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A graph theoretical approach to decisions in social choice, law, A.I.

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Abstract
Graph theory proved to be a quite powerful mathematical tool to model several problems in a discrete setting. In this paper the authors illustrate how a general graph framework can be used to model multi-dimensional alternatives decision processes in three different contexts: social choice, judgment aggregation and automata decisions.

Keywords Directed graphs · Multi-dimensional alternatives · Decision process · Judgment aggregation · CP-net

Mathematics Subject Classification 05C20 · 05C85

JEL Classification D71 · D72

1 Introduction

Graph theory, as many other mathematical tools, was born from a real problem: the Seven Bridges of Krönigsberg. Krönigsberg was a city in the ancient Prussia crossed by a river including two large islands which were connected to each other and to the mainland by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once. Its negative resolution by Leonhard Euler in 1736 laid the foundations and the formalization of the graph theory.

Either we talk about networks, decisions, connections or any other problem which involves a datum that can be codified in a point and an exchange of data, or information, among those points, we are in the presence of a graph (think, for instance, to the graph of railways with vertices the stations and edges their connections). One of the real problem, that can be

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efficiently modeled using graphs, is related to choices or decisions among different objects, situations or people.

In this paper we illustrate a way to model via graph theory, a multi-dimensional alternative decision process which can involve social choices (i.e. voting, group decisions, etc.), judgments (i.e. judges who have to decide if someone is guilty or not of something) or automata decisions. The latter is the new contribution of the paper.

The classical social choice literature studies the aggregation of individual preferences (usually represented as partial orders or sets of ordered pairs, i.e graphs) into collective "social" preferences. The judgement aggregation literature studies the aggregation of individual judgements (assignments of True or False to sets of logical propositions) into a collective (e.g. of a jury or a committee) judgement. Finally CP-nets were introduced in [3] to model automated decision processes in order to reach a key goal in the study of computer-based decision support: the construction of tools that allow the preference elicitation process to be automated, either partially or fully. While classical social choice has an old history which trace back to French revolution and the famous Condorcet paradox on voting (see Example 2.1), the judgement aggregation theory is much younger and essentially started with [9] which identified the Doctrinal paradox (see Example 2.2). Formalized by List and Pettit [11] by means of propositional logic, it has been studied by many authors (see, for instance, [6, 12] and, for a review on the argument, [16]).

More recently Marengo et al. [15] showed how the propositional logic framework introduced in List and Pettit [11] can be equivalently formalized by means of a multi-dimensional graph model which allows to extend to the Judgment aggregation framework the classical results in the social choice literature and vice-versa. Their attempt to model the judgment aggregation framework by means of graphs is a generalization of the decision process introduced in [13] and modeled via graphs in [1, 14].

In this paper, following their example, we model the automata decision framework by means of the graph model described in [14]. In particular this graph-based description allows to slightly generalize the CP-nets model largely used in the automata decision literature. We point out that the collective decision making over multidimensional alternatives has been largely studied over the past years with dozens of papers (see, for instance [10] for a partial survey on the argument). In this paper we are mainly interested in the new contribution added to this literature by Marengo and Pasquali [13], in particular in the decision process via objects scheme and in the concept of local optimum they introduced (see Sect. 3.1).

Our goal is double, from one hand we show a Category-like approach to decisions by exhibiting a general graph framework that model multi-dimensional alternatives decision processes in three different contexts. On the other hand the description of the CP-nets in terms of graphs allows to introduce the decision process of Marengo and Pasquali [13] and the known results related to the concept of local optimum in the automata decisions framework. In particular we remark that the CP-nets based decision process is a special case of the Marengo and Pasquali [13] decision process (see Remark 4.3).

The paper is organized as follows. In Sect. 2, we recall the preference aggregation framework with its associated graph, the judgement aggregation framework in terms of propositional logic and CP-net model. In Sect. 3 we illustrate the decision process introduced by Marengo and Pasquali [13] to model the case of multi-dimensional alternatives and we introduce a slightly different version of the *graph-theoretic framework* described in [15] for judgment aggregation. Finally in Sect. 4, the original part of our paper, we rewrite the CP-net model in terms of graphs opening the way to connections between the CP-net decision framework and the decision process introduced in [13] and developed in [15].

Table 1 Preferences $\succ_1$, $\succ_2$ and $\succ_3$

$\succ_1$	$x \succ_1 y \succ_1 z$
$\succ_2$	$y \succ_2 z \succ_2 x$
$\succ_3$	$z \succ_3 x \succ_3 y$

2 Preliminaries

2.1 The preference aggregation framework: graphs

Consider a population of ν *agents*, each agent characterized by a *system of transitive preferences* $\succeq_i$ over the set of social outcomes X. The set of systems of transitive preferences $\succeq$ is denoted by $\mathcal{P}$. The social choice literature studies social preference functions also called *social decision rules*

$$\begin{array}{rccc} \mathcal{R}: & \mathcal{P}^\nu & \longrightarrow & \overline{\mathcal{P}} \\ & (\succeq_1, \ldots, \succeq_\nu) & \longmapsto & \succeq_{\mathcal{R}} (\succeq_1, \ldots, \succeq_\nu) \end{array}$$

which aggregate the preferences of ν individual agents into a *system of social preferences* or *social rules* $\succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_\nu)}$. Here $\overline{\mathcal{P}}$ denotes the set of systems of (non-necessarily transitive) social preferences; as a matter of fact, we note that the social rule $\succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_\nu)}$ is not, in general, transitive anymore.

Example 2.1 (Majority rule) Let's consider the aggregation by majority rule, i.e. for any $x, y \in X$, $x \succ y$ if and only if the number m of agents i for which $x \succ_i y$ is $m > \frac{\nu}{2}$. Assume there are three agents with preferences $\succ_1$, $\succ_2$ and $\succ_3$ on the set $X = \{x, y, z\}$ as listed in Table 1.

Then the aggregation by majority rule provides $x \succ y \succ z \succ x$ which is clearly NOT transitive anymore. This is known as the *Condorcet Paradox*.

Notice that the social choice literature requires, in general, individual and social preferences to satisfy some properties. The most common ones include:

- (Pareto) A social preference function is *Pareto* if $x \succeq_i y$ for any $1 \le i \le \nu$ then $x \succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_\nu)} y$.
- (IIA / Indifference of Irrelevant Alternatives) If there are two profiles of individual preferences $(\succeq_1, \ldots, \succeq_\nu)$ and $(\succeq_1^*, \ldots, \succeq_\nu^*)$ such that $x \succeq_i y$ if and only if $x \succeq_i^* y$ for all i, then we have $x \succeq_{\mathcal{R}(\succeq_1,\ldots,\succeq_\nu)} y$ if and only if $x \succeq_{\mathcal{R}(\succeq_1^*,\ldots,\succeq_\nu^*)} y$.

