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POLYNOMIAL INEQUALITIES AND EMBEDDING THEOREMS WITH EXPONENTIAL WEIGHTS ON $(-1, 1)$

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Dedicated to Professor Giuseppe Mastroianni on the occasion of his 70th birthday

Abstract. Letting $0 < p \leqq \infty$, we prove Remez-, Bernstein–Markoff-, Schur- and Nikolskii-type inequalities for algebraic polynomials with exponential weights on $(-1, 1)$ multiplied by another weight function, which will satisfy the doubling or the A^* condition at different occurrences. Moreover, we state embedding theorems between some function spaces related with exponential weights.

1. Introduction

The aim of this paper is to extend some classical polynomial inequalities, considering weights of the form σu, where σ is a doubling weight and u is a weight defined by

$$u(x) = \left(1 - x^2\right)^{\beta} e^{-(1-x^2)^{-\alpha}}, \quad \alpha > 0 \quad \text{and} \quad \beta \geqq 0.$$

We emphasize that the weight u violates not only the doubling condition, but also the Szegő condition

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}}\, dx > -\infty$$

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for $\alpha \geqq 1/2$, and it belongs to a wide class of exponential weights defined in [9,10] (the definition is given at the beginning of Section 2).

To be more precise we will state Remez-, Bernstein–Markoff-, Schur- and Nikolskii-type inequalities with the weight σu, extending the results proved in [9,10,12] for exponential weights and in [6,16] for doubling weights.

Then we will prove some embedding theorems between function spaces related with the weight u, defined in [15].

The paper is structured as follows. In Section 2 we recall some basic facts. In Sections 3 and 4 we state some polynomial inequalities and embedding theorems, respectively. Finally, in Section 5 we prove our main results.

In the sequel, we will denote by $\mathbb{P}_m$ the collection of polynomials of degree at most m and by $\mathcal{C}$ a positive constant which can assume different values in different formulae. We shall write $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ when $\mathcal{C}$ is independent of $a, b, \dots$. Furthermore, if A and B are positive quantities depending on some parameters, then $A \sim B$ will mean that there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leqq \mathcal{C}$.

By a slight abuse of notation, in the sequel we denote by $\|\cdot\|_p$ the quasi-norm of the L^p-spaces for $0 < p < 1$, defined in the usual way.

2. Basic facts and preliminary results

Exponential weights. First of all, we recall the definition of a class of exponential weights, given by A. L. Levin and D. S. Lubinsky in [9, p. 5]. If ϱ is a weight function in $(-1,1)$, we will say that the weight $\varrho(x) = e^{-Q(x)}$, $x \in (-1,1)$, belongs to the class $\hat{\mathcal{W}}$ and write $\varrho \in \hat{\mathcal{W}}$ if and only if the function $Q : (-1,1) \to \mathbb{R}$ fulfills the following conditions:

(i) Q is even and twice continuously differentiable, with $\lim_{x \to 1} Q(x) = +\infty$;

(ii) $Q'(x) \geqq 0$ $Q''(x) \geqq 0$ for $x \in (0,1)$;

(iii) the function

$$T(x) = 1 + \frac{xQ''(x)}{Q'(x)}$$

is increasing in $[0,1)$ with $T(0) > 1$;

(iv) For some $A \in (0,1)$, the function T satisfies $T(x) \sim \frac{Q'(x)}{Q(x)}$ for $x \in [A,1)$.

The Mhaskar–Rahmanov–Saff number $a_m = a_m(\varrho)$, related to the weight ϱ, is defined as the positive root of the equation

$$m = \frac{2}{\pi} \int_0^1 a_m t Q'(a_m t) \frac{dt}{\sqrt{1-t^2}} \tag{2.1}$$

and the equivalence (see [11])

$$Q'(a_m) \sim m\sqrt{T(a_m)} \tag{2.2}$$

can lead to an approximation of a_m.

Let us now consider the weight function

$$u(x) = v^\beta(x)w(x) = \left(1-x^2\right)^\beta e^{-(1-x^2)^{-\alpha}}, \quad x \in (-1,1), \quad \alpha > 0, \quad \beta \geqq 0. \tag{2.3}$$

It is easily seen that the weight w belongs to the class $\hat{\mathcal{W}}$, while u can be considered as a logarithmic perturbation of w. In [15] it has been shown that the weight u belongs to the class $\hat{\mathcal{W}}$ and its related function T fulfills

$$T(x) \sim \frac{1}{1-x^2}, \tag{2.4}$$

for x close enough to 1.

In the sequel we will denote by $a_m = a_m(u)$ the Mhaskar–Rahmanov–Saff number related to the weight u. By (2.2) and (2.4), we get

$$1 - a_m \sim m^{-\frac{1}{\alpha+1/2}}. \tag{2.5}$$

If $x, y \in [-a_m, a_m]$, with $|x-y| \leqq (K/m)\sqrt{1-x^2}$, we have (see [14])

$$u(y) \sim u(x). \tag{2.6}$$

Moreover, setting

$$b_m = b_m(u) = a_m(1-\lambda\delta_m), \tag{2.7}$$

where $\lambda > 0$ is a constant and

$$\delta_m = \delta_m(u) = \left(mT(a_m)\right)^{-2/3} \sim m^{-\frac{2}{3}\left(\frac{2\alpha+3}{2\alpha+1}\right)}, \tag{2.8}$$

by (2.4) and (2.5), the following restricted range inequality has been proved by D. S. Lubinsky and E. B. Saff [12] for $p = \infty$ and by A. L. Levin and D. S. Lubinsky for $0 < p \leqq \infty$.

THEOREM 2.1 [10, pp. 95–97]. *Let $0 < p \leqq \infty$ and let $u(x)$ be defined by* (2.3). *For every polynomial $P_m \in \mathbb{P}_m$ we have*

$$\|P_m u\|_p \leqq \mathcal{C}\|P_m u\|_{L^p[-b_m, b_m]}, \tag{2.9}$$

where $\mathcal{C}$ is independent of m and P_m.

The next lemma will be crucial in order to prove our polynomial inequalities.

LEMMA 2.2. *Let $u(x)$ be defined by (2.3). Then, for sufficiently large m, there exists a polynomial $R_{lm} \in \mathbb{P}_{lm}$, with a fixed integer l, satisfying the properties*

$$\frac{1}{2}\, u(x) \leqq R_{lm}(x) \leqq \frac{3}{2}\, u(x) \tag{2.10}$$

and

$$\frac{\left| R'_{lm}(x) \right| \varphi(x)}{m} \leqq \mathcal{C} u(x) \tag{2.11}$$

for $|x| \leqq a_{sm}$, $s \geqq 1$ a fixed integer, with $\mathcal{C}$ independent of m, u and R_{lm}, where $\varphi(x) = \sqrt{1-x^2}$.

Lemma 2.2 was proved in [14] for the weight $w(x) = e^{-(1-x^2)^{-\alpha}}$, i.e. for $\beta = 0$. We will omit the proof, since it is similar to that in [14], taking into account (2.4).

