bar{\chi}(\hat{\mu}) = \hat{\chi}$, here we use a *Chebyshev regression of order n* (Algorithm 6.2 on p. 223 in [16]) on the line tangent to the optimal policy $\bar{\chi}(\mu)$ on the steady state $(\hat{\mu}, \hat{\chi})$ with coordinates defined in (43). The slope of $\bar{\chi}(\mu)$ on the steady state $(\hat{\mu}, \hat{\chi})$ is the positive solution of the quadratic equation obtained through l'Hôpital's rule, according to [4], pp. 595–596.

[11]All loci are plotted as thin black curves. Both the vertical line at μ^*, with μ^* defined in (34), and the flat increasing curve, defined as $\chi = \theta\mu^\alpha$, crossing points $(\hat{\mu}, \hat{\chi})$ and (μ^s, χ^s) determine the $\dot{\mu} = 0$ locus for system (32). The curve with more pronounced concavity crossing all three steady states is the unique $\dot{\chi} = 0$ locus for system (32), defined by (60) in [19]. Note that the curve $\chi = \theta\mu^\alpha$ defines the $\dot{\mu} = 0$ locus for system (39) as well. Finally, the vertical line at $\hat{\mu}$, with $\hat{\mu}$ defined in (43), is the $\dot{\chi} = 0$ locus for system (39).

[12]The steady state (μ^s, χ^s), with coordinates defined in (42), cannot be classified analytically as the Jacobian matrix of (32) evaluated at (μ^s, χ^s) has some elements that diverge either to $-\infty$ or to $+\infty$, the sign of infinity depending on the direction along which (μ^s, χ^s) is approached. For this reason, such point has been called 'supersingular' by Privileggi [19]. As a matter of fact, it turns out to be harmless, as the optimal policy $\tilde{\chi}(\mu)$ simply crosses it. See Remark 1 on p. 266 and the discussion on p. 267 in [19] for a more thorough description.

converge; that is, it is the unique representation in the (μ,χ) space of all steady states $\left(A,\hat{k}\left(A\right)\right)$ – *i.e.*, all points on the stagnation line $\hat{k}\left(A\right)$ – for the optimal dynamics of problem (38) in the (A,k) space.[13]

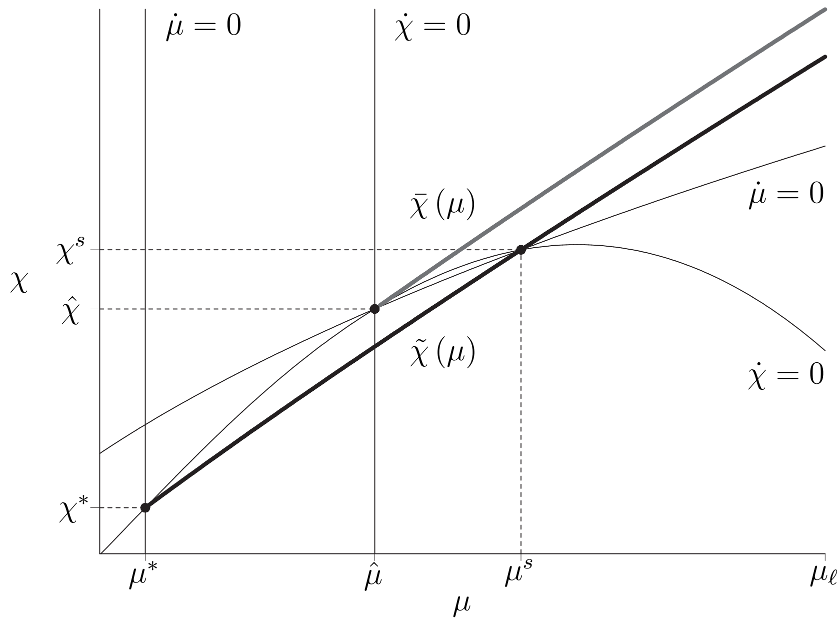

FIGURE 2: phase diagram for the policy along the turnpike, $\tilde{\chi}\left(\mu\right)$ (thick black curve), and for the policy toward stagnation, $\bar{\chi}\left(\mu\right)$ (thick dark grey curve), including all loci and steady states, for the parameters' values used in Section 7.

While Figure 2 shows the whole optimal policy $\tilde{\chi}\left(\mu\right)$ (the black thick curve) starting on any point on the turnpike and evolving along it to eventually converge to its steady state (μ^*,χ^*), only the upper right branch of the saddle-path stable arm crossing the steady state $(\hat{\mu},\hat{\chi})$ of the optimal dynamics defined by (39) (the dark grey thick curve) is reported. That is, only paths starting from initial values $\mu_0>\hat{\mu}$ and $\chi_0>\hat{\chi}$ are considered here for the optimal policy $\bar{\chi}\left(\mu\right)$. This is because, from Proposition 2, the relevant range for A_0 is the interval $\left(3/2,\hat{A}\right)$, where $\hat{A}$ is the knowledge value at which the turnpike $\tilde{k}\left(A\right)$ intersects the stagnation line $\hat{k}\left(A\right)$ from above, as calculated in (26); therefore, $\tilde{k}\left(A\right)>\hat{k}\left(A\right)$ for all $3/2<A<\hat{A}$, so that we are considering only monotonically decreasing time-path capital trajectories, $k\left(t\right)$, converging from above to the steady state $\left(A_0,\hat{k}\left(A_0\right)\right)$, all corresponding to the unique optimal policy $\bar{\chi}\left(\mu\right)$ starting from any value (μ_0,χ_0) to the north-east of the unique steady state $(\hat{\mu},\hat{\chi})$ at $t=0$ and then converging to it through a decreasing pattern of both time-path trajectories $\bar{\mu}\left(t\right)$ and $\bar{\chi}\left(t\right)$.[14]

[13]From (22) and the definition $\mu=k/A$ it is clear that any steady capital value on $\hat{k}\left(A\right)$ in the (A,k) space must correspond to a point on the vertical line at $\hat{\mu}$ in the (μ,χ) space, with $\hat{\mu}$ defined in (43). On the other hand, it is immediately seen from the system of ODEs describing the optimal dynamics for problem (38) in the (k,c) space – corresponding to system (39) in the (μ,χ) space – that all steady consumption values on the stagnation line are given by $\hat{c}\left(A\right)=\theta\left(\theta\alpha/\rho\right)^{\alpha/(1-\alpha)}A$. Hence, under the transformation $\chi=c/A$, all such steady consumption values correspond to the unique $\hat{\chi}$ value in the (μ,χ) space defined in (43).

[14]Note that all possible optimal policies for problem (38) – each depending on a different initial condition A_0 and belonging to a different phase diagram in the (k,c) space – after being transformed into the dynamics defined by system (39) have the unique phase diagram representation in the (μ,χ) space as in Figure 2, portraying the

4.4 Welfare Matching

To find the point A_m corresponding to the unique Skiba-point lying on the turnpike – that is, the knowledge level satisfying $k^{sk}(A_m) = \tilde{k}(A_m)$ – we must solve the equation $\tilde{V}(A) = \bar{V}(A)$, where $\tilde{V}$ and $\bar{V}$ are the value functions defined in (27) and (38) respectively.

To approximate the latter, note that, for any given (constant) A, (38) is a standard Ramsey model in the state variable k and control variable c with initial condition $k_0 = \tilde{k}(A)$. Hence, under Assumptions A.2 and A.3 it is a concave problem, so that the assumptions in Theorem 2 of Section 2 are satisfied and we can compute the derivative of the value function at the initial stock of capital value through (5):

$$\bar{V}'(k_0) = \bar{V}'\left[\tilde{k}(A)\right] = u'\left\{\theta\left[\tilde{k}(A)\right]^{\alpha} A^{1-\alpha} - \dot{\bar{k}}\left(0^+; \tilde{k}(A)\right)\right\} = \left[\bar{c}\left(A, \tilde{k}(A)\right)\right]^{-\sigma}, \tag{44}$$

where $\dot{\bar{k}}\left(0^+; \tilde{k}(A)\right)$ denotes the optimal initial investment toward stagnation and $\bar{c}\left(A, \tilde{k}(A)\right)$ is the estimated value of the optimal policy at $(A, k_0) = \left(A, \tilde{k}(A)\right)$ numerically obtained in (41). By replacing (44) into the Hamilton-Jacobi-Bellman equation (6) of Corollary 1, we obtain the value function of problem (38) directly as a function of A:

$$\begin{aligned}\bar{V}(A) &= (1/\rho)\left\{u\left[\bar{c}(A, k_0)\right] + \bar{V}'(k_0)\left[\theta k_0^{\alpha} A^{1-\alpha} - \bar{c}(A, k_0)\right]\right\} \\ &= \frac{1}{\rho}\left\{\frac{\left[\bar{c}\left(A, \tilde{k}(A)\right)\right]^{1-\sigma} - 1}{1-\sigma} + \frac{\theta\left[\tilde{k}(A)\right]^{\alpha} A^{1-\alpha} - \bar{c}\left(A, \tilde{k}(A)\right)}{\left[\bar{c}\left(A, \tilde{k}(A)\right)\right]^{\sigma}}\right\}.\end{aligned} \tag{45}$$

Because problem (27) turns out to be not concave in early-time dynamics, Theorem 2 is not directly applicable to approximate the value $\tilde{V}(A)$. Therefore, we rely on an ad-hoc approach based on guessing a candidate value function and then checking that it satisfies the assumptions of Theorem 1 in Section 2. As Theorem 2 provides only sufficient conditions for the differentiability of the value function, we are allowed to build our guess candidate by assuming that it is differentiable with derivative given by (5), and then again define $\tilde{V}(A)$ according to the the Hamilton-Jacobi-Bellman equation (6), as we did for $\bar{V}(A)$ in (45). Hence, we first set

$$\begin{aligned}w'(A) &= u'\left\{\theta\left[\tilde{k}(A)\right]^{\alpha} A^{1-\alpha} - \left[\tilde{k}'(A) + \varphi(A)\right]\dot{\tilde{A}}\left(0^+; A\right)\right\}\left[\tilde{k}'(A) + \varphi(A)\right] \\ &= \frac{\tilde{k}'(A) + \varphi(A)}{\left[\tilde{c}(A)\right]^{\sigma}},\end{aligned} \tag{46}$$

where $\dot{\tilde{A}}(0^+; A)$ denotes the optimal knowledge change on the turnpike at A according to (28), and $\tilde{c}(A)$ is the estimated value of the optimal policy along the turnpike at A numerically obtained in (37). Next, we use (28) to replace (46) into (6) and obtain our candidate guess for

same qualitative properties of all different phase diagrams in the (k, c) space, each characterized by a different steady state, stable arm, etc., depending on A_0.

the value function of problem (27):

$$w(A) = \frac{1}{\rho}\left\{u[\tilde{c}(A)] + w'(A)\frac{\theta\left[\tilde{k}(A)\right]^{\alpha}A^{1-\alpha} - \tilde{c}(A)}{\tilde{k}'(A) + \varphi(A)}\right\}$$

$$= \frac{1}{\rho}\left\{\frac{[\tilde{c}(A)]^{1-\sigma} - 1}{1-\sigma} + \frac{\theta\left[\tilde{k}(A)\right]^{\alpha}A^{1-\alpha} - \tilde{c}(A)}{[\tilde{c}(A)]^{\sigma}}\right\}. \tag{47}$$

Note that (47) yields the same expression of (45), only with the optimal policy value along the turnpike, $\tilde{c}(A)$, in place of the optimal policy value toward stagnation, $\bar{c}\left(A, \tilde{k}(A)\right)$.

Proposition 4 *Whenever the parameters' values in Assumptions A.1–A.3 satisfy*

$$\theta\alpha\left[\beta(1-\alpha)/\alpha\right]^{1-\alpha} < (1+\sigma)\rho, \tag{48}$$

$w(A)$ as defined in (47) is the value function $\tilde{V}(A)$ of problem (27).

Proof. See the Appendix. ■

Remark 2 *Under condition (48) of Proposition 4, using (47) and (45) in order to estimate welfare for both the model converging to the ABGP and the model converging toward stagnation require only one numerical step for each model: the projection method to approximate the optimal policies according to (37) and (41) as discussed in Subsections 4.1 and 4.2. Specifically, in the following Algorithm 1 no time-path trajectories estimations are needed.*

Remark 3 *Proposition 4 indirectly establishes that when (48) holds the value function of problem (27) is differentiable although assumption (ii) of Theorem 2 does not hold, as the instantaneous felicity $U(\cdot,\cdot)$ turns out to be not concave for small values of A; specifically, for those contained in the range $\left(3/2, \hat{A}\right]$. Figure 3 below shows that the value function $\tilde{V}(A) = w(A)$ itself turns out to be convex over $\left(3/2, \hat{A}\right]$, while it becomes concave for larger values of A.*

4.5 Skiba-Point Estimation

Assume that condition (48) holds and let

$$f(A) = w(A) - \bar{V}(A), \tag{49}$$

with $w(A)$ and $\bar{V}(A)$ defined in (47) and (45) respectively, be the function whose unique zero is to be found. In order to bracket this zero we must take an initial interval large enough to contain it; such a choice requires to find a left endpoint value $A_\ell > 3/2$ sufficiently close to[15] $3/2$ so that $f(A_\ell) < 0$, while as right endpoint a good value is given by $\hat{A}$ in (26), because Proposition 2 implies that $f\left(\hat{A}\right) > 0$. To A_ℓ corresponds the right endpoint $\mu_\ell = \tilde{k}(A_\ell)/A_\ell$ of the range for variable μ in the phase diagram of Figure 2. All steps to numerically approximate the unique Skiba-point on the turnpike are summarized in Algorithm 1 below.

[15] It has been observed in [20] that the $\tilde{\chi}(\mu)$ approximation becomes less reliable for larger μ values; thus, the choice of A_ℓ, should not be too small so to keep the error in the projection method described in Subsection 4.1 sufficiently small.