Preferences can be *weak*, denoted by $\succeq$, or *strict*, denoted by $\succ$, that is the two conditions $x \succeq_{\mathcal{R}} y$ and $y \succeq_{\mathcal{R}} x$ can hold both or be mutually exclusive, respectively. Moreover given the sets

$$\mathcal{Y}_{0,\succeq_{\mathcal{R}}} = \{x \in X \mid \exists y \in X \text{ such that } (x, y) \in \mathcal{Y}_{1,\succeq_{\mathcal{R}}} \text{ or } (y, x) \in \mathcal{Y}_{1,\succeq_{\mathcal{R}}}\},$$

and

$$\mathcal{Y}_{1,\succeq_{\mathcal{R}}} = \{(y, x) \in X \times X \mid x \neq y \text{ and } x \succeq_{\mathcal{R}} y\},$$

we say that the preferences are *complete* if $\mathcal{Y}_{0,\succeq_{\mathcal{R}}} = X$ and for any two elements $x, y \in \mathcal{Y}_{0,\succeq_{\mathcal{R}}}$ either $(x, y) \in \mathcal{Y}_{1,\succeq_{\mathcal{R}}}$ or $(y, x) \in \mathcal{Y}_{1,\succeq_{\mathcal{R}}}$.

The graph: A common representation of preferences is by means of directed graphs. The sets $\mathcal{Y}_{0,\succ}$ and $\mathcal{Y}_{1,\succ}$ correspond, respectively, to the sets of vertices and edges of a graph $\mathcal{Y}_{\succ}$.

Edges are directed from y to x if $x \succ y$. In the graph framework, a Condorcet Paradox of the form $x_1 \succ x_2 \succ \cdots \succ x_h \succ x_1$ corresponds to a cycle γ having $x_1, x_2, \cdots, x_h$ as vertices. For this reasons they are called cycle *à la* Condorcet-Arrow.

Note that the completeness assumption on social rules guarantees that the graph $\mathcal{Y}_{\succ}$ is connected and complete while for the preferences to be strict or weak implies to have either only one or possibly two edges between two vertices.

2.2 The judgement aggregation framework: logic

In the judgement aggregation framework, we have n *judges* $\mathcal{J} = \{j_1, j_2, \dots, j_n\}$ who are called to express judgements on a set $\Xi = \Xi_a \cup \Xi_c$ of $(m+s)$ logical propositions, which is the (disjoint) union of the set $\Xi_a = \{P_1, \dots, P_m\}$ of *atomic* propositions and the set $\Xi_c = \{C_1, \dots, C_s\}$ of *compound* propositions. A *compound* proposition is something in the Boolean algebra generated by logical operations on atomic propositions, such as $P \wedge Q$, $P \vee Q, \neg P \rightarrow Q$, etc.

Atomic propositions generally represent basic facts (i.e. P: *the suspect is guilty of "breaking"*; Q: *the suspect is guilty of "entering"*), compound propositions represent laws or rules (e.g. $P \wedge Q$: *the suspect is guilty of "breaking and entering" when (the suspect is guilty of "breaking") and (the suspect is guilty of "entering")*). A judgement is *logical* if and only if given the judgement's atomic propositions, the value of the judgement's compound propositions agrees with the respective rules.

In this framework a judge's judgment is an $(m+s)$-vector

$$J = (J_a || J_c) = (J(P_1), \dots, J(P_m), J(C_1), \dots, J(C_s))$$

(the symbol || denotes concatenation) that assigns 1 (True) or 0 (False), i.e. a truth value, to each proposition. The first m coordinates forming a vector J_a corresponding to atomic propositions and the last s coordinates forming a vector J_c corresponding to compound propositions (for details see [5, 7]). Since we want judges to be logically consistent, values of J_c are uniquely determined by values of J_a.

As an example, suppose we have $m = 2$ atomic propositions $P_1 = P$ and $P_2 = Q$ and $s = 1$ compound proposition $C_1 = P \wedge Q$. Suppose that a judge assigns value $J_a = (0, 0)$ to atomic propositions, then in order for her to be logic, $J_c = (0 \wedge 0) = (0)$. Thus, the final logical judgement is $J = (0, 0, 0)$.

If $\mathbb{U}_\Xi$ is the set of *logical judgements* then a *judgement aggregation function* is a function $f: \mathbb{U}_\Xi^n \rightarrow \mathbb{U}_\Xi$ which aggregates n logical judgements into a single logical judgement and is the analogous of a social preference function in the social choice framework. Like social preference functions, also judgement aggregation functions are usually required to satisfy some additional properties:

- (Unanimity) If $J_i(P) = x$ for all judgments J_i, then $J(P) = f(J_1, \dots, J_n)(P) = x$. Notice that unanimity implies $f(J, \dots, J) = J$.
- (Propositionwise Independence) Following Dietrich and List [7], a judgement aggregation function f is propositionwise independent if there exist $(m+s)$ functions $f_i: \{0,1\}^n \rightarrow \{0,1\}$, $1 \leq i \leq (m+s)$, such that whenever $f(J_1, \dots, J_n) = J$, for each i we have $(J)_i = f_i((J_1)_i, \dots, (J_n)_i)$.

Example 2.2 (Majority rule) Let the judgement aggregation function f be the majority rule and three judges $\{1, 2, 3\}$ judge about atomic propositions $\{P, Q\}$, and consistently on compound proposition $\{R(= P \wedge Q)\}$, as in Table 2.

Table 2 Judgements leading to a doctrinal paradox

	P (obligation)	Q (action)	R (liability)(: $\iff$ $(P \wedge Q)$)
Judge 1	True	True	True
Judge 2	False	True	False
Judge 3	True	False	False
Majority	True	True	False

Then the majority rule (last row of Table 2) aggregates single judges's judgments in a NOT logically consistent judgment. This is known as the *Doctrinal Paradox* and it is in judgment theory the analogous of the Condorcet Paradox in Example 2.1. The doctrinal paradox was presented in [9] and had some important real life examples such as the famous US Supreme Court case *Arizona vs. Fulminante*.