Doubling weights. In the sequel, we will consider, in addition to the exponential weight u in (2.3), another function σ, which will be a doubling, an A_∞ or an A^* weight at different occurrences. We recall that a weight σ is said to be doubling if

$$\sigma(2I) \leqq \mathcal{C}\sigma(I),$$

for all intervals $I \subset (-1,1)$, where

$$\sigma(I) := \int_I \sigma(t)\, dt,$$

$2I$ is the interval twice the length of I and with midpoint at the midpoint of I, $\mathcal{C}$ is independent of I. Moreover, σ is an A_∞ weight if, for every $a > 0$, there exists a $b > 0$ such that

$$\sigma(E) \geqq b\sigma(I)$$

for any interval $I \subset (-1,1)$ and any measurable set $E \subset I$ with $|E| \geqq a|I|$, where $|E|$ denotes the Lebesgue measure of E. Finally, we call σ an A^* weight if there exists a constant $\mathcal{C}$ such that

$$\sigma(x) \leqq \frac{\mathcal{C}}{|I|} \int_I \sigma(t)\, dt$$

for all intervals $I \subset (-1,1)$ and $x \in I$. As is well-known, the A^* property implies the A_∞ condition and the latter implies the doubling condition.

If σ is a doubling weight, we set

$$\sigma_m(x) = \frac{1}{\Delta_m(x)} \int_{x-\Delta_m(x)}^{x+\Delta_m(x)} \sigma(t)\,dt, \tag{2.12}$$

where

$$\Delta_m(x) = \frac{\varphi(x)}{m} + \frac{1}{m^2} = \frac{\sqrt{1-x^2}}{m} + \frac{1}{m^2}.$$

By means of σ_m we can characterize the weight σ (see [16, Lemma 7.1]). In fact σ is doubling if and only if for some $s > 0$ we have

$$\sigma_m(y) \leqq \mathcal{C}\big(1 + m|x-y| + m\big|\sqrt{1-x^2} - \sqrt{1-y^2}\big|\big)^s \sigma_m(x),$$

for any $x, y \in [-1,1]$ and $m \in \mathbb{N}$. In particular, if σ is a doubling weight, for any $m \in \mathbb{N}$ and $x, y \in [-1,1]$, with $|x-y| \leqq K\Delta_m(x)$, we get

$$\sigma_m(y) \sim \sigma_m(x), \tag{2.13}$$

where the constants in "$\sim$" are independent of K.

The following theorem, stating a Remez-type inequality for A_∞ weights, has been proved by G. Mastroianni and V. Totik [16] for $1 \leqq p < \infty$, and then extended for $0 < p < 1$ by T. Erdélyi [6].

Theorem 2.3 [6,16]. *Let $0 < p < \infty$ and σ be an A_∞ weight. For any $P_m \in \mathbb{P}_m$ and for every $\Lambda > 0$ there is a constant $\mathcal{C} = \mathcal{C}(\Lambda)$ such that, if $E \subset [-1,1]$ is a measurable set with*

$$\big|\cos^{-1}(E)\big| = \int_E \frac{dx}{\sqrt{1-x^2}} \leqq \frac{\Lambda}{m},$$

then

$$\int_{-1}^{1} \big|P_m(x)\sigma(x)\big|^p\,dx \leqq \mathcal{C} \int_{[-1,1]\setminus E} \big|P_m(x)\sigma(x)\big|^p\,dx. \tag{2.14}$$

For $p = \infty$, if σ is an A^* weight, we have

$$\|P_m\sigma\|_\infty \leqq \mathcal{C} \sup_{[-1,1]\setminus E} \big|P_m(x)\sigma(x)\big|. \tag{2.15}$$

If the weight σ satisfies the less restrictive doubling condition, Theorem 2.3 holds with stronger assumptions on the subset E, namely if $E \subset [-1,1]$ such that $E = \bigcup_{k=1}^N I_k$, where I_k are intervals, $N \in \mathbb{N}$ is fixed, and

$$\big|\cos^{-1}(I_k)\big| = \int_{I_k} \frac{dx}{\sqrt{1-x^2}} \geqq \frac{c}{m}, \quad c > 0, \qquad \sum_{k=1}^{N} \big|\cos^{-1}(I_k)\big| \leqq \frac{\Lambda}{m}.$$

In analogy with Lemma 2.2, the following lemma holds for doubling weights.

LEMMA 2.4 [16]. *Let σ be a doubling weight and let σ_m be defined by* (2.12). *Then there exist polynomials $q_m \in \mathbb{P}_m$ such that*

$$q_m(x) \sim \sigma_m(x) \tag{2.16}$$

and

$$\frac{|\,q_m'(x)|\,\varphi(x)}{m} \leqq \mathcal{C}\sigma_m(x), \tag{2.17}$$

for $x \in (-1,1)$, where $\mathcal{C}$ and the constants in "$\sim$" are independent of m.

Note that if σ is an A^* weight, then $\sigma(x) \leqq \mathcal{C}\sigma_m(x)$ for $x \in (-1,1)$.

Further tools in our proofs will be the following two theorems, proved in [16] for $1 \leqq p \leqq \infty$ and in [6] for $0 < p < 1$.

THEOREM 2.5 [6,16]. *Let $0 < p < \infty$, σ be a doubling weight on $(-1,1)$ and let σ_m be given by* (2.12). *Then for every $P_m \in \mathbb{P}_m$ we have*

$$\int_{-1}^{1} |\,P_m(x)\sigma(x)|^p\,dx \sim \int_{-1}^{1} |\,P_m(x)\sigma_m(x)|^p\,dx,$$

where the constant in "$\sim$" depends only on the doubling constant and p.

THEOREM 2.6 [16]. *Let σ be an A^* weight on $(-1,1)$ and let σ_m be given by* (2.12). *Then for every $P_m \in \mathbb{P}_m$ we have*

$$\|P_m\sigma\|_\infty \sim \|P_m\sigma_m\|_\infty,$$

where the constant in "$\sim$" is independent of m and P_m.

Finally, we recall the following Schur-type inequality, proved in [16] for $1 \leqq p < \infty$ and in [6] for $0 < p < 1$.

THEOREM 2.7 [6,16]. *Let $0 < p < \infty$, σ be a doubling weight and h be a generalized Jacobi weight of the form*

$$h(x) = \prod_{i=1}^{N} |x - x_i|^{\gamma_i}, \qquad \gamma_i > 0, \quad x, x_i \in [-1,1]. \tag{2.18}$$

Then, for any polynomial $P_m \in \mathbb{P}_m$, we have

$$\|P_m\sigma\|_p \leqq \mathcal{C}m^\Gamma\,\|P_m\sigma h\|_p \tag{2.19}$$

with $\mathcal{C}$ independent of m and P_m, where $\Gamma = \max_i \gamma_i^*$, $\gamma_i^* = \gamma_i$ if $x_i \neq \pm 1$ and $\gamma_i^* = 2\gamma_i$ if $x_i = \pm 1$.

To complete Theorem 2.7, inequality (2.19) holds also for $p=\infty$ if σ is an A^* weight.

3. Polynomial inequalities

Now, we are going to state polynomial inequalities related to the weight σu, where σ is a doubling weight and u is given by (2.3). In their proofs we will use the restricted range inequality (2.9), and approximate the weight u in $[-a_m, a_m]$ by means of a polynomial, by arguments analogous to those in [13,16].

First, we prove a Remez-type inequality, generalizing the results holding for exponential or A_∞ weights (see Theorems 2.1 and 2.3). We set $b_m = a_m(1-\lambda\delta_m)$, $\lambda>0$, with $a_m = a_m(u)$ and $\delta_m = \delta_m(u)$ satisfying (2.5) and (2.8).

