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A NUMERICAL SCHEME FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH DISTRIBUTIONAL DRIFT

TIZIANO DE ANGELIS, MAXIMILIEN GERMAIN, ELENA ISSOGLIO

ABSTRACT. In this paper we introduce a scheme for the numerical solution of one-dimensional stochastic differential equations (SDEs) whose drift belongs to a fractional Sobolev space of negative regularity (a subspace of Schwartz distributions). We obtain a convergence rate in a suitable L^1 -norm and, as a by-product, a convergence rate for a numerical scheme applied to SDEs with drift in L^p -spaces with $p \in (1, \infty)$.

1. INTRODUCTION

The aim of our paper is to obtain a numerical algorithm (and its convergence rate) capable of approximating the solution of a one-dimensional SDE of the form

$$(1) \quad dX_t = b(t, X_t)dt + dW_t, \quad X_0 = x, \quad t \in [0, T],$$

where W is a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and b is a distributional drift. In particular, $b(t)$ takes values in a fractional Sobolev space of negative order (defined in Section 2.1) for each $t \in [0, T]$, and $t \mapsto b(t)$ is Hölder continuous (i.e., $b \in \mathcal{C}^\kappa([0, T]; H_{\tilde{q}_0, q_0}^{-\beta_0})$ for $\kappa \in (1/2, 1)$, $\beta_0 \in (0, 1/4)$ and suitable \tilde{q}_0 and q_0). Existence and uniqueness of solutions for d -dimensional versions of (1) were first derived in Flandoli et al. [9], where the authors give a mathematical meaning to the term $\int_0^t b(s, X_s)ds$ by introducing the concept of *virtual solution*. The latter is needed since pointwise evaluation of $b(t, \cdot)$ is meaningless. Further theoretical work on equations of a similar kind can be found, for example, in Cannizzaro and Chouk [4], Delarue and Diel [6], Flandoli et al. [10], Issoglio and Jing [17], Issoglio and Russo [18]. Besides their theoretical interest in the context of regularisation by noise, these singular SDEs usually provide models for random irregular media. For example, Russo and Trutnau [37] use them in the study of singular Stochastic Partial Differential Equations (SPDEs). Other works such as [4, 6] explain how certain SDEs with distributional drift can be used to describe the infinitesimal behaviour of the so-called polymer measure (a singular measure on the space of continuous functions) with links to the celebrated KPZ equation. Another example of SDE with distributional drift is contained in Hu et al. [15] which studies the so-called Brox diffusion, introduced by Brox [2] as an example of random process in a random medium. In the framework of [15] the drift has the regularity of the distributional derivative of a two-sided Brownian motion.

For practical implementation, mathematical models as the ones mentioned above require numerical schemes designed to handle distributional drifts. Our work provides results and

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tools in that direction. We devise a 2-step algorithm for the numerical solution of (1): we first regularise the drift and then apply Euler-Maruyama scheme. This produces a sequence $(X^N)_{N \geq 1}$ of solutions of SDEs with a smooth drift and a sequence $(X^{N,m})_{m \geq 1}$ of corresponding Euler-Maruyama approximations for each $N \geq 1$. We prove in Theorem 3.5 that the scheme converges to the original virtual solution of (1) and obtain a strong L^1 -rate of convergence when we let $(N, m) \rightarrow \infty$ simultaneously, i.e., we obtain a rate of convergence for the limit

$$\lim_{(N,m) \rightarrow \infty} \sup_{0 \leq t \leq T} \mathbb{E} \left[|X_t - X_t^{N,m}| \right].$$

In order to regularise the drift we first pick a sequence of functions $(b^N)_{N \geq 1}$ that converges to b in the appropriate norm for the fractional Sobolev space of negative regularity (i.e., it converges in $\mathcal{C}^\kappa([0, T]; H_{q_0, q_0}^{-\beta_0})$). Then, we further mollify the functions b^N by convolution with the heat kernel (we refer to it as *randomisation* procedure). The first step removes the difficulty of working with distributions, while the randomisation allows us to control the convergence rate of the overall scheme as $(N, m) \rightarrow \infty$ simultaneously. As explained in detail at the beginning of Section 6, this approach gives us a better convergence rate than the one we would obtain if we omitted the randomisation procedure and relied upon known (tight) bounds in Euler-Maruyama schemes for SDEs with bounded measurable drift.

Due to the distributional nature of our drift, the actual numerical implementation of the scheme is non-trivial and, in particular, the choice of the approximating functions b^N , in the first step of the algorithm, needs to be addressed carefully. In general, an explicit representation of b can be provided in terms of an infinite sum of Haar wavelets (see Appendix A) and we construct the sequence $(b^N)_{N \geq 1}$ by suitably truncating such series representation. The main reasons for this choice are: (i) these wavelets form a basis for the fractional Sobolev spaces of negative order which are needed to accommodate the original drift b ; (ii) they enjoy the so-called *multi-resolution property*, which improves the computational efficiency of the algorithm; (iii) since Haar wavelets are piecewise constant functions, their convolution with the heat kernel only requires knowledge of the cumulative Gaussian distribution, hence requiring no additional computational effort for our randomisation procedure. Crucially, thanks to property (i) above, we are able to determine the convergence rate of b^N to b (see Proposition 3.3).

A special case of distributional drift is obtained when b is of Dirac-delta type. That leads to one-dimensional SDEs with local-time and the celebrated skew Brownian motion (see Harrison and Shepp [14]; see also Étoré and Martinez [7] for the time-inhomogeneous case). Such SDEs have been widely studied in the literature, including several works on numerical approximations (see, e.g., Étoré and Martinez [8] and various contributions by Lejay and co-authors [24, 25, 26]). Properties of the transition density and resolvent of the process, together with links to Itô and McKean's theory of one-dimensional diffusions ([20]), enable efficient numerical methods. Those methods cannot be applied in our setting, where the process X solution of (1) is not necessarily a semi-martingale, as shown in Flandoli et al. [11, Cor. 5.11].

Except for the case of skew diffusions, our work seems to be the first to address numerical methods for a class of SDEs whose drift is merely a distribution. This advancement on all the existing results hinges on the concept of virtual solution given by Flandoli et al. [9], which links the SDE in (1) to a class of partial differential equations (PDEs) with distributional drift studied in Issoglio [16]. It is worth emphasising that our algorithm *does not* require a numerical solution of the PDE and instead it deals directly with the SDE in (1). Hence, the methods that we use here can be adopted to complement/extend the existing studies on numerical schemes for SDEs whose drift is a *function* with low regularity.

The literature on Euler-Maruyama approximation of SDEs whose drift is some function with low regularity is very vast and here we only provide a short overview. Early contributions are due to Gyöngy and Krylov [12] who obtain convergence in probability for SDEs with continuous coefficients. A few years later Yan [45] proves weak convergence for SDEs with particular discontinuities of the coefficients and an L^1 -rate of convergence under the assumption of a Lipschitz-continuous drift and a Hölder-continuous diffusion coefficient, in a one-dimensional setting. Also Halidias and Kloeden [13] prove strong convergence in L^2 (but with no rate) when the coefficients allow certain types of discontinuity.

More recent results include work by Müller-Gronbach and Yaroslavtseva [31], who obtain an L^p -rate of $1/2$ (for any $p \geq 1$) for one-dimensional SDEs with discontinuous drift, and work by Neuenkirch et al. [33] where analogous results are obtained in a multi-dimensional setting with respect to an L^2 -norm. Neuenkirch and Szölgényi [32] instead find an L^2 -rate of up to $3/4$ for one-dimensional diffusions with possibly discontinuous drift (with Sobolev-Slobodeckij type regularity). Further related results can also be found in Leobacher and Szölgényi [27] where an L^2 -rate of convergence of $1/2$ is obtained for (possibly degenerate) multi-dimensional SDEs. Notice that in [27] the Euler-Maruyama scheme is applied to a process obtained as a suitable transformation of the solution of the SDE. Similar ideas were also used in another paper by the same authors ([28]) to find an L^2 -rate of convergence of $1/4$ but, differently from [27], the convergence in [28] is for the approximation of the original SDE. Numerical schemes for non-degenerate SDEs with irregular coefficients are also addressed in works by Ngo and Taguchi [34] (multi-dimensional setting, rate $1/4$) and [35] (one-dimensional setting, rate $1/2$).

Our approach is close in spirit to the one adopted by Dareiotis and Gerencsér [5], who use the regularising effect of the Brownian noise to obtain a strong convergence rate of $1/2$ for multidimensional SDEs with continuous drift and, in the one-dimensional case, for SDEs with bounded drift. We discuss extensively differences between their approach and ours at the beginning of Section 6. Prior to their work, Menoukeu-Pamen and Taguchi [29] had obtained strong rate of convergence in L^p of order $p\beta/2$ for d -dimensional SDEs with β -Hölder continuous coefficients. Finally, we would like also to mention a new approach developed by Butkovsky et al. [3], who use regularisation by noise and a so-called stochastic sewing lemma to obtain convergence rates for SDEs driven by fractional Brownian motion and a convergence rate of $1/2$ for SDEs with continuous drift and multiplicative Brownian noise.

There are also numerous results on weak convergence of Euler-Maruyama approximation of SDEs, however a detailed review falls outside the scopes of our paper. For example, when coefficients are smooth, convergence with rate up to 1 was obtained by Bally and Talay [1] (also work by Mikulevicius and Platen [30] contains further results in that direction). In the case of irregular drift, a scheme in two steps is analysed by Kohatsu-Higa et al. [22]. They first regularise the drift of their SDE and then apply Euler-Maruyama scheme to the more regular process and obtain a rate of weak convergence.

We note that a direct comparison of the rate we obtain in the case of distributional drift and the rates obtained in the various papers mentioned above is not necessarily meaningful: the methods used in those papers require that the drift be a function and do not allow easy extensions to the distributional case. Finally, it is worth noticing that our results also apply to SDEs with drift in L^p -spaces with $p \in (1, \infty)$ (Remark 3.6) for which no rate is known yet.

The paper is organised as follows. In Section 2.1 we introduce the necessary notation, including the fractional Sobolev spaces of negative order that our drift belongs to. Then in Section 2.2 we introduce the numerical scheme. In Section 3 we present the main results of the paper, whose proofs are then provided in Sections 5 and 6. Background material on SDEs with distributional drift, which is needed to understand our arguments of proof, is presented in Section 4. The paper is completed by a technical appendix that accounts for important

properties of Haar wavelets and a short appendix on standard estimates for the (killed) heat semigroup.

2. SETTING AND NUMERICAL SCHEME

2.1. Notation. Here we introduce the functional analytic framework needed for the well-posedness of equation (1). Throughout the paper we will use ∇ and Δ for the spatial gradient and Laplacian of a function, respectively, and ∂_t for its partial derivative with respect to time.

For any Banach space $(B, \|\cdot\|_B)$ we denote by $\mathcal{C}([0, T]; B)$ the space of B -valued continuous functions of time. This is again a Banach space when endowed with the norm $\|f\|_{\infty, B} = \sup_{0 \leq t \leq T} \|f(t)\|_B$. For future reference we also introduce on $\mathcal{C}([0, T]; B)$ the family of equivalent norms

$$(2) \quad \|f\|_{\infty, B}^{(\rho)} := \sup_{0 \leq t \leq T} e^{-\rho t} \|f(t)\|_B, \quad \text{for } \rho \geq 0.$$

For $\alpha \in (0, 1)$ we introduce the subspace $\mathcal{C}^\alpha([0, T]; B)$ of functions $f \in \mathcal{C}([0, T]; B)$ such that

$$(3) \quad [f]_{\alpha, B} := \sup_{s \neq t \in [0, T]} \frac{\|f(t) - f(s)\|_B}{|t - s|^\alpha} < \infty.$$

This is also a Banach space when endowed with the norm $\|f\|_{\alpha, B} = \|f\|_{\infty, B} + [f]_{\alpha, B}$.

For $1 \leq r \leq \infty$ we have the usual $L^r(\mathbb{R})$ -spaces endowed with the norms $\|\cdot\|_{L^r}$ and we use the short-hand notation L^r . We denote by $\mathcal{C}^{0,0}(\mathbb{R})$ and $\mathcal{C}^{1,0}(\mathbb{R})$ the closure of the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions with respect to the norms $\|f\|_{L^\infty}$ and $\|f\|_{L^\infty} + \|\nabla f\|_{L^\infty}$, respectively. For simplicity of notation we just write $\mathcal{C}^{0,0}$ and $\mathcal{C}^{1,0}$. Further, we define the space of continuous functions (respectively, continuously differentiable functions) which are α -Hölder continuous (respectively, with α -Hölder continuous first derivatives) for $0 < \alpha < 1$, that is the spaces

$$\begin{aligned} \mathcal{C}^{0,\alpha}(\mathbb{R}) &:= \{f \in \mathcal{C}^{0,0}(\mathbb{R}) \text{ such that } \|f\|_{\mathcal{C}^{0,\alpha}} < \infty\}, \\ \mathcal{C}^{1,\alpha}(\mathbb{R}) &:= \{f \in \mathcal{C}^{1,0}(\mathbb{R}) \text{ such that } \|f\|_{\mathcal{C}^{1,\alpha}} < \infty\}, \end{aligned}$$

where the norms are defined as

$$\begin{aligned} \|f\|_{\mathcal{C}^{0,\alpha}} &:= \|f\|_{L^\infty} + \sup_{x \neq y \in \mathbb{R}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}, \\ \|f\|_{\mathcal{C}^{1,\alpha}} &:= \|f\|_{L^\infty} + \|\nabla f\|_{L^\infty} + \sup_{x \neq y \in \mathbb{R}} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha}. \end{aligned}$$

For simplicity of notation we write $\mathcal{C}^{0,\alpha}$ and $\mathcal{C}^{1,\alpha}$ instead of $\mathcal{C}^{0,\alpha}(\mathbb{R})$ and $\mathcal{C}^{1,\alpha}(\mathbb{R})$, respectively.

For all $s \in \mathbb{R}$ and $r \in (1, \infty)$ we denote by $H_r^s(\mathbb{R})$ the fractional Sobolev spaces (or Bessel-potential spaces) defined as the image of $L^r(\mathbb{R})$ through fractional powers of $A := I - \frac{1}{2}\Delta$, i.e., $H_r^s(\mathbb{R}) := A^{-s/2}(L^r(\mathbb{R}))$ (for more details on fractional powers of A see [43, Remark 1.2]). This representation corresponds to

$$H_r^s(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) \mid \mathfrak{F}^{-1} \left((1 + |\xi|^2)^{\frac{s}{2}} \mathfrak{F} f \right) \in L^r \right\},$$

where \mathfrak{F} is the Fourier transform and \mathcal{S}' the space of Schwartz distributions ([40, Sec. 2.2.2, Eq. (11)]). For instance, $H_r^s(\mathbb{R})$ with $s = \alpha - 1$ contains the distributional derivative of α -Hölder-continuous functions with compact support (see, e.g., [16, Prop. 4.1]). These spaces are Banach spaces when equipped with the norm

$$\|f\|_{H_r^s(\mathbb{R})} := \|A^{s/2} f\|_{L^r(\mathbb{R})},$$

and $A^{\nu/2}$ is an isomorphism from $H_r^s(\mathbb{R})$ to $H_r^{s-\nu}(\mathbb{R})$ for all $\nu \in \mathbb{R}$, see again [43].

We observe that if $s < 0$ then $H_r^s(\mathbb{R})$ does actually contain distributions, while when $s \geq 0$ it only contains (measurable) functions. For $s = 0$ we have the special case $H_r^0(\mathbb{R}) = L^r(\mathbb{R})$.

These spaces enjoy the following inclusion property: for $1 < r \leq u < \infty$ and $-\infty < t \leq s < \infty$ such that $s - \frac{1}{r} \geq t - \frac{1}{u}$ we have

$$(4) \quad H_r^s(\mathbb{R}) \subset H_u^t(\mathbb{R})$$

(see [41, Theorem 2.8.1]) and, in particular, $H_r^s(\mathbb{R}) \subset H_r^{s-\nu}(\mathbb{R})$ for $\nu > 0$. Setting $H_{p,q}^s(\mathbb{R}) := H_p^s(\mathbb{R}) \cap H_q^s(\mathbb{R})$, by interpolation we have that if $f \in H_{p,q}^s(\mathbb{R})$ then f is an element of all spaces $H_r^s(\mathbb{R})$ for $p \wedge q < r < p \vee q$. For simplicity we will use H_r^s for the space $H_r^s(\mathbb{R})$ and, analogously, we denote the associated norm by $\|\cdot\|_{H_r^s}$.

We denote by $(P_t)_{t \geq 0}$ the (killed) heat semigroup on L^r generated by $-A$, that is, the semigroup with kernel $p(t, x) = e^{-t}(2\pi t)^{-1/2} \exp\{-\frac{x^2}{2t}\}$. This is a bounded analytic semigroup and $D(A^{s/2}) = H_r^s$ for $s \in \mathbb{R}$, see [18, 39] for details. Moreover for all $t > 0$ and all $s \in \mathbb{R}$ the operator P_t maps H_r^s into itself and, furthermore, for any $f \in H_r^{-s}$ and for any $\varepsilon > 0$ there is a constant $c = c_\varepsilon > 0$ such that

$$(5) \quad \begin{aligned} \|P_t f\|_{H_r^{-s}} &\leq \|f\|_{H_r^{-s}}, \\ \|P_t f - f\|_{H_r^{-(s+\varepsilon)}} &\leq c t^{\varepsilon/2} \|f\|_{H_r^{-s}}. \end{aligned}$$

These bounds can be obtained (as in, e.g., [19]) using the following facts: (i) P_t is a contraction on L^r , (ii) for $s > 0$ the operators $A^{s/2}$ and P_t commute by [36, Theorem II.6.13] and (iii) $A^{s/2}$ is an isomorphism as mentioned above. For more details, see Appendix B.

We will also need estimates for the L^∞ -norm of $P_t f$ and of its gradient. To get those estimates, we use the fractional Morrey inequality ([39, Theorem 2.8.1, Remark 2]) which guarantees the embeddings

$$(6) \quad H_r^\nu(\mathbb{R}) \subset C^{0,\alpha}(\mathbb{R}) \quad \text{and} \quad H_r^{1+\nu}(\mathbb{R}) \subset C^{1,\alpha}(\mathbb{R}),$$

if $\alpha := \nu - 1/r > 0$. Then, using arguments similar to those for (5), combined with fractional Morrey inequality, one obtains that for all $f \in H_r^{-s}$, $t \in (0, T]$ and $\nu > 1/r$ we have

$$(7) \quad \begin{aligned} \|P_t f\|_{L^\infty} &\leq c t^{-(s+\nu)/2} \|f\|_{H_r^{-s}}, \\ \|\nabla(P_t f)\|_{L^\infty} &\leq c t^{-(1+s+\nu)/2} \|f\|_{H_r^{-s}}, \end{aligned}$$

where $c > 0$ varies from line to line and depends on T . For more details, see Appendix B.