2.3 The automated decision framework: CP-Net

In this last subsection we introduce a different mathematical object largely used to study automated decision making: CP-net. CP-nets are used to study, in computer-based decision support, how to make the preference elicitation process automated, either partially or fully. For an introduction on the subject see [3] (see also [8] and, more recently, [4]).

In [3] authors assume existence of a set of *variables* (or features or attributes) $\mathbf{V} = \{X_1, \ldots, X_n\}$ over which the decision maker has preferences. Each variable X_i can assume several values in its domain $Dom(X_i) = \{x^i_1, ..., x^i_{n_i}\}$ and the *preference ranking* is defined by a total order $\succeq$ on the space $\mathcal{O} = \prod_{i=1}^{n} Dom(X_i)$ of all possible alternatives. An assignment $\mathbf{x}$ of values to a set $\mathbf{X} \subset \mathbf{V}$ of variables is a function that maps each variable in $\mathbf{X}$ to an element of its domain; if $\mathbf{X} = \mathbf{V}$, $\mathbf{x}$ is a complete assignment, otherwise $\mathbf{x}$ is called a partial assignment. Boutilier et al. [3] denoted by $Asst(\mathbf{X})$ the set of all assignments to $\mathbf{X} \subset \mathbf{V}$ and defined a set of variables $\mathbf{X}$ to be *preferentially independent* of its complement $\mathbf{Y} = \mathbf{V} \setminus \mathbf{X}$ if and only if for all $\mathbf{x}_1, \mathbf{x}_2 \in Asst(\mathbf{X})$ and $\mathbf{y}_1, \mathbf{y}_2 \in Asst(\mathbf{Y})$

$$\mathbf{x}_1\mathbf{y}_1 \succeq \mathbf{x}_2\mathbf{y}_1 \quad \text{iff} \quad \mathbf{x}_1\mathbf{y}_2 \succeq \mathbf{x}_2\mathbf{y}_2. \tag{2.3}$$

More generally, if $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ partition $\mathbf{V}$, $\mathbf{X}$ is said to be *conditionally preferentially independent* of $\mathbf{Y}$ given an assignment $\mathbf{z} \in Asst(\mathbf{Z})$ if and only if for all $\mathbf{x}_1, \mathbf{x}_2 \in Asst(\mathbf{X})$ and $\mathbf{y}_1, \mathbf{y}_2 \in Asst(\mathbf{Y})$

$$\mathbf{x}_1\mathbf{y}_1\mathbf{z} \succeq \mathbf{x}_2\mathbf{y}_1\mathbf{z} \quad \text{iff} \quad \mathbf{x}_1\mathbf{y}_2\mathbf{z} \succeq \mathbf{x}_2\mathbf{y}_2\mathbf{z}. \tag{2.4}$$

$\mathbf{X}$ is said to be *conditionally preferentially independent* of $\mathbf{Y}$ given $\mathbf{Z}$ if Eq. (2.4) holds for any $\mathbf{z} \in Asst(\mathbf{Z})$.

For each variable $X_i \in \mathbf{V}$, Boutilier et al. [3] ask the user to identify a set of parent variables $P(X_i) \subset \mathbf{V}$ that can affect her preference over various values of X_i. Formally a set $P(X_i) \subset \mathbf{V}$ such that X_i is conditionally preferentially independent of $\mathbf{V} \setminus (P(X_i) \cup \{X_i\})$ given $P(X_i)$. Hence X_i is annotated with a conditional preference table (CPT) describing the user's preferences over the values of the variable X_i given every combination of parent values. In other words, letting $P(X_i) = \mathbf{U}$, for each assignment $\mathbf{u} \in Asst(\mathbf{U})$ they assume that a total preorder $\succeq^i_{\mathbf{u}}$ is provided over the domain of X_i: for any two values $x, x' \in Dom(X_i)$,

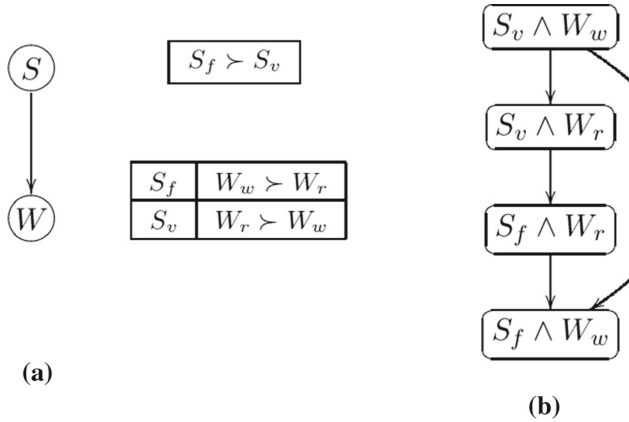


Fig. 1 **a** CP-net for soup and wine; **b** induced preference graph

either $x \succ^i_{\mathbf{u}} x'$ or $x' \succ^i_{\mathbf{u}} x$ or $x \simeq^i_{\mathbf{u}} x'$. Boutilier et al. [3] call these structures *conditional preference networks* or *CP-net*. Essentially they provide the following definition.

Definition 2.5 A *CP-net* over variables $\mathbf{V} = \{X_1, \ldots, X_n\}$ is a directed graph G over $X_1, \ldots, X_n$ whose nodes are annotated with conditional preference table $CPT(X_i)$ for each $X_i \in \mathbf{V}$. Each conditional preference table $CPT(X_i)$ associates a total order $\succeq^i_{\mathbf{u}}$ with each instantiation $\mathbf{u}$ of X_i's parent $P(X_i) = \mathbf{U}$.

Example 2.6 (My Dinner I) Consider the following example over dinner preferences (see [3]). The choice is among two preferences, soup and wine, each one taking two values: fish and vegetables for the soup and red and white for the wine. The preference on the latter is conditioned by the choice of the first. That is $\mathbf{V} = \{X_1, X_2\}$, X_1 = soup and X_2 = wine, $Dom(X_1) = \{$fish, vegetables$\}$, $Dom(X_2) = \{$red,white$\}$ and while the choice of the soup is not conditioned, i.e. $P(X_1) = \emptyset$, we have that $P(X_2) = \{X_1\}$.

If we denote the alternatives in $Dom(X_1)$ and $Dom(X_2)$, respectively by S_f, S_v, W_w and W_r, the CP-net described in Fig. 1a is equivalent to the fact that the fish soup is preferred to the vegetables one and that the preferred wine conditioned to the fish soup (resp. the vegetables soup) is the white wine (resp. the red wine). The preference graph induced by the CP-net in Fig. 1a is represented in Fig. 1b.