THEOREM 3.1. *Let $0<p\leqq\infty$, u be the weight in (2.3), σ be a doubling weight and σ_m be as in (2.12). For any $P_m\in\mathbb{P}_m$ and for every fixed $\Lambda>0$, if $E\subset\big[-b_{2m}, b_{2m}\big]$ is a measurable set with*

$$\int_E \frac{dx}{\sqrt{b_{2m}^2-x^2}} \leqq \frac{\Lambda}{m},$$

then

$$\|P_m\sigma_m u\|_p \leqq \mathcal{C}\|P_m\sigma_m u\|_{L^p([-b_{2m},b_{2m}]\setminus E)}, \tag{3.1}$$

where $b_{2m}=a_{2m}(1-\lambda\delta_{2m})$, $\lambda>0$, $\mathcal{C}$ is independent of m, P_m and depends only on Λ, λ.

Taking into account Theorems 2.1 and 2.3, a natural question arising is whether it is possible to replace the weight σ_m by σ in (3.1). For instance, it is easily seen that, if σ is a doubling weight, for $0<p\leqq\infty$ and for any $P_m\in\mathbb{P}_m$, the inequality

$$\|P_m\sigma u\|_p \leqq \mathcal{C}\|P_m\sigma u\|_{L^p[-1+m^{-2},1-m^{-2}]} \tag{3.2}$$

holds with $\mathcal{C}$ independent of m and P_m. Obviously this inequality is weaker than (3.1) near ±1. On the other hand, with more assumptions on the weight σ, as a consequence of Theorem 3.1, we can prove the following inequalities.

COROLLARY 3.2. *For $0<p<\infty$, under the assumptions of Theorem 3.1, if σ is an A_∞ weight, for any polynomial $P_m\in\mathbb{P}_m$ and for every $\Lambda>0$, if $E\subset\big[-b_{3m}, b_{3m}\big]$ is a measurable set with*

$$\int_E \frac{dx}{\sqrt{b_{3m}^2-x^2}} \leqq \frac{\Lambda}{m},$$

then

$$\int_{-1}^{1} |P_m(x)\sigma_m(x)u(x)|^p\,dx \leqq \mathcal{C}\int_{[-b_{3m},b_{3m}]\setminus E} |P_m(x)\sigma(x)u(x)|^p\,dx. \tag{3.3}$$

Moreover, for $0 < p \leqq \infty$, *if* σ *is an* A^* *weight, then*

$$\|P_m\sigma u\|_p \leqq \mathcal{C}\|P_m\sigma u\|_{L^p([-b_{3m},b_{3m}]\setminus E)}, \tag{3.4}$$

where $b_{3m} = a_{3m}(1-\lambda\delta_{3m})$, $\lambda > 0$, *and in both cases* $\mathcal{C}$ *depends only on* Λ *and* λ.

For instance, from Theorem 3.1, letting u be the weight in (2.3) and $\sigma(x) = |x|^\gamma$, with $\gamma > -1/p$ if $1 \leqq p < \infty$ and $\gamma \geqq 0$ if $p = \infty$, and for any $P_m \in \mathbb{P}_m$, it follows that

$$\| P_m| \cdot |^\gamma u\|_p \leqq \mathcal{C}\| P_m| \cdot |^\gamma u\|_{L^p\{\frac{c}{m}\leqq|x|\leqq a_{3m}(1-\lambda\delta_{3m})\}}, \qquad 1 \leqq p \leqq \infty,$$

with c, λ, $\mathcal{C}$ independent of m and P_m.

In the following theorems, we state the Bernstein–Markov inequalities related to the weight $\sigma_m u$ and as before, in their corollaries, we replace the weight σ_m by σ.

THEOREM 3.3. *Let* $0 < p \leqq \infty$, $u(x)$ *the weight in* (2.3), σ *a doubling weight and* σ_m *as in* (2.12). *Then, for any polynomial* $P_m \in \mathbb{P}_m$, *we have*

$$\|P_m'\varphi\sigma_m u\|_p \leqq \mathcal{C}m\|P_m\sigma_m u\|_p, \tag{3.5}$$

where $\mathcal{C}$ *is independent of* m *and* P_m.

In particular, from Theorem 3.3, for any $P_m \in \mathbb{P}_m$, we deduce

$$\sup_{x\in(-1,1)} \left| P_m'(x)\left(\sqrt{1-x^2}+\frac{1}{m}\right)^{\gamma+1} w(x)\right|$$

$$\leqq \mathcal{C}m \sup_{x\in(-1,1)} \left| P_m(x)\left(\sqrt{1-x^2}+\frac{1}{m}\right)^{\gamma} w(x)\right|,$$

with $\gamma \in \mathbb{R}$, $w(x) = e^{-(1-x^2)^{-\alpha}}$, $\alpha > 0$, and $\mathcal{C} \neq \mathcal{C}(m, P_m)$. The previous inequality, useful in various contexts, generalizes a result of Khalilova [8] (see also [17]), holding for Jacobi weights.

COROLLARY 3.4. *Under the assumptions of Theorem* 3.3, *for any* $P_m \in \mathbb{P}_m$ *and for* $0 < p < \infty$,

$$\|P_m'\varphi\sigma_m u\|_p \leqq \mathcal{C}m\|P_m\sigma u\|_p. \tag{3.6}$$

Moreover, if σ *is an* A^* *weight, then for any* $P_m \in \mathbb{P}_m$ *and for* $0 < p \leqq \infty$,

$$\|P_m' \varphi \sigma u\|_p \leqq \mathcal{C} m \|P_m \sigma u\|_p, \tag{3.7}$$

where in both cases $\mathcal{C}$ *is independent of* m *and* P_m.

THEOREM 3.5. *Let* $0 < p \leqq \infty$, $u(x)$ *the weight in* (2.3), σ *a doubling weight and* σ_m *as in* (2.12). *Then, for any polynomial* $P_m \in \mathbb{P}_m$, *we have*

$$\|P_m' \sigma_m u\|_p \leqq \mathcal{C} \frac{m}{\sqrt{1-a_m}} \|P_m \sigma_m u\|_p \tag{3.8}$$

with $\mathcal{C}$ *independent of* m *and* P_m.

COROLLARY 3.6. *Let* $0 < p < \infty$. *Under the assumptions of Theorem* 3.5, *for any* $P_m \in \mathbb{P}_m$,

$$\|P_m' \sigma_m u\|_p \leqq \mathcal{C} \frac{m}{\sqrt{1-a_m}} \|P_m \sigma u\|_p. \tag{3.9}$$

Moreover, if σ *is an* A^* *weight, then for any* $P_m \in \mathbb{P}_m$ *and* $0 < p \leqq \infty$,

$$\|P_m' \sigma u\|_p \leqq \mathcal{C} \frac{m}{\sqrt{1-a_m}} \|P_m \sigma u\|_p, \tag{3.10}$$

where in both cases $\mathcal{C}$ *is independent of* m *and* P_m.

Notice that the Bernstein-type inequality (3.7) has the same form as that proved in [14] for the weight u and also the one related to doubling or A^* weights, for $p < \infty$ or $p = \infty$ respectively, proved in [16]. Moreover, inequality (3.7) can be easily iterated as

$$\left\|P_m^{(r)} \varphi^r \sigma u\right\|_p \leqq \mathcal{C} m^r \|P_m \sigma u\|_p,$$

for $1 \leqq r \in \mathbb{Z}$.