2.2. Description of the scheme. Our numerical scheme for (1) is based on two subsequent approximations and a randomisation procedure. In order to justify pointwise evaluation of the distributional coefficient b , we approximate it by a sequence of bounded functions $(b^N)_{N \geq 1}$ that converges to b in a suitable norm (see Assumption 2). We further mollify the sequence $(b^N)_{N \geq 1}$ by convolution with the (killed) heat kernel and then we apply a generalised Euler-Maruyama scheme. The mollification can be interpreted as a randomisation procedure in space and it allows us to obtain a uniform rate of convergence for the overall scheme (see the discussion at the beginning of Section 6).

To fix notation, let us consider a bounded measurable function $b^N : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and fix a constant $\eta_N > 0$. Then the SDE

$$(8) \quad dX_t^N = (P_{\eta_N} b^N)(t, X_t^N) dt + dW_t, \quad X_0^N = x,$$

admits a unique strong solution. Note that in (8) we slightly abuse the notation, because the solution X^N depends both on N and η_N but we only indicate the dependence on N . Here and in what follows we always consider $\eta_N \rightarrow 0$ as $N \rightarrow \infty$.

Let $m \in \mathbb{N}$ and let us consider an equally-spaced partition of $[0, T]$ by setting $t_k := \frac{kT}{m}$ for $k = 0, 1, \dots, m$. Further, let us define

$$k(t) := \sup\{k : t_k \leq t\}.$$

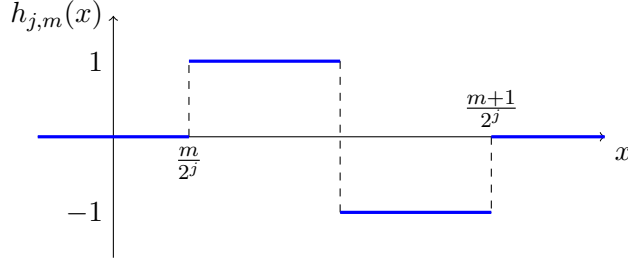


FIGURE 1. The Haar wavelet $h_{j,m}$ for $j \in \mathbb{N}$ and $m \in \mathbb{Z}$.

Then the Euler-Maruyama approximation of the solution X^N is given by

$$(9) \quad X_t^{N,m} = x + \int_0^t (P_{\eta_N} b^N) \left(t_{k(s)}, X_{t_{k(s)}}^{N,m} \right) ds + W_t,$$

and it is computed numerically according to

$$(10) \quad X_t^{N,m} = X_{t_{k(t)}}^{N,m} + (P_{\eta_N} b^N) \left(t_{k(t)}, X_{t_{k(t)}}^{N,m} \right) (t - t_{k(t)}) + \sqrt{t - t_{k(t)}} \varepsilon_k,$$

with $(\varepsilon_k)_k$ i.i.d. standard Gaussian random variables.

In general, the numerical implementation of the scheme is more complicated than a standard Euler scheme since the mollified drift $P_{\eta_N} b^N$ may not be easily computable. However, we choose b^N as a finite linear combination (with time-dependent coefficients) of Haar wavelets, which are piecewise constant functions with a very simple structure. Haar wavelets are convenient because they form an unconditional Schauder basis for the space H_r^s for $-\frac{1}{2} < s < \frac{1}{r}$ and $P_{\eta_N} b^N$ reduces to a finite sum of terms of the form

$$(11) \quad (P_{\eta_N} \mathbb{1}_{[x_1, x_2]})(x) = e^{-\eta_N} \Phi((x_2 - x)\eta_N^{-1/2}) - e^{-\eta_N} \Phi((x_1 - x)\eta_N^{-1/2}),$$

where Φ is the cumulative distribution of a standard normal and $x_1 < x_2$ are suitable real numbers (see Appendix A for details and Figure 1 for an illustration of a Haar wavelet). This procedure introduces no additional numerical complication and suggests that Haar wavelets are a natural candidate for a numerical implementation of the scheme.

3. MAIN THEORETICAL RESULTS

The main theoretical result of the paper, given in Theorem 3.5, states a rate of convergence of the numerical scheme in an L^1 -norm. To prove this result, we first find the rate of convergence of X^N to X in terms of the rate of convergence of b^N to b (Proposition 3.1 and Proposition 3.3). Then, for fixed N , we obtain the rate of convergence of the Euler-Maruyama scheme (Proposition 3.4). Finally, combining the two we obtain a global rate of convergence for the scheme (Theorem 3.5).

Let us start by introducing the main assumptions on b and b^N which are needed for the results of this section.

Assumption 1. Let $\beta_0 \in (0, \frac{1}{4})$ and $q_0 \in (4, \frac{1}{\beta_0})$ and fix $\tilde{q}_0 := (1 - \beta_0)^{-1}$. For some $\kappa \in (\frac{1}{2}, 1)$ we take $b \in \mathcal{C}^\kappa([0, T]; H_{\tilde{q}_0, q_0}^{-\beta_0})$.

Notice in particular that $b \in \mathcal{C}^\kappa([0, T]; H_2^{-\beta_0})$ by interpolation, since $1 < \tilde{q}_0 < 4/3$. For future reference we also set

$$(12) \quad \gamma_0 := 1 - \beta_0 - \frac{1}{q_0}$$

and notice that under Assumption 1 we have $\gamma_0 > \frac{1}{2}$.

Example. A simple example of a (time-homogeneous) drift b that satisfies our Assumption 1 is constructed as follows: $b(t, x) = f'(x)$, where f' is the distributional derivative of a α -Hölder continuous function f with compact support and $\alpha \in (\frac{3}{4}, 1)$. Indeed, arguing as in [16, Proposition 4.1] it can be shown that any α -Hölder continuous function f , with compact support and $\alpha \in (0, 1)$ belongs to $H_p^{\alpha'}(\mathbb{R})$ for any $1 < p < +\infty$ and $0 < \alpha' < \alpha$. Therefore, f' is a distribution in $H_p^{\alpha'-1}(\mathbb{R})$ for all $1 < p < \infty$ and $0 < \alpha' < \alpha$ and so it satisfies our Assumption 1 if $\alpha > \alpha' > \frac{3}{4}$. In that case $\beta_0 = \alpha' - 1 \in (0, \frac{1}{4})$, as needed, and $f' \in H_{\tilde{q}_0, q_0}^{-\beta_0} = H_{\tilde{q}_0}^{-\beta_0} \cap H_{q_0}^{-\beta_0}$.

For completeness, it is worth mentioning the two key steps in the proof of [16, Proposition 4.1]. Let f be any α -Hölder continuous function with compact support and $\alpha \in (0, 1)$. First, using an equivalent norm (see [42, Eq. (10.19)]) it is shown that $f \in B_{p', 2}^{\alpha'}(\mathbb{R})$ for $0 < \alpha' < \alpha$ and $p' \geq 1$, where $B_{p', 2}^{\alpha'}(\mathbb{R})$ is a Besov space. Then, the proof is completed by the embedding $B_{p', 2}^{\alpha'}(\mathbb{R}) \subset H_{p''}^{\alpha''}(\mathbb{R})$ (see [42, Eq. (11.17)]) for all $p'' > p'$ and $\alpha'' = \alpha' - \frac{1}{p'} + \frac{1}{p''}$. \square

Assumption 2. Let $(b^N)_{N \geq 1} \subset \mathcal{C}^{\frac{1}{2}}([0, T]; H_{\tilde{q}_0, q_0}^0)$ be such that

$$\lim_{N \rightarrow \infty} b^N = b \quad \text{in } \mathcal{C}^{\frac{1}{2}}([0, T]; H_{q_0}^{-\beta_0}).$$

The rate of convergence of X^N to X is given in the next proposition.

Proposition 3.1. Let Assumptions 1 and 2 hold. Take any (β, q) such that $\beta \in (\beta_0, \frac{1}{2})$ and $q_0 \geq q > \tilde{q} \geq \tilde{q}_0$, where $\tilde{q} := (1 - \beta)^{-1}$. Then, for any $1/2 < \gamma < \gamma_0$ there is a constant $C_\gamma > 0$ such that

$$(13) \quad \sup_{0 \leq t \leq T} \mathbb{E} [|X_t^N - X_t|] \leq C_\gamma \|P_{\eta_N} b^N - b\|_{\infty, H_q^{-\beta}}^{2\gamma-1}$$

as $N \rightarrow \infty$.

The proof of this result builds on a number of lemmas and we give it in Section 5 (the constant $C_\gamma > 0$ is found explicitly). It is worth noticing that on the right-hand side of (13) we use the $H_q^{-\beta}$ -norm with (β, q) possibly different from (β_0, q_0) . The reasons for this will become clear later (see in particular Proposition 3.3) and in the next remark we show that the right-hand side of (13) is well-defined.

Remark 3.2. Let us fix (β_0, q_0) according to Assumption 1 and let us pick (β, q) such that $\beta \in [\beta_0, \frac{1}{2})$ and $q_0 \geq q > \tilde{q} \geq \tilde{q}_0$, where $\tilde{q} := (1 - \beta)^{-1}$. This is always possible thanks to Assumption 1 and, as a special case, we can pick $q = 2$ as needed in Proposition 3.3 below. In this setting:

- (i) The embedding $H_{\tilde{q}_0, q_0}^{-\beta_0} \subset H_{\tilde{q}, q}^{-\beta}$ holds. Indeed $H_{\tilde{q}_0, q_0}^{-\beta_0} \subset H_{\tilde{q}_0, q_0}^{-\beta}$ by embedding of Sobolev spaces of negative order, see (4), and we also have $H_{\tilde{q}_0, q_0}^{-\beta} \subset H_{\tilde{q}, q}^{-\beta}$ by interpolation of L^p -spaces. Combining the above we have

$$b \in \mathcal{C}^\kappa([0, T]; H_{\tilde{q}_0, q_0}^{-\beta_0}) \subset \mathcal{C}^\kappa([0, T]; H_{\tilde{q}, q}^{-\beta})$$

and, in particular, $b \in \mathcal{C}^\kappa([0, T]; H_q^{-\beta})$. Thanks to (5) also $P_{\eta_N} b$ belongs to the same space as b . Similarly, $b^N, P_{\eta_N} b^N \in \mathcal{C}^{\frac{1}{2}}([0, T]; H_{\tilde{q}_0, q_0}^0)$.

- (ii) For $(b^N)_{N \geq 1}$ as in Assumption 2 we have $P_{\eta_N} b^N \rightarrow b$ in $\mathcal{C}([0, T]; H_q^{-\beta})$ as $N \rightarrow \infty$. Indeed, $b, P_{\eta_N} b$ and $P_{\eta_N} b^N$ belong to $H_q^{-\beta}$ due to the item above and, thanks to (5),

we have

$$\begin{aligned} \|P_{\eta_N} b^N - b\|_{\infty, H_q^{-\beta}} &\leq \|P_{\eta_N} (b^N - b)\|_{\infty, H_q^{-\beta}} + \|P_{\eta_N} b - b\|_{\infty, H_q^{-\beta}} \\ &\leq \|b^N - b\|_{\infty, H_q^{-\beta}} + \|P_{\eta_N} b - b\|_{\infty, H_q^{-\beta}} \\ &\leq \|b^N - b\|_{\infty, H_q^{-\beta}} + c \eta_N^{\frac{\beta-\beta_0}{2}} \|b\|_{\infty, H_q^{-\beta_0}}. \end{aligned}$$

If $\beta > \beta_0$, the last term clearly goes to zero as $\eta_N \rightarrow 0$. Moreover, since $-\beta_0 - \frac{1}{q_0} \geq -\beta - \frac{1}{q}$ by assumption, using (4) we have $H_{q_0}^{-\beta_0} \subset H_q^{-\beta}$. Hence

$$\|b^N - b\|_{\infty, H_q^{-\beta}} \leq c \|b^N - b\|_{\infty, H_{q_0}^{-\beta_0}} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

In order to obtain a convergence rate in our scheme as we let $(N, m) \rightarrow \infty$ simultaneously we need to write the right-hand side of (13) explicitly in terms of N . For that we define a specific sequence $(b^N)_{N \geq 1}$ that satisfies Assumption 2. In particular, the approximating sequence $(b^N)_{N \geq 1}$ is defined via a suitable truncation of the series expansion of b in Haar wavelets. Let

$$\{h_{j,m}, j \in \mathbb{N} \cup \{-1\}, m \in \mathbb{Z}\}$$

be a system of Haar wavelets on \mathbb{R} (Definition A.1). Since $b(t) \in H_q^{-\beta}$ for any $\beta \in [\beta_0, \frac{1}{2}]$, $q \in [\tilde{q}_0, q_0]$ and all $t \in [0, T]$, by Remark 3.2 part (i), then we have (see [43, Theorem 2.9] or (69) and Theorem A.2)

$$b(t) = \sum_{j=-1}^{+\infty} \sum_{m \in \mathbb{Z}} \mu_{j,m}(t) 2^{-j(-\beta-\frac{1}{q})} h_{j,m},$$

with $\mu_{j,m}(t)$ defined via the dual pairing of $b(t)$ and $h_{j,m}$ as

$$(14) \quad \mu_{j,m}(t) = 2^{j(-\beta-\frac{1}{q}+1)} \langle b(t), h_{j,m} \rangle,$$

for each $t \in [0, T]$. We remark that $\mu_{j,m}(t) = \mu_{j,m}(t; \beta, q)$ only depends on β and q via the exponential term and not via the dual pairing (see Remark A.4). Later on we will use

$$(15) \quad \mu_{j,m}(t; \beta, q) = 2^{-j(\beta-\beta_0)} \mu_{j,m}(t; \beta_0, q).$$

From [43, Theorem 2.9 and eq. (2.114), Sec. 2.2.3] we have

$$(16) \quad \|b\|_{\infty, H_q^{-\beta}} < \infty \iff \sup_{t \in [0, T]} \|\boldsymbol{\mu}(t; \beta, q) |f_{q,2}^{-}\| < \infty,$$

where $\boldsymbol{\mu}(t; \beta, q) := \{\mu_{j,m}(t; \beta, q)\}_{j,m}$ and a definition of its $f_{q,2}^{-}$ -norm with further details is provided in Appendix A.1 for completeness. In particular, if $q = 2$ we have

$$(17) \quad \sup_{t \in [0, T]} \|\boldsymbol{\mu}(t; \beta, 2) |f_{2,2}^{-}\| = \sup_{t \in [0, T]} \left[\sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} |\mu_{j,m}(t; \beta, 2)|^2 \right]^{\frac{1}{2}} < \infty.$$

Thanks to (17), for $\beta \in (\beta_0, \frac{1}{2})$ and $N \in \mathbb{N}$ fixed, we can define $\tau(N) \in \mathbb{N}$ as the smallest integer for which

$$(18) \quad \sup_{t \in [0, T]} \sum_{|m| > \tau(N)} |\mu_{j,m}(t; \beta, 2)|^2 \leq \frac{2^{-(N+1)(\beta-\beta_0)}}{N+1} \sup_{t \in [0, T]} \|\boldsymbol{\mu}(t; \beta_0, 2) |f_{2,2}^{-}\|^2,$$

for all $j = 0, \dots, N$. Then we set

$$(19) \quad b^N(t) := \sum_{j=-1}^N \sum_{m=-\tau(N)}^{\tau(N)} \mu_{j,m}(t) 2^{-j(-\beta-\frac{1}{q})} h_{j,m},$$

where for $j = -1$ the sum in m only takes the term with $m = 0$.

Clearly $b^N(t) \in H_{\tilde{q}_0, q_0}^0 \subset H_{\tilde{q}_0, q_0}^{-\beta_0}$ by construction. Since b is κ -Hölder continuous in time with values in $H_{\tilde{q}_0, q_0}^{-\beta_0}$, then also the coefficients $t \mapsto \mu_{j,m}(t)$ are κ -Hölder continuous with values in \mathbb{R} . Thus we have $b^N \in \mathcal{C}^\kappa([0, T]; H_{\tilde{q}_0, q_0}^0)$. Now we have a simple way of estimating the rate of convergence of b^N to b in the space $\mathcal{C}([0, T]; H_2^{-\beta})$, as illustrated below.

Proposition 3.3. *Let Assumption 1 hold and let the sequence $(b^N)_{N \geq 1}$ be defined as in (19). Then $(b^N)_{N \geq 1}$ satisfies Assumption 2 and for any $\beta \in (\beta_0, \frac{1}{2})$ we have*

$$(20) \quad \|b^N - b\|_{\infty, H_2^{-\beta}} \leq c 2^{-(N+1)(\beta-\beta_0)} \|b\|_{\infty, H_2^{-\beta_0}}.$$

Proof. As already observed $(b^N)_{N \geq 1} \subset \mathcal{C}^\kappa([0, T]; H_{\tilde{q}_0, q_0}^0)$ by construction. Recall also that $b(t), b^N(t) \in H_2^{-\beta}$ for all $\beta \in (\beta_0, \frac{1}{2})$ by Remark 3.2, part (i).