In the following section we will describe how both, preferences and judgments have been modeled by means of a very similar graph structure introduced in [14] to study preferences in multi-dimensional alternatives cases. The goal of this paper is to extend this graph framework to CP-net.

3 A graph model in multi-dimensional decisions

In this section we describe the graph framework introduced in [14] and [1] in order to model the multi-dimensional alternatives decision process described in [13]. We also describe its extension to the judgments aggregation introduced in [15].

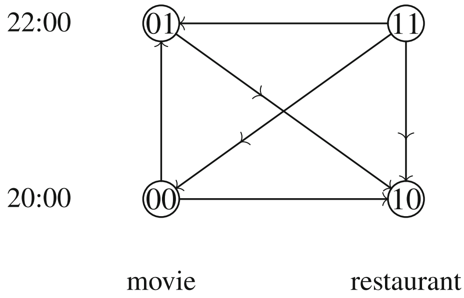


Fig. 2 The graph associated to the go out example

3.1 Multi-dimensional alternatives case in preferences

In real life situations, choices are often made among bundles of interdependent elements.

Example 3.1 Let's consider the example of a group of friends deciding *"what shall we do tonight?"* in which they have to decide upon where and when to go as in the following two-dimensional case:

Where?	movie (0), restaurant (1)	First feature f_1
When?	20:00 (0), 22:00 (1)	Second feature f_2

There are $2 \times 2 = 4$ possibilities and each alternative is a bundle of interdependent elements. The sub-alternatives are grouped into features and, in each group, denoted by numbers (starting from 0). So, for instance, "movie at 20:00" is preferred to "restaurant at 22:00", and this preference is denoted by $00 \succ 11$.

If all the preferences are expressed and an aggregation rule is established (e.g. majority), it is possible to aggregate all the preferences of the group in a single graph as the one depicted in Fig. 2. In this graph, for instance, the edge from 11 to 01 says that the group prefers the option 01 to 11.

In order to study those cases [13] considered a bundle of elements $F = \{f_1, \ldots, f_n\}$ that they called *features*, the ith of which takes m_i values, i.e. $\{0, 1, 2, \ldots, m_i - 1\}$ with $i = 1, \ldots, n$. In this framework a *social outcome* became an n-sequence $v_1 \cdots v_n$ of values such that $0 \le v_i < m_i$ and the set $X = \prod_{i=1}^n f_i$ of all social outcomes has cardinality $\prod_{i=1}^n m_i$. They hence described a decision method which can take into account only a given subset of features at any step of the process.

Let's consider as in Sect. 2.1 a population of ν *agents* which preferences are aggregated in a a social preference $\succeq$ over the set of social outcomes $X = \prod_{i=1}^n f_i$. The graph $\mathcal{Y}_{\succeq}$ of the social preference is then the graph having vertices $\mathcal{Y}_{0,\succeq} = X$ and edges $\mathcal{Y}_{1,\succeq} = \{(x, y) \in X \times X \setminus \Delta \mid y \succeq x\}$.

Object Schemes and Best Neighbors An *objects scheme* is defined as a set of objects $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}, \mathcal{A}_{I_j} = \{f_i \mid i \in I_j\}, I_j \subseteq \{1, \ldots, n\}$, such that $\bigcup_{j=1}^k I_j = \{1, \ldots, n\}$. So, for instance, in the Example 3.1 there are two possible objects scheme: $A_1 = \{\{\text{where,when}\}\}$ and $A_2 = \{\{\text{where}\}, \{\text{when}\}\}$.

A social outcome y is said to be a *preferred neighbor* of a social outcome x with respect to an object $\mathcal{A}_{I_h} \in A$ if the following conditions hold:

1. $y \succ x$,
2. in x and y the features $f_j \notin A_{I_h}$ have the same value,

3. x and y have different values for at least one feature $f_j \in A_{I_h}$.

For instance, from Fig. 2 we notice that the preferred neighbor with respect to the only object {where,when} of A_1 is 11 for any vertex. Meanwhile if we consider the objects scheme A_2 then, for example, 00 is the preferred neighbor of 10 with respect to the object {where} and the preferred neighbor of 01 with respect to the object {when}.

The set of all preferred neighbors of the social outcome x with respect to $\mathcal{A}_{I_h} \in A$ and A, respectively, are denoted by $\Phi(x, \mathcal{A}_{I_h})$ and $\Phi(x, A) = \bigcup_{j=1}^{k} \Phi(x, \mathcal{A}_{I_j})$.

A preferred neighbor $y \in \Phi(x, \mathcal{A}_{I_h})$ of x is said to be a *best neighbor* if

$$y \succ w \quad \forall w \in \Phi(x, \mathcal{A}_{I_h}).$$

The set of all best neighbors of the social outcome x with respect to $\mathcal{A}_{I_h} \in A$ is denoted by $B(x, \mathcal{A}_{I_h})$. When preferences are strict, either $B(x, \mathcal{A}_{I_h})$ is empty or $B(x, \mathcal{A}_{I_h})$ contains one social outcome only.

Domination paths and local optima A *domination path* $DP(x, y, A)$ *through* A, *starting from* x *and ending in* y, is a path in the social graph $\mathcal{Y}_{\succeq}$ connecting sequence of best neighbors with respect to objects in A, i.e. a sequence

$$x = x_0 \prec x_1 \prec \cdots \prec x_s = y$$

such that there exist objects, not necessarily distinct, $\mathcal{A}_{I_{h_1}}, \ldots, \mathcal{A}_{I_{h_s}} \in A$ with $x_i \in B(x_{i-1}, \mathcal{A}_{I_{h_i}})$ for all $1 \leq i \leq s$.

A social outcome x is said to be a *local optimum for* A if $\Phi(x, A)$ is empty and, simply, a local optimum if there exists at least an object scheme A such that $\Phi(x, A)$ is empty. Notice that if $\Phi(x, A)$ is empty then $B(x, A) = \bigcup_{j=1}^{k} B(x, \mathcal{A}_{I_j})$ is empty that is any domination path through A which contains x ends in x. So, for instance, if we consider the objects scheme A_1 in the Example 3.1, we only have the local optimum 11 while, if we consider the objects scheme A_2, also 00 is a local optimum.

In [14], authors proved that a social outcome x is a local optimum if and only if it is preferred to all the social outcome y which differs from x by only one feature.