Concerning the Markoff-type inequality, in case of weights belonging to the class $\hat{\mathcal{W}}$, it has been obtained in [12] for $p = \infty$ and in [10, p. 294] for $0 < p \leqq \infty$. While, with respect to the Markoff inequality proved in [16] for doubling weights, in inequality (3.8) the factor 2 is replaced by a smaller one. To be more precise, from Theorem 3.5 and equivalence (2.5), it follows that

$$\|P_m' u\|_p \leqq \mathcal{C} m^{\frac{2\alpha+2}{2\alpha+1}} \|P_m u\|_p, \tag{3.11}$$

and then the factor 2 in the classical Markoff inequality is replaced by $\frac{2\alpha+2}{2\alpha+1}$.

Now, we state some Schur-type inequalities.

THEOREM 3.7. *Let $0 < p < \infty$, $u(x)$ the weight in (2.3), σ a doubling weight and σ_m as in (2.12). Furthermore, let h be a generalized Jacobi weight of the form (2.18). Then, for any polynomial $P_m \in \mathbb{P}_m$, we have*

$$\|P_m\sigma_m u\|_p \leqq \mathcal{C}m^{\Gamma}\|P_m\sigma_m uh\|_p \tag{3.12}$$

with $\mathcal{C}$ independent of m and P_m, where $\Gamma = \max_i \gamma_i^*$, $\gamma_i^* = \gamma_i$ if $x_i \neq \pm 1$ and $\gamma_i^* = \gamma_i/(\alpha + 1/2)$ if $x_i = \pm 1$.

COROLLARY 3.8. *Let $0 < p < \infty$. Under the assumptions of Theorem 3.7, for any $P_m \in \mathbb{P}_m$,*

$$\|P_m\sigma_m u\|_p \leqq \mathcal{C}m^{\Gamma}\|P_m\sigma uh\|_p. \tag{3.13}$$

Moreover, if σ is an A^* weight, for any $P_m \in \mathbb{P}_m$ and $0 < p \leqq \infty$, then

$$\|P_m\sigma u\|_p \leqq \mathcal{C}m^{\Gamma}\|P_m\sigma uh\|_p. \tag{3.14}$$

where in both cases $\mathcal{C}$ is independent of m and P_m, $\Gamma = \max_i \gamma_i^*$, $\gamma_i^* = \gamma_i$ if $x_i \neq \pm 1$ and $\gamma_i^* = \gamma_i/(\alpha + 1/2)$ if $x_i = \pm 1$.

To conclude this section, we state some Nikolskii-type inequalities. By a slight abuse of notation, for $q = \infty$ we will set $1/q = 0$.

THEOREM 3.9. *Let $1 \leqq p < q \leqq \infty$, $u(x)$ the weight in (2.3), σ a doubling weight and σ_m as in (2.12). Then, for any $P_m \in \mathbb{P}_m$,*

$$\|P_m\sigma_m u\|_q \leqq \mathcal{C}\left(\frac{m}{\sqrt{1-a_m}}\right)^{\frac{1}{p}-\frac{1}{q}}\|P_m\sigma_m u\|_p \tag{3.15}$$

and

$$\left\|P_m\varphi^{\frac{1}{p}-\frac{1}{q}}\sigma_m u\right\|_q \leqq \mathcal{C}m^{\frac{1}{p}-\frac{1}{q}}\|P_m\sigma_m u\|_p, \tag{3.16}$$

where $\mathcal{C}$ does not depend on m and P_m.

COROLLARY 3.10. *Under the assumptions of Theorem 3.9, for any $P_m \in \mathbb{P}_m$ and for $1 \leqq p < q < \infty$,*

$$\|P_m\sigma_m u\|_q \leqq \mathcal{C}\left(\frac{m}{\sqrt{1-a_m}}\right)^{\frac{1}{p}-\frac{1}{q}}\|P_m\sigma u\|_p \tag{3.17}$$

and

$$\left\|P_m\varphi^{\frac{1}{p}-\frac{1}{q}}\sigma_m u\right\|_q \leqq \mathcal{C}m^{\frac{1}{p}-\frac{1}{q}}\|P_m\sigma u\|_p. \tag{3.18}$$

Moreover, if σ is an A^* weight, then for any $P_m \in \mathbb{P}_m$ and for $0 < p < q \leqq \infty$,

$$\|P_m \sigma u\|_q \leqq \mathcal{C} \left(\frac{m}{\sqrt{1-a_m}} \right)^{\frac{1}{p}-\frac{1}{q}} \|P_m \sigma u\|_p \tag{3.19}$$

and

$$\left\| P_m \varphi^{\frac{1}{p}-\frac{1}{q}} \sigma u \right\|_q \leqq \mathcal{C} m^{\frac{1}{p}-\frac{1}{q}} \|P_m \sigma u\|_p, \tag{3.20}$$

where in each of the previous inequalities $\mathcal{C}$ is independent of m and P_m.

4. Embedding theorems

Here, using the Nikolskii inequalities (3.19) and (3.20), we prove some embedding theorems, connecting function spaces related to the weight (2.3). These spaces were introduced in [14] for $\beta = 0$ (see also [2,3] for the case $\alpha = 1/2$ and $\beta = 0$).

Let us first define the spaces. By L^p_u, $1 \leqq p < \infty$, we denote the set of all measurable functions f such that

$$\|f\|_{L^p_u} := \|fu\|_p = \left(\int_{-1}^1 |fu|^p(x)\,dx \right)^{1/p} < \infty,$$

while, for $p = \infty$, by a slight abuse of notation, we set

$$L^\infty_u = C_u = \left\{ f \in C^0(-1,1) : \lim_{x \to \pm 1} f(x)u(x) = 0 \right\}$$

and we equip this space with the norm

$$\|f\|_{L^\infty_u} := \|fu\|_\infty = \sup_{x \in [-1,1]} \big| f(x)u(x) \big|.$$

For $1 \leqq p \leqq \infty$, $r \geqq 1$ and $0 < t < t_0$, we define the main part of the rth modulus of smoothness as

$$\Omega^r_\varphi(f,t)_{u,p} = \sup_{0 < h \leqq t} \big\| \Delta^r_{h\varphi}(f) u \big\|_{L^p(\mathcal{I}_h(B))},$$

where $\mathcal{I}_h = \left[-1 + B\,h^{1/(\alpha+\frac{1}{2})}, 1 - B\,h^{1/(\alpha+\frac{1}{2})} \right]$, $B > 1$ is a fixed constant, and

$$\Delta^r_{h\varphi} f(x) = \sum_{i=0}^r \binom{r}{i} (-1)^i f\left(x + (r-2i) \frac{h\varphi(x)}{2} \right).$$

Then we define the complete rth modulus of smoothness by

$$\omega_\varphi^r(f,t)_{u,p} = \Omega_\varphi^r(f,t)_{u,p} + \inf_{P\in\mathbb{P}_{r-1}} \left\| (f-P)u \right\|_{L^p[-1,-t^*]} + \inf_{P\in\mathbb{P}_{r-1}} \left\| (f-P)u \right\|_{L^p[t^*,1]} \tag{4.1}$$

with $t^* = 1 - B\,t^{1/(\alpha+\frac{1}{2})}$ and $B > 1$ is a fixed constant. We remark that the behaviour of $\omega_\varphi^r(f,t)_{u,p}$ is independent of the constant B.