Thanks to (16) and (19) (see also (77) and (78) in Appendix A.1) it is immediate to see that $\|b^N(t) - b(t)\|_{H_{q_0}^{-\beta_0}}$ decreases to zero as $N \rightarrow \infty$. Moreover $t \mapsto \|b^N(t) - b(t)\|_{H_{q_0}^{-\beta_0}}$ is continuous and therefore by Dini's theorem $b^N \rightarrow b$ in $\mathcal{C}([0, T]; H_{q_0}^{-\beta_0})$ as $N \rightarrow \infty$. Thanks to the equivalence of the norms (16) it is immediate to see that

$$\sup_{t \neq s \in [0, T]} \frac{\|b^N(t) - b^N(s)\|_{H_{q_0}^{-\beta_0}}}{|t - s|^\kappa} \sim \sup_{t \neq s \in [0, T]} \frac{\|\boldsymbol{\mu}^N(t; \beta_0, q_0) - \boldsymbol{\mu}^N(s; \beta_0, q_0)\|_{f_{q_0, 2}^-}}{|t - s|^\kappa},$$

where we use the symbol “ \sim ” to indicate equivalence of the norms and $\boldsymbol{\mu}^N(\cdot; \beta_0, q_0) = \{\mu_{j,m}(\cdot; \beta_0, q_0), j = -1, \dots, N, m = -\tau(N), \dots, \tau(N)\}$ contains the coefficients that appear in the expression for $b^N(\cdot)$ in (19). Since the series expansion of b^N contains a finite subset of the terms in the series expansion of b , then

$$\begin{aligned} & \sup_{t \neq s \in [0, T]} \frac{\|\boldsymbol{\mu}^N(t; \beta_0, q_0) - \boldsymbol{\mu}^N(s; \beta_0, q_0)\|_{f_{q_0, 2}^-}}{|t - s|^\kappa} \\ & \leq \sup_{t \neq s \in [0, T]} \frac{\|\boldsymbol{\mu}(t; \beta_0, q_0) - \boldsymbol{\mu}(s; \beta_0, q_0)\|_{f_{q_0, 2}^-}}{|t - s|^\kappa} \leq c [b]_{\kappa, H_{q_0}^{-\beta_0}}, \end{aligned}$$

for some constant $c > 0$ that arises from the equivalence of norms. Then $(b^N)_{N \geq 1}$ is also a bounded subset of $\mathcal{C}^\kappa([0, T]; H_{q_0}^{-\beta_0})$. Recalling that $\kappa \in (\frac{1}{2}, 1)$ we can conclude that $b^N \rightarrow b$ in $\mathcal{C}^{\frac{1}{2}}([0, T]; H_{q_0}^{-\beta_0})$ as $N \rightarrow \infty$ since the embedding $\mathcal{C}^{\frac{1}{2}} \subset \mathcal{C}^\kappa$ is compact. Hence, $(b^N)_{N \geq 1}$ satisfies Assumption 2.

It remains to prove (20). For $q = 2$ and a suitable constant $c > 0$, using (15), (17) and (18) we have

$$\begin{aligned} & \|b^N(t) - b(t)\|_{H_2^{-\beta}}^2 \\ & \leq c \left(\sum_{j > N} \sum_{m \in \mathbb{Z}} |\mu_{j,m}(t; \beta, 2)|^2 + \sum_{j=0}^N \sum_{|m| > \tau(N)} |\mu_{j,m}(t; \beta, 2)|^2 \right) \\ & \leq c \left(2^{-2(N+1)(\beta-\beta_0)} \sum_{j > N} \sum_{m \in \mathbb{Z}} |\mu_{j,m}(t; \beta_0, 2)|^2 + 2^{-2(N+1)(\beta-\beta_0)} \|\boldsymbol{\mu}(t; \beta_0, 2)\|_{f_{2,2}^-}^2 \right) \\ & \leq 2c 2^{-2(N+1)(\beta-\beta_0)} \|\boldsymbol{\mu}(t; \beta_0, 2)\|_{f_{2,2}^-}^2. \end{aligned}$$

Taking supremum over $t \in [0, T]$ and recalling (16) we conclude by incorporating the factor 2 into the constant c . \square

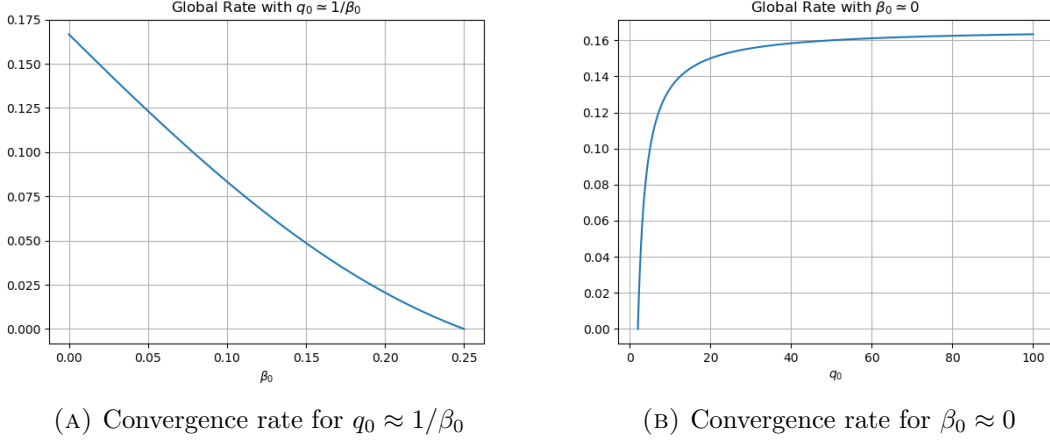


FIGURE 2. Convergence rate of the scheme as function of the parameter $\beta_0 \in (0, 1/4)$ (left panel) and $q_0 > 4$ (right panel) when the other parameter is fixed.

Since the drift in the SDE for X^N is Lipschitz in space and $1/2$ -Hölder continuous in time we expect a standard strong convergence rate of $1/2$ for the Euler-Maruyama scheme. This is confirmed in the next proposition where, however, we are particularly interested in the dependence of the multiplicative constants on N . By controlling those constants, later on, we will establish an overall rate of convergence for the scheme as we let $(N, m) \rightarrow \infty$ at the same time.

Proposition 3.4. *Let Assumption 1 hold and let $b^N \in \mathcal{C}^{\frac{1}{2}}([0, T]; H_{q_0, q_0}^0)$ for some fixed N . Then, as $m \rightarrow \infty$, we have*

$$(21) \quad \sup_{0 \leq t \leq T} \mathbb{E} \left[|X_t^{N, m} - X_t^N| \right] \leq C_2(N)m^{-1} + C_3(N)m^{-\frac{1}{2}}$$

with

$$(22) \quad \begin{aligned} C_2(N) &:= c \|P_{\eta_N} b^N\|_{\infty, L^\infty} \left(1 + \|\nabla(P_{\eta_N} b^N)\|_{\infty, L^\infty} \right), \\ C_3(N) &:= c' \left(\|\nabla(P_{\eta_N} b^N)\|_{\infty, L^\infty} + [P_{\eta_N} b^N]_{\frac{1}{2}, L^\infty} \right) \end{aligned}$$

and $c, c' > 0$ constants independent of (N, m) .

The proof of the proposition is given in Section 6.

Combining the results above we obtain the full convergence result, that summarises the theoretical findings in the paper.

Theorem 3.5. *Let Assumption 1 hold, let $(b^N)_{N \geq 1}$ be defined as in (19) (so that Assumption 2 holds too) and let $\theta_* := \frac{1}{2} \left[\frac{3}{4} - \beta_0(\gamma_0 - \frac{1}{2}) \right]^{-1}$ with γ_0 as in (12). Then, as $m \rightarrow \infty$, taking $\eta_N = m^{-\theta_*}$ and $N = \lfloor 2\theta_* \log_2 m \rfloor$ we have*

$$(23) \quad \sup_{0 \leq t \leq T} \mathbb{E} \left[|X_t^{N, m} - X_t| \right] \leq c_\varepsilon \left(m^{-\theta_* \left(\frac{1}{2} - \beta_0 \right) (\gamma_0 - \frac{1}{2}) + \varepsilon} \right),$$

where $\varepsilon > 0$ can be arbitrarily small and $c_\varepsilon > 0$ is a constant depending on ε .

Before proving the theorem, we offer some basic insight into the meaning of the rate in (23) (in Figure 2 we also plot two examples).

Remark 3.6.

- If $q_0 \approx 1/\beta_0$, which is the largest possible q_0 , then the rate decreases as β_0 increases, and the best rate is obtained when β_0 is close to zero. The rate is illustrated in Figure 2a as a function of $\beta_0 \in (0, \frac{1}{4})$.
- For $\beta_0 \approx 0$ we have $\tilde{q}_0 \approx 1$ and we can pick any $4 < q_0 < \infty$. Then we obtain a convergence rate for SDEs with drift in $\mathcal{C}^\kappa([0, T]; L^1 \cap L^{q_0})$ for $\kappa > \frac{1}{2}$. The rate is illustrated in Figure 2b as a function of $q_0 > 4$. While existence of strong solutions for SDEs with drift in $L^p([0, T]; L^q)$ -spaces was obtained by Krylov and Röckner in [23], we are not aware of convergence rates results for numerical schemes if $b \in \mathcal{C}^\kappa([0, T]; L^{q_0})$ and $1 < q_0 < \infty$.
- In the extreme case when $\beta_0 \approx 0$ and $q_0 \rightarrow \infty$ (i.e. $b \in \mathcal{C}^\kappa([0, T]; L^1 \cap L^\infty)$ with $\kappa > \frac{1}{2}$) we obtain a convergence rate of $m^{-1/6}$. However, if we assumed $b \in \mathcal{C}^\kappa([0, T]; L^\infty)$ from the start, we would have been able to obtain a better rate from the Euler-Maruyama scheme using ideas from Dareiotis and Genrenčsér [5] (notice though that the constants in [5, Lemma 2.2] depend exponentially on $\|b\|_{L^\infty}$). The approach we take in Section 6 allows us to avoid that the constants in Proposition 3.4 depend exponentially on the L^∞ -norm of b^N , which is essential when b is a distribution. Clearly, if $b \in \mathcal{C}^\kappa([0, T]; L^\infty)$ that caution is no longer needed.

Proof of Theorem 3.5. Fix $\varepsilon > 0$ and let $c_\varepsilon > 0$ be a constant that may vary from line to line, possibly depending on $\|b\|_{\infty, H_{\tilde{q}_0, q_0}^{-\beta_0}}$ and $\varepsilon > 0$ but independent of N and m . In the rest of this proof we will use Proposition 3.1 with $\gamma = \gamma_0 - \frac{1}{2}\varepsilon$ so that the constant C_γ is absorbed in $c_\varepsilon > 0$.

Using triangular inequality, (13) from Proposition 3.1 (with $q = 2$) and (21) from Proposition 3.4 we obtain

$$(24) \quad \sup_{0 \leq t \leq T} \mathbb{E} \left[|X_t^{N,m} - X_t| \right] \leq c_\varepsilon \left(\|P_{\eta_N}(b^N - b)\|_{\infty, H_2^{-\beta}}^{2\gamma_0 - 1 - \varepsilon} + \|P_{\eta_N} b - b\|_{\infty, H_2^{-\beta}}^{2\gamma_0 - 1 - \varepsilon} + C_2(N)m^{-1} + C_3(N)m^{-\frac{1}{2}} \right).$$

The L^∞ -norms appearing in the constants $C_2(N)$ and $C_3(N)$ in (22) can be estimated further by using (7). Recall that $(b^N)_{N \geq 1}$ fulfils Assumptions 2 thanks to Proposition 3.3. The most favourable estimates in (7) are obtained for $r = q_0$, $s = \beta_0$ and $\nu = \frac{1}{q_0} + \varepsilon$. Then, we have

$$\begin{aligned} \|P_{\eta_N} b^N\|_{\infty, L^\infty} &\leq c \eta_N^{-\frac{1}{2}(\beta_0 + \frac{1}{q_0} + \varepsilon)} \|b^N\|_{\infty, H_{q_0}^{-\beta_0}} \leq c \eta_N^{-\frac{1}{2}(1 - \gamma_0 + \varepsilon)} \|b\|_{\infty, H_{q_0}^{-\beta_0}} \\ \|\nabla(P_{\eta_N} b^N)\|_{\infty, L^\infty} &\leq c \eta_N^{-\frac{1}{2}(1 + \beta_0 + \frac{1}{q_0} + \varepsilon)} \|b^N\|_{\infty, H_{q_0}^{-\beta_0}} \leq c \eta_N^{-\frac{1}{2}(2 - \gamma_0 + \varepsilon)} \|b\|_{\infty, H_{q_0}^{-\beta_0}}, \\ [P_{\eta_N} b^N]_{\frac{1}{2}, L^\infty} &\leq c \eta_N^{-\frac{1}{2}(\beta_0 + \frac{1}{q_0} + \varepsilon)} [b^N]_{\frac{1}{2}, H_{q_0}^{-\beta_0}} \leq c \eta_N^{-\frac{1}{2}(1 - \gamma_0 + \varepsilon)} [b]_{\frac{1}{2}, H_{q_0}^{-\beta_0}}, \end{aligned}$$

where the final inequality in each of the above expressions follows from the convergence $\lim_{N \rightarrow \infty} b^N = b$ in $\mathcal{C}^{\frac{1}{2}}([0, T]; H_{q_0}^{-\beta_0})$ and we used $\beta_0 + 1/q_0 = 1 - \gamma_0$ and $1 + \beta_0 + 1/q_0 = 2 - \gamma_0$.

Using also (5) to bound the first two terms on the right-hand side of (24) (where c_ε may change from line to line) we have

$$(25) \quad \begin{aligned} & \sup_{0 \leq t \leq T} \mathbb{E} \left[|X_t^{N,m} - X_t| \right] \\ & \leq c_\varepsilon \left(\|b^N - b\|_{\infty, H_2^{-\beta}}^{2\gamma_0 - 1 - \varepsilon} + \eta_N^{(\beta - \beta_0)(\gamma_0 - \frac{1}{2}) - \varepsilon} \right. \\ & \quad \left. + \eta_N^{-\frac{1}{2}(3 - 2\gamma_0 + \varepsilon)} m^{-1} + \eta_N^{-\frac{1}{2}(2 - \gamma_0 + \varepsilon)} m^{-\frac{1}{2}} \right), \end{aligned}$$

where for the final two terms we selected the leading order in η_N by using that $\eta_N \in (0, 1)$, with no loss of generality, and $1 + \beta_0 + \frac{1}{q_0} \geq 2(\beta_0 + \frac{1}{q_0})$, since $\beta_0 \in (0, \frac{1}{4})$.

Thanks to Proposition 3.3, and abusing slightly the notation by letting $\varepsilon > 0$ vary from the first to the second inequality, we have

$$(26) \quad \|b^N - b\|_{\infty, H_2^{-\beta}}^{2\gamma_0 - 1 - \varepsilon} \leq c 2^{-(N+1)(\beta - \beta_0)(2\gamma_0 - 1 - \varepsilon)} \leq c 2^{-N\frac{1}{2}(\beta - \beta_0)(\gamma_0 - \frac{1}{2}) + \varepsilon}.$$

The aim is to let N and m diverge to infinity and $\eta_N \rightarrow 0$ at the same time. In order to do so we choose suitable N and η_N depending on m . Take $\eta_N = m^{-\theta}$ for some $\theta > 0$ to be determined. The last three terms in (25) read

$$(27) \quad m^{-\theta(\beta - \beta_0)(\gamma_0 - \frac{1}{2}) + \varepsilon} + m^{-1 + \frac{1}{2}\theta(3 - 2\gamma_0 + \varepsilon)} + m^{-\frac{1}{2} + \frac{1}{2}\theta(2 - \gamma_0 + \varepsilon)}$$

and, as $m \rightarrow \infty$, the leading terms are the first and last one. By comparing (26) and (25) we notice that there is no loss of generality in choosing $N = \lfloor 2\theta \log_2 m \rfloor$. Finally, plugging (26) and (27) back into (25), ignoring terms of lower order in m , we obtain

$$\begin{aligned} \sup_{0 \leq t \leq T} \mathbb{E} \left[|X_t^{N,m} - X_t| \right] & \leq c_\varepsilon \left(m^{-\theta(\beta - \beta_0)(\gamma_0 - \frac{1}{2}) + \varepsilon} + m^{-\frac{1}{2} + \frac{1}{2}\theta(2 - \gamma_0 + \varepsilon)} \right) \\ & \leq c_\varepsilon \left(m^{-\theta(\frac{1}{2} - \beta_0)(\gamma_0 - \frac{1}{2}) + \varepsilon} + m^{-\frac{1}{2} + \frac{1}{2}\theta(2 - \gamma_0 + \varepsilon)} \right), \end{aligned}$$

where for the second inequality we have chosen the best possible β , which is just below $\frac{1}{2}$, and with a slight abuse of notation we have allowed ε to vary from line to line. It remains to select $\theta > 0$ that gives the fastest convergence rate. Notice that the first term on the right-hand side of the expression above is decreasing in θ whereas the second one is increasing. Then the optimum is attained when the exponents are equal and we get

$$\theta_* = \frac{1}{2} \left[\frac{3}{4} - \beta_0(\gamma_0 - \frac{1}{2}) \right]^{-1},$$

as claimed. \square

Remark 3.7. *For practical use of our numerical scheme one must compute the coefficients $\mu_{j,m}$ from (14). It is shown in Theorem A.9 and Remark A.10 that such computation is very easy when $b(t, \cdot)$ is supported on a bounded interval on \mathbb{R} .*

Remark 3.8. *When implementing the scheme, the coefficients in the formula for b^N are computed offline and stored in the memory at the beginning of the algorithm. The complexity of the algorithm is determined by the number of operations involving such coefficients and the Haar functions (multiplications and summation) and by the number of time steps in the Euler scheme. In particular, we count the number of terms in the sum and we multiply that by the number of time-steps in the Euler scheme. From (10) and (19), the overall complexity of the algorithm is $O(mN\tau(N))$. Hence, by taking $N = 2\theta_* \log_2 m$ as in the statement of Theorem 3.5 we obtain a complexity of $O(m \log_2 m \tau(\lfloor 2\theta_* \log_2 m \rfloor))$. Unfortunately $\tau(N)$ can*

be difficult to compute in general but, in the special case of b supported on a bounded interval I , we have

$$b(t) = \mu_0(t)h_0 + \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} \mu_{j,m}(t)2^{-j\left(-\beta-\frac{1}{q}\right)} h_{j,m}$$

(see (73) in Appendix A, where we take $I = (0, 1)$ for simplicity and with no loss of generality). Then we can define b^N as

$$b^N(t) = \mu_0(t)h_0 + \sum_{j=0}^N \sum_{m=0}^{2^j-1} \mu_{j,m}(t)2^{-j\left(-\beta-\frac{1}{q}\right)} h_{j,m},$$

and, in the proof of Proposition 3.3, we have

$$\|b^N(t) - b(t)\|_{H_2^{-\beta}}^2 \leq c \sum_{j>N} \sum_{m=0}^{2^j-1} |\mu_{j,m}(t; \beta, 2)|^2 \leq c 2^{-2(N+1)(\beta-\beta_0)} \|\boldsymbol{\mu}(t; \beta_0, 2)|f_{2,2}^-\|^2.$$

In that case the complexity is $O(m2^N)$ hence by taking $N = \lfloor 2\theta^* \log_2 m \rfloor$ we obtain a complexity of $O(m^2)$. Notice that the computation of the semigroup in (11) does not modify the complexity.