The decision process Finally we can describe the decision process as follows.
A domination path is said to be *maximal* if it ends in either a local optimum or a limit domination cycle[1] An *agenda* α of an object scheme $A = \{\mathcal{A}_{I_1}, \ldots, \mathcal{A}_{I_k}\}$ is an ordered t-uple of indices $(h_1, \ldots, h_t)$ with $t \geq k$ such that $\{h_1, \ldots, h_t\} = \{1, \ldots, k\}$. An agenda α states the order in which the objects $\mathcal{A}_{I_i}$ are decided upon.

The decision process consists in moving from an initial social outcome x_0, called *status quo*, along the maximal path through a fixed object scheme A, ordered by an agenda α. If the maximal path ends up in a local optimum, then this will be the preferred choice of the society.

Thanks to this graph description Amendola and Settepanella builded an algorithm that allowed them to obtain numerical results described in [1] and to prove (see [2]) that the probability to get a local optimum in the binary case, i.e. f_i's take only value 0 or 1, converge to a value close to 63% when the number of features increases.

The following questions arise naturally: can this result be extended to CP-net? Is this decision process compatible with CP-net structure? The first step to answer to those questions is to re-write the CP-net framework in terms of graphs (see Sect. 4).

[1] More precisely, either x_s is a local optimum or x_{s-t} belongs to $B(x_s, \mathcal{A}_{I_{h_s+1}})$, where h_s is the remainder of the division of $s-1$ by t .

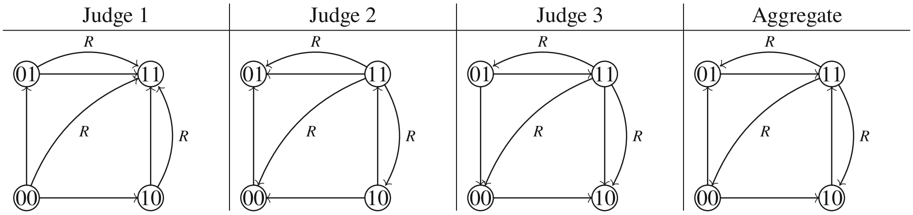


Fig. 3 The graphs G_i of preferences of judges $j_i, i = 1, 2, 3$ in Table 2 and their aggregated graph (under majority rule). Curved edges labelled R represent preferences on the compound proposition $R = P \wedge Q$

3.2 Graph theoretical approach to Judgment

In [15], authors show how the doctrinal paradox in Example 2.2 can be represented via a graph similar to the one in Example 3.1. Indeed if we assume proposition-wise consistency, i.e. we assume that what a judge think about the proposition P is not influenced by the value of the proposition Q and vice versa, then the judgements of judges 1, 2 and 3 in Table 2 correspond to graphs in Fig. 3. For instance, via the correspondence True-1 and False-0, it is an easy check that since the preference of the Judge 1 is True-True, i.e. 11, then the value 1 is preferred in each single proposition and in the compound one, that is we obtain the first graph in Fig. 3 in which the vertex 11 is a sink.[2]

The aggregation by majority rule gives rise to a cycle involving vertices (0, 0), (0, 1) and (1, 1) corresponding to a cycle between False-False, False-True and True-True. This is a Condorcet cycle which corresponds to the doctrinal paradox (also called discursive dilemma) in Table 2. Starting from this example, Marengo, Settepanella and Zhang provided a general proof of the equivalence between the two paradoxes beyond this particular example (see [15]).

In the rest of this Section we will describe the graph-theoretic framework defined in [15][3] and which generalizes the multi-dimensional alternatives model described in [14].

A multi-graphs model Let's consider n individuals (e.g. voters or judges), a set of N alternatives, where each alternative is an m-dimensional object, and a set of s *labels* for s-tuple of directed graphs. Each individual is characterized by an s-tuple of *preference graphs*, each one with N vertices corresponding to the alternatives. In the case of preferences described in [14], $s = 1$ and the s-tuple of graphs will contain only one element.

Recall from Sect. 2.2 that Ξ contains atomic propositions $P_j, j = 1, \ldots, m$ and compound propositions C_j, $j = 1, \ldots, s$. Given a judgement $J = (J_a||J_c) \in \{0, 1\}^{m+s}$, we build s associated graphs $G_i(J) = (V(J), E_i(J))$, each of them corresponding to a compound proposition C_i, as follows:

1. each $G_i(J)$ has the same set of vertices $V(J)$, which consists of the 2^m vertices $v \in \{0, 1\}^m$ of the m-dimensional cube in $\mathbb{R}^m$. As discussed in Sect. 2.2, these correspond to the 2^m potential choices of J_a.
2. An edge (v, w) from v to w,$v \neq w$, is in $E_i(J)$ if and only if one of the following occurs:

2 Recall that a sink in a directed graph is a vertex in which all edges enter or arrive and no edge exits.

3 Notice that in [15] authors give a slightly different description of this framework by means of a unique graph. This description as s-tuple of directed graphs has been chosen since it is more consistent with the purpose of this paper and it first appears here.

(i) they differ only for the value of one entry t, and the value $(J)_t$ of the tth entry of J equals the tth entry $(w)_t$ of w (and hence $(v)_t = \neg(J)_t$)[4]. We call such an edge an *atomic* edge.
(ii) $C_i(v) \neq C_i(w)$, $C_i(w) = (J_c)_i$, $C_i(v) \neq (J_c)_i$. We call such an edge a *compound* edge.

Note that all the graphs G_i have the same vertices and atomic edges.

Example 3.2 Reconsider Example 2.2 from Table 2. This is a case with only one compound proposition, that is $s = 1$ and each judge j_i has only one associated graph G_i, $i = 1, 2, 3$ as described in Fig. 3.

We conclude this Section with a Theorem equivalent to Theorem 4.4 in [15] and which states the equivalence between the logic framework used to model Judgment aggregation and the graph-theoretic framework.

Denote by $Gr(m)$ the set of all graphs on 2^m vertices, by $G : \mathbb{U}_\Xi \to (Gr(m))^s$ the map that sends a logical judgement J to the s-tuple $(G_1, \ldots, G_s)$ of associated graphs and by $Gr(m)^G = G(\{0, 1\}^{m+s}) \subset Gr(m)^s$ the set of s-tuples of graphs $(G_1, \ldots, G_s)$ obtainable from any judgement (even non-logical ones) in $\{0, 1\}^{m+s}$. The following Theorem holds (see [15]).