By means of the main part of the modulus of smoothness, for $1 \leqq p \leqq \infty$, we can define the Zygmund-type spaces

$$Z_s^p(u) := Z_{s,r}^p(u) = \left\{ f \in L_u^p : \sup_{t>0} \frac{\omega_\varphi^r(f,t)_{u,p}}{t^s} < \infty,\ r > s \right\}, \quad s \in \mathbb{R}^+,$$

equipped with the norm

$$\|f\|_{Z_{s,r}^p(u)} = \|f\|_{L_u^p} + \sup_{t>0} \frac{\omega_\varphi^r(f,t)_{u,p}}{t^s}.$$

In the sequel we will denote these subspaces briefly by $Z_s^p(u)$, without the second index r and with the assumption $r > s$. We remark that, since for $f \in Z_s^p(u)$, we have $\Omega_\varphi^r(f,t)_{u,p} \sim \omega_\varphi^r(f,t)_{u,p}$, in the definition of the Zygmund-type spaces $\omega_\varphi^r(f,t)_{u,p}$ can be replaced by $\Omega_\varphi^r(f,t)_{u,p}$ (see [14]).

Let us denote by $E_m(f)_{u,p} = \inf_{P\in\mathbb{P}_m} \left\| (f-P)u \right\|_p$ the error of best polynomial approximation in L_u^p, $1 \leqq p \leqq \infty$. An element realizing the infimum in the previous definition is called polynomial of best approximation for $f \in L_u^p$. Moreover, we say that $P \in \mathbb{P}_m$ is a polynomial of quasi best approximation for $f \in L_u^p$ if

$$\left\| (f-P)u \right\|_p \leqq \mathcal{C}\, E_m(f)_{u,p},$$

with $\mathcal{C}$ independent of m and f.

Since the weights w and u in (2.3) satisfy the same properties, in particular they fulfill (2.4), proceeding as in [14] for the weight $w(x) = e^{-(1-x^2)^{-\alpha}}$, $\alpha > 0$, we can obtain the following Jackson- and Stechkin-type inequalities.

THEOREM 4.1. *Let $u(x)$ be as in (2.3). For any $f \in L_u^p$, $1 \leqq p \leqq \infty$, we get*

$$E_m(f)_{u,p} \leqq \mathcal{C}\, \omega_\varphi^r\left(f, \frac{1}{m}\right)_{u,p} \tag{4.2}$$

and

$$\omega_\varphi^r\left(f,\frac{1}{m}\right)_{u,p} \leqq \frac{\mathcal{C}}{m^r}\sum_{i=0}^{m}(1+i)^{r-1}E_i(f)_{u,p}, \tag{4.3}$$

where in both cases $\mathcal{C}$ *is independent of* m *and* f.

From the previous inequalities we deduce $f \in L_u^p$ if and only if $E_m(f)_{u,p} \to 0$.

Moreover, a weak Jackson-type inequality in the next theorem can be proved (see [14]).

THEOREM 4.2. *Let* $1 \leqq p \leqq \infty$ *and* $u(x)$ *as in* (2.3). *Assume* $f \in L_u^p$ *with* $\Omega_\varphi^r(f,t)_{u,p}t^{-1} \in L^1[0,1]$. *Then*

$$E_m(f)_{u,p} \leqq \mathcal{C}\int_0^{1/m}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t}\,dt, \qquad r<m, \tag{4.4}$$

with $\mathcal{C}$ *independent of* m *and* f.

Now we can state some embedding theorems among the previously introduced spaces, extending some results due to Ul'yanov [18] (see also [4, 7]).

THEOREM 4.3. *Let* $1 \leqq p < \infty$ *and* $u(x)$ *as in* (2.3). *For any* $f \in L_u^p$, *such that*

$$\int_0^1 \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\eta/p}}\,dt < \infty, \tag{4.5}$$

where $\eta = (2\alpha+2)/(2\alpha+1)$, *we have*

$$E_m(f)_{u,\infty} \leqq \mathcal{C}\int_0^{1/m}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\eta/p}}\,dt, \tag{4.6}$$

$$\Omega_\varphi^r\left(f,\frac{1}{m}\right)_{u,\infty} \leqq \mathcal{C}\int_0^{1/m}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\eta/p}}\,dt \tag{4.7}$$

and

$$\|fu\|_\infty \leqq \mathcal{C}\left\{\|fu\|_p + \int_0^1\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\eta/p}}\,dt\right\}, \tag{4.8}$$

where $\mathcal{C}$ *depends only on* r.

THEOREM 4.4. *Let* $1 \leqq p < \infty$ *and* $u(x)$ *as in* (2.3). *For any* $f \in L^p_u$, *such that*

$$\int_0^1 \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,dt < \infty, \tag{4.9}$$

we have

$$E_m(f)_{\varphi^{1/p}u,\infty} \leqq \mathcal{C} \int_0^{1/m} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,dt, \tag{4.10}$$

$$\Omega^r_\varphi\left(f,\frac{1}{m}\right)_{\varphi^{1/p}u,\infty} \leqq \mathcal{C} \int_0^{1/m} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,dt \tag{4.11}$$

and

$$\|f\varphi^{1/p}u\|_\infty \leqq \mathcal{C}\left\{\|fu\|_p + \int_0^1 \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,dt\right\}, \tag{4.12}$$

where $\mathcal{C}$ *depends only on* r.

From Theorem 4.4 we can easily deduce the following corollary, useful in several contexts.

COROLLARY 4.5. *Let* $1 \leqq p < \infty$ *and* $u(x)$ *as in* (2.3). *If* $f \in L^p_u$ *is such that*

$$\int_0^1 \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,dt < \infty, \tag{4.13}$$

then f *is continuous on* $(-1,1)$.

5. Proofs

We recall that we denote by $\|\cdot\|_p$ the quasinorm of the L^p-spaces for $0<p<1$.

PROOF OF THEOREM 3.1. Let $0<p\leqq\infty$. By Lemma 2.4, there exists a polynomial $q_m \in \mathbb{P}_m$ such that $q_m \sim \sigma_m$ in $[-1,1]$. Moreover, by Lemma 2.2, we can choose a polynomial $R_{lm} \in \mathbb{P}_{lm}$, l sufficiently large, satisfying $R_{lm} \sim u$ in $\big[-b_{2m}, b_{2m}\big]$, where $b_{2m} = a_{2m}(1-\lambda\delta_{2m})$, $\lambda>0$, $1-a_{2m} \sim (2m)^{-\frac{2}{2\alpha+1}}$ and $\delta_{2m} \sim (2m)^{-\frac{2(2\alpha+3)}{3(2\alpha+1)}}$. Hence, by Theorem 2.1, for any $P_m \in \mathbb{P}_m$, we get

$$\begin{aligned}\|P_m\sigma_m u\|_p \sim \|P_m q_m u\|_p &\leqq \mathcal{C}\|P_m q_m u\|_{L^p[-b_{2m},b_{2m}]}\\ &\leqq \mathcal{C}\|P_m q_m R_{lm}\|_{L^p[-b_{2m},b_{2m}]}.\end{aligned}$$