4. BACKGROUND MATERIAL ON VIRTUAL SOLUTIONS

As anticipated, the proofs of Proposition 3.1 and Proposition 3.4 rely upon a few technical lemmas. To set out clearly our arguments and keep the exposition self-contained it is convenient to review and complement some results from [9].

We will work in the framework of [9] but we restrict our attention to $[0, T] \times \mathbb{R}$ rather than working with $[0, T] \times \mathbb{R}^d$ as in the original paper. Throughout this section we make the following standing assumption.

Assumption 3. Let $\beta \in (0, \frac{1}{2})$, fix $\tilde{q} := \frac{1}{1-\beta}$ and let $q \in (\tilde{q}, \frac{1}{\beta})$. We take $b \in \mathcal{C}([0, T]; H_{\tilde{q}, q}^{-\beta})$.

Notice that Assumption 3 is implied by Assumption 1 (with (β_0, q_0) instead of (β, q) and $q > 4$). It was shown in [9, Theorem 28] that under Assumption 3 for every $x \in \mathbb{R}$ there exists a unique in law *virtual solution* of (1). A virtual solution of (1) is given in terms of a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W)$ and an \mathbb{F} -adapted, continuous stochastic process $X := (X_t)_{t \in [0, T]}$ (shortened as (X, \mathbb{F})) such that the integral equation

$$(28) \quad \begin{aligned} X_t &= x + u(0, x) - u(t, X_t) + (\lambda + 1) \int_0^t u(s, X_s) ds \\ &\quad + \int_0^t (\nabla u(s, X_s) + 1) dW_s \end{aligned}$$

holds for all $t \in [0, T]$, with probability one. Here u is the mild solution of the following parabolic Kolmogorov-type PDE

$$(29) \quad \begin{cases} \partial_t u + \frac{1}{2} \Delta u + b \nabla u - (\lambda + 1)u = -b & \text{on } [0, T] \times \mathbb{R} \\ u(T) = 0 & \text{on } \mathbb{R} \end{cases}$$

with $\lambda > 0$. The mild solution u is unique in $\mathcal{C}([0, T]; H_p^{1+\delta})$, for any $(\delta, p) \in \mathcal{K}(\beta, q)$, where the set $\mathcal{K}(\beta, q)$ is defined as

$$\mathcal{K}(\beta, q) := \{(\delta, p) \mid \beta < \delta < 1 - \beta, \frac{1}{\delta} < p < q\}.$$

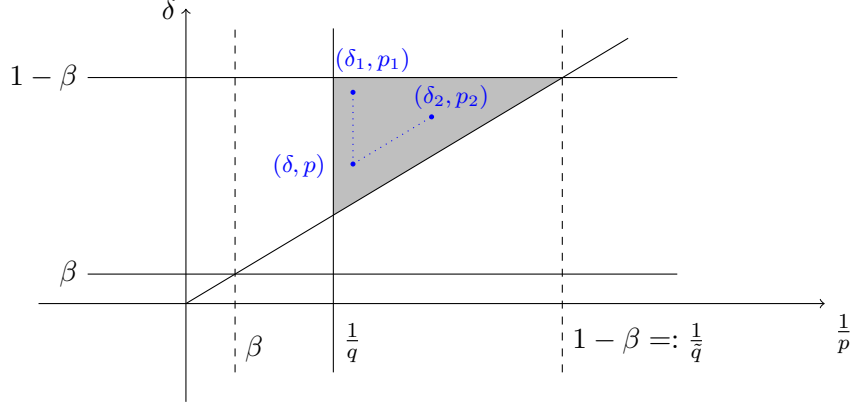


FIGURE 3. An illustration of the set $\mathcal{K}(\beta, q)$ (figure modified from [9]). Given a couple (β, q) according to Assumption 3, there exists a unique mild solution $u \in \mathcal{C}([0, T]; H_p^{1+\delta})$ for the PDE (29), for all (δ, p) in the grey triangle. For any two points $(\delta_1, p_1), (\delta_2, p_2) \in \mathcal{K}(\beta, q)$, it is always possible to find $(\delta, p) \in \mathcal{K}(\beta, q)$ such that $H_{p_1}^{1+\delta_1} \subset H_p^{1+\delta}$ and $H_{p_2}^{1+\delta_2} \subset H_p^{1+\delta}$, see dotted lines for embeddings.

The set $\mathcal{K}(\beta, q)$ is drawn in Figure 3 for the reader's convenience and it is not empty thanks to Assumption 3. Notice that the stochastic integral that appears in (28) is well-defined thanks to fractional Morrey's inequality (6).

Remark 4.1 (Uniqueness). *We remark that, thanks to the shape of $\mathcal{K}(\beta, q)$ and to the embedding (4), given two couples $(\delta_1, p_1), (\delta_2, p_2) \in \mathcal{K}(\beta, q)$, it is always possible to find $(\delta, p) \in \mathcal{K}(\beta, q)$ such that $H_{p_1}^{1+\delta_1} \subset H_p^{1+\delta}$ and $H_{p_2}^{1+\delta_2} \subset H_p^{1+\delta}$, see Figure 3. Since the solution u to (29) is unique in the space $H_p^{1+\delta}$, it follows that it must coincide with the solutions found in the spaces $H_{p_1}^{1+\delta_1}$ and $H_{p_2}^{1+\delta_2}$. Hence, the solution of (29) is unique in the whole triangle $\mathcal{K}(\beta, q)$.*

It is worth noticing that the concept of virtual solution follows a Zvonkin-type transformation based on heuristic application of Itô's formula to $u(t, X_t)$. This allows to replace the drift term $b(t, X_t)dt$ in (1) with the terms in (28) depending on u and ∇u . The reader might have noticed that the PDE (29) and the virtual SDE (28) depend on an extra parameter λ , while the original SDE (1) does not. This is due to a technical step in the proof, that leads to good properties of u . However, it is possible to show that the virtual solution is independent of λ , as shown in [9, Section 3.3].

For the numerical scheme illustrated in Section 2.2 we also need to consider the approximating PDE

$$(30) \quad \begin{cases} \partial_t u^N + \frac{1}{2} \Delta u^N + a^N \nabla u^N - (\lambda + 1)u^N = -a^N & \text{on } [0, T] \times \mathbb{R} \\ u^N(T) = 0 & \text{on } \mathbb{R}, \end{cases}$$

where $a^N := P_{\eta^N} b^N$, for each $N \geq 1$.

We will now review the arguments that guarantee existence, uniqueness and regularity of the solutions to (29) and (30). Under Assumption 3 and for $(\delta, p) \in \mathcal{K}(\beta, q)$, [9, Theorem 14] guarantees that for each $\lambda > 0$ there exists a unique solution $u_\lambda \in \mathcal{C}([0, T]; H_p^{1+\delta})$ to (29). Since the time derivative and the second spatial derivative of u_λ are not well defined, u_λ is a so-called *mild* solution (for details see, e.g., [16]), and it is obtained as a fixed point in the space $\mathcal{C}([0, T]; H_p^{1+\delta})$ equipped with the norm $\|\cdot\|_{\infty, H_p^{1+\delta}}^{(\rho)}$, with $\rho > \lambda$ sufficiently large. Using

fractional Morrey's inequality (6) it is possible to embed the fractional Sobolev space $H_p^{1+\delta}$ in smoother spaces. In particular we have

$$(31) \quad u_\lambda \in \mathcal{C}([0, T]; \mathcal{C}^{1, \gamma}), \quad \text{with } \gamma = \delta - 1/p.$$

Analogously, (30) admits a unique solution $u_\lambda^N \in \mathcal{C}([0, T]; H_p^{1+\delta})$ (regularity of u_λ^N could of course be upgraded to $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}) \cap \mathcal{C}([0, T] \times \mathbb{R})$ by virtue of higher regularity of a^N but this will not be needed for our purposes).

Next, [9, Lemma 20] gives useful bounds for the gradient of u_λ and u_λ^N . We give a statement which is adapted to our notation¹.

Lemma 4.2. *Let $(\delta, p) \in \mathcal{K}(\beta, q)$. There exists $\lambda_0 > 0$ such that, given any $\lambda > \lambda_0$, letting $u = u_\lambda$ and $u^N = u_\lambda^N$ be the mild solutions in $\mathcal{C}([0, T]; H_p^{1+\delta})$ to the corresponding problems (29) and (30), respectively, we have*

$$\sup_{(t,x) \in [0, T] \times \mathbb{R}} |\nabla u(t, x)| \leq \frac{1}{2} \quad \text{and} \quad \sup_{(t,x) \in [0, T] \times \mathbb{R}} |\nabla u^N(t, x)| \leq \frac{1}{2}.$$

Furthermore, [9, Lemma 21] also guarantees that

$$(32) \quad u, u^N, \nabla u, \nabla u^N \in \mathcal{C}([0, T] \times \mathbb{R}).$$

The next result is a refined statement of [9, Lemma 23]. In particular our equation (33) is contained in the final part of the original proof in [9].

Lemma 4.3. *Let $(\delta, p) \in \mathcal{K}(\beta, q)$ and let $u = u_\lambda$ and $u^N = u_\lambda^N$ be the mild solutions in $\mathcal{C}([0, T]; H_p^{1+\delta})$ to (29) and (30), respectively. Then, if $a^N \rightarrow b$ in $\mathcal{C}([0, T]; H_q^{-\beta})$, there is a constant $c_0 > 0$ such that*

$$(33) \quad \|u - u^N\|_{\infty, H_p^{1+\delta}}^{(\rho)} \leq c_0 \frac{\left(\|u\|_{\infty, H_p^{1+\delta}}^{(\rho)} + 1 \right) \rho^{\frac{\delta+\beta-1}{2}}}{1 - c_0 \left(\|b\|_{\infty, H_q^{-\beta}} \rho^{\frac{\delta+\beta-1}{2}} + \lambda \rho^{-1} \right)} \|b - a^N\|_{\infty, H_q^{-\beta}},$$

for any $\rho > \lambda$ that is sufficiently large to guarantee that the denominator above is positive.

For future reference we define

$$(34) \quad c(\rho) := \frac{c_0 \left(\|u\|_{\infty, H_p^{1+\delta}}^{(\rho)} + 1 \right) \rho^{\frac{\delta+\beta-1}{2}}}{1 - c_0 \left(\|b\|_{\infty, H_q^{-\beta}} \rho^{\frac{\delta+\beta-1}{2}} + \lambda \rho^{-1} \right)},$$

for $\rho > 0$ large enough so that the denominator is positive.

Remark 4.4. *Notice that in Lemma 4.2 we can choose $N \geq N_0$, sufficiently large, so that λ_0 depends only on δ, β and $\|b\|_{\infty, H_q^{-\beta}}$, because $a^N \rightarrow b$ in $\mathcal{C}([0, T]; H_q^{-\beta})$. Then, in Lemma 4.3 we can choose $\rho > \rho_0$ so that the denominator in (33) is positive and $\rho_0 > \lambda_0$ (as needed for the fixed point in [9, Theorem 14]).*

From now on we will simplify our notation and set $u = u_\lambda$, for some λ sufficiently large so that Lemma 4.2 holds. In order to solve equation (28) and find a virtual solution of (1), one has to transform the SDE (28) into a more standard one. This is achieved by setting $Y_t := \varphi(t, X_t)$, where

$$(35) \quad \varphi(t, x) := x + u(t, x).$$

¹We note that there is a typo in the statement of [9, Lemma 20]. Indeed it can be easily checked from the proof that the condition $\rho < \lambda$ is not needed therein.

Notice that $\varphi \in \mathcal{C}([0, T]; \mathcal{C}^1)$ thanks to (32). Moreover by Lemma 4.2 $x \mapsto \varphi(t, x)$ is invertible for each fixed $t \in [0, T]$, with its inverse denoted by

$$(36) \quad \psi(t, \cdot) := \varphi^{-1}(t, \cdot).$$

By Lemma 4.2, $\psi(t, \cdot)$ is 2-Lipschitz, uniformly in t . Then, solving (28) is equivalent to solving the standard SDE for Y below

$$(37) \quad Y_t = y_0 + (\lambda + 1) \int_0^t u(s, \psi(s, Y_s)) ds + \int_0^t (\nabla u(s, \psi(s, Y_s)) + 1) dW_s,$$

where $y_0 = \varphi(0, x)$. Existence of a weak solution for (37) is guaranteed by [38, Theorem 10.2.2] since its coefficients $\tilde{b}(t, y) := (\lambda + 1)u(t, \psi(t, y))$ and $\tilde{\sigma}(t, y) := \nabla u(t, \psi(t, y)) + 1$ are bounded continuous with $\tilde{\sigma}$ uniformly non-degenerate (see [9, Proposition 27] for details).

Likewise, letting $\varphi^N(t, x) := x + u^N(t, x)$, $y_0^N := \varphi^N(0, x)$ and $\psi^N(t, \cdot) := (\varphi^N)^{-1}(t, \cdot)$, the analogue of (37) for the approximated SDE (8) is given by an SDE for $Y^N := \psi^N(t, X_t^N)$. That is

$$(38) \quad \begin{aligned} Y_t^N = & y_0^N + (\lambda + 1) \int_0^t u^N(s, \psi^N(s, Y_s^N)) ds \\ & + \int_0^t (\nabla u^N(s, \psi^N(s, Y_s^N)) + 1) dW_s. \end{aligned}$$

Moreover, $\psi^N(t, \cdot)$ is 2-Lipschitz, uniformly in t , by Lemma 4.2.

Remark 4.5. In [9] the authors work in d dimensions and find weak solutions for the SDE for Y . However, for $d = 1$ both equations (37) and (38) admit a unique strong solution if $\gamma := \delta - 1/p > 1/2$. That holds because the diffusion coefficient is γ -Hölder continuous (see (31)) and the drift is Lipschitz continuous. This result is used in the proof of Proposition 3.1 to justify the use of the same Brownian motion when estimating $Y^N - Y$.

It then follows that $X_t = \psi(t, Y_t)$ and $X_t^N = \psi^N(t, Y_t^N)$ are adapted to the Brownian filtration and, in that sense, they are ‘strong’ virtual solutions to (1) and (8), respectively. Moreover, they are unique up to indistinguishability because of the one-to-one mapping between Y, Y^N and X, X^N : for example, if two different solutions X and \tilde{X} of (1) exist, they give rise to two different solutions Y and \tilde{Y} of (37), which is impossible by uniqueness of the solution to (37).

We conclude this section with some further remarks on the set $\mathcal{K}(\beta, q)$ and on the different choices (δ, p) , and we explain the implications for the solution u . To facilitate the discussion, let us consider (β_0, q_0) as in Assumption 1 and let us define

$$(39) \quad \mathcal{H}(\beta_0, q_0) := \{(\delta, p) \in \mathcal{K}(\beta_0, q_0) \mid \delta - 1/p > 1/2\}.$$

The sets $\mathcal{H}(\beta_0, q_0)$ and $\mathcal{K}(\beta_0, q_0)$ are illustrated in Figure 4.

Remark 4.6. Let $b \in \mathcal{C}^\kappa([0, T], H_{q_0, q_0}^{-\beta_0})$, where (β_0, q_0) is fixed and satisfies Assumption 1. Recall that Assumption 3 is automatically satisfied.

- (i) The set $\mathcal{H}(\beta_0, q_0)$ defined in (39) is not empty (see the black triangle in Figure 4).
- (ii) Although our drift b satisfies Assumption 1, in Propositions 3.1 and 3.3 we chose to look at it as an element of a larger space. In particular, we used the embedding (see Remark 3.2)

$$b \in \mathcal{C}([0, T]; H_{q_0, \tilde{q}_0}^{-\beta_0}) \subset \mathcal{C}([0, T]; H_{q_1, \tilde{q}_1}^{-\beta_1}),$$

for $0 < \beta_0 < \beta_1 < \frac{1}{2}$ and $q_0 > q_1 > \tilde{q}_1 > \tilde{q}_0$.

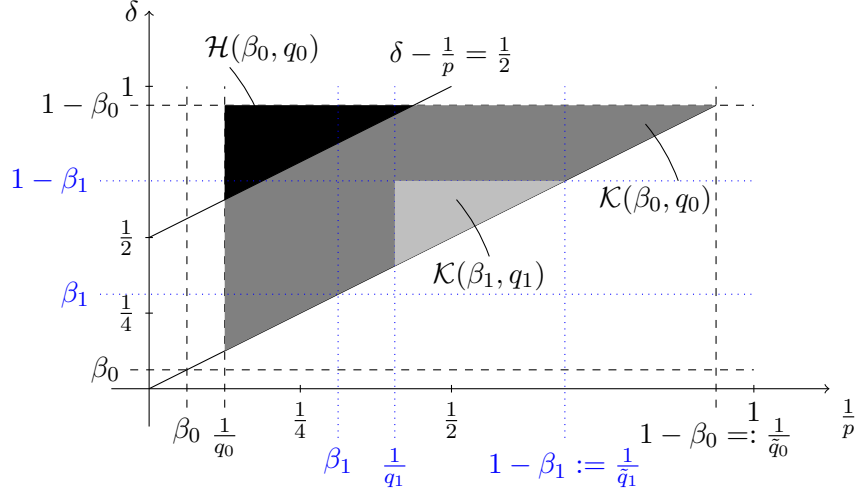


FIGURE 4. The sets $\mathcal{K}(\beta_0, q_0)$ (large grey triangle), $\mathcal{K}(\beta_1, q_1)$ (small light grey triangle) and $\mathcal{H}(\beta_0, q_0)$ (small black triangle) are drawn here.