Algorithm 1 (Finds A_m satisfying $k^{sk}(A_m) = \tilde{k}(A_m)$)

Step 1: *Choose a value $A_\ell > 3/2$ sufficiently close to $3/2$; the range for the $\tilde{\chi}(\mu)$ policy approximation in the (μ, χ) space is $[\mu^*, \mu_\ell]$, with μ^* as in (34) and, according to (30), $\mu_\ell = \{\alpha/[\beta(1-\alpha)]\}[1 + 2/(2A_\ell - 3)]$. The range for the $\bar{\chi}(\mu)$ policy approximation is $[\hat{\mu}, \mu_\ell]$, with $\hat{\mu}$ defined in (43), corresponding to $\left[A_\ell, \hat{A}\right]$ in the (A, k) space.*

Step 2: *Apply the OLS-Projection method discussed in Subsection 4.1 to estimate the optimal policy along the turnpike, $\tilde{\chi}(\mu)$, on the range $[\mu^*, \mu_\ell]$.*

Step 3: *Apply the Collocation-Projection method discussed in Subsection 4.2 to estimate the optimal policy toward stagnation, $\bar{\chi}(\mu)$, on the range $[\hat{\mu}, \mu_\ell]$.*

Step 4: *Use policies $\tilde{\chi}(\mu)$ and $\bar{\chi}(\mu)$ evaluated in steps 2 and 3 to compute $\tilde{c}(A)$ as in (37) and $\bar{c}\left(A, \tilde{k}(A)\right)$ as in (41) so to get $w(A)$ and $\bar{V}(A)$ according to (47) and (45) respectively; define $f(A)$ as in (49).*

Step 5: *Apply the standard* Maple 16 'fsolve' routine *to equation (49) using the range $\left[A_\ell, \hat{A}\right]$ to find A_m satisfying $f(A_m) = 0$.*

Step 6: *Report the solution, A_m, and evaluate the Skiba-point, $k^{sk}(A_m) = \tilde{k}(A_m)$.*

The Maple 16 code for Algorithm 1 is available from the author upon request.

Figure 3(a) plots both value functions $\tilde{V}(A) = w(A)$ (in black) and $\bar{V}(A)$ (in dark grey) of problems (27) and (38) respectively as approximated through Algorithm 1 on the range $\left[A_\ell, \hat{A}\right]$ for the parameters' values considered in Section 7. As problem (27) is not concave for small values of the stock of knowledge A, consistently, $\tilde{V}(A)$ turns out to be convex on such initial values range. Do not be misled by the convexity of the value function $\bar{V}(A)$ of the model leading to stagnation: problem (38) is a standard concave Ramsey problem in its state variable, which is physical capital, k; in fact, its value function as a function of k is definitely concave. Function $\bar{V}(A)$, as a function of A, represents a whole family of Ramsey problems, each indexed by the value A (initial stock of knowledge) which remains constant as physical capital, k, evolves through time toward its steady value $\hat{k}(A)$. It turns out that the value functions of these problems evolve in a convex fashion as A increases in the range $\left[A_\ell, \hat{A}\right]$. Figure 3(b) plots the same value functions for a range of A-values larger than $\left[A_\ell, \hat{A}\right]$; they are obtained through Algorithm 1 where a larger interval $\left[\underline{\mu}, \mu_\ell\right]$, with $\underline{\mu} < \hat{\mu}$, has been chosen in step 3. It is clearly seen that $\tilde{V}(A)$ becomes concave as A increases, that is, when the unit cost of knowledge production $\varphi(A)$ in (12) approaches its asymptotic constant value $1/\beta$, or, equivalently, when the turnpike $\tilde{k}(A)$ approaches the asymptotic turnpike $\tilde{k}_\infty(A)$.

5 Skiba-Points Above the Turnpike

According to Proposition 3, to the left of the knowledge level A_m found in the previous section the Skiba-point necessarily must lie strictly 'above' the turnpike, *i.e.*, $k^{sk}(A_0) > \tilde{k}(A_0)$ for all $A_\ell \leq A_0 < A_m$. Thus, for values of the initial stock of knowledge $A_0 < A_m$ we must characterize optimal trajectories starting above the turnpike at $t = 0$ and entering the turnpike at some

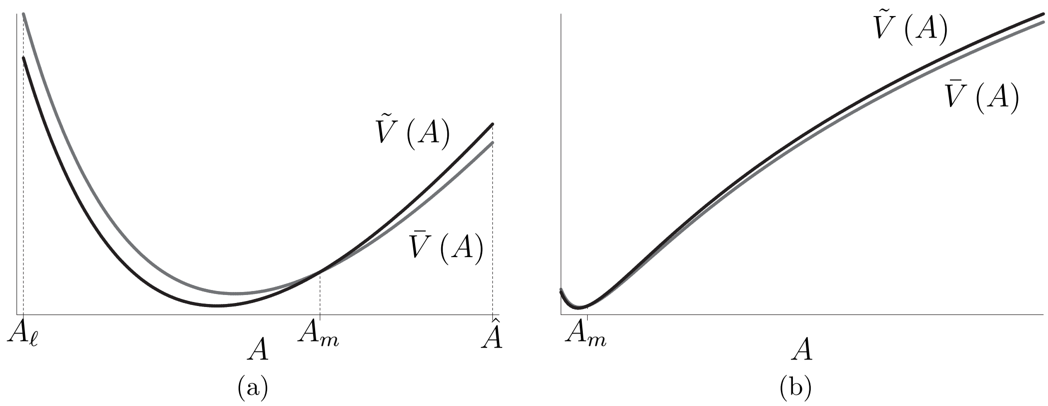

FIGURE 3: value functions $\tilde{V}(A)$ (in black) and $\bar{V}(A)$ (in dark grey) of problems (27) and (38) for the parameters' values used in Section 7, (a) on the range $\left(A_\ell, \hat{A}\right]$ and (b) for larger values of A.

later instant $t_0 > 0$, after which the economy continues its evolution according to optimal trajectories of the sort discussed in Subsection 4.1.

Any optimal trajectory above the turnpike must satisfy the last necessary condition in (19), $\delta/\varphi(A) > \lambda$, corresponding to the largest possible investment in R&D activities by the social planner:[16] $J = y = \theta k^\alpha A^{1-\alpha}$. In other words, along such early-transition trajectories it is optimal to invest all the output into the production of new knowledge. Hence, on the time interval $[0, t_0]$ problem (17) simplifies to one in two state, A and k, and one control, c, variables:

$$\max_{[c]} \int_0^{t_0} e^{-\rho t} \frac{c^{1-\sigma} - 1}{1-\sigma} dt \tag{50}$$

$$\text{subject to} \begin{cases} \dot{A} = \theta k^\alpha A^{1-\alpha}/\varphi(A) \\ \dot{k} = -c \\ A(t_0) = A_r,\ k(t_0) = \tilde{k}(A_r),\ c(t_0) = \tilde{c}(A_r), \end{cases}$$

with the additional constraint $0 \le c \le k$, where again the time argument has been dropped for simplicity, $A_r > A_0$ is the knowledge level corresponding to instant $t_0 > 0$ at which the turnpike is hit from above, $\tilde{k}(A_r)$ is the corresponding capital value on the turnpike and $\tilde{c}(A_r)$ is the optimal policy value for consumption on the turnpike at A_r according to (37). Instead of initial conditions, three terminal conditions are given for problem (50) that bound the optimal trajectories to land on the turnpike at the point $\left(A_r, \tilde{k}(A_r)\right)$ at instant t_0. While the first two are obvious, the last one, $c\left(\tilde{t}\right) = \tilde{c}(A_r)$, stating that the terminal value of consumption must match the optimal consumption value on the turnpike, holds because the control c of problem (17) must be continuous for all $t \ge 0$, as noted in Remark 1.

The Skiba-point corresponding to some initial stock of knowledge $A_\ell \le A_0 < A_m$, is the initial capital value $k_0 = k^{sk}(A_0)$ that equates the welfare produced by the whole optimal consumption time-path trajectory, for $t \in [0, +\infty)$, that starts on (A_0, k_0) at $t = 0$ and it is the piecewise union of the optimal early transition trajectory above the turnpike over $[0, t_0]$ with the optimal transition trajectory along the turnpike over $(t_0, +\infty)$, with the welfare generated, according to $\delta/\varphi(A) < \lambda$ in (19), by a constant zero-R&D investment policy, $J \equiv 0$, starting from the same initial point (A_0, k_0) and leading the economy toward the steady state $\left(A_0, \hat{k}(A_0)\right)$ on the stagnation line. Clearly, the former trajectory is defined by the intersection

[16] See Proposition 1 on p. 3464 in [24].

value A_r at the positive instant t_0. Thus, our aim is to build an iterative algorithm to estimate A_r and the corresponding positive instant $t_0 > 0$ that determines an optimal whole time-path trajectory toward steady growth starting at (A_0, k_0) yielding the same welfare of that starting as well from (A_0, k_0) but leading to stagnation. In other words, we start by an arbitrary choice of A_r and study the union of the optimal early transition trajectory originating from $\left(A_r, \tilde{k}(A_r)\right)$ at some instant $t_0 > 0$ and, by going backward in time, defines a pair of initial values (A_0, k_0) at $t = 0$, with its continuation along the turnpike for $t > t_0$; next, we compare the welfare generated by such whole trajectory with that produced by the optimal trajectory that starts from the same initial point (A_0, k_0) and leads to the steady state $\left(A_0, \hat{k}(A_0)\right)$.

Substituting J with $\theta k^\alpha A^{1-\alpha}$ in the necessary conditions for the current-value Hamiltonian (18) we are led to the following optimal dynamics associated to (50):

$$\begin{cases} \dot{A} = \theta k^\alpha A^{1-\alpha} / \varphi(A) \\ \dot{k} = -c \\ \dot{c} = c\left[\theta\alpha\left(k/A\right)^{\alpha-1} - \rho\right]/\sigma, \end{cases} \tag{51}$$

which, together with the three terminal conditions, is a Cauchy problem in the three variables k, A and c. To solve system (51) again we eliminate time by taking the ratios $\dot{k}/\dot{A}$ and $\dot{c}/\dot{A}$ and study the following system of two ODEs in the functions $k(A)$ and $c(A)$:

$$\begin{cases} k'(A) = -\dfrac{c(A)\,\varphi(A)}{\theta\left[k(A)\right]^\alpha A^{1-\alpha}} \\ c'(A) = \dfrac{c(A)\,\varphi(A)\left\{\theta\alpha\left[k(A)/A\right]^{\alpha-1} - \rho\right\}}{\sigma\theta\left[k(A)\right]^\alpha A^{1-\alpha}}. \end{cases} \tag{52}$$

To solve (52) we first choose the initial stock of knowledge $A_\ell < A_0 < A_m$ – with A_ℓ being the lower bound used in Algorithm 1 and A_m the estimate generated by the same Algorithm – and a value $A_r > A_0$. We then apply a *projection method* based on *Chebyshev Orthogonal Collocation* on n collocation points over the interval $[A_0, A_r]$ applied to the two residual functions – one for each policy $k^{ab}(A)$ and $c^{ab}(A)$ to be estimated – built upon approximation functions which are linear combinations of n Chebyshev polynomials.[17] Thus, our estimates of the two policies $k^{ab}(A)$ and $c^{ab}(A)$ are polynomials of degree n, possibly with a different n than those used in the previous subsections. To approximate the optimal time-path trajectory of the stock of knowledge $A^{ab}(t)$ along this early transition dynamic for the economy, $k^{ab}(A)$ is substituted into the first equation of (51) so to obtain a ODE with respect to time which can be numerically solved through the standard *Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant* method available in Maple 16. The corresponding optimal time-path trajectories $k^{ab}(t)$, $y^{ab}(t)$, $J^{ab}(t)$ and $c^{ab}(t)$ are then computed as $k^{ab}(t) = k^{ab}\left[A^{ab}(t)\right]$, $y^{ab}(t) = J^{ab}(t) = \theta\left[k^{ab}(t)\right]^\alpha\left[A^{ab}(t)\right]^{1-\alpha}$ and $c^{ab}(t) = c^{ab}\left[A(t)\right]$ respectively. Finally, the instant t_0 at which the optimal trajectories just evaluated hit the turnpike on the point $\left(A_r, \tilde{k}(A_r)\right)$ is approximated by solving $A^{ab}(t) = A_r$ with respect to t through the *Maple 16 'fsolve' routine*. The initial

[17] As initial condition for the *Maple 16 'fsolve' routine* used to numerically solve the system of $2n+2$ equations setting the two residual functions equal to zero on each collocation node plus the two terminal conditions $k^{ab}(A_r) = \tilde{k}(A_r)$ and $c^{ab}(A_r) = \tilde{c}(A_r)$, we use a *Chebyshev regression of order n* (Algorithm 6.2 on p. 223 in [16]) on the lines crossing the pairs of points $\left(A_0, \tilde{k}(A_0)\right)$, $\left(A_r, \tilde{k}(A_r)\right)$ and $(A_0, \tilde{c}(A_0))$, $(A_r, \tilde{c}(A_r))$ for the $k^{ab}(A_r)$ and $c^{ab}(A)$ policies respectively.

capital level corresponding to A_0 at $t = 0$ along the backward-in-time trajectory starting from $\left(A_r, \tilde{k}(A_r)\right)$ is thus computed as

$$k_0(A_0, A_r) = k^{ab}(A_0), \tag{53}$$

where $k^{ab}(A)$ is the capital optimal policy solving (52).

The whole optimal transition time-path trajectories $\tilde{A}^{ab}(t)$, $\tilde{k}^{ab}(t)$, $\tilde{y}^{ab}(t)$, $\tilde{J}^{ab}(t)$ and $\tilde{c}^{ab}(t)$ for all $t \geq 0$ when the economy starts at $t = 0$ from the initial conditions $(A_0, k_0(A_0, A_r))$, with $k_0(A_0, A_r)$ as in (53), can be built as *piecewise functions* by joining each trajectory above the turnpike over $[0, t_0]$ with its 'continuation' along the turnpike over $(t_0, +\infty)$ at the instant $t = t_0$,

$$\tilde{z}^{ab}(t) = \begin{cases} z^{ab}(t) & \text{for } t \in [0, t_0] \\ \tilde{z}(t) & \text{for } t \in (t_0, +\infty), \end{cases} \tag{54}$$

with $z^{ab} \in \{A^{ab}, k^{ab}, y^{ab}, c^{ab}, J^{ab}\}$, while all the $\tilde{z} \in \left\{\tilde{A}, \tilde{k}, \tilde{y}, \tilde{c}, \tilde{J}\right\}$ time-path trajectories are built according to the method discussed at the end of Subsection 4.1 on the range $[\mu^*, \mu_r]$, with μ^* as in (34) and $\mu_r = \tilde{k}(A_r)/A_r$. As $\tilde{J}^{ab}(t) = y^{ab}(t)$ for $t \in [0, t_0]$ while $\tilde{J}^{ab}(t) = \tilde{J}(t) < \tilde{y}(t)$ for $t \in (t_0, +\infty)$, with $\tilde{J}(t)$ given by (23), we expect to observe a discontinuity 'jump' for the optimal control $\tilde{J}^{ab}$ at instant t_0, as postulated by necessary conditions (19), while all other trajectories must exhibit a kink on t_0, where they are not differentiable. This pattern is confirmed in Figure 6 of Section 7.