Then, for any $E \subset \left[-b_{2m}, b_{2m}\right]$, with $\int_E \left(b_{2m}^2 - x^2\right)^{-1/2} dx \leqq \Lambda/m$, using the unweighted Remez-type inequality (see [1, pp. 227–274]), by Lemmas 2.4 and 2.2, we have

$$\|P_m \sigma_m u\|_p \leqq \mathcal{C} \|P_m q_m R_{lm}\|_{L^p[-b_{2m}, b_{2m}]}$$

$$\leqq \mathcal{C} \|P_m q_m R_{lm}\|_{L^p([-b_{2m}, b_{2m}] \setminus E)} \leqq \mathcal{C} \|P_m \sigma_m u\|_{L^p([-b_{2m}, b_{2m}] \setminus E)},$$

that is (3.1). □

Proof of Corollary 3.2. We are going to prove only inequality (3.3). Proceeding as in the proof of Theorem 3.1, by Lemmas 2.4 and 2.2, there exist polynomials $q_m \in \mathbb{P}_m$ and $R_{lm} \in \mathbb{P}_{lm}$, l sufficiently large, such that $q_m \sim \sigma_m$ in $[-1, 1]$ and $R_{lm} \sim u$ in $\left[-b_{3m}, b_{3m}\right]$.

Now, in order to use Theorem 2.5, observe that for $x \in \left[-b_{2m}, b_{2m}\right]$, from $\sqrt{1-x^2} \sim \sqrt{b_{3m}^2 - x^2}$ we deduce $\sigma_m(x) \sim \tilde{\sigma}_m(x)$, where

$$\tilde{\sigma}_m(x) = \frac{1}{\tilde{\Delta}_m(x)} \int_{x-\tilde{\Delta}_m(x)}^{x+\tilde{\Delta}_m(x)} \sigma(t)\, dt, \qquad \tilde{\Delta}_m(x) = \frac{\sqrt{b_{3m}^2 - x^2}}{m} + \frac{1}{m^2}.$$

Then, for $0 < p < \infty$, by Theorem 2.5, we have

$$\|P_m \sigma_m u\|_p \leqq \mathcal{C} \|P_m \sigma_m R_{lm}\|_{L^p[-b_{2m}, b_{2m}]} \leqq \mathcal{C} \|P_m \tilde{\sigma}_m R_{lm}\|_{L^p[-b_{2m}, b_{2m}]}$$

$$\leqq \mathcal{C} \|P_m \tilde{\sigma}_m R_{lm}\|_{L^p[-b_{3m}, b_{3m}]} \leqq \mathcal{C} \|P_m \sigma R_{lm}\|_{L^p[-b_{3m}, b_{3m}]}.$$

Using Theorem 2.3, inequality (3.3) follows. □

We will omit the proofs of inequality (3.4) and also of Corollaries 3.6, 3.8 and 3.10, since they follow from Theorems 3.5, 3.7 and 3.9 by using arguments analogous to those in the previous proof and in the proof of Corollary 3.4.

Proof of Theorem 3.3. We use arguments analogous to those in [14]. Let $0 < p \leqq \infty$. By Lemma 2.4, there exist two polynomials $q_m, r_m \in \mathbb{P}_m$ such that $\sigma_m \sim q_m$ and $\varphi < \mathcal{C} \varphi_m \sim r_m$ in $[-1, 1]$, where φ_m is defined as in (2.12). Hence, by the restricted range inequality (2.9), we have

$$\left\| P_m' \varphi \sigma_m u \right\|_p \leqq \mathcal{C} \left\| P_m' r_m q_m u \right\|_p \leqq \mathcal{C} \left\| P_m' r_m q_m u \right\|_{L^p[-a_{3m}, a_{3m}]}.$$

By Lemma 2.2, there exists $R_{lm} \in \mathbb{P}_{lm}$, satisfying (2.10) and (2.11) in $\left[-a_{4m}, a_{4m}\right]$. Note that $\left[-a_{4m}, a_{4m}\right] \subset \left[-1 + \frac{c}{m^2}, 1 - \frac{c}{m^2}\right]$ for m sufficiently large and then $r_m \sim \varphi$ in $\left[-a_{4m}, a_{4m}\right]$. Hence, by (2.10) and (2.16), it follows that

$$\left\| P_m' \varphi \sigma_m u \right\|_p \leqq \mathcal{C} \left\| P_m' q_m R_{lm} \varphi \right\|_{L^p[-a_{3m}, a_{3m}]} \tag{5.1}$$

$$\leqq \mathcal{C}\left\|\left(P_m q_m R_{lm}\right)'\varphi\right\|_{L^p[-a_{3m},a_{3m}]}+\mathcal{C}\left\|P_m q_m' R_{lm}\varphi\right\|_{L^p[-a_{3m},a_{3m}]}$$

$$+\,\mathcal{C}\left\|P_m q_m R_{lm}'\varphi\right\|_{L^p[-a_{3m},a_{3m}]}.$$

Let us consider the first summand in (5.1). Since $\varphi(x)\sim\sqrt{a_{4m}^2-x^2}$ for $x\in\left[-a_{3m},a_{3m}\right]$, using the unweighted Bernstein inequality in $\left[-a_{4m},a_{4m}\right]$, by (2.16) and (2.10), we get

(5.2)

$$\left\|\left(P_m q_m R_{lm}\right)'\varphi\right\|_{L^p[-a_{3m},a_{3m}]}\leqq \mathcal{C}\left\|\left(P_m q_m R_{lm}\right)'\sqrt{a_{4m}^2-\cdot^2}\right\|_{L^p[-a_{4m},a_{4m}]}$$

$$\leqq \mathcal{C}m\left\|P_m q_m R_{lm}\right\|_{L^p[-a_{4m},a_{4m}]}\leqq \mathcal{C}m\|P_m\sigma_m u\|_{L^p[-a_{4m},a_{4m}]}.$$

Concerning the second summand in (5.1), by (2.17) and (2.10), we have

$$\left\|P_m q_m' R_{lm}\varphi\right\|_{L^p[-a_{3m},a_{3m}]}\leqq \mathcal{C}m\left\|P_m\sigma_m R_{lm}\right\|_{L^p[-a_{3m},a_{3m}]} \tag{5.3}$$

$$\leqq \mathcal{C}m\|P_m\sigma_m u\|_{L^p[-a_{3m},a_{3m}]}.$$

Finally, for the third summand in (5.1), by (2.11) and (2.16), we get

$$\left\|P_m q_m R_{lm}'\varphi\right\|_{L^p[-a_{3m},a_{3m}]}\leqq \mathcal{C}m\|P_m q_m u\|_{L^p[-a_{3m},a_{3m}]} \tag{5.4}$$

$$\leqq \mathcal{C}m\|P_m\sigma_m u\|_{L^p[-a_{3m},a_{3m}]}.$$

Combining (5.2), (5.3) and (5.4) with (5.1), we obtain the Bernstein-type inequality (3.5) for $0<p\leqq\infty$. □

PROOF OF COROLLARY 3.4. We first prove inequality (3.6). Proceeding as in the proof of Theorem 3.3, we choose a polynomial $R_{lm}\in\mathbb{P}_{lm}$, l sufficiently large, satisfying (2.10) in $\left[-a_{5m},a_{5m}\right]$, and then we have

$$\left\|P_m'\varphi\sigma_m u\right\|_p\leqq \mathcal{C}m\left\|P_m\sigma_m R_{lm}\right\|_{L^p[-a_{4m},a_{4m}]},\qquad 0<p<\infty,$$

where

$$\sigma_m(x)=\frac{1}{\Delta_m(x)}\int_{x-\Delta_m(x)}^{x+\Delta_m(x)}\sigma(t)\,dt,\qquad \Delta_m(x)=\frac{\sqrt{1-x^2}}{m}+\frac{1}{m^2}.$$