- (iii) *Since the solution of (29) is unique in $\mathcal{K}(\beta_0, q_0)$ (see large triangle in Figure 4), then we can always pick $(\delta, p) \in \mathcal{H}(\beta_0, q_0)$ such that $u \in \mathcal{C}([0, T]; H_p^{1+\delta})$ and $\nabla u(t, \cdot) \in \mathcal{C}^\gamma$ for $\gamma = \delta - \frac{1}{p} > 1/2$ and all $t \in [0, T]$. The largest Hölder exponent we can find in $\mathcal{H}(\beta_0, q_0)$ is $\gamma_0 - \varepsilon$, where $\gamma_0 = 1 - \beta_0 - 1/q_0$ and $\varepsilon > 0$ is arbitrarily small. This means that for the solution of (29) we have $\nabla u(t, \cdot) \in \mathcal{C}^\gamma$ for all $t \in [0, T]$ and any $\gamma < \gamma_0$.*

5. CONVERGENCE RATE OF $X^N - X$

In this section we prove Proposition 3.1. It turns out that in order to show the convergence rate of X^N to X stated in Proposition 3.1 we must provide an upper bound for the local time at zero of $Y - Y^N$. Recall that for any real-valued continuous semi-martingale \bar{Y} , the local time $L_t^0(\bar{Y})$ is defined as

$$(40) \quad L_t^0(\bar{Y}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{|\bar{Y}_s| \leq \varepsilon\}} d\langle \bar{Y} \rangle_s, \quad \mathbb{P}\text{-a.s.}$$

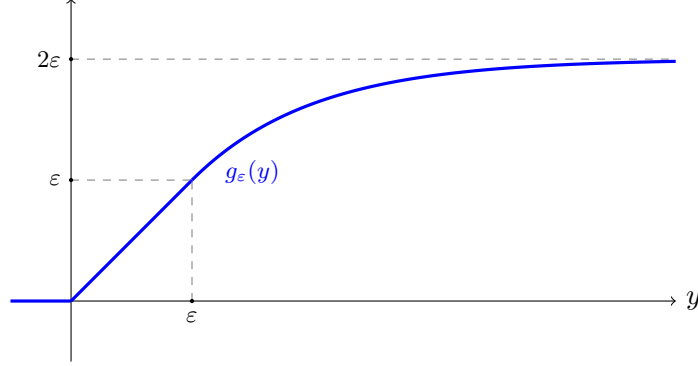
for all $t \geq 0$. Now we derive a bound on (40) that will be needed later on.

Lemma 5.1. *For any $\varepsilon \in (0, 1)$ and any real-valued, continuous semi-martingale \bar{Y} we have*

$$(41) \quad \begin{aligned} \mathbb{E} [L_t^0(\bar{Y})] &\leq 4\varepsilon - 2\mathbb{E} \left[\int_0^t \left(\mathbf{1}_{\{\bar{Y}_s \in (0, \varepsilon)\}} + \mathbf{1}_{\{\bar{Y}_s \geq \varepsilon\}} e^{1-\bar{Y}_s/\varepsilon} \right) d\bar{Y}_s \right] \\ &\quad + \frac{1}{\varepsilon} \mathbb{E} \left[\int_0^t \mathbf{1}_{\{\bar{Y}_s > \varepsilon\}} e^{1-\bar{Y}_s/\varepsilon} d\langle \bar{Y} \rangle_s \right]. \end{aligned}$$

Proof. For $\varepsilon \in (0, 1)$ and $y \in \mathbb{R}$ we define (see Figure 5)

$$g_\varepsilon(y) := 0 \cdot \mathbf{1}_{\{y < 0\}} + y \mathbf{1}_{\{0 \leq y < \varepsilon\}} + \varepsilon \left[2 - e^{1-y/\varepsilon} \right] \mathbf{1}_{\{y \geq \varepsilon\}}.$$

FIGURE 5. The function $y \mapsto g_\varepsilon(y)$.

Straightforward calculations allow to show that $g_\varepsilon \in \mathcal{C}^1(\mathbb{R} \setminus \{0\})$ and it is semi-concave, in the sense that $y \mapsto g_\varepsilon(y) - y^2$ is concave. Moreover, we have

$$(42) \quad \begin{aligned} 0 \leq g_\varepsilon(y) \leq 2\varepsilon, \quad & \text{for } y \in \mathbb{R} \\ g'_\varepsilon(y) = \mathbf{1}_{\{0 \leq y < \varepsilon\}} + e^{1-y/\varepsilon} \mathbf{1}_{\{y \geq \varepsilon\}}, \quad & \text{for } y \in \mathbb{R} \\ g''_\varepsilon|_{(-\infty, 0)} = g''_\varepsilon|_{(0, \varepsilon)} = 0, \\ (43) \quad g''_\varepsilon(y) = -\varepsilon^{-1} e^{1-y/\varepsilon}, \quad & \text{for } y > \varepsilon. \end{aligned}$$

Now, an application of Itô-Tanaka formula gives

$$\begin{aligned} & g_\varepsilon(\bar{Y}_t) - g_\varepsilon(\bar{Y}_0) \\ &= \int_0^t g'_\varepsilon(\bar{Y}_s) \mathbf{1}_{\{\bar{Y}_s \neq 0\}} d\bar{Y}_s + \frac{1}{2} \int_0^t g''_\varepsilon(\bar{Y}_s) \mathbf{1}_{\{\bar{Y}_s \neq 0\} \cap \{\bar{Y}_s \neq \varepsilon\}} d\langle \bar{Y} \rangle_s \\ & \quad + \frac{1}{2} [g'_\varepsilon(0+) - g'_\varepsilon(0-)] L_t^0(\bar{Y}) \end{aligned}$$

where $g'_\varepsilon(0\pm)$ denotes the left/right limit of the derivative at zero. Rearranging terms, taking expectations and using (42)–(43) gives (41). \square

The next lemma controls the approximation error between u and u^N . We recall that $a^N = P_{\eta_N} b^N$.

Lemma 5.2. *Let Assumption 3 hold and fix $(\delta, p) \in \mathcal{K}(\beta, q)$. Let $u, u^N \in \mathcal{C}([0, T]; H_p^{1+\delta})$ be the mild solutions to (29) and (30), respectively. Then, for $\rho > \rho_0$ and $N > N_0$ as in Remark 4.4, and all $t \in [0, T]$ we have*

$$(44) \quad \begin{cases} \|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho \|b - a^N\|_{\infty, H_q^{-\beta}} \\ \|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho \|b - a^N\|_{\infty, H_q^{-\beta}} \end{cases}$$

with

$$\kappa_\rho := c \cdot c(\rho) \cdot e^{\rho T},$$

where $c(\rho) > 0$ is given in (34) and $c > 0$.

Proof. Let $\gamma := \delta - 1/p$. Since $\gamma > 0$, by the fractional Morrey's inequality (6) we have $H_p^{1+\delta} \subset \mathcal{C}^{1, \gamma}$ and we can find $c > 0$ such that for all $t \in [0, T]$ it holds

$$(45) \quad \begin{cases} \|u^N(t) - u(t)\|_{L^\infty} & \leq \|u^N(t) - u(t)\|_{\mathcal{C}^{1, \gamma}} \leq c \|u^N(t) - u(t)\|_{H_p^{1+\delta}} \\ \|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} & \leq \|u^N(t) - u(t)\|_{\mathcal{C}^{1, \gamma}} \leq c \|u^N(t) - u(t)\|_{H_p^{1+\delta}}. \end{cases}$$

Then, recalling (2) and (33) we easily obtain

$$(46) \quad \|u^N - u\|_{\infty, H_p^{1+\delta}} \leq e^{\rho T} \|u^N - u\|_{\infty, H_p^{1+\delta}}^{(\rho)} \leq c(\rho) e^{\rho T} \|b - a^N\|_{\infty, H_q^{-\beta}}.$$

Combining (45) and (46) gives (44). \square

Now we provide a bound on the difference $\psi - \psi^N$, where ψ is defined in (36).

Lemma 5.3. *Take $\rho > \rho_0$ and $N > N_0$ as in Remark 4.4 and κ_ρ as in Lemma 5.2. Under Assumption 3 we have*

$$(47) \quad \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\psi(t,x) - \psi^N(t,x)| \leq 2\kappa_\rho \|b - a^N\|_{\infty, H_q^{-\beta}}.$$

Proof. Recall that $\varphi \in \mathcal{C}([0,T]; \mathcal{C}^1)$ was defined in (35). For any $y, y' \in \mathbb{R}$ we have

$$\begin{aligned} & |\varphi(t,y) - \varphi(t,y')|^2 \\ & \geq |u(t,y) - u(t,y')|^2 + |y - y'|^2 - 2|u(t,y) - u(t,y')||y - y'| \\ & = (|y - y'| - |u(t,y) - u(t,y')|)^2 \geq \frac{1}{4}|y - y'|^2 \end{aligned}$$

where the final inequality uses Lemma 4.2. Now, taking $y = \psi(t,x)$ and $y' = \psi^N(t,x)$ in the above inequality gives

$$|\varphi(t, \psi(t,x)) - \varphi(t, \psi^N(t,x))| \geq \frac{1}{2} |\psi(t,x) - \psi^N(t,x)|.$$

The latter implies

$$\begin{aligned} |\psi(t,x) - \psi^N(t,x)| & \leq 2 |\varphi(t, \psi(t,x)) - \varphi(t, \psi^N(t,x))| \\ & = 2 |\varphi^N(t, \psi^N(t,x)) - \varphi(t, \psi^N(t,x))| \end{aligned}$$

where the final equality uses $\varphi(t, \psi(t,x)) = x = \varphi^N(t, \psi^N(t,x))$. By definition of φ and φ^N and (44), from Lemma 5.2 we also obtain

$$\begin{aligned} & |\varphi^N(t, \psi^N(t,x)) - \varphi(t, \psi^N(t,x))| \\ & = |u^N(t, \psi^N(t,x)) - u(t, \psi^N(t,x))| \\ & \leq \|u(t) - u^N(t)\|_{L^\infty} \leq \kappa_\rho \|b - a^N\|_{\infty, H_q^{-\beta}}. \end{aligned}$$

Combining the above expressions we get (47). \square

In the next proposition we provide an upper bound for the local time at zero of $Y - Y^N$. For $r > 0$ we denote by $o(r)$ a generic function with $o(r)/r \rightarrow 0$ as $r \rightarrow 0$.

Proposition 5.4. *Let Assumption 1 and 2 hold. Take arbitrary (β, q) that satisfy Assumption 3 and such that $\beta \in [\beta_0, \frac{1}{2})$ and $q_0 \geq q > \tilde{q} \geq \tilde{q}_0$. Then, for any $1/2 < \gamma < \gamma_0$, we have*

$$(48) \quad \begin{aligned} \mathbb{E}[L_T^0(Y^N - Y)] & \leq c_\gamma \|a^N - b\|_{\infty, H_q^{-\beta}}^{2\gamma-1} + c' \mathbb{E} \left[\int_0^T |Y_s^N - Y_s| ds \right] \\ & \quad + o(\|a^N - b\|_{\infty, H_q^{-\beta}}^{2\gamma-1}) \end{aligned}$$

as $N \rightarrow \infty$, with $c_\gamma, c' > 0$ given constants (c_γ depending on γ).

Proof. It is clear that $\mathcal{H}(\beta_0, q_0) \neq \emptyset$ by Remark 4.6 and that we can choose (β, q) as indicated, thanks to Assumption 1. Let us also recall Remark 4.6, part (iii), so that for the solution u of (29) we have $\nabla u(t, \cdot) \in \mathcal{C}^\gamma$ for all $1/2 < \gamma < \gamma_0$.

Thanks to (37) and (38) it is easy to derive the dynamics of $\bar{Y} := Y^N - Y$ (recall that Y and Y^N are strong solutions by Remark 4.5). Then, applying Lemma 5.1 we obtain

$$(49) \quad \mathbb{E} [L_t^0(Y^N - Y)] \leq 4\varepsilon \\ -2(1+\lambda)\mathbb{E} \left[\int_0^t \left(\mathbf{1}_{\{\bar{Y}_s \in (0, \varepsilon)\}} + \mathbf{1}_{\{\bar{Y}_s \geq \varepsilon\}} e^{1-\bar{Y}_s/\varepsilon} \right) (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))) \, ds \right] \\ + \frac{1}{\varepsilon} \mathbb{E} \left[\int_0^t \mathbf{1}_{\{Y_s^N - Y_s > \varepsilon\}} e^{1-(Y_s^N - Y_s)/\varepsilon} (\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)))^2 \, ds \right]$$

where we have removed the martingale term. Adding and subtracting terms we have

$$\begin{aligned} & |u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))| \\ & \leq |u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N))| + |u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N))| \\ & \quad + |u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s))|. \end{aligned}$$

In order to estimate the right-hand side of the expression above we use Lemma 4.2, Lemma 5.2 and Lemma 5.3, upon recalling that $a^N \rightarrow b$ in the space $\mathcal{C}([0, T]; H_q^{-\beta})$, by Remark 3.2 part (ii). Since $\psi(s, \cdot)$ and $\psi^N(s, \cdot)$ are 2-Lipschitz, and $u(s, \cdot)$ and $u^N(s, \cdot)$ are $\frac{1}{2}$ -Lipschitz, uniformly in $s \in [0, T]$ we have

$$(50) \quad |u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))| \leq 2\kappa_\rho \|b - a^N\|_{\infty, H_q^{-\beta}} + |Y_s^N - Y_s|,$$

for $\rho > \rho_0$ and $N > N_0$ as in Remark 4.4 and κ_ρ as in Lemma 5.2. Similarly, for the term in (49) involving the gradient of u and u^N we get

$$(51) \quad |\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))| \\ \leq \kappa_\rho \|b - a^N\|_{\infty, H_q^{-\beta}} + 2^{\gamma+1} \kappa_\rho^\gamma \|u\|_{\infty, \mathcal{C}^{1, \gamma}} \|b - a^N\|_{\infty, H_q^{-\beta}}^\gamma \\ + |\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))|,$$

where for the second term on the right-hand side above we used that $\nabla u(t, \cdot)$ is γ -Hölder continuous for any $1/2 < \gamma < \gamma_0$, and then used Lemma 5.3. Now, plugging (50) and (51) into (49) and using the well-known inequality

$$(x_1 + x_2 + \dots + x_k)^2 \leq k(x_1^2 + \dots + x_k^2)$$

we obtain

$$(52) \quad \mathbb{E} [L_t^0(Y^N - Y)] \\ \leq 4\varepsilon + 4(1+\lambda) \left(2\kappa_\rho t \|a^N - b\|_{\infty, H_q^{-\beta}} + \mathbb{E} \left[\int_0^t |Y_s^N - Y_s| \, ds \right] \right) \\ + \frac{1}{\varepsilon} 3t \|a^N - b\|_{\infty, H_q^{-\beta}} \left(\kappa_\rho^2 \|a^N - b\|_{\infty, H_q^{-\beta}} + 4(2\kappa_\rho)^{2\gamma} \|u\|_{\infty, \mathcal{C}^{1, \gamma}}^2 \|a^N - b\|_{\infty, H_q^{-\beta}}^{2\gamma-1} \right) \\ + \frac{1}{\varepsilon} 3\mathbb{E} \left[\int_0^t \mathbf{1}_{\{Y_s^N - Y_s > \varepsilon\}} e^{1-(Y_s^N - Y_s)/\varepsilon} |\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))|^2 \, ds \right].$$

For simplicity, we denote by $I_t^{N, \varepsilon}$ the last term in (52). To find an upper bound for $I_t^{N, \varepsilon}$ we pick $\zeta \in (0, 1)$ such that $\gamma\zeta > 1/2$ and recall that $\varepsilon \in (0, 1)$ so that $\varepsilon^\zeta > \varepsilon$. Using the fact that $\nabla u(s, \cdot)$ is γ -Hölder continuous uniformly in $s \in [0, T]$ with constant $\|u\|_{\infty, \mathcal{C}^{1, \gamma}}$, that ψ is

2-Lipschitz and that ∇u is uniformly bounded by $1/2$ thanks to Lemma 4.2, we get

$$\begin{aligned}
 (53) \quad I_t^{N,\varepsilon} &\leq \frac{1}{\varepsilon} 3\mathbb{E} \left[\int_0^t \mathbf{1}_{\{\varepsilon < Y_s^N - Y_s \leq \varepsilon\zeta\}} e^{1-(Y_s^N - Y_s)/\varepsilon} 2^{2\gamma} \|u\|_{\infty, \mathcal{C}^{1,\gamma}}^2 |Y_s^N - Y_s|^{2\gamma} ds \right] \\
 &\quad + \frac{1}{\varepsilon} 3\mathbb{E} \left[\int_0^t \mathbf{1}_{\{Y_s^N - Y_s > \varepsilon\zeta\}} e^{1-(Y_s^N - Y_s)/\varepsilon} (2\|\nabla u\|_{L^\infty})^2 ds \right] \\
 &\leq \frac{1}{\varepsilon} \left(3T 2^{2\gamma} \|u\|_{\infty, \mathcal{C}^{1,\gamma}}^2 \varepsilon^{2\gamma\zeta} + 3T e^{1-\varepsilon\zeta^{-1}} \right).
 \end{aligned}$$

With no loss of generality we can take $\varepsilon = \|a^N - b\|_{\infty, H_q^{-\beta}}$. Combining (52) and (53) we then find

$$\begin{aligned}
 &\mathbb{E} [L_t^0(Y^N - Y)] \\
 &\leq c_1 \|a^N - b\|_{\infty, H_q^{-\beta}} + c_2 \|a^N - b\|_{\infty, H_q^{-\beta}}^{2\gamma-1} + c_3 \|a^N - b\|_{\infty, H_q^{-\beta}}^{2\gamma\zeta-1} \\
 &\quad + c_4 \|a^N - b\|_{\infty, H_q^{-\beta}}^{-1} \exp \left(1 - \|a^N - b\|_{\infty, H_q^{-\beta}}^{\zeta-1} \right) \\
 &\quad + c_5 \mathbb{E} \left[\int_0^t |Y_s^N - Y_s| ds \right],
 \end{aligned}$$

where

$$\begin{aligned}
 c_1 &= 4 + 8T(1 + \lambda)\kappa_\rho + 3T\kappa_\rho^2, & c_2 &= 12T(2\kappa_\rho)^{2\gamma} \|u\|_{\infty, \mathcal{C}^{1,\gamma}}^2, \\
 c_3 &= 3T 2^{2\gamma} \|u\|_{\infty, \mathcal{C}^{1,\gamma}}^2, & c_4 &= 3T, & c_5 &= 4(1 + \lambda).
 \end{aligned}$$

Since $\zeta \in (0, 1)$, the term containing the exponential goes to zero faster than any polynomial as $N \rightarrow \infty$. Moreover, ζ can be taken arbitrarily close to one and $1/2 < \gamma < \gamma_0$ was also arbitrary, hence (48) holds and the proof is complete with $c_\gamma = c_3$ and $c' = c_5$. \square

We are now ready to prove Proposition 3.1, which we recall below for the reader's convenience.

Proposition 3.1. *Let Assumptions 1 and 2 hold. Take any (β, q) such that $\beta \in (\beta_0, \frac{1}{2})$ and $q_0 \geq q > \tilde{q} \geq \tilde{q}_0$, where $\tilde{q} := (1 - \beta)^{-1}$. Then, for any $1/2 < \gamma < \gamma_0$ there is a constant $C_\gamma > 0$ such that*

$$(54) \quad \sup_{0 \leq t \leq T} \mathbb{E} [|X_t^N - X_t|] \leq C_\gamma \|P_{\eta_N} b^N - b\|_{\infty, H_q^{-\beta}}^{2\gamma-1}$$

as $N \rightarrow \infty$.