To estimate welfare when the economy follows its path along the turnpike toward the ABGP starting at $t = 0$ from $(A_0, k_0(A_0, A_r))$, with $k_0(A_0, A_r)$ defined by (53), we apply Lemma 1 of Section 2 and again Proposition 4. Specifically, under condition (48) we conveniently split it as the sum of two terms:

$$\tilde{V}^{ab}(A_0, A_r) = \int_0^{t_0} e^{-\rho t} \frac{\left[c^{ab}(t)\right]^{1-\sigma} - 1}{1 - \sigma} dt + e^{-\rho t_0} w(A_r), \tag{55}$$

where $w(A_r) = \tilde{V}(A_r)$ is the value function of problem (27) according to (47) of Subsection 4.4 evaluated at the intersection point A_r. That is, at $t = t_0$ we consider the welfare generated by the economy along the turnpike when it starts with initial stock of knowledge A_r, and discount this value in $t = 0$. The first integral on the RHS of (55) is approximated through a *Gauss-Legendre quadrature* routine on a large number of nodes over the time range $[0, t_0]$, using the time-path trajectory value of optimal consumption, $c^{ab}(t)$, defined before on each node.

To calculate welfare when the economy converges to the steady state $\left(A_0, \hat{k}(A_0)\right)$ on the stagnation line when starting at $t = 0$ from the same initial point $(A_0, k_0(A_0, A_r))$, with $k_0(A_0, A_r)$ defined by (53), we restate problem (38) according to

$$\bar{V}^{ab}(A_0, A_r) = \max_{[c]} \int_0^\infty e^{-\rho t} \frac{c^{1-\sigma} - 1}{1 - \sigma} dt \tag{56}$$

$$\text{subject to } \begin{cases} \dot{k} = \theta A_0^{1-\alpha} k^\alpha - c, \\ k(0) = k_0(A_0, A_r). \end{cases}$$

Because the 'detrended' system in the ratio variables $\mu = k/A_0$ and $\chi = c/A_0$ associated to the optimal dynamics of (56) turns out to be the same as in (39), the optimal policy for (56) is obtained according to the same approximation procedure discussed in Subsection 4.2 by means of (41), that is, $\bar{c}(A_0, k) = \bar{\chi}(k/A_0) A_0$, using $k(0) = k_0(A_0, A_r)$, with $k_0(A_0, A_r)$ defined by (53), as initial condition. More precisely, as we are going to study trajectories starting from initial capital values above the turnpike, $k_0(A_0, A_r) > \tilde{k}(A_0)$, the Collocation-Projection method must be performed over a range $[\hat{\mu}, \mu_0]$ larger than the interval $[\hat{\mu}, \mu_\ell]$ used in Subsection 4.5. Specifically, when the initial stock of knowledge equates the lower bound $A_0 = A_\ell$ used in Algorithm 1, setting $\mu_0 = k_0(A_\ell, A_r)/A_\ell$ implies that $\mu_0 > \mu_\ell = \tilde{k}(A_\ell)/A_\ell$ whenever $k_0(A_0, A_r) > \tilde{k}(A_\ell)$, as will be the case in one of our simulations of Section 7.

To approximate $\bar{V}^{ab}(A_0, A_r)$ in (56) we apply the same technique explained in Subsection 4.4, based on Theorem 2 and Corollary 1, and again exploit the Hamilton-Jacobi-Bellman equation:

$$\bar{V}^{ab}(A_0, A_r) = \frac{\{\bar{c}[A_0, k_0(A_0, A_r)]\}^{1-\sigma} - 1}{\rho(1-\sigma)} + \frac{\theta[k_0(A_0, A_r)]^\alpha A_0^{1-\alpha} - \bar{c}[A_0, k_0(A_0, A_r)]}{\rho\{\bar{c}[A_0, k_0(A_0, A_r)]\}^\sigma}, \quad (57)$$

where $\bar{c}[A_0, k_0(A_0, A_r)]$ is the estimated value of the optimal consumption policy at the initial point $(A_0, k_0(A_0, A_r))$ numerically obtained by (41).

Algorithm 2 below summarizes all the steps discussed above. Because a complex pointwise estimation for each welfare value $\tilde{V}^{ab}(A_0, A_r)$ required by the integral approximation in (55), it relies on a standard Bisection Method (see, *e.g.*, Algorithm 5.1 on p. 148 in [16]). Fix a given initial stock of knowledge $A_\ell \leq A_0 < A_m$ and let

$$f^{ab}(A_r) = \tilde{V}^{ab}(A_0, A_r) - \bar{V}^{ab}(A_0, A_r), \quad (58)$$

with $\tilde{V}^{ab}(A_0, A_r)$ and $\bar{V}^{ab}(A_0, A_r)$ defined in (55) and (57) respectively, be the function whose unique zero is the target of our search routine. For each given A_0, the unique value A_r^* such that $f^{ab}(A_r^*) = 0$ yields our estimation of the Skiba-point as

$$k^{sk}(A_0) = k_0(A_0, A_r^*), \quad (59)$$

where $k_0(A_0, A_r)$ is defined according to (53). As $k^{sk}(A_0) > \tilde{k}(A_0)$, $f^{ab}(A_0) < 0$ must hold; hence, A_0 plays a useful role as left endpoint of the initial interval bracketing the unique zero of f^{ab}. The following result is useful to estimate the right endpoint of such interval.

Proposition 5 *For any initial stock of knowledge A_0 such that $A_\ell \leq A_0 < A_m$, the unique A_r^* such that $f^{ab}(A_r^*) = 0$ must satisfy $A_r^* \geq A_m$.*

Proof. Suppose, on the contrary, that $A_r^* < A_m$. Let $k^{sk}(A_0)$ as in (59) be the Skiba-point associated to the initial knowledge level A_0, then the trajectory starting at $t = 0$ on the initial point $\left(A_0, k^{sk}(A_0)\right)$, with $A_0 < A_m$, and hitting the turnpike at a later instant $t_0 > 0$ on $\tilde{k}(A_r^*)$ yields the same welfare as the trajectory leading to stagnation from $\left(A_0, k^{sk}(A_0)\right)$. In Section 4 A_m has been defined as the unique value satisfying $k^{sk}(A_m) = \tilde{k}(A_m)$; therefore, according to Proposition 3, $A_r^* < A_m$ implies that the trajectory converging to the steady state $\left(A_r^*, \hat{k}(A_r^*)\right)$ on the stagnation line according to a zero-R&D investment, $J \equiv 0$, policy for $t > t_0$ yields a larger welfare than the trajectory continuing along the turnpike toward steady growth. This contradicts the assumption that $k^{sk}(A_0)$ is the Skiba-point. ■

In view of Proposition 5, before starting the true Bisection Method we will perform a number of preliminary iterations to estimate the right endpoint of the interval bracketing the zero of f^{ab}, starting from $A_R = A_m$ and then increasing this value by a (small) constant increment after each iteration until a value A_R such that $f^{ab}(A_R) \geq 0$ is found.

Algorithm 2 (Finds the Skiba-point when $A_0 < A_m$)

Step 1: *Set the range* $[\mu^*, \mu_\ell]$ *as in step 1 of Algorithm 1 for the* $\tilde{\chi}(\mu)$ *policy approximation in the* (μ, χ) *space. The range for the* $\bar{\chi}(\mu)$ *policy approximation is* $[\hat{\mu}, \mu_0]$*, with* $\hat{\mu}$ *as in (43) and* $\mu_0 = \mu_\ell + \vartheta$*, with* ϑ *sufficiently large to allow the estimation of* $\bar{c}[A_0, k_0(A_0, A_r)]$ *according to (53) in the following step 4.2.7 when* $k_0(A_0, A_r) > \tilde{k}(A_\ell)$*; i.e.,* μ_0 *must satisfy* $k_0(A_0, A_R) \leq \mu_0 A_0$*.*

Step 2: *Apply the OLS-Projection method discussed in Subsection 4.1 to estimate the optimal policy along the turnpike,* $\tilde{\chi}(\mu)$*, on the range* $[\mu^*, \mu_\ell]$*.*

Step 3: *Apply the Collocation-Projection method discussed in Subsection 4.2 to estimate the optimal policy toward stagnation,* $\bar{\chi}(\mu)$*, on the range* $[\hat{\mu}, \mu_0]$*.*

Step 4: *Find* A_r *satisfying* $f^{ab}(A_r) = 0$ *for* f^{ab} *defined in (58).*

Step 4.1 (Initialization): *Set* $[A_L, A_R] = [A_0, A_m]$*, with* A_m *being the output of Algorithm 1, as the initial interval for searching the interval bracketing the zero of* f^{ab} *in (58), set a (switch) variable* $B = 1$*, choose* $t_{\max} > 0$ *for the range of the Runge-Kutta routine in the following step 4.2.3, choose* $N > 0$ *as the number of nodes for the Gauss-Legendre quadrature routine in the following step 4.2.5, choose an increment* $\epsilon > 0$*, choose stopping rule parameters* $0 < \varepsilon, \eta < 1$*, and set (fake) initial values* $f^{ab}(A_r) = f^{ab}(A_R) = 1 > \eta$*.*

Step 4.2 (Bisection loop): *While* $A_R - A_L > \varepsilon$ *and* $\left|f^{ab}(A_m)\right| > \eta$ *do:*

1. *if* $B = 1$ *then set* $A_R = A_R + \epsilon$ *(increase right bound) and* $A_r = A_R$*, else set* $A_r = (A_R - A_L)/2$ *(compute midpoint),*
2. *approximate policies* $k^{ab}(A)$ *and* $c^{ab}(A)$ *over* $[A_0, A_r]$ *by solving (52) through the Collocation-Projection method described above,*
3. *use* $k^{ab}(A)$ *and* $c^{ab}(A)$ *from step 4.2.2 to build the time-path trajectories* $A^{ab}(t)$ *and* $c^{ab}(t)$ *over* $[0, t_{\max}]$ *through the Runge-Kutta routine as explained above,*
4. *find* t_0 *by solving* $A^{ab}(t) = A_r$ *through Maple 16 'fsolve' routine over* $[0, t_{\max}]$*,*
5. *apply the Gauss-Legendre quadrature routine explained before to approximate the integral in (55), use* $\tilde{\chi}(\mu)$ *from step 2 to evaluate* $\tilde{c}(A)$ *through (37), compute* $w(A_r)$ *according to (47) and evaluate* $\tilde{V}^{ab}(A_0, A_r)$ *as in (55),*
6. *evaluate* $k_0(A_0, A_r)$ *using* $k^{ab}(A)$ *from step 4.2.2 to according to (53),*
7. *use* $\bar{\chi}(\mu)$ *from step 3 to evaluate* $\bar{c}[A_0, k_0(A_0, A_r)]$ *through (41) and evaluate* $\bar{V}^{ab}(A_0, A_r)$ *by means of (57),*
8. *update* $f^{ab}(A_r)$ *by setting* $f^{ab}(A_r) = \tilde{V}^{ab}(A_0, A_r) - \bar{V}^{ab}(A_0, A_r)$*,*
9. *if* $B = 1$ *and* $f^{ab}(A_r) < 0$ *then (keep searching for bracket right endpoint) go to step 4.2, else (bisection loop)*
   - if $B = 1$ *set* $B = 0$ *(stop searching for bracket),*
   - refine the bounds: if $f^{ab}(A_r) f^{ab}(A_R) < 0$ *then set* $A_L = A_r$*, else set* $A_R = A_r$ *and update* $f^{ab}(A_R)$ *by setting* $f^{ab}(A_R) = f^{ab}(A_r)$*.*

Step 5: *Report the Skiba-point from step 4.2.6,* $k^{sk}(A_0) = k_0(A_0, A_r)$*.*

Remark 4

1. *The choice of* $t_{\max}$ *in step 4.1 is a delicate issue, because it depends on the range* $[A_0, A_r]$ *over which the Projection Method approximates the policies* $k^{ab}(A)$ *and* $c^{ab}(A)$ *in step 4.2.2. If it is too large, the Runge-Kutta algorithm in step 4.2.3 stops too early yielding an error message because it tries to estimate a trajectory continuing beyond the intersection point* $\left(A_r, \tilde{k}(A_r)\right)$ *on the turnpike, which after a short while ceases to be defined. On the other hand, if it is too small it fails to catch the* t_0 *value, which, indeed, happens to be close to* $t_{\max}$*. Hence a suitable* $t_{\max}$ *value should be chosen through some guess-and-tries.*

2. *The degree of approximation,* n*, in the Collocation-Projection method performed in step 4.2.2 must be smaller for* A_0 *values closer to* A_m*, as too many Chebyshev polynomials in a small interval cause the algorithm to stall.*

The Maple 16 code for Algorithm 2 is available from the author upon request.

6 Skiba-Points Below the Turnpike

For values of initial stock of knowledge to the right of A_m Proposition 3 implies that the Skiba-point must lie strictly 'below' the turnpike, *i.e.*, $k^{sk}(A_0) < \tilde{k}(A_0)$ for all $A_m < A_0 < \hat{A}$, where A_m is given by Algorithm 1 and $\hat{A}$ is defined in (26). This type of scenario forecasts optimal early transition trajectories starting below the turnpike at $t = 0$ and entering the turnpike from below at some later instant $t_0 > 0$, after which the economy continues its evolution along the turnpike according to optimal trajectories of the sort discussed in Subsection 4.1. The former trajectories are characterized by $\delta/\varphi(A) < \lambda$ in (19) and thus envisage a zero investment policy in new knowledge production: $J(t) \equiv 0$ and $A(t) \equiv A_0$ for all $t \in [0, t_0]$. In other words, like along trajectories converging to the point $\left(A_0, \hat{k}(A_0)\right)$ on the stagnation line, the economy evolves through time along the vertical line $A \equiv A_0$ in the (A, k) space, though in the opposite direction (*i.e.*, moving upward), accumulating physical capital until the turnpike is reached.