Now, for $x\in\left[-a_{4m},a_{4m}\right]$, $\sigma_m(x)\sim\tilde{\sigma}_m(x)$, where

$$\tilde{\sigma}_m(x)=\frac{1}{\tilde{\Delta}_m(x)}\int_{x-\tilde{\Delta}_m(x)}^{x+\tilde{\Delta}_m(x)}\sigma(t)\,dt,\qquad \tilde{\Delta}_m(x)=\frac{\sqrt{a_{5m}^2-x^2}}{m}+\frac{1}{m^2}.$$

Hence, by Theorem 2.5 and inequality (2.10), we get

$$\|P'_m\varphi\sigma_m u\|_p \leqq \mathcal{C}m\|P_m\tilde{\sigma}_m R_{lm}\|_{L^p[-a_{4m},a_{4m}]}$$

$$\leqq \mathcal{C}m\|P_m\sigma R_{lm}\|_{L^p[-a_{5m},a_{5m}]} \leqq \mathcal{C}m\|P_m\sigma u\|_{L^p[-a_{5m},a_{5m}]}$$

for $0<p<\infty$, i.e. (3.6).

Finally, to prove inequality (3.7), we can proceed as in the first part of this proof, using Theorem 2.6 in place of Theorem 2.5, in order to obtain (3.6) also in the case where $p=\infty$ and σ is an A^* weight. Then, since $\sigma \leqq \mathcal{C}\sigma_m$ if σ is an A^* weight, from (3.6) we deduce (3.7) for $0<p\leqq\infty$. □

Proof of Theorem 3.5. By Lemma 2.4, there exists $q_m \in \mathbb{P}_m$ such that $\sigma_m \sim q_m$ in $(-1,1)$. Hence, by the restricted range inequality (2.9), we have

$$\|P'_m\sigma_m u\|_p \leqq \mathcal{C}\|P'_m q_m u\|_{L^p[-a_{2m},a_{2m}]}, \qquad 0<p\leqq\infty.$$

Thus, multiplying and dividing by φ, and proceeding as in the proof of Theorem 3.3, we get

$$\|P'_m\sigma_m u\|_p \leqq \frac{\mathcal{C}}{\sqrt{1-a_{2m}^2}}\|P'_m q_m u\varphi\|_{L^p[-a_{2m},a_{2m}]} \leqq \mathcal{C}\frac{m}{\sqrt{1-a_m}}\|P_m\sigma_m u\|_p,$$

which was our claim. □

Proof of Theorem 3.7. By Theorem 3.1, letting $E=\bigcup_{x_i\neq\pm1}\left[x_i-m^{-1},x_i+m^{-1}\right]$, we have

$$\|P_m\sigma_m u\|_p \leqq \mathcal{C}\|P_m\sigma_m u\|_{L^p([-a_{2m},a_{2m}]\setminus E)} \leqq \mathcal{C}m^\Gamma\|P_m\sigma_m uh\|_{L^p([-a_{2m},a_{2m}]\setminus E)},$$

where $\Gamma=\max_i\gamma_i^*$, $\gamma_i^*=\gamma_i$ if $x_i\neq\pm1$ and $\gamma_i^*=\gamma_i/(\alpha+1/2)$ if $x_i=\pm1$, taking into account that, for $x_i=\pm1$ and $x\in\left[-a_{2m},a_{2m}\right]$, by (2.5), we have

$$|x_i\pm x|^{-\gamma_i}=(1\pm x)^{-\gamma_i}\leqq\left(1-a_{2m}^2\right)^{-\gamma_i}\leqq\mathcal{C}m^{\frac{\gamma_i}{\alpha+1/2}}. \quad \square$$

Proof of Theorem 3.9. We first prove (3.16) for $q=\infty$. Taking into account that, by Theorem 3.1 and Corollary 3.2, we deduce

$$\|P_m\varphi^{1/p}\sigma_m u\|_\infty \leqq \mathcal{C}\|P_m\varphi^{1/p}\sigma_m u\|_{L^\infty[a_{3m},a_{3m}]},$$

we can assume $x\in[-a_{3m},a_{3m}]$, with $a_{3m}=a_{3m}(u)$. Then, by the equality

$$\int_a^b f(t)\,dt=(b-a)\begin{Bmatrix}f(a)\\f(b)\end{Bmatrix}+\int_a^b\begin{Bmatrix}(b-t)\\(a-t)\end{Bmatrix}f'(t)\,dt,$$

since $\varphi(x) \sim \varphi(t)$ for $t \in I_x := \big[x - \varphi(x)/m, x + \varphi(x)/m\big]$, we have

$$(5.5) \qquad |P_m(x)\varphi(x)| \leqq \mathcal{C}m\Big[\int_{I_x} |P_m(t)|\,dt + \frac{\varphi(x)}{m}\int_{I_x} |P_m'(t)|\,dt\Big]$$

$$\leqq \mathcal{C}m\Big[\int_{I_x} |P_m(t)|\,dt + \frac{1}{m}\int_{I_x} |P_m'(t)\varphi(t)|\,dt\Big].$$

Since, by (2.6) and (2.13), $u(x) \sim u(t)$ and $\sigma_m(x) \sim \sigma_m(t)$ for $t \in I_x$, by using the Hölder inequality, we get

$$|P_m(x)\varphi(x)\sigma_m(x)u(x)|$$

$$\leqq \mathcal{C}m\left(\frac{\varphi(x)}{m}\right)^{1-1/p}\left[\left(\int_{I_x} |P_m\sigma_m u|^p(t)\,dt\right)^{1/p} + \frac{1}{m}\left(\int_{I_x} |P_m'\varphi\sigma_m u|^p(t)\,dt\right)^{1/p}\right]$$

$$\leqq \mathcal{C}m\left(\frac{\varphi(x)}{m}\right)^{1-1/p}\left[\left(\int_{-1}^{1} |P_m\sigma_m u|^p(t)\,dt\right)^{1/p} + \frac{1}{m}\left(\int_{-1}^{1} |P_m'\varphi\sigma_m u|^p(t)\,dt\right)^{1/p}\right].$$

Hence, by using the Bernstein-type inequality (3.5) and inequality (5.5), we obtain

$$\|P_m\varphi^{1/p}\sigma_m u\|_\infty \leqq \mathcal{C}m^{1/p}\|P_m\sigma_m u\|_p,$$

i.e. (3.16) for $q = \infty$. Otherwise, if $1 \leqq p < q < \infty$, it follows that

$$\big\|P_m\varphi^{\frac{1}{p}-\frac{1}{q}}\sigma_m u\big\|_q^q \leqq \|P_m\varphi^{1/p}\sigma_m u\|_\infty^{q-p}\|P_m\sigma_m u\|_p^p \leqq \mathcal{C}m^{\frac{q-p}{p}}\|P_m\sigma_m u\|_p^q,$$

i.e. (3.16).