Proof. Recall the embedding $b \in \mathcal{C}([0, T]; H_{\tilde{q}_0, q_0}^{-\beta_0}) \subset \mathcal{C}([0, T]; H_{\tilde{q}, q}^{-\beta})$ (Remark 3.2 part (i)) and notice that $a^N = P_{\eta_N} b^N \rightarrow b$ in $\mathcal{C}([0, T]; H_q^{-\beta})$ as $N \rightarrow \infty$ (Remark 3.2 part (ii)). Next we note that

$$\begin{aligned}
 |X_t^N - X_t| &= |\psi^N(t, Y_t^N) - \psi(t, Y_t)| \\
 &\leq |\psi^N(t, Y_t^N) - \psi(t, Y_t^N)| + |\psi(t, Y_t^N) - \psi(t, Y_t)| \\
 &\leq 2\kappa_\rho \|b - a^N\|_{\infty, H_q^{-\beta}} + 2|Y_t^N - Y_t|,
 \end{aligned}$$

where in the final inequality we have used Lemma 5.3 and that $\psi(t, \cdot)$ is 2-Lipschitz, uniformly in $t \in [0, T]$ (Lemma 4.2). Therefore it is sufficient to find a bound for $|Y_t^N - Y_t|$.

From Itô-Tanaka formula we get

$$|Y_t^N - Y_t| = |y_0^N - y_0| + \int_0^t \text{sign}(Y_s^N - Y_s) d(Y_s^N - Y_s) + \frac{1}{2} L_t^0(Y^N - Y).$$

Taking expectation, using (37) and (38) and removing the martingale term we obtain

$$\begin{aligned}
& \mathbb{E} [|Y_t^N - Y_t|] \\
&= |u^N(0, x) - u(0, x)| + \frac{1}{2} \mathbb{E} [L_t^0(Y^N - Y)] \\
&\quad + (1 + \lambda) \mathbb{E} \left[\int_0^t \text{sign}(Y_s^N - Y_s) (u^N(s, \psi^N(s, Y_s^N)) - u^N(s, \psi(s, Y_s))) ds \right] \\
&\leq |u^N(0, x) - u(0, x)| + \frac{1}{2} \mathbb{E} [L_t^0(Y^N - Y)] \\
&\quad + 2(1 + \lambda)t\kappa_\rho \|b - a^N\|_{\infty, H_q^{-\beta}} + (1 + \lambda) \mathbb{E} \left[\int_0^t |Y_s^N - Y_s| ds \right],
\end{aligned}$$

where the inequality follows from the bound (50) used in Proposition 5.4. Using (44) we have

$$|u^N(0, x) - u(0, x)| \leq \kappa_\rho \|a^N - b\|_{\infty, H_q^{-\beta}}.$$

Thanks to Proposition 5.4, we have an upper bound for the local time and, in particular, for any $1/2 < \gamma < \gamma_0$ we have

$$\mathbb{E} [|Y_t^N - Y_t|] \leq C \|a^N - b\|_{\infty, H_q^{-\beta}}^{2\gamma-1} + C' \mathbb{E} \left[\int_0^t |Y_s^N - Y_s| ds \right] + o(\|a^N - b\|_{\infty, H_q^{-\beta}}^{2\gamma-1}),$$

as $N \rightarrow \infty$, where

$$C = \frac{3}{2} 2^{2\gamma} T \|u\|_{\infty, \mathcal{C}^{1,\gamma}}^2, \quad C' = 3(1 + \lambda).$$

By an application of Gronwall's lemma we conclude the proof and the constant $C_\gamma > 0$ in (54) can be taken as $C_\gamma = (1 + C)e^{TC'}$ since $o(\|a^N - b\|_{\infty, H_q^{-\beta}}^{2\gamma-1}) \leq \|a^N - b\|_{\infty, H_q^{-\beta}}^{2\gamma-1}$ for N large enough. \square

6. CONVERGENCE RATE FOR EULER-MARUYAMA SCHEME

In this section we prove Proposition 3.4, which gives a bound for

$$\mathbb{E} [|X_t^{N,m} - X_t^N|],$$

where N is fixed and m tends to infinity. Let us start with some initial considerations.

For each N the drift in the equation for X^N is certainly bounded, $\frac{1}{2}$ -Hölder continuous in time and Lipschitz continuous in space, since b^N is bounded measurable. Therefore we could use classical results (see, e.g., [21]) on the convergence of the Euler-Maruyama scheme in order to obtain a rate of convergence of order $m^{-1/2}$. Following this approach, the multiplicative constant in front of the rate $m^{-1/2}$ depends exponentially on the Lipschitz constant of the drift, i.e., in our case on the exponential of $\|\nabla(P_{\eta_N} b^N)\|_{\infty, L^\infty}$.

Substantially more refined results for SDEs with additive noise were obtained recently by [5], who found a strong convergence rate of order $m^{-1/2}$ under the sole requirement of a bounded (time-homogeneous) drift; see [5, Theorem 1.2]. In that theorem the multiplicative constant that appears in front of the rate $m^{-1/2}$ depends exponentially on the L^∞ -norm of the drift ([5, Lemma 2.2]), i.e., in our case on the exponential of $\|P_{\eta_N} b^N\|_{\infty, L^\infty}$.

Both $\|\nabla(P_{\eta_N} b^N)\|_{\infty, L^\infty}$ and $\|P_{\eta_N} b^N\|_{\infty, L^\infty}$ explode as $N \rightarrow \infty$ (and $\eta_N \rightarrow 0$) at a rate depending on the inverse of η_N , hence increasing the overall approximation error exponentially as we send $(N, m) \rightarrow \infty$ at the same time. Of course one could let $\eta_N \rightarrow 0$ very slowly, in order to compensate for the exponential explosion, but this would produce a very slow rate of convergence of $X^N \rightarrow X$ (see Proposition 3.1) hence deteriorating, once again, the overall convergence rate. Here we find a compromise by contenting ourselves with a convergence rate

for the Euler Maruyama scheme of order $m^{-1/2}$ (see (21)) but with a multiplicative constant (see (22)) which grows polynomially with the inverse of η_N (rather than exponentially).

We use a transformation which is the analogue of the one used to define the virtual solutions. That is, we transform the processes X^N and $X^{N,m}$ into new processes Y^N and $Y^{N,m}$ whose dynamics are expressed in terms of Itô's diffusions with 'nice' coefficients.

Throughout this section Assumptions 1 and 2 are enforced. Since the index N is fixed, it is convenient to simplify the notation and write

$$\widehat{X} := X^N, \quad \widehat{X}^m := X^{N,m}, \quad \widehat{a} := a^N = P_{\eta_N} b^N, \quad \widehat{u} := u^N,$$

and denote

$$\widehat{\varphi}(t, x) := \varphi^N(t, x) = x + \widehat{u}(t, x) \quad \text{and} \quad \widehat{\psi}(t, x) := \psi^N(t, x).$$

Using this notation we can define $\widehat{Y}_t := \widehat{\varphi}(t, \widehat{X}_t)$ and $\widehat{Y}_t^m := \widehat{\varphi}(t, \widehat{X}_t^m)$ so that $\widehat{X}_t = \widehat{\psi}(t, \widehat{Y}_t)$ and $\widehat{X}_t^m = \widehat{\psi}(t, \widehat{Y}_t^m)$. Recalling that $\widehat{\psi}(t, \cdot)$ is 2-Lipschitz, uniformly in t , we obtain

$$(55) \quad \mathbb{E} \left[|\widehat{X}_t^m - \widehat{X}_t| \right] \leq 2\mathbb{E} \left[|\widehat{Y}_t^m - \widehat{Y}_t| \right].$$

In order to estimate the right-hand side in the expression above, we first find the dynamics of \widehat{Y} and \widehat{Y}^m in the next lemma.

Lemma 6.1. *The dynamics of \widehat{Y} is given by*

$$(56) \quad \widehat{Y}_t = \widehat{\varphi}(0, x) + (1 + \lambda) \int_0^t \widehat{u}(s, \widehat{\psi}(s, \widehat{Y}_s)) ds + \int_0^t (\nabla \widehat{u}(s, \widehat{\psi}(s, \widehat{Y}_s)) + 1) dW_s,$$

for all $t \in [0, T]$. Moreover, for any $m \geq 1$ the dynamics of \widehat{Y}^m is given by

$$(57) \quad \widehat{Y}_t^m = \widehat{\varphi}(0, x) + (1 + \lambda) \int_0^t \widehat{u}(s, \widehat{\psi}(s, \widehat{Y}_s^m)) ds + \int_0^t (\nabla \widehat{u}(s, \widehat{\psi}(s, \widehat{Y}_s^m)) + 1) dW_s + E_t^m,$$

for all $t \in [0, T]$, where the 'error process' E^m can be written in terms of \widehat{X}^m as

$$E_t^m := \int_0^t \left(\widehat{a}(t_{k(s)}, \widehat{X}_{t_{k(s)}}^m) - \widehat{a}(s, \widehat{X}_s^m) \right) \left(1 + \nabla \widehat{u}(s, \widehat{X}_s^m) \right) ds.$$

Proof. We start by proving (56). Since $\widehat{a} \in \mathcal{C}^1(\mathbb{R})$, the unique mild solution \widehat{u} of the associated PDE (30) must be a classical solution, i.e., $\widehat{u} \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$. Then, applying Itô's formula we obtain

$$\begin{aligned} \widehat{u}(t, \widehat{X}_t) &= \widehat{u}(0, x) + \int_0^t \left(\partial_t \widehat{u} + \frac{1}{2} \Delta \widehat{u} + \widehat{a} \nabla \widehat{u} \right) (s, \widehat{X}_s) ds + \int_0^t \nabla \widehat{u}(s, \widehat{X}_s) dW_s \\ &= \widehat{u}(0, x) + \int_0^t \left((1 + \lambda) \widehat{u}(s, \widehat{X}_s) - \widehat{a}(s, \widehat{X}_s) \right) ds + \int_0^t \nabla \widehat{u}(s, \widehat{X}_s) dW_s. \end{aligned}$$

Plugging this into the definition of \widehat{Y} and using the SDE (8) for \widehat{X} we get

$$\begin{aligned} \widehat{Y}_t &= \widehat{\varphi}(t, \widehat{X}_t) \\ &= \widehat{u}(t, \widehat{X}_t) + \widehat{X}_t \\ &= \widehat{\varphi}(0, x) + (1 + \lambda) \int_0^t \widehat{u}(s, \widehat{\psi}(s, \widehat{Y}_s)) ds + \int_0^t (\nabla \widehat{u}(s, \widehat{\psi}(s, \widehat{Y}_s)) + 1) dW_s, \end{aligned}$$

upon recalling that $\widehat{X}_s = \widehat{\psi}(s, \widehat{Y}_s)$.

The proof of (57) follows the same ideas but we have the additional error term E_t^m , due to the special drift of \widehat{X}^m in (9). By Itô's formula indeed we obtain

$$(58) \quad \begin{aligned} \hat{u}(t, \widehat{X}_t^m) &= \hat{u}(0, x) + \int_0^t \left[(\partial_t \hat{u} + \frac{1}{2} \Delta \hat{u})(s, \widehat{X}_s^m) + \hat{a}(t_{k(s)}, \widehat{X}_{t_{k(s)}}^m) \nabla \hat{u}(s, \widehat{X}_s^m) \right] ds \\ &\quad + \int_0^t \nabla \hat{u}(s, \widehat{X}_s^m) dW_s. \end{aligned}$$

Using the PDE (30) we can substitute

$$(\partial_t \hat{u} + \frac{1}{2} \Delta \hat{u})(s, \widehat{X}_s^m) = [(1 + \lambda) \hat{u} - \hat{a} \nabla \hat{u} - \hat{a}](s, \widehat{X}_s^m)$$

in (58) to obtain

$$\begin{aligned} \hat{u}(t, \widehat{X}_t^m) &= \hat{u}(0, x) + \int_0^t \left((1 + \lambda) \hat{u}(s, \widehat{X}_s^m) - \hat{a}(s, \widehat{X}_s^m) \right) ds \\ &\quad + \int_0^t \nabla \hat{u}(s, \widehat{X}_s^m) dW_s + E_t^{1,m}, \end{aligned}$$

with

$$E_t^{1,m} := \int_0^t \left(\hat{a}(t_{k(s)}, \widehat{X}_{t_{k(s)}}^m) - \hat{a}(s, \widehat{X}_s^m) \right) \nabla \hat{u}(s, \widehat{X}_s^m) ds.$$

Using the SDE (8) for \widehat{X}^m and the definition of \widehat{Y}^m we get

$$\begin{aligned} \widehat{Y}_t^m &= \widehat{\varphi}(0, x) + (1 + \lambda) \int_0^t \hat{u}(s, \widehat{\psi}(s, \widehat{Y}_s^m)) ds + \int_0^t (\nabla \hat{u}(s, \widehat{\psi}(s, \widehat{Y}_s^m)) + 1) dW_s \\ &\quad + E_t^{1,m} + E_t^{2,m} \end{aligned}$$

where

$$E_t^{2,m} := \int_0^t \left(\hat{a}(t_{k(s)}, \widehat{X}_{t_{k(s)}}^m) - \hat{a}(s, \widehat{X}_s^m) \right) ds.$$

Hence, (57) follows by setting $E_t^m = E_t^{1,m} + E_t^{2,m}$. \square

To find a bound for (55) and prove the rate of convergence of the scheme, we will proceed similarly to the proof of Proposition 3.1. Indeed, we will apply Itô-Tanaka formula to $|\widehat{Y}^m - \widehat{Y}|$ and estimate the resulting terms. Preliminary bounds are obtained in the next two lemmas.

Lemma 6.2. *Let $s \mapsto |dE_s^m|$ be the infinitesimal variation of E^m . Then*

$$(59) \quad \mathbb{E} \left[\int_0^t |dE_s^m| \right] \leq c_1(N) m^{-1} + c_2(N) m^{-\frac{1}{2}},$$

with

$$\begin{aligned} c_1(N) &= 6T \|a^N\|_{\infty, L^\infty} + \frac{3}{4} T^2 \|\nabla a^N\|_{\infty, L^\infty} \|a^N\|_{\infty, L^\infty} \quad \text{and} \\ c_2(N) &= T^{\frac{3}{2}} \|\nabla a^N\|_{\infty, L^\infty} + \frac{3}{2} [a^N]_{\frac{1}{2}, L^\infty} T^{\frac{3}{2}}. \end{aligned}$$

Proof. By definition of E_t^m and using $|\nabla \hat{u}(s, \hat{X}_s^m)| \leq \frac{1}{2}$ from Lemma 4.2 we have

$$\begin{aligned} & \mathbb{E} \left[\int_0^t |dE_s^m| \right] \\ & \leq \frac{3}{2} \mathbb{E} \left[\int_0^t \left| \hat{a}(t_{k(s)}, \hat{X}_{t_{k(s)}}^m) - \hat{a}(s, \hat{X}_s^m) \right| ds \right] \\ & \leq \frac{3}{2} \mathbb{E} \left[\int_0^{t_1} \left| \hat{a}(0, x) - \hat{a}(s, \hat{X}_s^m) \right| ds + \sum_{k=1}^{k(t)-1} \int_{t_k}^{t_{k+1}} \left| \hat{a}(t_k, \hat{X}_{t_k}^m) - \hat{a}(t_k, \hat{X}_s^m) \right| ds \right] \\ & \quad + \frac{3}{2} \mathbb{E} \left[\sum_{k=1}^{k(t)-1} \int_{t_k}^{t_{k+1}} \left| \hat{a}(t_k, \hat{X}_s^m) - \hat{a}(s, \hat{X}_s^m) \right| ds + \int_{t_{k(t)}}^t \left| \hat{a}(t_{k(s)}, \hat{X}_{t_{k(s)}}^m) - \hat{a}(s, \hat{X}_s^m) \right| ds \right]. \end{aligned}$$

The first and last term in the final expression above are bounded by $6 \frac{T}{m} \|\hat{a}\|_{\infty, L^\infty}$, recalling that $0 \leq t - t_{k(t)} \leq T/m$. In the third term we recall the Hölder seminorm (3) and obtain

$$\begin{aligned} & \mathbb{E} \left[\sum_{k=1}^{k(t)-1} \int_{t_k}^{t_{k+1}} \left| \hat{a}(t_k, \hat{X}_s^m) - \hat{a}(s, \hat{X}_s^m) \right| ds \right] \leq \sum_{k=1}^{k(t)-1} \int_{t_k}^{t_{k+1}} \mathbb{E} [\|\hat{a}(t_k) - \hat{a}(s)\|_{L^\infty}] ds \\ & \leq [\hat{a}]_{\frac{1}{2}, L^\infty} \sum_{k=1}^{k(t)-1} \int_{t_k}^{t_{k+1}} (s - t_k)^{\frac{1}{2}} ds \leq [\hat{a}]_{\frac{1}{2}, L^\infty} T^{\frac{3}{2}} m^{-\frac{1}{2}}, \end{aligned}$$

where in the final inequality we use that there are at most m terms in the sum.

Finally, for the second term we have

$$\mathbb{E} \left[\sum_{k=1}^{k(t)-1} \int_{t_k}^{t_{k+1}} \left| \hat{a}(t_k, \hat{X}_{t_k}^m) - \hat{a}(t_k, \hat{X}_s^m) \right| ds \right] \leq \|\nabla \hat{a}\|_{\infty, L^\infty} \sum_{k=1}^{k(t)-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[\left| \hat{X}_{t_k}^m - \hat{X}_s^m \right| \right] ds.$$

Thanks to (9), each term in the sum above can be easily estimated as

$$\begin{aligned} & \int_{t_k}^{t_{k+1}} \mathbb{E} \left[\left| \hat{X}_{t_k}^m - \hat{X}_s^m \right| \right] ds \\ & = \int_{t_k}^{t_{k+1}} \mathbb{E} \left[\left| \hat{a}(t_k, \hat{X}_{t_k}^m)(s - t_k) + (W_s - W_{t_k}) \right| \right] ds \\ & \leq \int_{t_k}^{t_{k+1}} \left(\|\hat{a}\|_{\infty, L^\infty} (s - t_k) + \mathbb{E} [|W_s - W_{t_k}|] \right) ds \\ & = \frac{1}{2} \|\hat{a}\|_{\infty, L^\infty} (t_{k+1} - t_k)^2 + \mathbb{E} [|W_1|] \int_{t_k}^{t_{k+1}} (s - t_k)^{\frac{1}{2}} ds \\ & = \frac{1}{2} \|\hat{a}\|_{\infty, L^\infty} \left(\frac{T}{m} \right)^2 + \frac{2}{3} \mathbb{E} [|W_1|] \left(\frac{T}{m} \right)^{\frac{3}{2}}, \end{aligned}$$

where again we used $t_{k+1} - t_k = T/m$.