Because A remains constant at the A_0 level when $t \in [0, t_0]$, these optimal dynamics restated in terms of ratio variables $\mu = k/A$ and $\chi = c/A$ must satisfy the associated short-run necessary conditions described by (39) in the (μ, χ) space, so that they can be represented in the same phase diagram of the $\tilde{\chi}(\mu)$ and $\bar{\chi}(\mu)$ policies discussed in Subsections 4.1 and 4.2 respectively. Specifically, setting

$$\tilde{\mu}_0 = \tilde{k}(A_0)/A_0, \tag{60}$$

as the μ value corresponding to the point at which our trajectory hits the turnpike from below at $t = t_0$, the terminal condition

$$\chi^{be}(\tilde{\mu}_0) = \tilde{\chi}(\tilde{\mu}_0) \tag{61}$$

establishes a well defined Cauchy problem for the single ODE (40). According to Remark 1, terminal condition (61) is justified by the continuity of the optimal control $\chi^{be} = c/A$ at the intersection point with the turnpike.

After fixing an initial stock of knowledge A_0 such that $A_m < A_0 < \hat{A}$ we solve this problem by means of a *projection method* based on *OLS* applied to the integral of a residual function built upon an approximation function which is a linear combination of n Chebyshev polynomials.[18]

[18] As usual, the integral of the residual function is itself approximated by means of *Gauss-Chebyshev quadrature* on the relevant interval.

Therefore, our estimate of the early time transition policy $\chi^{be}(\mu)$ turns out to be a polynomial of degree n, possibly with a different n than those used in the previous sections. We opt for the OLS rather than the Collocation method here because, while losing some degree of precision upon the latter, the former appears to be more flexible and better equipped to adapt to stiff ODEs. Indeed, some trajectories defined by (40) for our Cauchy problem turn out to pass very close to the unique steady state $(\hat{\mu}, \hat{\chi})$, with coordinates defined in (43), of system (39), so that they exhibit nearly a kink in proximity of $(\hat{\mu}, \hat{\chi})$. As initial condition for the *Maple 16 nonlinear programming (NLP) solver* we use a Chebyshev regression of order n (Algorithm 6.2 on p. 223 in [16]) on the line joining the two points $\left(\mu^{sk}, \bar{\chi}\left(\mu^{sk}\right)\right)$ and $(\tilde{\mu}_0, \tilde{\chi}(\tilde{\mu}_0))$, where μ^{sk} is the midpoint defined in the Bisection loop of the following iterative Algorithm 3, $\tilde{\mu}_0$ is defined in (60) and $\tilde{\chi}(\mu)$, $\bar{\chi}(\mu)$ are the policies defined in Subsections 4.1 and 4.2 respectively.

To approximate the optimal time-path trajectory $\mu^{be}(t)$ along this early transition dynamic, the $\chi^{be}(\mu)$ policy just estimated is substituted into the first equation of (39) so to obtain a ODE with respect to time which can be numerically solved through the standard *Runge-Kutta method* available in Maple 16. The time-path trajectory $\chi^{be}(t)$ is then computed as $\chi^{be}(t) = \chi^{be}\left[\mu^{be}(t)\right]$, and the corresponding optimal time-path trajectories are given by $k^{be}(t) = \mu^{be}(t) A_0$, $y^{be}(t) = \theta\left[k^{be}(t)\right]^{\alpha} A_0^{1-\alpha}$ and $c^{be}(t) = \chi^{be}(t) A_0$, while $J(t) \equiv 0$ and $A(t) \equiv A_0$ for all $t \in [0, t_0]$. The instant t_0 at which the optimal trajectories just evaluated hit the turnpike from below on the point $\left(A_0, \tilde{k}(A_0)\right)$ is approximated by solving $\mu^{be}(t) = \tilde{\mu}_0$ with respect to t through the *Maple 16 'fsolve' routine*. The whole optimal transition time-path trajectories $\tilde{A}^{be}(t)$, $\tilde{k}^{be}(t)$, $\tilde{y}^{be}(t)$, $\tilde{J}^{be}(t)$ and $\tilde{c}^{be}(t)$ for all $t \geq 0$ when the economy starts at $t = 0$ from any initial conditions (A_0, k_0) with $A_m < A_0 < \hat{A}$ and $0 < k_0 < \tilde{k}(A_0)$, are thus built as *piecewise functions* by joining each trajectory above the turnpike over $[0, t_0]$ with its 'continuation' along the turnpike over $(t_0, +\infty)$ at the instant $t = t_0$:

$$\tilde{z}^{be}(t) = \begin{cases} z^{be}(t) & \text{for } t \in [0, t_0] \\ \tilde{z}(t) & \text{for } t \in (t_0, +\infty), \end{cases} \tag{62}$$

with $z^{be} \in \left\{A \equiv A_0, k^{be}, y^{be}, c^{be}, J \equiv 0\right\}$, while all the $\tilde{z} \in \left\{\tilde{A}, \tilde{k}, \tilde{y}, \tilde{c}, \tilde{J}\right\}$ time-path trajectories are built according to the method discussed at the end of Subsection 4.1 on the range[19] $[\mu^*, \mu_m]$, with μ^* as in (34) and $\mu_m = \tilde{k}(A_m)/A_m$, where A_m is the output of Algorithm 1. As $\tilde{J}^{be}(t) \equiv 0$ for $t \in [0, t_0]$ while $\tilde{J}^{be}(t) = \tilde{J}(t) > 0$ for $t \in (t_0, +\infty)$, with $\tilde{J}(t)$ given by (23), we expect to observe a discontinuity 'jump' for the optimal control $\tilde{J}^{be}$ at instant t_0, as postulated by necessary conditions (19), while all other trajectories must exhibit a kink on t_0, where they are not differentiable. This pattern is confirmed in Figure 7 of Section 7.

As we did in Section 5, to estimate welfare when the economy is driven toward the ABGP starting at $t = 0$ from (A_0, k), we apply Lemma 1 and Proposition 4 so that, under condition (48), we can split it as the sum of two terms:

$$\tilde{V}^{be}(A_0, k_0) = \int_0^{t_0} e^{-\rho t} \frac{\left[c^{be}(t)\right]^{1-\sigma} - 1}{1 - \sigma} dt + e^{-\rho t_0} w(A_0), \tag{63}$$

where $w(A_0) = \tilde{V}(A_0)$ is the value function of problem (27) according to (47) of Subsection 4.4 evaluated at the initial point A_0. The first integral on the RHS of (63) is approximated through a *Gauss-Legendre quadrature* routine on a large number of nodes over the time range $[0, t_0]$, using the time-path trajectory value of optimal consumption, $c^{be}(t)$, on each node.

[19] This range is shorter than that used in previous sections as there is no point of considering trajectories along the turnpike starting to the right of μ_m, or, equivalently, to the left of A_m. Incidentally, in Section 7 it will be seen that this slightly improves the $\tilde{\chi}(\mu)$ estimation.

Because now we are confronting trajectories starting anywhere at $t = 0$ on the vertical segment $A \equiv A_0$, with $0 < k_0 < \tilde{k}(A_0)$, in the (A, k) space, either moving upward to intersect the turnpike and then continue along it toward steady growth or converging to the steady state $\left(A_0, \hat{k}(A_0)\right)$ on the stagnation line, we restate problem (56) as

$$\bar{V}^{be}(A_0, k_0) = \max_{[c]} \int_0^\infty e^{-\rho t} \frac{c^{1-\sigma} - 1}{1 - \sigma} dt \tag{64}$$
$$\text{subject to } \begin{cases} \dot{k} = \theta A_0^{1-\alpha} k^\alpha - c, \\ k(0) = k_0, \end{cases}$$

where A_0 and k_0 can be any values satisfying $A_m < A_0 < \hat{A}$ and $0 < k_0 < \tilde{k}(A_0)$. Clearly, the 'detrended' system in the ratio variables $\mu = k/A_0$ and $\chi = c/A_0$ associated to the optimal dynamics of (64) is again (39). Hence, the optimal policy for (64) is obtained according to the same approximation procedure discussed in Subsection 4.2 by means of (41), that is, $\bar{c}(A_0, k) = \bar{\chi}(k/A_0) A_0$, using $k(0) = k_0$ as initial condition. Because here we are considering any initial capital stock value below the turnpike, k_0 can be arbitrarily small, which, in turn, means that the left endpoint of the range over which the Collocation-Projection method is performed can be arbitrarily small. Thus, in principle we should consider the whole half stable arm of the unique steady state $(\hat{\mu}, \hat{\chi})$, with coordinates defined in (43), over the range $(0, \mu_m]$, with right endpoint corresponding to the left endpoint of the A range considered in this section: $\mu_m = \tilde{k}(A_m)/A_m$. However, below a (positive) lower threshold value for μ the Collocation-Projection method performances rapidly degenerate. Therefore, we set a lower bound, $\bar{\mu}_L > 0$, which implies that the estimation of Skiba-points close to the right endpoint of the initial knowledge level range considered in this section, $\hat{A}$, are ruled out.

Moreover, to build a Bisection algorithm similar to those developed in the previous section it is important to distinguish intersection points between early transition paths and the turnpike at instant t_0 lying above the *unstable arm*, $\bar{\chi}^{unst}(\mu)$, associated to the steady state $(\hat{\mu}, \hat{\chi})$ in the (μ, χ) space from the same intersection points lying below the unstable arm. This is because the two types of early transition paths lie in different basins of the same phase diagram and thus exhibit quite different features, starting from the range over which they are defined, which, in turn require a different fine tuning of the following iterative Algorithm 3. For our purposes it is sufficient to consider only the initial part of the unstable half arm originating at the steady state $(\hat{\mu}, \hat{\chi})$ and pointing south-east. Specifically, we consider its portion over the range $[\hat{\mu}, \mu_m]$, which is approximated through a *projection method* based on *Chebyshev Orthogonal Collocation* on n collocation points applied to the residual function built upon an approximation function which is a linear combination of n Chebyshev polynomials, in a fashion similar to that discussed in Subsection 4.2.[20]

Figure 4 reports, in the same phase diagram of Figure 2, two examples drawn from Section 7 of the paths that trajectories built according to (62) must follow when starting below the turnpike and leading the economy toward steady growth. Figure 4(a) shows, in thick black, the detail of the early transition generated by $\chi^{be}(\mu)$ as the finite trait of a path hitting the turnpike policy $\tilde{\chi}(\mu)$ at some point above the unstable arm of $(\hat{\mu}, \hat{\chi})$ (the thick decreasing

[20]As initial condition for the *Maple 16 'fsolve' routine* used to numerically solve the system of $n+1$ equations setting the residual function equal to zero on each collocation node plus the steady state constraint, $\bar{\chi}^{unst}(\hat{\mu}) = \hat{\chi}$, we use a *Chebyshev regression of order n* (Algorithm 6.2 on p. 223 in [16]) on the line tangent to the unstable arm $\bar{\chi}^{unst}(\mu)$ on the steady state $(\hat{\mu}, \hat{\chi})$ with coordinates defined in (43). The slope of $\bar{\chi}^{unst}(\mu)$ on the steady state $(\hat{\mu}, \hat{\chi})$ is the negative solution of the quadratic equation obtained through l'Hôpital's rule, according to [4], pp. 595–596.

curve in dark grey), that is, such that $\tilde{\chi}_0 = \tilde{\chi}(\tilde{\mu}_0) > \bar{\chi}^{unst}(\tilde{\mu}_0)$, with $\tilde{\mu}_0$ defined in (60). It is a portion of the decreasing branch of one of the paths lying on the right of the steady state $(\hat{\mu}, \hat{\chi})$ in a typical Ramsey-model phase diagram, *i.e.*, those coming from north-east just below the top-right branch of the stable arm of $(\hat{\mu}, \hat{\chi})$ (the increasing thick curve in dark grey), initially heading toward south-west but then, at some point in a neighborhood of $(\hat{\mu}, \hat{\chi})$, turning toward south-east, hitting the $\tilde{\chi}(\mu)$ policy at instant t_0 on $(\tilde{\mu}_0, \tilde{\chi}_0)$. The whole path $\tilde{\chi}^{be}(\mu)$, in thick black, is the union of $\chi^{be}(\mu)$ with the path defined by policy $\tilde{\chi}(\mu)$ along the turnpike eventually heading south-west toward the steady state (μ^*, χ^*) (outside the figure). Figure 4(b) exhibits, in thick black, the early transition $\chi^{be}(\mu)$ as the finite trait of a path entering the turnpike policy at some point below the unstable arm of $(\hat{\mu}, \hat{\chi})$ (the short decreasing thick curve in dark grey on top right), that is, such that $\tilde{\chi}_0 = \tilde{\chi}(\tilde{\mu}_0) < \bar{\chi}^{unst}(\tilde{\mu}_0)$, with $\tilde{\mu}_0$ defined in (60). It is a portion of a path lying below the steady state $(\hat{\mu}, \hat{\chi})$ in a typical Ramsey-model phase diagram, coming from south-west just below the bottom-left branch of the stable arm of $(\hat{\mu}, \hat{\chi})$ [the increasing thick curve in dark grey, indistinguishable from the black curve representing $\chi^{be}(\mu)$ in the figure], initially heading toward north-east but then, at some point in a neighborhood of $(\hat{\mu}, \hat{\chi})$, turning toward south-east and hitting the $\tilde{\chi}(\mu)$ policy at instant t_0 on $(\tilde{\mu}_0, \tilde{\chi}_0)$. The whole path $\tilde{\chi}^{be}(\mu)$, in thick black, is the union of $\chi^{be}(\mu)$ with the path defined by policy $\tilde{\chi}(\mu)$ along the turnpike eventually heading south-west toward the steady state (μ^*, χ^*). In Figure 4(b) the whole path representation starts on a point $\left(\mu^{sk}, \chi^{sk}\right)$ below the bottom-left branch of the stable arm of $(\hat{\mu}, \hat{\chi})$, keeps very close to it as it moves toward north-east, performs a steep turn toward south-east while passing by $(\hat{\mu}, \hat{\chi})$ and, as it proceeds below the unstable arm, intercepts the policy $\tilde{\chi}(\mu)$ just to eventually end up on the steady state (μ^*, χ^*), which in the (μ, χ) space happens to be close to the starting point $\left(\mu^{sk}, \chi^{sk}\right)$.