Let us now prove inequality (3.15) for $q = \infty$. Using Lemma 2.4 and Theorem 3.1, by (3.16) and (2.5), we have

$$\|P_m\sigma_m u\|_\infty \leqq \mathcal{C}\|P_m\sigma_m u\|_{L^\infty[-a_{2m},a_{2m}]}$$

$$\leqq \mathcal{C}\left(\frac{1}{\sqrt{1-a_{2m}}}\right)^{1/p}\|P_m\varphi^{1/p}\sigma_m u\|_\infty \leqq \mathcal{C}\left(\frac{m}{\sqrt{1-a_m}}\right)^{1/p}\|P_m\sigma_m u\|_p.$$

Proceeding as before, for $1 \leqq p < q < \infty$, we get

$$\|P_m\sigma_m u\|_q^q \leqq \|P_m\sigma_m u\|_\infty^{q-p}\|P_m\sigma_m u\|_p^p$$

$$\leqq \mathcal{C}\left(\frac{m}{\sqrt{1-a_m}}\right)^{\frac{q-p}{p}}\|P_m\sigma_m u\|_p^q \leqq \mathcal{C}\left(\frac{m}{\sqrt{1-a_m}}\right)^{\frac{q-p}{p}}\|P_m\sigma_m u\|_p^q,$$

which completes the proof. □

Proof of Theorem 4.3. Let us denote by $\{P_m\}_m$, where $P_m \in \mathbb{P}_m$, a sequence of polynomials of quasi best approximation for $f \in L_u^p$, $1 \leqq p < \infty$. By Theorem 4.2, the assumption (4.5) implies $\lim_m E_m(f)_{u,p} = 0$. Then the equalities

$$(5.6) \qquad (f - P_m)u = \lim_{n\to\infty}\left(P_{2^n m} - P_m\right)u$$

$$= \lim_{n\to\infty}\left\{\sum_{k=0}^{n-1}\left(P_{2^{k+1}m} - P_{2^k m}\right)u\right\} = \sum_{k=0}^{\infty}\left(P_{2^{k+1}m} - P_{2^k m}\right)u$$

hold a.e. in $(-1,1)$.

By the Nikolskii-type inequality (3.19), with $\eta = (2\alpha+2)/(2\alpha+1)$, and the restricted range inequality (2.9), and using the inequality (see [14])

$$(5.7) \qquad \widetilde{E}_m(f)_{u,p} = \inf_{P_m\in\mathbb{P}_m}\left\|(f-P_m)u\right\|_{L^p[-a_m,a_m]} \leqq \mathcal{C}\,\Omega_\varphi^r\left(f,\frac{1}{m}\right)_{u,p},$$

we get

$$\left\|\left(P_{2^{k+1}m} - P_{2^k m}\right)u\right\|_\infty \leqq \mathcal{C}\left(2^{k+1}m\right)^{\eta/p}\left\|\left(P_{2^{k+1}m} - P_{2^k m}\right)u\right\|_p$$

$$\leqq \mathcal{C}\left(2^{k+1}m\right)^{\eta/p}\left\|\left(P_{2^{k+1}m} - P_{2^k m}\right)u\right\|_{L^p[-a_{2^{k+1}m},a_{2^{k+1}m}]}$$

$$\leqq \mathcal{C}\left(2^k m\right)^{\eta/p}\Omega_\varphi^r\left(f,\frac{1}{2^k m}\right)_{u,p},$$

in analogy with (5.7). Hence, since

$$1 = \frac{1}{2^k m}\int_{\frac{1}{2^{k+1}m}}^{\frac{1}{2^k m}}\frac{dt}{t^2} \leqq 2\int_{\frac{1}{2^{k+1}m}}^{\frac{1}{2^k m}}\frac{dt}{t}$$

we get

$$(5.8) \qquad \sum_{k=0}^{\infty}\left\|\left(P_{2^{k+1}m} - P_{2^k m}\right)u\right\|_\infty \leqq \mathcal{C}\sum_{k=0}^{\infty}\left(2^k m\right)^{\eta/p}\Omega_\varphi^r\left(f,\frac{1}{2^k m}\right)_{u,p}$$

$$\leqq \mathcal{C}\sum_{k=0}^{\infty}\left(2^k m\right)^{\eta/p}\int_{\frac{1}{2^{k+1}m}}^{\frac{1}{2^k m}}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t}\,dt$$

$$\leqq \mathcal{C}\sum_{k=0}^{\infty}\int_{\frac{1}{2^{k+1}m}}^{\frac{1}{2^k m}}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\eta/p}}\,dt \leqq \mathcal{C}\int_0^{\frac{1}{m}}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\eta/p}}\,dt.$$

Therefore, by (4.5), we deduce that, in (5.6), the series is uniformly convergent (in particular $f\in C_u$) and the equalities hold everywhere in $(-1,1)$. Thus, by (5.6) and (5.8), we obtain (4.6).

In order to prove inequality (4.7), we denote by $P_m\in\mathbb{P}_m$ a polynomial of quasi best approximation for $f\in L^p_u$. Since the equivalence

$$\omega_\varphi^r\left(f,\frac{1}{m}\right)_{w,p}\sim\inf_{P_m\in\mathbb{P}_m}\left\{\left\|(f-P_m)w\right\|_p+\frac{1}{m^r}\left\|P_m^{(r)}\varphi^r w\right\|_p\right\}$$

holds true for any $1\leqq p\leqq\infty$ (see [14]), we have

$$\Omega_\varphi^r\left(f,\frac{1}{m}\right)_{u,p}\leqq\mathcal{C}\left\{\left\|(f-P_m)u\right\|_p+\frac{1}{m^r}\left\|P_m^{(r)}\varphi^r u\right\|_p\right\}.$$

Moreover, proceeding as in [5, p. 99] and as was done [14] to prove the weak Jackson-type inequality (4.4), one can show that

$$\left\|P_m^{(r)}\varphi^r u\right\|_p\leqq\mathcal{C}\,m^r\int_0^{\frac{1}{m}}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t}\,dt.$$

Hence, by (3.19), we obtain

$$\Omega_\varphi^r\left(f,\frac{1}{m}\right)_{u,\infty}\leqq\mathcal{C}\left\{\left\|(f-P_m)u\right\|_\infty+\frac{m^{\eta/p}}{m^r}\left\|P_m^{(r)}\varphi^r u\right\|_p\right\}$$

$$\leqq\mathcal{C}\left\{E_m(f)_{u,\infty}+\int_0^{\frac{1}{m}}\frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+\eta/p}}\,dt\right\},$$

and then (4.7), by (4.6).

Finally, (4.8) follows by applying (4.6) with $m=r+1$. In fact, denoting by $P_{r+1}\in\mathbb{P}_{r+1}$ a polynomial of quasi best approximation for $f\in L^p_u$ and using (3.19) with $m=r+1$, we have

$$\|fu\|_\infty\leqq\left\|(f-P_{r+1})u\right\|_\infty+\|P_{r+1}u\|_\infty$$

$$\leqq\left\|(f-P_{r+1})u\right\|_\infty+\mathcal{C}\|P_{r+1}u\|_p\leqq\mathcal{C}\left\{E_{r+1}(f)_{u,\infty}+\|fu\|_p\right\},$$

where $\mathcal{C}$ depends only on r. This completes the proof, taking into account (4.6). □

We omit the proof of Theorem 4.4, which follows by similar arguments of the previous one, using inequality (3.20) in place of (3.19).

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