Combining all of the above estimates we conclude

$$\begin{aligned} & \mathbb{E} \left[\int_0^t |dE_s^m| \right] \\ & \leq 6 \|\hat{a}\|_{\infty, L^\infty} \frac{T}{m} + \frac{3}{2} \|\nabla \hat{a}\|_{\infty, L^\infty} \left(\frac{1}{2} \|\hat{a}\|_{\infty, L^\infty} T^2 m^{-1} + \frac{2}{3} T^{\frac{3}{2}} m^{-\frac{1}{2}} \right) + \frac{3}{2} [\hat{a}]_{\frac{1}{2}, L^\infty} T^{\frac{3}{2}} m^{-\frac{1}{2}}. \end{aligned}$$

Rearranging terms gives (59). \square

Lemma 6.3. *The following holds*

$$\mathbb{E} \left[L_T^0(\widehat{Y} - \widehat{Y}^m) \right] \leq 4(1 + \lambda) \mathbb{E} \left[\int_0^T |\widehat{Y}_s^m - \widehat{Y}_s| ds \right] + 2 \mathbb{E} \left[\int_0^T |dE_s^m| \right].$$

Proof. This estimate uses arguments analogous to those in the proof of Proposition 5.4. By Lemma 5.1 and using the dynamics (56) and (57), and the fact that $\hat{u}(t, \cdot)$ and $\hat{\psi}(t, \cdot)$ are Lipschitz (uniformly in $t \in [0, T]$) we obtain

$$(60) \quad \mathbb{E} \left[L_T^0(\widehat{Y} - \widehat{Y}^m) \right] \\ \leq 4\varepsilon + 4(1 + \lambda) \mathbb{E} \left[\int_0^T |\widehat{Y}_s - \widehat{Y}_s^m| ds \right] + 2 \mathbb{E} \left[\int_0^T |dE_s^m| \right] \\ + \frac{1}{\varepsilon} \mathbb{E} \left[\int_0^T \mathbb{1}_{\{\widehat{Y}_s - \widehat{Y}_s^m > \varepsilon\}} e^{(1 - (\widehat{Y}_s - \widehat{Y}_s^m)/\varepsilon)} \left| \nabla \hat{u}(s, \hat{\psi}(s, \widehat{Y}_s)) - \nabla \hat{u}(s, \hat{\psi}(s, \widehat{Y}_s^m)) \right|^2 ds \right]$$

having removed the martingale term. Notice that the last term is analogous to $I_t^{N, \varepsilon}$ in (53) and with very similar calculations we get

$$(61) \quad \frac{1}{\varepsilon} \mathbb{E} \left[\int_0^T \mathbb{1}_{\{\widehat{Y}_s - \widehat{Y}_s^m > \varepsilon\}} e^{1 - (\widehat{Y}_s - \widehat{Y}_s^m)/\varepsilon} \left| \nabla \hat{u}(s, \hat{\psi}(s, \widehat{Y}_s)) - \nabla \hat{u}(s, \hat{\psi}(s, \widehat{Y}_s^m)) \right|^2 ds \right] \\ \leq \frac{1}{\varepsilon} \left(\|\hat{u}\|_{\infty, \mathcal{C}^{1, \gamma}}^2 2^{2\gamma} T \varepsilon^{2\gamma\zeta} + T \exp(1 - \varepsilon^\zeta) \right),$$

with $1/2 < \gamma < \gamma_0$ and any $\zeta \in (0, 1)$ such that $2\gamma\zeta > 1$. Hence the right-hand side of (61) tends to zero as $\varepsilon \rightarrow 0$. Noting that also $4\varepsilon \rightarrow 0$ in (60) concludes the proof. \square

We are now ready to prove Proposition 3.4, which we recall below for the reader's convenience.

Proposition 3.4. *Let Assumption 1 hold and let $b^N \in \mathcal{C}^{\frac{1}{2}}([0, T]; H_{q_0, q_0}^0)$ for some fixed N . Then, as $m \rightarrow \infty$, we have*

$$(62) \quad \sup_{0 \leq t \leq T} \mathbb{E} \left[|X_t^{N, m} - X_t^N| \right] \leq C_2(N) m^{-1} + C_3(N) m^{-\frac{1}{2}}$$

with

$$C_2(N) := c \|P_{\eta_N} b^N\|_{\infty, L^\infty} \left(1 + \|\nabla(P_{\eta_N} b^N)\|_{\infty, L^\infty} \right), \\ C_3(N) := c' \left(\|\nabla(P_{\eta_N} b^N)\|_{\infty, L^\infty} + [P_{\eta_N} b^N]_{\frac{1}{2}, L^\infty} \right)$$

and $c, c' > 0$ constants independent of (N, m) .

Proof. Since $\hat{\psi}(t, \cdot) = \psi^N(t, \cdot)$ is 2-Lipschitz we have

$$(63) \quad \mathbb{E} \left[\left| \widehat{X}_t^m - \widehat{X}_t \right| \right] = \mathbb{E} \left[\left| \hat{\psi}(t, \widehat{Y}_t^m) - \hat{\psi}(t, \widehat{Y}_t) \right| \right] \leq 2 \mathbb{E} \left[\left| \widehat{Y}_t^m - \widehat{Y}_t \right| \right].$$

We apply Itô-Tanaka formula to $|\widehat{Y}_t^m - \widehat{Y}_t|$, using (56) and (57), and removing the martingale term by taking expectation. Thus we obtain

$$\begin{aligned} & \mathbb{E} \left[\left| \widehat{Y}_t^m - \widehat{Y}_t \right| \right] \\ &= (1 + \lambda) \mathbb{E} \left[\int_0^t \text{sign}(\widehat{Y}_s^m - \widehat{Y}_s) \left(\hat{u}(s, \hat{\psi}(s, \widehat{Y}_s^m)) - \hat{u}(s, \hat{\psi}(s, \widehat{Y}_s)) \right) ds \right] \\ & \quad + \mathbb{E} \left[\int_0^t \text{sign}(\widehat{Y}_s^m - \widehat{Y}_s) dE_s^m \right] + \frac{1}{2} \mathbb{E} \left[L_t^0(\widehat{Y}^m - \widehat{Y}) \right] \\ & \leq (1 + \lambda) \mathbb{E} \left[\int_0^t \left| \widehat{Y}_s^m - \widehat{Y}_s \right| ds \right] + \mathbb{E} \left[\int_0^t |dE_s^m| \right] + \frac{1}{2} \mathbb{E} \left[L_t^0(\widehat{Y}^m - \widehat{Y}) \right], \end{aligned}$$

where we have also used that, $\psi^N(t, \cdot)$ is 2-Lipschitz, uniformly in $t \in [0, T]$ and \hat{u} is $\frac{1}{2}$ -Lipschitz. Applying Lemma 6.3 to the term featuring the local time we get

$$\mathbb{E} \left[\left| \widehat{Y}_t^m - \widehat{Y}_t \right| \right] \leq 3(1 + \lambda) \mathbb{E} \left[\int_0^t \left| \widehat{Y}_s^m - \widehat{Y}_s \right| ds \right] + 2 \mathbb{E} \left[\int_0^t |dE_s^m| \right].$$

Then, by Lemma 6.2 we obtain

$$\begin{aligned} & \mathbb{E} \left[\left| \widehat{Y}_t^m - \widehat{Y}_t \right| \right] \\ & \leq 3(1 + \lambda) \mathbb{E} \left[\int_0^t \left| \widehat{Y}_s^m - \widehat{Y}_s \right| ds \right] + 2c_1(N)m^{-1} + 2c_2(N)m^{-\frac{1}{2}} \end{aligned}$$

with $c_1, c_2 > 0$ as in Lemma 6.2. By Gronwall's inequality we get

$$(64) \quad \mathbb{E} \left[\left| \widehat{Y}_t^m - \widehat{Y}_t \right| \right] \leq \frac{1}{2} C_2(N)m^{-1} + \frac{1}{2} C_3(N)m^{-\frac{1}{2}}$$

where

$$C_2(N) := 4c_1(N)e^{3(1+\lambda)T} \quad \text{and} \quad C_3(N) := 4c_2(N)e^{3(1+\lambda)T}.$$

Then, finally plugging (64) into (63) we obtain (62). \square

APPENDIX A. HAAR AND FABER BASIS

In this appendix we introduce Haar and Faber functions and discuss some of their key properties. These functions form a basis for certain fractional Sobolev spaces, which we use throughout the paper. Roughly speaking, Haar functions are ‘step functions’ that form a basis for H_r^s with $-1/2 < s < 1/r$ and $2 \leq r < \infty$, while Faber functions are ‘hat functions’ (obtained by integrating Haar functions) that form a basis for H_r^s with $1/2 < s < 1/r + 1$ and $2 \leq r < \infty$.

Using these bases it is possible to represent an element f from either of those fractional Sobolev spaces in terms of infinite sums. Moreover, the sums can be cut to finite sums as a way of approximating the original function f . This procedure can be made rigorous thanks to the theory of fractional Sobolev spaces and to the properties of Haar and Faber function. The link between Haar and Faber representations is useful for numerical implementation and worth commenting on. Distributional drifts as those we consider in this paper can be often obtained as the (distributional) derivative of a function g in H_r^s with $1/2 < s < 1/r + 1$ and $2 \leq r < \infty$. In that case, it is easy to obtain the coefficients on the Haar basis expansion by evaluating g at a finite number of points (see Remark A.10). Below we recall the key results and definitions that we use in this paper.

Throughout the section we denote by \mathcal{S} the space of Schwartz functions, and by \mathcal{S}' its dual (the space of Schwartz distributions). Moreover we use $\mathcal{D} = \mathcal{C}_c^\infty$ to indicate \mathcal{C}^∞ -functions

with compact support and \mathcal{D}' for its dual. Next we introduce the Haar wavelet system on \mathbb{R} , see [43, equations (2.93)–(2.96)].

Definition A.1 (Haar wavelets on \mathbb{R}). *Let us define the mother wavelet $x \mapsto h_M(x)$ by $h_M := \mathbb{1}_{[0, \frac{1}{2})} - \mathbb{1}_{[\frac{1}{2}, 1)}$. The Haar wavelet system on \mathbb{R} is given by*

$$(65) \quad \{h_{j,m} : j \in \mathbb{N} \cup \{-1\}, m \in \mathbb{Z}\}$$

where $h_{-1,m}(x) := \sqrt{2}|h_M(x-m)|$ for $m \in \mathbb{Z}$, and $h_{j,m}(x) := h_M(2^j x - m)$ for $j \in \mathbb{N}$ and $m \in \mathbb{Z}$. Alternatively we can rearrange the system (65) as follows

$$(66) \quad \{h_{j,m}^k : j \in \mathbb{N} \cup \{-1\}, k \in \mathbb{Z}, m = 0, \dots, 2^j - 1\}$$

where $h_{-1,0}^k(x) := h_{-1,k}(x)$ for all $k \in \mathbb{Z}$, and $h_{j,m}^k(x) := h_{j,m}(x-k)$ for all $j \in \mathbb{N}$, $k \in \mathbb{Z}$ and $m = 0, \dots, 2^j - 1$.

See Figure 1 for the plot of a generic Haar function $h_{j,m}$. For future reference note that

$$(67) \quad h_{j,m}(x) = \begin{cases} 1 & \text{if } x \in [\frac{m}{2^j}, \frac{m+1/2}{2^j}) \\ -1 & \text{if } x \in [\frac{m+1/2}{2^j}, \frac{m+1}{2^j}) \\ 0 & \text{else.} \end{cases}$$

It turns out that the Haar wavelets system (65) (or equivalently (66)) is an unconditional basis for fractional Sobolev spaces on \mathbb{R} of order “close to zero” (i.e. $-1/2 < s < 1/r$), as detailed in the theorem below which is taken from [44, Theorem 3.3 and Remark 3.4].

Theorem A.2. *Let $0 \leq r < \infty$, $-1/2 < s < 1/r$, and let $f \in \mathcal{S}'(\mathbb{R})$. Then $f \in H_r^s(\mathbb{R})$ if and only if it can be represented as*

$$(68) \quad f = \sum_{j=-1}^{+\infty} \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^j-1} \mu_{j,m}^k 2^{-j(s-\frac{1}{r})} h_{j,m}^k,$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R})$ and locally in any space $H_r^\sigma(\mathbb{R})$ with $\sigma < s$. Here $\sum_{m=0}^{2^j-1}$ means $m = 0$ when $j = -1$.

The representation is unique, with the coefficients given by

$$\mu_{j,m}^k := 2^{j(s-\frac{1}{r}+1)} \int_{\mathbb{R}} f(x) h_{j,m}^k(x) dx,$$

where the integral is to be understood in the sense of dual pairing. Moreover the system

$$\left\{ 2^{-j(s-\frac{1}{r})} h_{j,m}^k : j \in \mathbb{N} \cup \{-1\}, k \in \mathbb{Z}, m = 0, \dots, 2^j - 1 \right\}$$

is an unconditional normalised basis of $H_r^s(\mathbb{R})$.

It is shown in [43, Theorem 2.9 and Remark 2.12] that (68) can be equivalently written as

$$(69) \quad f = \sum_{j=-1}^{+\infty} \sum_{m \in \mathbb{Z}} \mu_{j,m} 2^{-j(s-\frac{1}{r})} h_{j,m},$$

where $\mu_{j,m} = \mu_{j,m}^0$ and $h_{j,m}$ are as in (67).

Remark A.3. *Let us denote*

$$(70) \quad \tilde{\mu}_{j,m}^k := 2^j \int_{\mathbb{R}} f(x) h_{j,m}^k(x) dx = 2^{j(-s+\frac{1}{r})} \mu_{j,m}^k.$$

It follows from Theorem A.2 and (69) that $\tilde{\mu}_{j,m}^0 = \tilde{\mu}_{j,m}$, where $\tilde{\mu}_{j,m}$ is defined as in (70) but with $h_{j,m}^k$ replaced by $h_{j,m}$.

Moreover, from (68) we get the more compact representation

$$(71) \quad f = \sum_{j=-1}^{+\infty} \sum_{k \in \mathbb{Z}} \sum_{m=0}^{2^j-1} \tilde{\mu}_{j,m}^k h_{j,m}^k.$$

Remark A.4. The coefficients $\tilde{\mu}_{j,m}^k$ do not actually depend on r or s . More precisely, if $f \in H_r^s(\mathbb{R}) \cap H_{r'}^{s'}(\mathbb{R})$ for some $r \neq r'$ and $s \neq s'$ then the representation (71) is exactly the same in both spaces, with the same coefficients.

In what follows we analyse fractional Sobolev spaces on an open bounded interval $I \subset \mathbb{R}$. Let us recall [43, Definition 1.24 (i)]: let I be an open set in \mathbb{R} , then

$$H_r^s(I) := \{f \in \mathcal{D}'(I) : f = h|_I \text{ for some } h \in H_r^s(\mathbb{R})\}$$

endowed with the norm

$$\|f\|_{H_r^s(I)} = \inf\{\|h\|_{H_r^s(\mathbb{R})} : h \in H_r^s(\mathbb{R}) \text{ with } f = h|_I\}.$$

With no loss of generality we specialise to $I := (0, 1)$ for simplicity of exposition. Next we introduce the Haar wavelet system on I (see [43, equations (2.128) and (2.129)] for details) which is useful for the fractional Sobolev space $H_r^s(I)$.

Definition A.5 (Haar wavelets on $I = (0, 1)$). The Haar wavelet system on I is given by

$$(72) \quad \{h_0, h_{j,m} : j \in \mathbb{N}, m = 0, \dots, 2^j - 1\}$$

where

$$h_0(x) := \mathbb{1}_I(x)$$

and $h_{j,m}$ is as in (67).

Notice that the system (72) is essentially the restriction of (65) to the interval I . In particular, $h_{j,m}$ is now restricted to values of m between 0 and $2^j - 1$ rather than $m \in \mathbb{Z}$ as in Definition A.1. Moreover, the set of elements $h_{-1,m}$, defined on \mathbb{R} , with $m \in \mathbb{Z}$, has been replaced by h_0 , defined on I . For the fractional Sobolev spaces on $H_r^s(I)$ we have again a representation in terms of Haar functions, as illustrated below. For more details see [43, Theorem 2.13].

Theorem A.6. Let $2 \leq r < \infty$, $-\frac{1}{2} < s < \frac{1}{r}$, and let $f \in \mathcal{D}'(I)$. Then $f \in H_r^s(I)$ if and only if it can be represented as

$$(73) \quad f = \mu_0 h_0 + \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} \mu_{j,m} 2^{-j(s-\frac{1}{r})} h_{j,m},$$

with unconditional convergence in any space $H_r^\sigma(I)$ with $\sigma < s$. The representation is unique, with the coefficients given by

$$\mu_0 := \int_I f(x) h_0(x) dx$$

and, for $j \in \mathbb{N}$ and $m = 0, \dots, 2^j - 1$, by

$$\mu_{j,m} := 2^{j(s-\frac{1}{r}+1)} \int_I f(x) h_{j,m}(x) dx,$$

where the integrals are to be understood in the sense of dual pairing. Moreover the system

$$\left\{ h_0, 2^{-j(s-\frac{1}{r})} h_{j,m} : j \in \mathbb{N}, m = 0, \dots, 2^j - 1 \right\}$$

is an unconditional normalised basis of $H_r^s(I)$.

Notice that (73) can be written in terms of $\tilde{\mu}_{j,m}$ (see Remark A.3) as

$$(74) \quad f = \mu_0 h_0 + \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} \tilde{\mu}_{j,m} h_{j,m}.$$

Of course a distribution f defined on I can be seen as a distribution defined on \mathbb{R} but only supported on I (in the sense that $f(\phi) = 0$ for all $\phi \in \mathcal{D}$ supported on $\mathbb{R} \setminus I$). The link between the series representations on I and on \mathbb{R} is given in the next lemma.