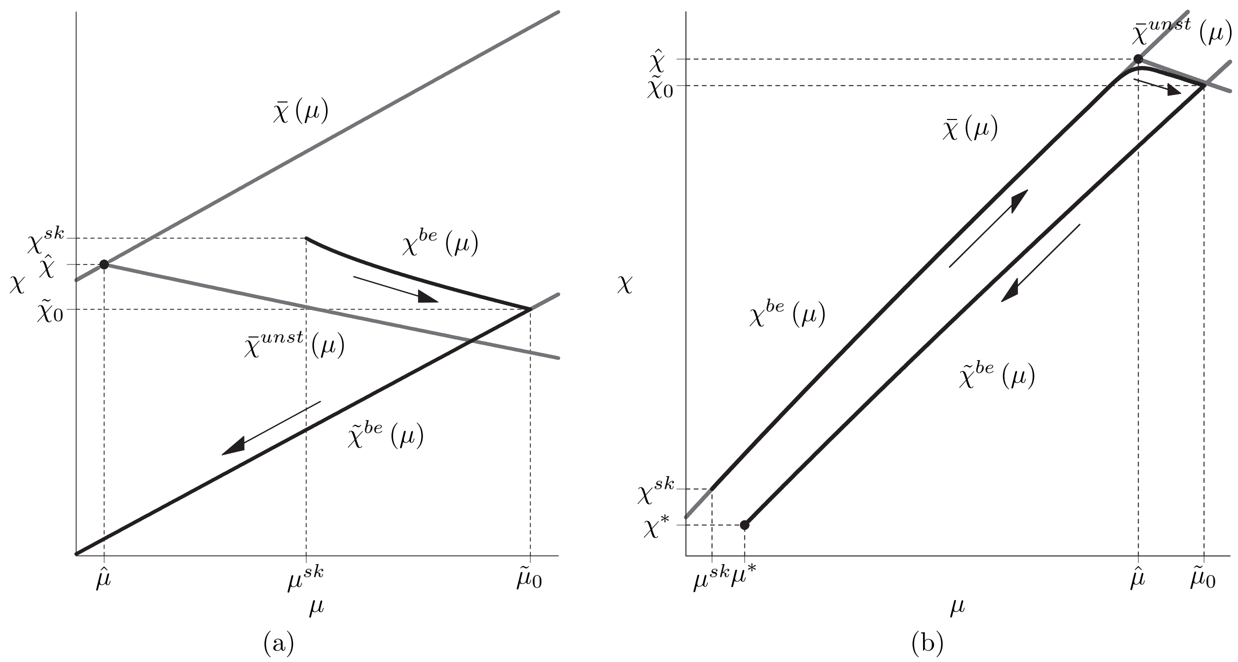

FIGURE 4: piecewise policies $\tilde{\chi}^{be}(\mu)$ (thick black curves) defining optimal paths from $\left(\mu^{sk}, \chi^{sk}\right)$ to (μ^*, χ^*); (a) when $\tilde{\chi}_0 = \tilde{\chi}(\tilde{\mu}_0) > \bar{\chi}^{unst}(\tilde{\mu}_0)$ and (b) when $\tilde{\chi}_0 = \tilde{\chi}(\tilde{\mu}_0) < \bar{\chi}^{unst}(\tilde{\mu}_0)$.

In view of Theorem 2 and Corollary 1, $\bar{V}^{be}(A_0, k_0)$ in (64) is again approximated by means of the Hamilton-Jacobi-Bellman equation:

$$\bar{V}^{be}(A_0,k_0)=\frac{1}{\rho}\left\{\frac{[\bar{c}(A_0,k_0)]^{1-\sigma}-1}{1-\sigma}+\frac{\theta k_0^{\alpha}A_0^{1-\alpha}-\bar{c}(A_0,k_0)}{[\bar{c}(A_0,k_0)]^{\sigma}}\right\}, \tag{65}$$

where $\bar{c}(A_0,k_0)$ is the estimated value of the optimal consumption policy at the initial point (A_0,k_0) numerically obtained by (41).

Algorithm 3 below summarizes all the steps discussed so far. Again it relies on a Bisection Method, due to the complexity of each pointwise estimation for the welfare $\tilde{V}^{be}(A_0,k_0)$ according to (63). For a given initial stock of knowledge $A_m < A_0 < \hat{A}$ and initial stock of capital $0 < k_0 < \tilde{k}(A_0)$ let

$$f^{be}(k_0)=\tilde{V}^{be}(A_0,k_0)-\bar{V}^{be}(A_0,k_0), \tag{66}$$

with $\tilde{V}^{be}(A_0,k_0)$ and $\bar{V}^{be}(A_0,k_0)$ defined in (63) and (65) respectively, be the function whose unique zero is the target of our search routine. For each given A_0, the unique value k_0^* such that $f^{be}(k_0^*)=0$ yields our estimation of the Skiba-point as $k^{sk}(A_0)=k_0^*$. The initial interval bracketing the unique zero of f^{be} depends on whether the choice of A_0 implies that the intersection point with the turnpike lies above or below the unstable arm $\bar{\chi}^{unst}(\mu)$, that is, on whether $\tilde{\chi}(\tilde{\mu}_0) > \bar{\chi}^{unst}(\tilde{\mu}_0)$ or $\tilde{\chi}(\tilde{\mu}_0) < \bar{\chi}^{unst}(\tilde{\mu}_0)$, with $\tilde{\mu}_0$ defined in (60). While in both scenarios $\tilde{\mu}_0$ definitely is a good choice as the right endpoint of such interval, the former case requires a careful and delicate preliminary search for the left endpoint, starting from $\tilde{\mu}_0$ and then (slowly) decreasing[21] it until a value μ_L is found such that $f^{be}(k_0)=f^{be}(\mu_L A_0)\leq 0$. Conversely, in the latter case a good choice for the left endpoint of the interval bracketing the zero of f^{be} is the lower bound, $\bar{\mu}_L$, used in the approximation of the $\bar{\chi}(\mu)$ policy for problem (64).

Algorithm 3 (Finds the Skiba-point when $A_m < A_0 < \hat{A}$)

Step 1: *Set the range $[\mu^*,\mu_m]$, with μ^* defined in (34) and $\mu_m=\tilde{k}(A_m)/A_m$, where A_m is the output of Algorithm 1, for the $\tilde{\chi}(\mu)$ policy approximation in the (μ,χ) space. The range for the $\bar{\chi}(\mu)$ policy approximation is set as $[\bar{\mu}_L,\mu_m]$, with $\bar{\mu}_L > 0$ not too small so that the Collocation-Projection of the following step 3 performs satisfactorily.*

Step 2: *Apply the OLS-Projection method discussed in Subsection 4.1 to estimate the optimal policy along the turnpike, $\tilde{\chi}(\mu)$, on the range $[\mu^*,\mu_m]$.*

Step 3: *Apply the Collocation-Projection method discussed in Subsection 4.2 to estimate the optimal policy toward stagnation, $\bar{\chi}(\mu)$, on the range $[\bar{\mu}_L,\mu_m]$.*

Step 4: *Apply a similar Collocation-Projection method to estimate the unstable arm, $\bar{\chi}^{unst}(\mu)$, on the range $[\hat{\mu},\mu_m]$, with $\hat{\mu}$ defined in (43).*

Step 5: *Find $k_0=\mu^{sk}A_0$ satisfying $f^{be}(k_0)=0$ for f^{be} defined in (66).*

Step 5.1 (Initialization): *Compute $\tilde{\mu}_0=\tilde{k}(A_0)/A_0$ and set $\mu_R=\tilde{\mu}_0$, choose $t_{\max}>0$ for the ranges of the Runge-Kutta routines in the following step 5.3.4, choose $N>0$ as the number of nodes for the Gauss-Legendre quadrature routine in the following step 5.3.6, choose a decreasing step $\epsilon>0$, choose stopping rule parameters $0<\varepsilon,\eta<1$, and set (fake) initial values $f^{be}(k_0)=f^{be}(k_R)=1>\eta$.*

[21]Especially for initial values A_0 close to the lower bound A_m the (constant) decrease must be small so to keep our approximation kit away from the vertical turn of the trajectory, on which it switches from a south-west to a south-east direction.

Step 5.2 (Select bracket type): *If* $\tilde{\chi}(\tilde{\mu}_0) \geq \bar{\chi}^{unst}(\tilde{\mu}_0)$ *then set a (switch) variable* $B = 1$ *and set* $\mu_L = \mu_R - \epsilon$*, else set* $B = 0$ *and set* $\mu_L = \bar{\mu}_L$*.*

Step 5.3 (Bisection loop): *While* $\mu_R - \mu_L > \varepsilon\mu_R$ *and* $\left|f^{be}(k_0)\right| > \eta$ *do:*

1. *if* $B = 1$ *then set* $\mu^{sk} = \mu_L$ *(search for bracket left endpoint), else set* $\mu^{sk} = (\mu_R - \mu_L)/2$ *(compute midpoint),*
2. *compute* $k_0 = \mu^{sk} A_0$*,*
3. *approximate policy* $\chi^{be}(\mu)$ *over* $\left[\mu^{sk}, \tilde{\mu}_0\right]$ *by solving the Cauchy problem defined by (40) plus the terminal condition (61) through the OLS-Projection method described above,*
4. *use* $\chi^{be}(\mu)$ *from step 5.3.3 to build the time-path trajectories* $\mu^{be}(t)$*,* $\chi^{be}(t)$ *and* $c^{be}(t)$ *over* $[0, t_{\max}]$ *through the Runge-Kutta routine as explained above,*
5. *find* t_0 *by solving* $\mu^{be}(t) = \tilde{\mu}_0$ *through Maple 16 'fsolve' routine over* $[0, t_{\max}]$*,*
6. *apply the Gauss-Legendre quadrature routine to approximate the integral in (63), use* $\tilde{\chi}(\mu)$ *from step 2 to evaluate* $\tilde{c}(A)$ *through (37), compute* $w(A_0)$ *according to (47) and evaluate* $\tilde{V}^{be}(A_0, k_0)$ *as in (63),*
7. *use* $\bar{\chi}(\mu)$ *from step 3 and* k_0 *from step 5.3.2 to evaluate* $\bar{c}(A_0, k_0)$ *through (41) and evaluate* $\bar{V}^{be}(A_0, A_r)$ *by means of (65),*
8. *update* $f^{be}(k_0)$ *by setting* $f^{be}(k_0) = \tilde{V}^{be}(A_0, k_0) - \bar{V}^{be}(A_0, k_0)$*,*
9. *if* $B = 1$ *and* $f^{be}(k_0) > 0$ *then (keep searching for bracket left endpoint) set* $\mu_L = \mu_L - \epsilon$ *and*[22] $\mu_R = \mu_R - \epsilon$ *and go to step 5.3, else (bisection loop)*
   - if $B = 1$ *set* $B = 0$ *(stop searching for bracket),*
   - refine the bounds: if $f^{be}(k_0) f^{be}(k_R) < 0$ *then set* $\mu_L = \mu^{sk}$*, else set* $\mu_R = \mu^{sk}$ *and update* $f^{be}(k_R)$ *by setting* $f^{be}(k_R) = f^{be}(k_0)$*.*

Step 6: *Report the Skiba-point from step 5.3.2,* $k^{sk}(A_0) = k_0$*.*

Remark 5

1. *The choice of* $t_{\max}$ *in step 5.1 depends on the range* $\left[\mu^{sk}, \tilde{\mu}_0\right]$*, thus the caveats in Remark 4.1 apply here as well. As a matter of fact, especially when* $\tilde{\chi}(\tilde{\mu}_0) < \bar{\chi}^{unst}(\tilde{\mu}_0)$*, in which case the bisection method operates on a large range (see step 5.2) and thus must approximate a wide variety of trajectories, the Runge-Kutta algorithm in step 5.3.4 inevitably stops too early in some iterations. However, as the* t_0 *value computed in step 5.3.5 is always smaller than* $t_{\max}$*, this does not prevent the whole algorithm to work properly.*

2. *All estimations in the* $\tilde{\chi}(\tilde{\mu}_0) < \bar{\chi}^{unst}(\tilde{\mu}_0)$ *scenario, that is for* A_0 *values closer to the upper bound* $\hat{A}$*, appear to be quite unreliable, often yielding results up to* 50% *apart from different runs of Algorithm 3 on identical data. This is confirmed by the poor performance of the OLS-Projection method run in step 5.3.3 and can be explained by two features characterizing this case: 1) the* $\chi^{be}(\mu)$ *approximation in step 5.3.3 builds trajectories that almost coincide with* $\bar{\chi}(\mu)$ *up to near the steady state* $(\hat{\mu}, \hat{\chi})$*, only to suddenly turn toward south-east just before reaching it, as can be clearly seen in Figure 4(b); 2) the* $\bar{V}^{be}(A_0, k_0)$ *approximation in step 5.3.7 relies on the* $\bar{\chi}(\mu)$ *policy approximation from step 3, which rapidly degenerates for smaller* $\bar{\mu}_L$ *values, that is, when* A_0 *approaches* $\hat{A}$*.*

The Maple 16 code for Algorithm 3 is available from the author upon request.

[22] Shifting the right endpoint μ_R to the left as well delivers a smaller initial bracket for the bisection loop.

7 A Numerical Example

We consider the same values for parameters α, β, ρ, σ and θ used in [19] and [20] satisfying both the necessary growth condition (24) and condition (48) of Proposition 4:

$$\alpha = 0.5, \qquad \beta = 0.0124, \qquad \rho = 0.04, \qquad \text{and} \qquad \sigma = \theta = 1, \tag{67}$$

where $\sigma = 1$ implies logarithmic instantaneous utility. Figure 1 shows the three characteristic curves, $\tilde{k}(A)$, $\hat{k}(A)$ and $\tilde{k}_\infty(A)$ in the (A, k) space, for the parameters' values in (67). The turnpike $\tilde{k}(A)$ converges from above to the linear asymptotic turnpike $\tilde{k}_\infty(A)$, corresponding to long-run balanced growth with (constant) growth rate given by (25): $\gamma = 0.0157$. According to (34), (42) and (43), the three relevant steady states of the phase diagram in the (μ, χ) space are thus computed as:

$$(\mu^*, \chi^*) = (80.6452, 6.4516), \quad (\mu^s, \chi^s) = (204.4503, 14.2986), \quad (\hat{\mu}, \hat{\chi}) = (156.25, 12.5). \tag{68}$$

It is easily seen that μ^* and μ^s in (68) satisfy condition (35), so that the three steady states' layout appears as in Figure 2. In our analysis the lower bound for the initial stock of knowledge given by $A_\ell = 1.86$ is chosen, corresponding to the upper bound $\mu_\ell = 304.6595$ for the range of variable μ in the phase diagram of Figure 2, while the upper bound, according to (26), turns out to be $\hat{A} = 2.5667$ [which identifies the intersection between the turnpike $\tilde{k}(A)$ and the stagnation line $\hat{k}(A)$ in Figure 1], corresponding to $\hat{\mu} = 156.25$ as in (68) in the (μ, χ) space. The initial capital values associated to these endpoints on the turnpike are $k_\ell = \tilde{k}(A_\ell) = 566.6667$ and $\hat{k} = \tilde{k}\left(\hat{A}\right) = \hat{k}\left(\hat{A}\right) = 401.0417$.