Theorem 3.3 *The function G that associates to each element $J \in \{0, 1\}^{m+s}$ its s-tuple of graphs $(G_1(J), \ldots, G_s(J))$ gives a bijection between $\{0, 1\}^{m+s}$ and $Gr(m)^G$. G naturally induces a bijection $\tilde{G}$ between sets of functions*

$$\{f \mid f : (\{0, 1\}^{m+s})^n \to \{0, 1\}^{m+s}\} \simeq \{f_{gr} \mid f_{gr} : (Gr(m)^G)^n \to (Gr(m)^G)\},$$

where f and f_{gr} are aggregation functions, respectively, on Judgments and Graphs.

4 The graph representation of CP-Nets

In this Section we show how the CP-net language can be translated in the multi-alternatives graph framework described in the Sect. 3. This open the question wether the decision process introduced in [13] and extended to graphs in [14] can be used to fruitfully model automated decision making.

4.1 The graph of preferences

Let's consider, as in Sect. 2.3, the space $\mathcal{O} = \prod_{i=1}^{n} Dom(X_i)$ of all complete assignments endowed with a total order $\succeq$. To be consistent with notations, if $\mathbf{X} \subset \mathbf{V}$, we use the symbol $\mathcal{O}_\mathbf{X} = \prod_{X_i \in \mathbf{X}} Dom(X_i)$ to denote assignments, instead of $Asst(\mathbf{X})$.

Completion of partial assignments If $\mathrm{pr}_\mathbf{X}$ denotes the projection map

$$\mathrm{pr}_\mathbf{X} : \mathcal{O} \to \mathcal{O}_\mathbf{X},$$

then elements $o \in \mathcal{O}$ such that $\mathrm{pr}_\mathbf{X}(o) = x$ are called *completions* of x.

[4] This captures the idea of *judge systematicity* introduced in [15], i.e. the idea that what a judge think with respect to proposition P is not affected by what she thinks on proposition Q.

Given $y \in \mathcal{O}_{\mathbf{Y}}$, $\mathbf{Y} = \mathbf{V} \setminus \mathbf{X}$, $xy \in \mathcal{O}$ will denote the only completion of x which satisfies $pr_{\mathbf{Y}}(xy) = y$. More in general, given a partition $\{\mathbf{Y}_1, \ldots, \mathbf{Y}_k\}$ of $\mathbf{V}$, $y_1 \ldots y_k \in \mathcal{O}$ will denote the only element which satisfies $pr_{\mathbf{Y}_j}(y_1 \ldots y_k) = y_j$ for any $j = 1, \ldots, k$.

The graph Analogously to Sect. 2.1, the space $\mathcal{O}$ with its assigned total order $\succeq$ can be represented by the graph $G = (V(G), E(G))$ having vertices $V(G) = \mathcal{O}$ and oriented edges $E(G) = \{(o, o') \in \mathcal{O} \mid o \neq o', o' \succeq o\}$. An example of such a graph is the one depicted in the Fig. 4 in which, for instance, the fact that the vertex 000 is a sink of the graph is equivalent to the fact that the assignment 000 is preferred to any other assignment.

Then, fixed a subset $\mathbf{X} \subset \mathbf{V}$, the set of completions of a partial assignment $x \in \mathcal{O}_{\mathbf{X}}$ corresponds to the subset of vertices

$$V(G_x) = \{o \in \mathcal{O} | \mathrm{pr}_{\mathbf{X}}(o) = x\},$$

which naturally defines the subgraph $G_x = (V(G_x), E(G_x))$ of G with

$$E(G_x) = \{(o, o') \in E(G) \mid o, o' \in V(G_x)\}.$$

For example, if we fix the subset $\mathbf{X} = \{X_1\} \subset \mathbf{V} = \{X_1, X_2, X_3\}$ in the Example 4.10 and we consider the partial assignment $x = 1 \in \mathcal{O}_{\mathbf{X}} = \{0, 1\}$, then the subgraph G_x is the graph in the right side of the cube in the Fig. 4, that is the subgraph involving the vertices $V(G_x) = \{100, 110, 111, 101\}$ in which the value of the first entry is always 1.

More in general, given a partition $\{\mathbf{Y}_1, \ldots, \mathbf{Y}_k\}$ of $\mathbf{V}$, the set of completions of a partial assignment $y_2 \ldots y_k$ corresponds to the subset of vertices

$$V(G_{y_2 \ldots y_k}) = \{o \in \mathcal{O} \mid pr_{\mathbf{Y}_j}(o) = y_j, j = 2, \ldots, k\}$$

which naturally defines the graph $G_{y_2 \ldots y_k}$.

Preferentially and conditionally independence Recall that two graphs G and H are isomorphic if and only if there is a bijection f between vertices which preserves the edges. Given a partition $\{\mathbf{Y}_1, \ldots, \mathbf{Y}_k\}$ of $\mathbf{V}$ and two partial assignments $y_2 \ldots y_k, z_2 \ldots z_k$, in this paper we are mainly interested in the graph isomorphism

$$G_{y_2 \ldots y_k} \simeq G_{z_2 \ldots z_k}$$

which sends the vertex $o \in V(G_{y_2 \ldots y_k})$, $pr_{\mathbf{Y}_1}(o) = x$ into the only vertex $o' \in V(G_{z_2 \ldots z_k})$ which satisfies $pr_{\mathbf{Y}_1}(o') = x$. For simplicity, in the rest of the paper we use the symbol $\simeq$ to refer to such isomorphisms.

We can easily check that, given a partition $\{\mathbf{X}, \mathbf{Y}\}$ of $\mathbf{V}$, $\mathbf{X}$ is preferentially independent of $\mathbf{Y}$ if and only if $G_{y_1} \simeq G_{y_2}$ for all $y_1, y_2 \in \mathcal{O}_{\mathbf{Y}}$. Indeed the Eq. (2.3) becomes

$$(x_2 y_1, x_1 y_1) \in E(G_{y_1}) \text{ if and only if } (x_2 y_2, x_1 y_2) \in E(G_{y_2}). \tag{4.1}$$

For instance, in the Fig. 5d the four edges oriented all in the same direction correspond to the fact that $G_{y_1} \simeq G_{y_2}$ for all $y_1, y_2 \in \mathcal{O}_{\mathbf{Y}}$ given the partition $\{\mathbf{X}, \mathbf{Y}\}$, $\mathbf{X} = \{X_1\}$, $\mathbf{Y} = \{X_2, X_3\}$ of $\mathbf{V}$. Indeed in this case $\mathcal{O}_{\mathbf{Y}} = \{00, 01, 10, 11\}$ and, for example, G_{00} is the graph having as vertices $V(G_{00}) = \{000, 100\}$ and as edges the only edge joining 100 to 000. It is then an easy remark that $G_{y_1} \simeq G_{y_2}$ for all $y_1, y_2 \in \mathcal{O}_{\mathbf{Y}}$.