Lemma A.7. *If $f \in H_r^s(\mathbb{R})$ and $\text{supp}(f) \subset I$ then its representation on \mathbb{R} given by (71) (or equivalently by (68)) coincides with its representation on I given by (74).*

Proof. First we remark that in this case the restriction of f to I (denoted again by f) belongs to $H_r^s(I)$ by definition of the latter space. Since $\text{supp}(f) \subset I$, it follows that $\tilde{\mu}_{j,m}^k = 0$ for all $k \neq 0$ because the functions $h_{j,m}^k$ are supported on $(k, k+1)$ while f is supported on $(0, 1)$ which implies that the dual pairing between $h_{j,m}^k$ and f is non-zero only if $k = 0$. Hence the Haar representation (71) becomes

$$\begin{aligned} f &= \sum_{j=-1}^{+\infty} \sum_{k=0}^{2^j-1} \sum_{m=0}^{2^j-1} \tilde{\mu}_{j,m}^k h_{j,m}^k \\ &= \sum_{j=-1}^{+\infty} \sum_{m=0}^{2^j-1} \tilde{\mu}_{j,m} h_{j,m} \\ &= \tilde{\mu}_{-1,0} h_{-1,0} + \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} \tilde{\mu}_{j,m} h_{j,m}, \end{aligned}$$

where we used the fact that $\tilde{\mu}_{j,m}^0 = \tilde{\mu}_{j,m}$. The proof can be concluded by noticing that

$$\begin{aligned} \tilde{\mu}_{-1,0} h_{-1,0} &= 2^{-1} \int_{\mathbb{R}} f(x) h_{-1,0}(x) dx h_{-1,0} \\ &= 2^{-1} \int_{\mathbb{R}} f(x) \sqrt{2} \mathbb{1}_{(0,1)}(x) dx \sqrt{2} \mathbb{1}_{(0,1)} \\ &= \int_I f(x) \mathbb{1}_{(0,1)}(x) dx \mathbb{1}_{(0,1)} = \mu_0 h_0. \quad \square \end{aligned}$$

Next we recall the definition of Faber functions. They are denoted by v_{jm} and are hat-functions, defined as the normalised integrals of the Haar functions h_{jm} on I . Notice that the Faber series representation holds in general only on bounded domains in \mathbb{R} . Here we only recall their definition on the unit interval $I = (0, 1)$. More details can be found in [44, Section 3.2.1].

Definition A.8 (Faber basis on I). *The Faber system on $(0, 1)$ is given by*

$$\{v_0, v_1, v_{j,m} : j \in \mathbb{N}, m = 0, \dots, 2^j - 1\}$$

where

$$v_0(x) := 1 - x \quad \text{and} \quad v_1(x) := x$$

for $0 \leq x \leq 1$ (and zero outside I), and the hat-functions are defined as

$$v_{j,m}(x) = 2^{j+1} \int_0^x h_{j,m}(y) dy,$$

that is

$$v_{j,m}(x) := \begin{cases} 2^{j+1}(x - 2^{-j}m) & \text{if } x \in [\frac{m}{2^j}, \frac{m+1/2}{2^j}) \\ 2^{j+1}(2^{-j}(m+1) - x) & \text{if } x \in [\frac{m+1/2}{2^j}, \frac{m+1}{2^j}) \\ 0 & \text{else.} \end{cases}$$

Using the Faber system on I it is possible to represent elements of fractional Sobolev spaces on domain I for $1/2 < s < 1 + 1/r$ and $2 \leq r < \infty$ as we see below. For a proof see [43, Theorem 3.1 and Corollary 3.3].

Theorem A.9. *Let $g \in H_r^s(I)$ for $2 \leq r < \infty$, and $\frac{1}{2} < s < 1 + \frac{1}{r}$. Then we have the unique Faber representation for g*

$$g = \bar{\mu}_0 v_0 + \bar{\mu}_1 v_1 + \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} \bar{\mu}_{j,m} v_{j,m}$$

with unconditional convergence in $\mathcal{C}(I)$ and in $H_r^\sigma(I)$ with $\sigma < s$. Here the coefficients $\bar{\mu}$ are explicitly given by

$$(75) \quad \begin{cases} \bar{\mu}_{j,m} &= -\frac{1}{2} (\Delta_{2^{-j-1}}^2 g)(2^{-j}m) \\ \bar{\mu}_0 &= g(0) \\ \bar{\mu}_1 &= g(1) \end{cases}$$

and where $(\Delta_h^2 g)(x) := g(x+2h) - 2g(x+h) + g(x)$.

This representation of g using Faber functions is fundamental to calculate the coefficients for the Haar representation of g' , as we see below.

Remark A.10. *In the proof of the above theorem (see [43, Theorem 3.1, Corollary 3.3]) the following expansion for $g' \in H_r^{s-1}(I)$ is derived*

$$g' = (\bar{\mu}_1 - \bar{\mu}_0)h_0 + \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} 2^{j+1} \bar{\mu}_{j,m} h_{j,m}.$$

Comparing it with (74) from Theorem A.6, with $f = g'$, we obtain an explicit representation of the coefficients $\tilde{\mu}_{j,m}$ and μ_0 that appear in (74), that is

$$(76) \quad \mu_0 = \bar{\mu}_1 - \bar{\mu}_0 \quad \text{and} \quad \tilde{\mu}_{j,m} = 2^{j+1} \bar{\mu}_{j,m}.$$

The link expressed in (76) together with the explicit expression (75) is crucial to evaluate numerically the coefficients in the Haar expansion of an element $f \in H_r^{s-1}(I)$ for $\frac{1}{2} < s < 1 + \frac{1}{r}$ and $2 \leq r < \infty$. Indeed to do so we only need to evaluate the associated function g at (a finite number of) mesh points

$$\begin{cases} \mu_0 &= g(1) - g(0) \\ \tilde{\mu}_{j,m} &= -2^j \left(g(\frac{m+1}{2^j}) - 2g(\frac{m+1/2}{2^j}) + g(\frac{m}{2^j}) \right). \end{cases}$$

A.1. Equivalent norms and coefficients of Haar series. For each $t \in [0, T]$ we have, from [43, Theorem 2.9 and eq. (2.114), Sec. 2.2.3], that the norms $\|b(t)\|_{H_q^{-\beta}}$ and $\|\boldsymbol{\mu}(t; \beta, q)\|_{f_{q,2}^-}$ are equivalent, where

$$\|\boldsymbol{\mu}(t; \beta, q)\|_{f_{q,2}^-} := \left[\int_{\mathbb{R}} \left(\sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} |\mu_{j,m}(t; \beta, q)| 2^{\frac{j}{q}} \mathbb{1}_{[2^{-j}m, 2^{-j}(m+1))}(x) \right)^{\frac{q}{2}} dx \right]^{\frac{1}{q}}.$$

We note that if $q = 2$ the expression above simplifies to

$$\begin{aligned}
(77) \quad \|\boldsymbol{\mu}(t; \beta, 2)|f_{2,2}^-\| &= \left[\int_{\mathbb{R}} \left(\sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} |\mu_{j,m}(t; \beta, 2)|^2 2^j \mathbb{1}_{[2^{-j}m, 2^{-j(m+1)}]}(x) \right) dx \right]^{\frac{1}{2}} \\
&= \left[\sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} |\mu_{j,m}(t; \beta, 2)|^2 2^j \int_{\mathbb{R}} \mathbb{1}_{[2^{-j}m, 2^{-j(m+1)}]}(x) dx \right]^{\frac{1}{2}} \\
&= \left[\sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} |\mu_{j,m}(t; \beta, 2)|^2 \right]^{\frac{1}{2}},
\end{aligned}$$

where we can swap the sums and the integral by monotone convergence. Since the norms are equivalent for all $t \in [0, T]$ then

$$(78) \quad \|b\|_{\infty, H_q^{-\beta}} < \infty \iff \sup_{t \in [0, T]} \|\boldsymbol{\mu}(t; \beta, q)|f_{q,2}^-\| < \infty.$$

APPENDIX B. SOME ESTIMATES FOR THE (KILLED) HEAT SEMIGROUP

Outline of the derivation of Eq. (5). Let us start from the first equation in (5). To show that P_t is a contraction on H_r^{-s} for $s > 0$ we write

$$\|P_t f\|_{H_r^{-s}} = \|A^{-s/2} P_t A^{s/2} A^{-s/2} f\|_{L^r} = \|P_t A^{-s/2} f\|_{L^r},$$

where in the final equality we use that for $s > 0$ the operators $A^{s/2}$ and P_t commute by [36, Theorem II.6.13]. Since $A^{-s/2} f \in L^r(\mathbb{R})$ and P_t is a contraction on $L^r(\mathbb{R})$ we get $\|P_t A^{-s/2} f\|_{L^r} \leq \|A^{-s/2} f\|_{L^r} = \|f\|_{H_r^{-s}}$, as needed. To prove that P_t is a contraction for H_r^s the idea is the same but we write $P_t f = A^{-s/2} A^{s/2} P_t f$.

Let us now prove the second equation in (5). First we recall that $\|P_t g - g\|_{L^r} \leq c t^{\varepsilon/2} \|A^{\varepsilon/2} g\|_{L^r}$ if $g \in D(A^{\varepsilon/2})$, from [36, Theorem II.6.13]. For $f \in H_r^{-s}$ we choose $g = A^{-(s+\varepsilon)/2} f$ and we get

$$\|P_t f - f\|_{H_r^{-(s+\varepsilon)}} = \|A^{-(s+\varepsilon)/2} (P_t f - f)\|_{L^r} = \|(P_t A^{-(s+\varepsilon)/2} f - A^{-(s+\varepsilon)/2} f)\|_{L^r},$$

where in the final equality we use again that $A^{s/2}$ and P_t commute. Since $A^{-s/2} f \in L^r$ then $g = A^{-(s+\varepsilon)/2} f \in H_r^{\varepsilon} = D(A^{\varepsilon/2})$ so we get

$$\|(P_t A^{-(s+\varepsilon)/2} f - A^{-(s+\varepsilon)/2} f)\|_{L^r} \leq c t^{\varepsilon/2} \|A^{-s/2} f\|_{L^r} = c t^{\varepsilon/2} \|f\|_{H_r^{-s}},$$

as needed.

Outline of the derivation of Eq. (7). Let us start from the first equation in (7). By [36, Theorem II.6.13], for $t \in (0, T)$, $\nu \geq 0$ and $g \in L^r$ we have that $\|A^{\alpha/2} P_t g\|_{L^r} \leq c t^{-\alpha/2} \|g\|_{L^r}$. Then choosing $\alpha = \nu + s$, $g = A^{-s/2} f$, and using again that P_t and $A^{s/2}$ commute, we get

$$\begin{aligned}
\|P_t f\|_{H_r^{\nu}} &= \|A^{\nu/2} P_t f\|_{L^r} = \|A^{(\nu+s)/2} P_t A^{-s/2} f\|_{L^r} \\
&\leq c t^{-(\nu+s)/2} \|A^{-s/2} f\|_{L^r} = c t^{-(\nu+s)/2} \|f\|_{H_r^{-s}}.
\end{aligned}$$

Now if $\nu > 1/r$ we can use fractional Morrey inequality and obtain

$$\|P_t f\|_{L^{\infty}} \leq \|P_t f\|_{C^{0,\alpha}} \leq \|P_t f\|_{H_r^{\nu}},$$

by which we conclude. Similarly for the second bound in (7), simply replace ν by $1 + \nu$ and use $\|\nabla P_t f\|_{L^{\infty}} \leq \|P_t f\|_{C^{1,\alpha}} \leq \|P_t f\|_{H_r^{1+\nu}}$ to conclude.

REFERENCES

- [1] V. Bally and D. Talay. The law of the Euler scheme for stochastic differential equations. *Probab. Theory Related Fields*, 104(1):43–60, 1996.
- [2] Th. Brox. A one-dimensional diffusion process in a Wiener medium. *Ann. Probab.*, 14(4):1206 – 1218, 1986.
- [3] O. Butkovsky, K. Dareiotis, and M. Gerencsér. Approximation of SDEs – a stochastic sewing approach. *Probab. Theory Relat. Fields*, 2021.
- [4] G. Cannizzaro and K. Chouk. Multidimensional sdes with singular drift and universal construction of the polymer measure with white noise potential. *Ann. Probab.*, 46(3):1710–1763, 2018.
- [5] K. Dareiotis and M. Gerencsér. On the regularisation of the noise for the Euler-Maruyama scheme with irregular drift. *Electron. J. Probab.*, 25:1–18, 2020.
- [6] F. Delarue and R. Diel. Rough paths and 1d SDE with a time dependent distributional drift: application to polymers. *Probab. Theory Related Fields*, 165(1):1–63, 2016.
- [7] P. Étoré and M. Martinez. On the existence of a time inhomogeneous skew Brownian motion and some related laws. *Electron. J. Probab.*, 17:1 – 27, 2012.
- [8] P. Étoré and M. Martinez. Exact simulation of one-dimensional stochastic differential equations involving the local time at zero of the unknown process. *Monte Carlo Methods Appl.*, 19(1):41–71, 2013.
- [9] F. Flandoli, E. Issoglio, and F. Russo. Multidimensional stochastic differential equations with distributional drift. *Trans. Amer. Math. Soc.*, 369 (3):1655–1688, 2017.
- [10] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift Part I: General calculus. *Osaka J. Math.*, 40(2):493 – 542, 2003.
- [11] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô’s formula and semimartingale characterization. *Random Oper. Stoch. Equ.*, 12(2):145–184, 2004.
- [12] I. Gyöngy and N. Krylov. Existence of strong solutions for Itô’s stochastic equations via approximations. *Probab. Theory Related Fields*, 105:143 – 158, 1996.
- [13] N. Halidias and P.E. Kloeden. A note on the Euler-Maruyama scheme for stochastic differential equations with a discontinuous monotone drift coefficient. *BIT Numerical Mathematics*, 48:51–59, 2008.
- [14] J.M. Harrison and L.A. Shepp. On skew brownian motion. *Ann. Probab.*, 9(2):309–313, 1981.
- [15] Y. Hu, K. Lê, and L. Mytnik. Stochastic differential equation for Brox diffusion. *Stoch. Process. Appl.*, 127(7):2281–2315, 2017.
- [16] E. Issoglio. Transport equations with fractal noise: existence uniqueness and regularity of the solution. *J. Anal. Appl.*, 32(1):37–53, 2013.
- [17] E. Issoglio and S. Jing. Forward–backward SDEs with distributional coefficients. *Stoch. Process. Appl.*, 130(1):47–78, 2020.
- [18] E. Issoglio and F. Russo. A Feynman–Kac result via Markov BSDEs with generalised drivers. *Bernoulli*, 26(1):728–766, 2020.
- [19] E. Issoglio and M. Zähle. Regularity of the solutions to SPDEs in metric measure spaces. *Stoch. Partial Differ. Equ. Anal. Comput.*, 3(2):272–289, 2015.
- [20] K. Itô and H.P. McKean, Jr. *Diffusion processes and their sample paths*. Die Grundlehren der mathematischen Wissenschaften, Band 125. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-New York, 1965.
- [21] P.E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*, volume 23. Springer Science & Business Media, 2013.
- [22] A. Kohatsu-Higa, A. Lejay, and K. Yasuda. Weak rate of convergence of the Euler-Maruyama scheme for stochastic differential equations with non-regular drift. *J. Comput. Appl. Math.*, 326:138–158, 2016.
- [23] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields*, 131(2):154–196, 2005.
- [24] A. Lejay. On the constructions of the skew Brownian motion. *Probab. Surv.*, 3:413 – 466, 2006.
- [25] A. Lejay, L. Lenôtre, and G. Pichot. An exponential timestepping algorithm for diffusion with discontinuous coefficients. *J. Comput. Phys.*, 396:888–904, 2019.
- [26] A. Lejay and M. Martinez. A scheme for simulating one-dimensional diffusion processes with discontinuous coefficients. *Ann. Appl. Probab.*, 16(1):107 – 139, 2006.
- [27] G. Leobacher and M. Szölgényi. A strong order 1/2 method for multidimensional SDEs with discontinuous drift. *Ann. Appl. Probab.*, 27(4):2383–2418, 2017.
- [28] G. Leobacher and M. Szölgényi. Convergence of the Euler-Maruyama method for multidimensional SDEs with discontinuous drift and degenerate diffusion coefficient. *Numer. Math.*, 138 (1):219–239, 2018.
- [29] O. Menoukeu-Pamen and D. Taguchi. Strong rate of convergence for the Euler-Maruyama approximation of SDEs with Hölder continuous drift coefficient. *Stoch. Process. Appl.*, 127(8):2542–2559, 2017.

- [30] R. Mikulevicius and E. Platen. Rate of convergence of the Euler approximation for diffusion processes. *Math. Nachr.*, 151(1):233–239, 1991.
- [31] T. Müller-Gronbach and L. Yaroslavl'tseva. On the performance of the Euler–Maruyama scheme for SDEs with discontinuous drift coefficient. *Ann. Inst. Henri Poincaré Probab. Stat.*, 56(2):1162–1178, 2020.
- [32] A. Neuenkirch and M. Szölgényi. The Euler–Maruyama scheme for SDEs with irregular drift: convergence rates via reduction to a quadrature problem. *IMA J. Numer. Anal.*, 41(2):1164–1196, 06 2020.
- [33] A. Neuenkirch, M. Szölgényi, and L. Szpruch. An adaptive Euler-Maruyama scheme for stochastic differential equations with discontinuous drift and its convergence analysis. *SIAM J. Numer. Anal.*, 57(1):378–403, 2019.
- [34] H.-L. Ngo and D. Taguchi. Strong rate of convergence for the Euler-Maruyama approximation of stochastic differential equations with irregular coefficients. *Math. Comp.*, 85:1793–1819, 2016.
- [35] H.-L. Ngo and D. Taguchi. On the Euler-Maruyama approximation for one-dimensional stochastic differential equations with irregular coefficients. *IMA J. Numer. Anal.*, 37(4):1864–1883, 2017.
- [36] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [37] F. Russo and G. Trutnau. Some parabolic PDEs whose drift is an irregular random noise in space. *Ann. Probab.*, 35(6):2213 – 2262, 2007.
- [38] D. W. Stroock and SR.S. Varadhan. *Multidimensional diffusion processes*. Springer, 2007.
- [39] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1978.
- [40] H. Triebel. *Theory of Function Spaces I*. Birkhäuser Basel, 1983.
- [41] H. Triebel. *Theory of Function Spaces II*. Birkhäuser Basel, 1992.
- [42] H. Triebel. *Fractals and Spectra*. Birkhäuser, Basel, 1997.
- [43] H. Triebel. *Bases in function spaces, sampling, discrepancy, numerical integration*. European Mathematical Society, 2010.
- [44] H. Triebel. *Faber systems and their use in sampling, discrepancy, numerical integration*. European Mathematical Society, 2012.
- [45] L. Yan. The Euler scheme with irregular coefficients. *Ann. Probab.*, 30(3):1172–1194, 2002.

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