Algorithms 1–3 are written in Maple 16 and run on an Intel Core i7-2670QM CPU machine. For Algorithms 2 and 3 we set stopping rules $\varepsilon = \eta = 10^{-7}$ and take $N = 1000$ nodes over $[0, t_{\max}]$ for the *Gauss-Legendre quadrature* routine to estimate the integrals in (55) and (63).

Following [20], in Algorithm 1 we choose a degree of approximation $n = 7$ for the *OLS-Projection method* applied to ODE (36) to approximate the policy along the turnpike $\tilde{\chi}(\mu)$ over $[\mu^*, \mu_\ell]$ as described in Subsection 4.1; the integral of the residual function is approximated through *Gauss-Chebyshev quadrature* over 57 nodes on $[\mu^*, \mu_\ell]$, while the two equality constraints $\tilde{\chi}(\mu^s) = \chi^s$ and $\tilde{\chi}(\mu^*) = \chi^*$ are being used in the *Maple 16 nonlinear programming (NLP) solver*. The resulting approximation of $\tilde{\chi}(\mu)$, the black curve in Figure 2, exhibits a maximum error of 3×10^{-4} and, as expected, the residual function does not oscillate around zero as it should,[23] while the 8 coefficients associated to each Chebyshev polynomial of the approximation function follow a uniformly decreasing pattern, from $a_0 = 13.4535$ to $a_7 = -4.7 \times 10^{-5}$. For the *Collocation-Projection* method applied to ODE (40) to approximate the policy toward stagnation $\bar{\chi}(\mu)$ over $[\hat{\mu}, \mu_\ell]$ as described in Subsection 4.2 we take $n = 13$ as degree of approximation. The resulting approximation of $\bar{\chi}(\mu)$, the dark grey curve in Figure 2, exhibits a much better performance with a maximum error of 10^{-9} and a residual function symmetrically oscillating around zero, while the 14 coefficients associated to each Chebyshev polynomial of the approximation function uniformly decrease from $a_0 = 17.1097$ to $a_{10} = -1.6 \times 10^{-9}$, starting to oscillate between -2.6×10^{-9} and 1.8×10^{-9} for a_{11}–a_{13}. For the parameterization in (67) Algorithm 1 yields the following Skiba-point value on the turnpike:

$$k_m^{sk} = k^{sk}(A_m) = \tilde{k}(A_m) = 416.6199, \tag{69}$$

corresponding to the stock of knowledge value

$$A_m = 2.3067. \tag{70}$$

[23] See [20] for a discussion on this type of behavior.

We run Algorithm 2 for seven values of initial stock of knowledge $A_\ell \leq A_0 < A_m$, starting with $A_0 = A_\ell$, so to estimate seven associated Skiba-points above the turnpike. If on one hand the policy along the turnpike $\tilde{\chi}(\mu)$ is still approximated over $[\mu^*, \mu_\ell]$, now the policy toward stagnation $\bar{\chi}(\mu)$ is approximated through a Collocation-Projection method applied to ODE (40) over the larger range $[\hat{\mu}, \mu_0] = [\hat{\mu}, \mu_\ell + 50] = [156.25, 354.6595]$, so to include optimal consumption paths toward stagnation starting above the turnpike. Using a degree of approximation $n = 12$ we manage to obtain an approximation with a maximum error of 4.3×10^{-10}, a residual function almost symmetrically oscillating around zero and all 13 coefficients associated to each Chebyshev polynomial of the approximation function uniformly decreasing from $a_0 = 18.6$ to $a_{12} = -2.5 \times 10^{-11}$. The increment for the search of the right bound of the bracket in step 4.1 is set at $\epsilon = 0.2$. The results are reported in Table 1, including the seven A_0 values, their corresponding Skiba-points $k^{sk}(A_0)$, the A_r values at which the early transition paths intersect the turnpike and the instants t_0 at which such intersections occur. The Collocation-Projection methods used to approximate the policies $k^{ab}(A)$ and $c^{ab}(A)$ over $[A_{0,}A_r]$ by solving (52) as described in Section 5 use degrees of approximation decreasingly ranging from $n = 20$ when $A_0 = A_\ell$ down to $n = 7$ when $A_0 = 2.2429$. Overall, these approximation routines perform quite well, with maximum errors in a range between 2.1×10^{-7} and 5.5×10^{-6}, all referred to residual functions for the $k^{ab}(A)$ policies [maximum errors for the $c^{ab}(A)$ policies are on average 10^{-2} smaller], residual functions almost symmetrically oscillating around zero and all $n + 1$ coefficients associated to each Chebyshev polynomial of the approximation function uniformly decreasing in each case. The choice of $t_{\max}$ in step 4.1 of Algorithm 2 ranges from $t_{\max} = 6$ when $A_0 = A_\ell$ to $t_{\max} = 4$ when $A_0 = 2.2429$, while the time elapsed for each Skiba-point computation runs from 45 seconds when $A_0 = 2.2429$ to 161 seconds when $A_0 = A_\ell$, with numbers of iterations ranging from 19 to 26.

A_0	$k^{sk}(A_0)$	A_r	t_0
1.8600	571.0748	2.8736	5.7158
1.9238	539.9772	2.7629	4.7628
1.9876	513.6633	2.6708	3.8924
2.0515	490.7823	2.5919	3.0862
2.1153	470.3927	2.5220	2.3254
2.1791	451.7583	2.4567	1.5901
2.2429	434.1761	2.3901	0.8467

TABLE 1: Skiba-points above the turnpike; A_0 = initial stock of knowledge, $k^{sk}(A_0)$ = associated Skiba-point, A_r = knowledge level at the intersection with the turnpike, t_0 = intersection instant.

Figure 5 shows in thick black the whole piecewise policies $\tilde{k}^{ab}(A)$ and $\tilde{c}^{ab}(A)$ starting above the turnpike and then hitting it at A_r and continuing along it for $A > A_r$ for $A_0 = A_\ell = 1.86$ when the economy initiates on the Skiba-point $k^{sk}(A_0) = 571.0748$ (as in the first row of Table 1); the thin black curves to the left of A_r are the first part of the optimal policies along the turnpike, $\tilde{k}(A)$ and $\tilde{c}(A)$, for $A_0 \leq A \leq A_r$. Figure 6 reports the whole optimal time-path trajectories $\tilde{A}^{ab}(t)$, $\tilde{k}^{ab}(t)$, $\tilde{c}^{ab}(t)$ and $\tilde{J}^{ab}(t)$ for all $t \geq 0$ defined according to (54) through $\tilde{k}^{ab}(A)$ and $\tilde{c}^{ab}(A)$ for $A_0 = 1.86$ and $k_0 = 571.0748$. Note the kink on A_r of the policies in Figure 5; the same kink occurs at t_0 for the time-path trajectories $\tilde{A}^{ab}(t)$, $\tilde{k}^{ab}(t)$ and $\tilde{c}^{ab}(t)$, while trajectory $\tilde{J}^{ab}(t)$ exhibits a discontinuity jump at t_0, as reported in Figure 6.

Algorithm 3 for searching Skiba-points below the turnpike for initial stock of knowledge values $A_m < A_0 < \hat{A}$ is the most problematic, as it is quite unstable when the intersection of the early transition $\chi^{be}(\mu)$ with the optimal policy along the turnpike $\tilde{\chi}(\mu)$ lies below the unstable

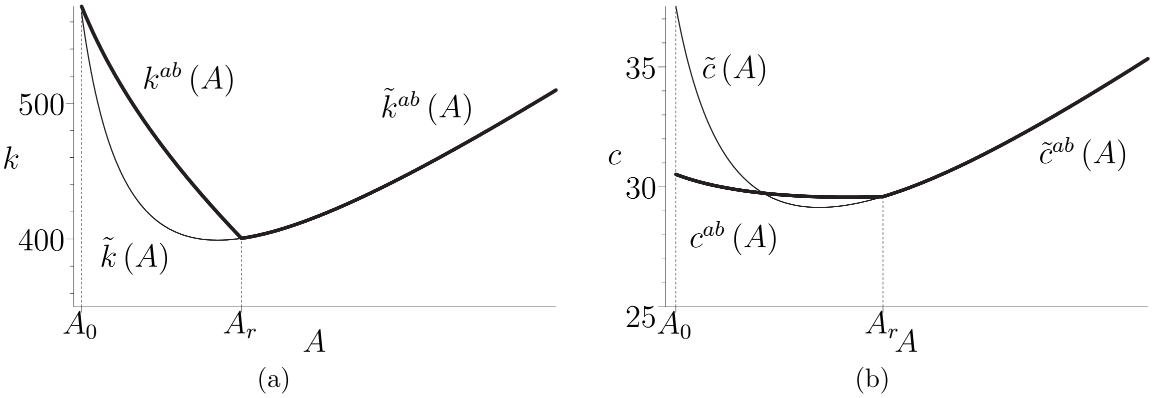

FIGURE 5: piecewise policies, (a) $\tilde{k}^{ab}(A)$ and (b) $\tilde{c}^{ab}(A)$ starting above the turnpike from $A_0 = 1.86$ and $k_0 = 571.0748$.

arm, $\bar{\chi}^{unst}(\mu)$, associated to the steady state $(\hat{\mu}, \hat{\chi})$ of the policy toward stagnation $\bar{\chi}(\mu)$, as it has been explained in Remark 5.2. In this case it is enough to approximate the policy along the turnpike $\tilde{\chi}(\mu)$ over the shorter range $[\mu^*, \mu_m] = [80.6452, 180.6105]$, with $\mu_m = \tilde{k}(A_m)/A_m$, where A_m is given by (70). This allows for a slightly better approximation through a OLS-Projection method applied to ODE (36) with degree of approximation $n = 16$, in which the integral of the residual function is approximated through Gauss-Chebyshev quadrature over 66 nodes on $[\mu^*, \mu_m]$, again using the two equality constraints $\tilde{\chi}(\mu^s) = \chi^s$ and $\tilde{\chi}(\mu^*) = \chi^*$ in the Maple 16 nonlinear programming (NLP) solver. Here the approximation of $\tilde{\chi}(\mu)$ exhibits a maximum error of 3.7×10^{-7} with a residual function symmetrically oscillating around zero, and with uniformly decreasing coefficients associated to each Chebyshev polynomial of the approximation function, from $a_0 = 9.7091$ to $a_{11} = 10^{-6}$, starting to oscillate between -1.8×10^{-7} and 3.2×10^{-7} for a_{12}–a_{16}. The policy toward stagnation $\bar{\chi}(\mu)$ is approximated through a Collocation-Projection method applied to ODE (40) over the range $[\bar{\mu}_L, \mu_m] = [50, 180.6105]$, where we have chosen the lower bound $\bar{\mu}_L = 50$. With a degree of approximation $n = 15$ we are able to obtain an approximation with a maximum error of 7.9×10^{-10}, a residual function almost symmetrically oscillating around zero and all 16 coefficients associated to each Chebyshev polynomial of the approximation function uniformly decreasing from $a_0 = 9.6832$ to $a_{15} = 2.6 \times 10^{-10}$. The unstable arm $\bar{\chi}^{unst}(\mu)$ is approximated over $[\hat{\mu}, \mu_m] = [156.25, 180.6105]$ through a Collocation-Projection method applied to ODE (40) with a degree of approximation $n = 6$, exhibiting a maximum error of 5.4×10^{-10}, a residual function almost symmetrically oscillating around zero and all 7 coefficients associated to each Chebyshev polynomial of the approximation function uniformly decreasing from $a_0 = 12.2114$ to $a_6 = 1.5 \times 10^{-9}$.

The decrease steps for the search of the left bound of the bracket in step 5.3 of Algorithm 3 are set between $\epsilon = 2$ and $\epsilon = 6$, with increasing values as A_0 increases from A_m. The results are reported in Table 2 where seven A_0 values such that $A_m < A_0 < \hat{A}$ are considered plus their corresponding Skiba-points $k^{sk}(A_0)$ and the instants t_0 at which the early transition paths intersect the turnpike from below. The OLS-Projection method used to approximate the policy $\chi^{be}(\mu)$ over $\left[\mu^{sk}, \tilde{\mu}_0\right]$, with μ^{sk} set in step 5.3.1 of Algorithm 3 and $\tilde{\mu}_0 = \tilde{k}(A_0)/A_0$, by solving the Cauchy problem (40) together with terminal condition (61) as discussed in Section 6, use degrees of approximation ranging from $n = 8$ for $A_0 = 2.337$ (the first row in Table 2) up to $n = 35$ when $A_0 = 2.410$ (the fifth row in Table 2), the latter corresponding to an early transition path passing below but very close to the steady state $(\hat{\mu}, \hat{\chi})$. The approximation routines perform well for the cases in which $\tilde{\chi}(\tilde{\mu}_0) \geq \bar{\chi}^{unst}(\tilde{\mu}_0)$ (see step 5.2 of Algorithm 3),