Similarly, given a partition $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$ of $\mathbf{V}$, $\mathbf{X}$ is said to be conditionally preferentially independent of $\mathbf{Y}$ given $z \in \mathcal{O}_{\mathbf{Z}}$ if $G_{y_1 z} \simeq G_{y_2 z}$ for any $y_1, y_2 \in \mathcal{O}_{\mathbf{Y}}$ and given $\mathbf{Z}$ if this holds for any $z \in \mathcal{O}_{\mathbf{Z}}$. Equation (2.4) becomes

$$(x_2 y_1 z, x_1 y_1 z) \in E(G_{y_1 z}) \text{ if and only if } (x_2 y_2 z, x_1 y_2 z) \in E(G_{y_2 z}). \tag{4.2}$$

Family of Parent sets Consider a set of parent variables $P(X_i) \subset \mathbf{V}$ defined in Sect. 2.3. If we call *parent sets* such subsets of $\mathbf{V}$ then , for any fixed variable $X_i \in \mathbf{V}$ the *family of parent sets* is

$$\mathcal{P}(X_i) = \{\mathbf{Z} \subset \mathbf{V} \mid G_{zy_1} \simeq G_{zy_2} \forall z \in \mathcal{O}_{\mathbf{Z}}, \forall y_1, y_2 \in \mathcal{O}_{\mathbf{Y}}, \mathbf{Y} = \mathbf{V} \setminus (\{X_i\} \cup \mathbf{Z})).$$

Notice that the two variable subsets $\mathbf{X} = \{X_i\}$ and $\mathbf{Y} = \mathbf{V} \setminus \{X_i\}$ are preferentially independent if and only if $\emptyset \in \mathcal{P}(X_i)$.

Remark 4.3 It is worthy to mention at this point that the variables correspond to the features described in the Sect. 3.1. Analogously the CP-nets decision process which fixes a single variable at each step, corresponds to the special case in which the objects scheme contains exactly as many objects as the number of variables and each object has exactly one element, i.e. $A = \{\{X_1\}, \ldots, \{X_n\}\}$. In particular an order of the variables corresponds to an agenda. This allows to retrieve in the automata decisions the concept of local optimum (see the Sect. 4.4) and the result on local optimum obtained in [2] and described in Sect. 3.1.

Finally notice that the agenda in the CP-nets decision process is simply an order on the variables and it is only repeated once. Meanwhile in the Marengo and Pasquali decision process the fixed agenda can be more general allowing several repetitions inside it and the agenda itself can be repeated infinitely many times. This implies that the known results on the local optimum have an higher degree of generality.

4.2 The Lattice of parent sets and the minimal elements

In this section we show that the family of parent sets is a lattice with respect to the operations of intersection and union of sets. Moreover the family $\mathcal{P}(X_i)$ can be partially ordered by inclusion and, in particular, it admits a minimum element which turn out to be the best choice in the CP-net decision process (see Sect. 4.3).

In order to prove that the family of parent sets is a lattice we need the following Lemma.

Lemma 4.4 *Let $X_i \in \mathbf{V}$ be a variable, $\mathcal{P}(X_i)$ its family of parent sets and $Z_1 \in \mathcal{P}(X_i)$. Then any subset $Z_2 \subset \mathbf{V}$ such that $Z_2 \supset Z_1$ belongs to $\mathcal{P}(X_i)$.*

Proof Denote by $Y_1 := V \setminus (\{X_i\} \cup Z_1)$ and $Y_2 := V \setminus (\{X_i\} \cup Z_2)$. Since $Z_1 \subset Z_2$, then any $z_2 \in \mathcal{O}_{Z_2}$ can be uniquely written as the completion $z_2 = z_1 z$ of an element $z_1 \in Z_1$ such that $pr_{Z_2 \setminus Z_1}(z_2) = z$. Hence, in particular, for any two elements $y_2, y_2' \in \mathcal{O}_{Y_2}$, the completions $y_1 = y_2 z$ and $y_1' = y_2' z$ are elements in $\mathcal{O}_{Y_1}$ and, by $Z_1 \in \mathcal{P}(X_i)$, we have that $G_{z_1 y_1} \simeq G_{z_1 y_1'}$. Finally, by the unicity of the completion, $z_1 y_1 = z_1 z y_2 = z_2 y_2$ and $z_1 y_1' = z_1 z y_2' = z_2 y_2'$, that is $G_{z_2 y_2} \simeq G_{z_1 y_1} \simeq G_{z_1 y_1'} \simeq G_{z_2 y_2'}$. □

Proposition 4.5 *Let $X_i \in \mathbf{V}$ be a variable, $\mathcal{P}(X_i)$ its family of parent sets and $Z_1, Z_2 \in \mathcal{P}(X_i)$. Then $Z_1 \cup Z_2 \in \mathcal{P}(X_i)$ and $Z_1 \cap Z_2 \in \mathcal{P}(X_i)$.*

Proof [$\cup$] It is a direct consequence of Lemma 4.4.
[$\cap$] The cases $Z_1 \subset Z_2$ and $Z_2 \subset Z_1$ are trivially true. Let's consider the case in which $Z_1 \not\subset Z_2$ and $Z_2 \not\subset Z_1$ and define $Z = Z_1 \cap Z_2$. If $Z \neq \emptyset$ then V is partitioned by Z, $Y_1 = Z_1 \setminus Z_2$, $Y_2 = Z_2 \setminus Z_1$ and $Y_3 = V \setminus (Z_1 \cup Z_2)$.