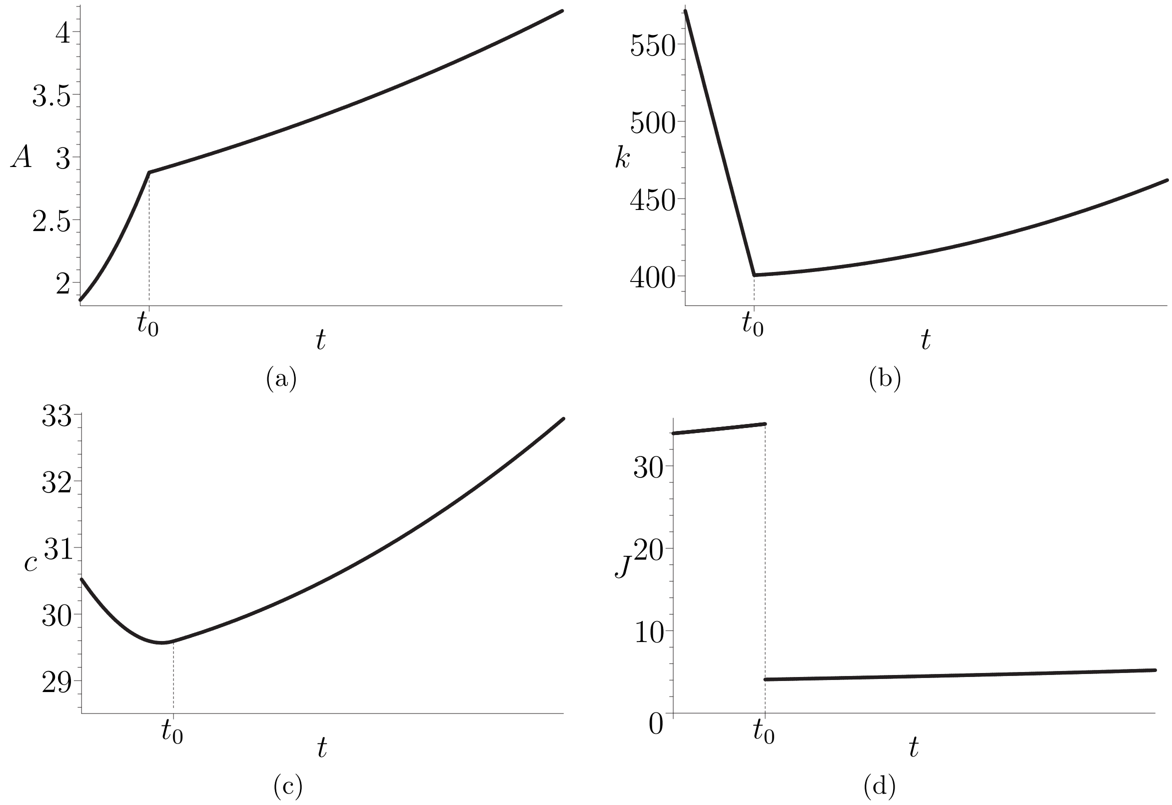

FIGURE 6: whole optimal transition time-path trajectories, (a) $\tilde{A}^{ab}(t)$, (b) $\tilde{k}^{ab}(t)$, (c) $\tilde{c}^{ab}(t)$ and (d) $\tilde{J}^{ab}(t)$, starting above the turnpike from $A_0 = 1.86$ and $k_0 = 571.0748$ in $t = 0$.

that is, for the first four rows in Table 2, with maximum errors in a range between 6.9×10^{-8} and 3.3×10^{-5}, residual functions almost symmetrically oscillating around zero and coefficients associated to each Chebyshev polynomial of the approximation functions almost uniformly decreasing in each case. As anticipated in Remark 5.2, a completely different picture is provided by the same approximations when $\tilde{\chi}(\tilde{\mu}_0) < \bar{\chi}^{unst}(\tilde{\mu}_0)$, yielding highly inaccurate results all with maximum errors of order 10^{-3}, not always oscillating behavior of the residual functions and definitely a non-decreasing pattern for the coefficients associated to each Chebyshev polynomial of the approximation functions, which indeed start oscillating already at low degrees. The choice of $t_{\max}$ in step 5.1 of Algorithm 3 ranges from $t_{\max} = 10$ when $A_0 = 2.337$ to $t_{\max} = 250$ when $A_0 = 2.410$, while the time elapsed for each Skiba-point computation runs from 82 seconds when $A_0 = 2.337$ to 953 seconds when $A_0 = 2.410$, with numbers of iterations ranging from 13 to 24. Figure 4(a) shows the detail of the initial part of $\tilde{\chi}^{be}(\mu)$ when $\tilde{\chi}_0 = \tilde{\chi}(\tilde{\mu}_0) > \bar{\chi}^{unst}(\tilde{\mu}_0)$ for $A_0 = 2.388$, *i.e.*, for the case considered in the third row of Table 2. Figure 4(b) exhibits the whole policy $\tilde{\chi}^{be}(\mu)$ when $\tilde{\chi}_0 = \tilde{\chi}(\tilde{\mu}_0) < \bar{\chi}^{unst}(\tilde{\mu}_0)$ for $A_0 = 2.414$, *i.e.*, for the case considered in the sixth row of Table 2.

Figure 7 shows the whole optimal time-path trajectories $\tilde{A}^{be}(t)$, $\tilde{k}^{be}(t)$, $\tilde{c}^{be}(t)$ and $\tilde{J}^{be}(t)$ for all $t \geq 0$ defined according to (62) through $\tilde{\chi}^{be}(\mu)$ for $A_0 = 2.414$ and $k_0 = k^{sk}(A_0) = 179.5518$ (as in the sixth row of Table 2). As expected, the time-path trajectories $\tilde{A}^{be}(t)$, $\tilde{k}^{be}(t)$ and $\tilde{c}^{be}(t)$ exhibit a kink at t_0, while trajectory $\tilde{J}^{be}(t)$ exhibits a discontinuity jump at t_0, suddenly shifting from a zero R&D-investment policy to $\tilde{J}(t) > 0$ as defined in (23).

Two key features are apparent from columns 2 and 3 of Table 2: 1) after the fourth row the Skiba-points plunge quite rapidly toward zero, a feature confirmed also by Figure 8 below, and

A_0	$k^{sk}(A_0)$	t_0
2.337	408.6140	3.9516
2.363	400.1418	9.5046
2.388	390.3523	18.7883
2.409	376.6959	72.2989
2.410	211.8432	232.0534
2.414	179.5518	206.8047
2.418	124.2926	199.7331

TABLE 2: Skiba-points below the turnpike; A_0 = initial stock of knowledge, $k^{sk}(A_0)$ = associated Skiba-point, t_0 = instant of intersection with the turnpike.

2) the time required to reach the turnpike increases for larger A_0 values in the first column but only up to the fifth row, starting to decrease for larger values of A_0. While the increasing pattern of t_0 up to $A_0 = 2.410$ is justified by the fact that a longer path to hit the turnpike requires more time, the decreasing pattern thereafter is explained by observing that the trajectories evolving below the steady state $(\hat{\mu}, \hat{\chi})$ go for a long while neck to neck with the trajectory defined by $\bar{\chi}(\mu)$ and leading toward the steady state $(\hat{\mu}, \hat{\chi})$, only to suddenly turn south-east just before hitting the steady state $(\hat{\mu}, \hat{\chi})$ to enter the turnpike policy $\tilde{\chi}(\mu)$ later on [see Figure 4(b)]. Because when close to $(\hat{\mu}, \hat{\chi})$ such trajectories are very similar to that defined by $\bar{\chi}(\mu)$, their speed must quite slow down in the proximity of $(\hat{\mu}, \hat{\chi})$, as if they were to eventually hit the steady state only to abruptly change their behavior just before touching $(\hat{\mu}, \hat{\chi})$. This pattern is confirmed by Figures 7(b) and 7(c), where it is evident that both capital and consumption follow a non-monotonic path through time, starting by growing fast at early times only to slow down after a while – *i.e.*, when they get closer to $(\hat{\mu}, \hat{\chi})$ in the (μ, χ)-phase diagram of Figure 4(b) – before hitting the turnpike at[24] t_0. For larger values of A_0 these trajectories initiate more far away from $(\hat{\mu}, \hat{\chi})$ and thus start deviating away from the stable arm defined by $\bar{\chi}(\mu)$ progressively before, which, in turn, implies that they pass more apart from $(\hat{\mu}, \hat{\chi})$; this allow for a speedier travel when they turn south-east and translates in a slightly smaller time span – *i.e.*, smaller t_0 – of the whole journey toward the turnpike policy $\tilde{\chi}(\mu)$.

Besides the initial portions of the turnpike $\tilde{k}(A)$ (in black) and the stagnation line $\hat{k}(A)$ (in dark grey) Figure 8 reports all simulated Skiba-points obtained so far as black dots. Note that the Skiba-point $\hat{k}^{sk} = k^{sk}(2.409) = 376.6959$, defined in the fourth row of Table 2 and corresponding to the initial stock of knowledge level $\hat{A}_0 = 2.409$, lies on the stagnation line $\hat{k}(A)$, that is, it is the unique Skiba-point below the turnpike $\tilde{k}(A)$ with the property that $\hat{k}^{sk} = \hat{k}\left(\hat{A}_0\right)$. This is not a coincidence, because, according to (60), $\hat{A}_0 = 2.409$ defines the unique upper bound $\tilde{\mu}_0 = \tilde{k}\left(\hat{A}_0\right)/\hat{A}_0$ corresponding to the intersection point between the unstable arm $\bar{\chi}^{unst}(\mu)$ and the turnpike policy $\tilde{\chi}(\mu)$, that is, $\tilde{\mu}_0 = 169.3708$ is the unique μ value such that $\tilde{\chi}(\tilde{\mu}_0) = \bar{\chi}^{unst}(\tilde{\mu}_0)$. The corresponding values of $\left(\hat{A}_0, \hat{k}^{sk}\right)$ in the ratio variables are $\hat{\mu}^{sk} = \hat{k}^{sk}/\hat{A}_0 = 156.3749$ and $\hat{\chi}^{sk} = 12.5081$, very close to the steady state $(\hat{\mu}, \hat{\chi}) = (156.25, 12.5)$. Indeed, this Skiba-point happens to be the unique initial point yielding

[24]Note that, consistently with Figure 4(b), capital $\tilde{k}^{be}(t)$ in Figure 7(b) maintains an increasing pattern until the turnpike is entered at t_0, although growth becomes slow at the turn of $\chi^{be}(\mu)$ in proximity of $(\hat{\mu}, \hat{\chi})$, accelerating before the intersection with $\tilde{\chi}(\mu)$ only to decrease for a while after the turnpike has been entered and then starting to grow again. According to the decreasing portion of the path $\chi^{be}(\mu)$ to the right of the steady state $(\hat{\mu}, \hat{\chi})$ in Figure 4(b), consumption $\tilde{c}^{be}(t)$ in Figure 7(c) begins to decrease before hitting the turnpike, while it starts to grow again only after evolving along the turnpike for some time.

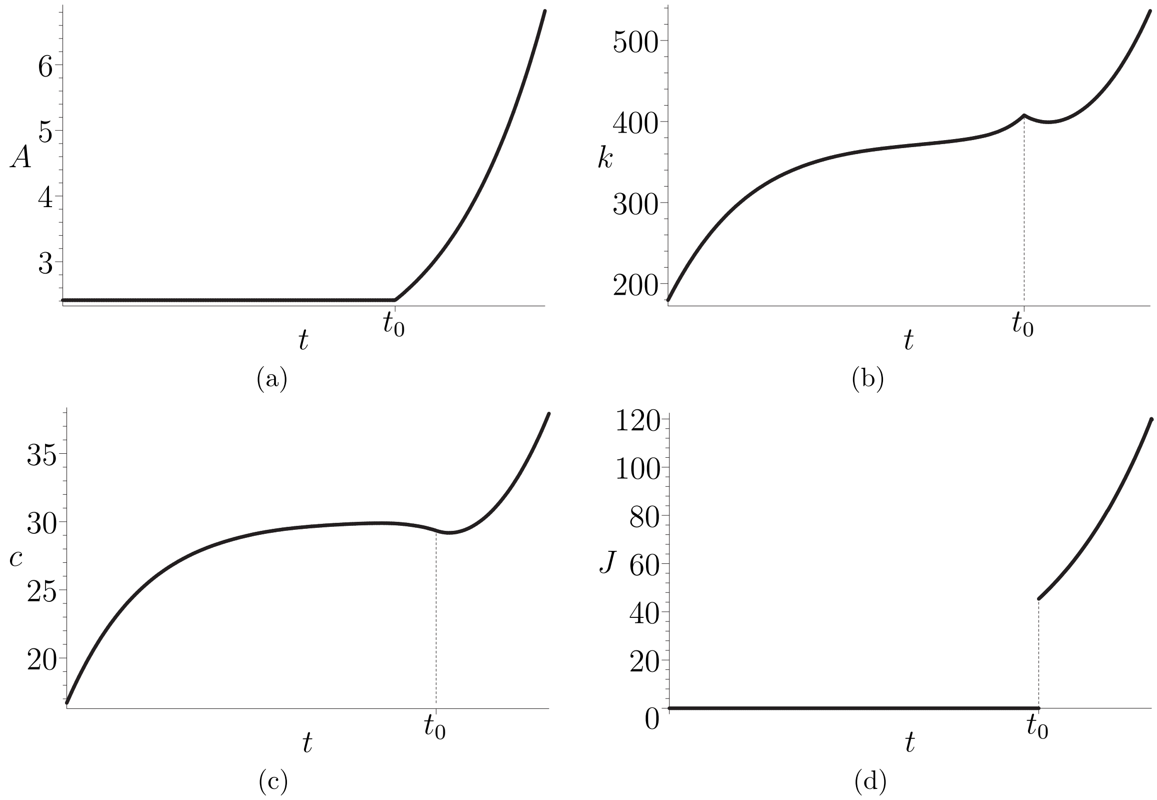

FIGURE 7: whole optimal transition time-path trajectories, (a) $\tilde{A}^{be}(t)$, (b) $\tilde{k}^{be}(t)$, (c) $\tilde{c}^{be}(t)$ and (d) $\tilde{J}^{be}(t)$, starting below the turnpike from $A_0 = 2.414$ and $k_0 = 179.5518$ in $t = 0$.

the same welfare either if the economy remains steady at its values $\hat{A}_0$ and $\hat{k}^{sk}$ forever or if it takes off from it first to move upward toward the turnpike – which lies above $\left(\hat{A}_0, \hat{k}^{sk}\right)$ in this case – and then toward steady growth along the turnpike. In the (μ, χ) space this scenario corresponds to either staying forever at the steady state[25] $(\hat{\mu}, \hat{\chi})$ or moving first along the unstable arm $\bar{\chi}^{unst}(\mu)$ and then entering the turnpike policy $\tilde{\chi}(\mu)$ at the intersection point $\tilde{\mu}_0$.

Finally, recall that we are not able to estimate Skiba-points further to the right of $A_0 = 2.418$ (last row in Table 2) because of the strictly positive lower bound $\bar{\mu}_L = 50$, which rules out the possibility of defining the optimal policy toward stagnation for initial stock of capital values lower than $k^{sk}(2.418) = 124.2926$. All in all, if on one hand the Bisection Algorithms built in the previous sections perform sufficiently well for initial stock of knowledge values in the range $\left[A_\ell, \hat{A}_0\right]$, with $A_\ell = 1.86$ and $\hat{A}_0 = 2.409$, for knowledge values values in the range $\left(\hat{A}_0, \hat{A}\right) = (2.409, 2.5667)$ Algorithm 3 either performs poorly or cannot be run at all.

8 Conclusions

In this paper three algorithms capable of approximating the Skiba-points of a hyperbolic-Cobb-Douglas-CIES specification of the continuous-time version proposed by Tsur and Zemel [24] of the endogenous recombinant growth model originally introduced by Weitzman [25] has been

[25]Recall from footnote 13 that all steady states on the stagnation line $\hat{k}(A)$ correspond to the unique steady $(\hat{\mu}, \hat{\chi})$ state in the (μ, χ) space.

thoroughly discussed and tested. Besides being based on results in [19] and [20], they heavily exploit Projection methods, either based on OLS or on Orthogonal Collocation, plus standard Runge-Kutta type algorithms to approximate the solutions of a variety of ODEs, and Gauss-Legendre quadrature routines to approximate the welfare of the economy in different scenarios, depending on where the initial stock of knowledge A_0 and initial stock of capital k_0 lie with respect to a characteristic turnpike curve in the (A, k) space – on, above or below such curve. For optimal dynamics either converging to a steady state or moving along the turnpike we estimated welfare through a direct application of the Hamilton-Jacobi-Bellman equation, thus containing the number of approximation steps.

Such techniques have been applied to a specific parameterization of the model in Section 7, for which we simulated several Skiba-points, covering almost all qualitative scenarios envisaged by the theoretical analysis, reporting the results in Tables 1 and 2, while Figure 8 shows all such points in the (A, k) space. The simulations produce quite different varieties of early-transition dynamic patterns, depending on whether the Skiba-point lie on, above or below the turnpike. Only trajectories starting below the stagnation line turned out to be hard to estimate because of the very long early transition passing close to their unique steady state in the 'detrended' variables space; for some of these types of trajectories we produced very unreliable approximations, while for trajectories starting close to the upper bound $\hat{A}$ for positive Skiba-points, Algorithm 3 fails to deliver any result.

From Figure 8 it can be inferred that the Skiba-points lie on a decreasing curve plunging to zero (obtained by connecting the dots in the figure) as the initial stock of knowledge approaches the intersection point between the turnpike and the stagnation line. Haunschmied et al. [11] have been the first who numerically computed the set of Skiba-points for a two-state capital accumulation model; it turned out to be a curve that they called '*DNS-curve*'.[26] In our analysis this curve separates two areas of the (A, k) space containing initial conditions that determine two different equilibria for our model: that above the curve (in white, denoted by 'Growth') is the set of all initial pairs of knowledge and capital, (A_0, k_0), for which the social planner finds convenient to invest into new knowledge production, thus allowing the economy to take off towards an ABGP envisaging constant growth; that below the curve (in light grey, denoted by 'Stagnation') contains all pairs (A_0, k_0) for which it turns out to be optimal not to invest in R&D activities, so that the economy is eventually bound to stagnation. Note that the latter set is characterized by low values of the initial knowledge stock, A_0; moreover, the decreasing pattern of our DNS-curve implies that lower values of A_0 require larger values of initial capital, k_0, to let the economy take off.

This peculiar Skiba behavior is explained by the role played by a decreasing unit cost of knowledge production, $\varphi(A)$ defined by (12), in the dynamic constraint (11): lower values of A_0 determine higher values of the unit cost $\varphi(A_0)$, which, in turn, in the RHS of (11), require a larger investment J in order to have a sufficiently large value for $\dot{A}$ in the LHS for the economy to grow; but a larger J can only be justified if the initial stock of capital, k_0, is large enough to let the initial sacrifice in consumption worthwhile. This pattern reflects the very nature of the Weitzman [25] recombinant process, which is very efficient when seed ideas (potential matchings among existing successful ideas) abound but requires a great effort in terms of physical resources when seed ideas are scarce – *i.e.*, according to (10), when the initial stock of knowledge is small. Specifically, for $A_0 < \hat{A}$, with $\hat{A}$ defined in (26), and for each given preferences parameterization (the inverse of the intertemporal elasticity of substitution, σ, and discount factor, ρ, in Assumption A.3) there is a threshold value for k_0 – the Skiba-point

[26] Haunschmied et al. [10] and Caulkins et al. [6] built other models exhibiting DNS-curves that separate different types of equilibrium behavior.

$k^{sk}(A_0)$ – below which such effort, borne in terms of early consumption renunciation, is not compensated by the larger consumption associated to growth in the distant future.

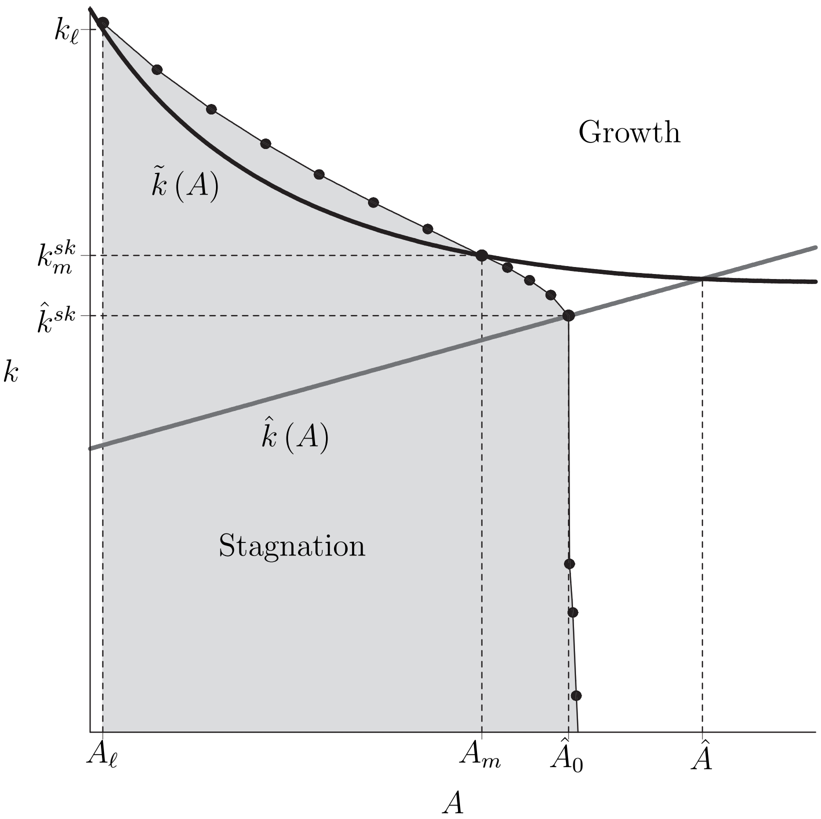

Figure 8: Skiba-points obtained by running Algorithms 1–3; connecting the dots a DNS-curve separating the basin toward growth from that toward stagnation emerges.

Appendix

Proof of Proposition 4. We must show that assumptions (i)–(iii) of Theorem 1 in Section 2 hold for the candidate function $w : \mathbb{R} \to \mathbb{R}$ defined in (47).

To prove (i) note that the function $u\left\{\theta\left[\tilde{k}(A)\right]^{\alpha} A^{1-\alpha} - \left[\tilde{k}'(A) + \varphi(A)\right] z\right\} + w'(A)\, z$ is strictly concave in z; hence, the unique value z^* such that

$$-u'\left\{\theta\left[\tilde{k}(A)\right]^{\alpha} A^{1-\alpha} - \left[\tilde{k}'(A) + \varphi(A)\right] z^*\right\}\left[\tilde{k}'(A) + \varphi(A)\right] + w'(A) = 0$$

satisfies the FOC, and thus attains the maximum in the RHS of (3). By construction, $z^* = \dot{\tilde{A}}(0^+; A)$ in (46) is such value, so that (i) holds.

Provided that $w'(A)$ in (46) is the derivative of $w(A)$ as defined in (47), assumption (ii) is trivially satisfied as, again by construction, (47) coincides with the Hamilton-Jacobi-Bellman equation (4). To check whether $w'(A)$ coincides with the derivative with respect to A of the RHS in (47), we simplify notation by setting $\tilde{y}(A) = \theta\left[\tilde{k}(A)\right]^{\alpha} A^{1-\alpha}$ and compute the latter to show that it is equal to the RHS of (46), that is,

$$\frac{1}{\rho}\left\{\frac{\tilde{c}'(A)}{\left[\tilde{c}(A)\right]^{\sigma}} + \frac{\left[\tilde{y}'(A) - \tilde{c}'(A)\right]\left[\tilde{c}(A)\right]^{\sigma} - \sigma\left[\tilde{y}(A) - \tilde{c}(A)\right]\left[\tilde{c}(A)\right]^{\sigma-1}\tilde{c}'(A)}{\left[\tilde{c}(A)\right]^{2\sigma}}\right\} = \frac{\tilde{k}'(A) + \varphi(A)}{\left[\tilde{c}(A)\right]^{\sigma}}$$

must hold. After some algebra, the equality above boils down to the following ODE in $\tilde{c}(A)$:

$$\tilde{c}'(A) = \frac{\rho\left[\tilde{k}'(A) + \varphi(A)\right] - \tilde{y}'(A)}{\sigma\left[1 - \tilde{y}(A)/\tilde{c}(A)\right]}. \tag{71}$$

Differentiating $\tilde{y}(A) = \theta\left[\tilde{k}(A)\right]^{\alpha} A^{1-\alpha}$ and, according to (20), repeatedly using $\tilde{k}(A) = [\alpha/(1-\alpha)]\,\varphi(A)\,A$, we get:

$$\begin{aligned}
\tilde{y}'(A) &= \theta\left[1 - \alpha + \alpha\frac{\tilde{k}'(A)}{\tilde{k}(A)}A\right]\left[\frac{\tilde{k}(A)}{A}\right]^{\alpha} \\
&= \theta\frac{\tilde{k}(A)}{A}\left[1 - \alpha + \alpha\frac{\tilde{k}'(A)}{\tilde{k}(A)}A\right]\left[\frac{\tilde{k}(A)}{A}\right]^{\alpha-1} \\
&= \theta\frac{\alpha\varphi(A)}{1-\alpha}\left[1 - \alpha + (1-\alpha)\frac{\tilde{k}'(A)}{\varphi(A)}\right]\left[\frac{\tilde{k}(A)}{A}\right]^{\alpha-1} \\
&= \theta\alpha\left[\tilde{k}'(A) + \varphi(A)\right]\left[\tilde{k}(A)/A\right]^{\alpha-1},
\end{aligned}$$

so that (71) can be rewritten as

$$\begin{aligned}
\tilde{c}'(A) &= \frac{\rho\left[\tilde{k}'(A) + \varphi(A)\right] - \theta\alpha\left[\tilde{k}'(A) + \varphi(A)\right]\left[\tilde{k}(A)/A\right]^{\alpha-1}}{\sigma\left[1 - \tilde{y}(A)/\tilde{c}(A)\right]} \\
&= \frac{\tilde{c}(A)\left\{\theta\alpha\left[\tilde{k}(A)/A\right]^{\alpha-1} - \rho\right\}\left[\tilde{k}'(A) + \varphi(A)\right]}{\sigma\left\{\theta\left[\tilde{k}(A)\right]^{\alpha} A^{1-\alpha} - \tilde{c}(A)\right\}}.
\end{aligned} \tag{72}$$

Note that (72) is the ODE characterizing the (undetrended) optimal policy along the turnpike obtained by eliminating time in the system of ODEs (29) – *i.e.*, taking the ratio of its equations, $c'(A) = \dot{c}/\dot{A}$. Therefore, we have just shown that whenever the optimal policy, $\tilde{c}(A)$, for the undetrended optimal dynamics (29) is used both in (46) and in (47), the former turns out to be the derivative of $w(A)$. Because, by construction, $\tilde{c}(A)$ in (37) is such policy, assumption (ii) of Theorem 1 is fully satisfied.

To show (iii) first note that the RHS in (47) is decreasing in $\tilde{c}$. Choose a constant $\varepsilon > 0$ sufficiently small so that the (feasible) constant path along the turnpike defined as $c(t; A) \equiv \varepsilon$ satisfies $c(t; A) \equiv \varepsilon \leq \tilde{c}(A(t))$ for all $t \geq 0$. Then, letting $\tilde{y}(t) = \theta\left\{\tilde{k}[A(t)]\right\}^{\alpha}[A(t)]^{1-\alpha}$, for all $t \geq 0$ we have

$$\begin{aligned}
w[A(t)] &= \frac{1}{\rho}\left\{\frac{[\tilde{c}(A(t))]^{1-\sigma} - 1}{1-\sigma} + \frac{\tilde{y}(t) - \tilde{c}(A(t))}{[\tilde{c}(A(t))]^{\sigma}}\right\} \\
&\leq \frac{1}{\rho}\left[\frac{\varepsilon^{1-\sigma} - 1}{1-\sigma} + \frac{\tilde{y}(t) - \varepsilon}{\varepsilon^{\sigma}}\right] \\
&< \frac{1}{\rho}\left[\frac{\varepsilon^{1-\sigma} - 1}{1-\sigma} + \frac{\tilde{y}(t)}{\varepsilon^{\sigma}}\right].
\end{aligned}$$

Hence, $\lim_{t\to\infty} e^{-\rho t} w[A(t)] < [1/(\rho\varepsilon^{\sigma})] \lim_{t\to\infty} e^{-\rho t}\tilde{y}(t)$, and a sufficient condition for the last limit to be zero is $\lim_{t\to\infty}\left[\dot{\tilde{y}}(t)/\tilde{y}(t)\right] < \rho$, provided that such limit exists. From (25) of

Proposition 1 (ii) we know that, when the economy grows towards its ABGP along the turnpike, $\lim_{t\to\infty}\left[\dot{\tilde{y}}(t)/\tilde{y}(t)\right] = \left\{\theta\alpha\left[\beta(1-\alpha)/\alpha\right]^{1-\alpha} - \rho\right\}/\sigma$. Therefore, a sufficient condition for $\lim_{t\to\infty} e^{-\rho t} w\left[A(t)\right] = 0$ is $\left\{\theta\alpha\left[\beta(1-\alpha)/\alpha\right]^{1-\alpha} - \rho\right\}/\sigma < \rho$, which is (48), and the proof is complete. ■

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