Let's denote by Y the set $Y = Y_1 \cup Y_2 \cup Y_3 = V \setminus Z$. Then any two elements $y, y' \in \mathcal{O}_Y$ are written in an unique way as $y = y_1 y_2 y_3$, $y' = y_1' y_2' y_3'$ with $y_i, y_i' \in \mathcal{O}_{Y_i}$ and we have

$$G_{zy} = G_{zy_1 y_2 y_3} = G_{z_1 y_2 y_3} \simeq G_{z_1 y_2' y_3'} = G_{z, y_1 y_2' y_3'},$$

where the isomorphism holds by $z_1 = zy_1 \in Z_1 \in \mathcal{P}(X_i)$. Analogously, by $z_2 = zy_2' \in Z_2 \in \mathcal{P}(X_i)$, we have

$$G_{zy} \simeq G_{zy_1 y_2' y_3'} = G_{z_2 y_1 y_3'} \simeq G_{z_2 y_1' y_3'} = G_{zy_1' y_2' y_3'} = G_{zy'}$$

which conclude the proof when $Z \neq \emptyset$. Exactly same argument applies to the case $Z = \emptyset$ considering the partition of V given by Y_1, Y_2, Y_3. □

If we denote by

$$P_{\mathrm{m}}(X_i) = \bigcap_{Z \in \mathcal{P}(X_i)} Z \quad \text{and} \quad P_M(X_i) = \bigcup_{Z \in \mathcal{P}(X_i)} Z \tag{4.6}$$

by Proposition 4.5 $P_{\mathrm{m}}(X_i), P_M(X_i) \in \mathcal{P}(X_i)$ and the following theorem holds.

Theorem 4.7 *For any variable $X_i \in \mathbf{V}$, its family of parent sets $\mathcal{P}(X_i)$ is a bounded lattice with $P_m(X_i)$ and $P_M(X_i)$ as, respectively, minimum and maximum elements.*

4.3 The minimal CP-graph

In this Subsection we are going to define a graph structure that we call *minimal CP-graph*, equivalent to the CP-net model. We will need the following definition.

Definition 4.8 Let $\mathbf{V} = \{X_1, \dots, X_n\}$ be a set of variables. Then for any fixed element $\mathbf{P} = (P(X_1), \dots, P(X_n)) \in \prod_{i \in \{1,\dots,n\}} \mathcal{P}(X_i)$, we call **P**-*variable graph* the graph $\mathcal{G}_{\mathbf{P}}(\mathbf{V})$ having $\mathbf{V}$ as set of vertices and $\{(X_j, X_i) \in \mathbf{V} \times \mathbf{V} \mid X_j \in P(X_i)\}$ as set of edges.

Let $V = \{X_1, \dots, X_n\}$ be a set of variables, $\mathcal{O} = \prod_{i \in \{1,\dots,n\}} Dom(X_i)$ the space of all possible alternatives ranked by the preference order $\succ$, $G = (\mathcal{O}, \succ)$ the associated preference graph defined in the Sect. 4.1 and $\mathcal{P}(X_i)$, $i = 1, \dots, n$ families of parent sets.

For any element $\mathbf{Z} \in \mathcal{P}(X_i)$, $X_i \in \mathbf{V}$ is conditionally preferentially independent of $\mathbf{Y} = \mathbf{V} \setminus (\{X_i\} \cup \mathbf{Z})$ and, in particular, for any fixed $z \in \mathcal{O}_{\mathbf{Z}}$, all graphs in $\{G_{zy} \mid y \in \mathcal{O}_{\mathbf{Y}}\}$ are isomorphic. Then, without restriction of generality, we can pick any representative $G_{zy} \in \{G_{zy} \mid y \in \mathcal{O}_{\mathbf{Y}}\}$ and give the following definition equivalent to the definition of CP-net.

Definition 4.9 Let $V = \{X_1, \dots, X_n\}$ be a set of variables, $G = (\mathcal{O}, \succ)$ the associated preference graph and $\mathcal{P}(X_i), i = 1, \dots, n$ families of parent sets. Fixed an element $\mathbf{P} = (P(X_1), \dots, P(X_n)) \in \prod_{i \in \{1,\dots,n\}} \mathcal{P}(X_i)$, we call *CP-graph* the couple $(\mathcal{G}_{\mathbf{P}}(\mathbf{V}), \{G_P(X_i)\}_{i=1,\dots,n})$ where

$$G_P(X_i) = \bigcup_{z \in \mathcal{O}_{P(X_i)}} G_{zy_i}, \quad \text{for a chosen } y_i \in \mathcal{O}_{\mathbf{V} \setminus (\{X_i\} \cup P(X_i))}.$$

In particular if $\mathbf{P} = (P_m(X_1), \dots, P_m(X_n))$ is the n-tuple of minimal parent sets, we will call $(\mathcal{G}_{\mathbf{P}}(\mathbf{V}), \{G_{P_m}(X_i)\}_{i=1,\dots,n})$ the *minimal* CP-graph and it will be denoted by $(\mathcal{G}_m(\mathbf{V}), \{G_m(X_i)\}_{i=1,\dots,n})$.

We provide the following example.

Example 4.10 Let's consider the acyclic complete graph G represented in Fig. 4 associated to the variable set $\mathbf{V} = \{X_1, X_2, X_3\}$ with binary variables X_i, i.e. $\mathcal{O}_{X_i} = \{0, 1\}$, and families of parent sets represented in Table 3.

Let's describe the graphs $G_P(X_1)$ for $P(X_1) \in \mathcal{P}(X_1)$. In the case that $P(X_1) = \emptyset$, if we choose $y = 00 \in \mathcal{O}_{\mathbf{V} \setminus (\{X_1\} \cup \emptyset)}$ as representative, $G_P(X_1)$ is the graph represented in Fig. 5a.

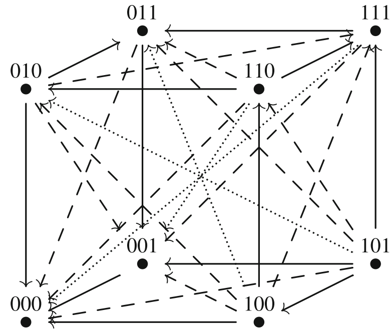
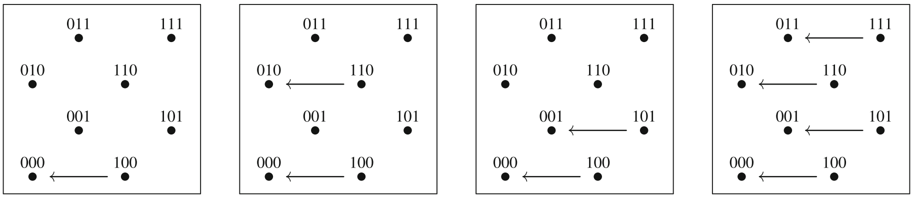


Fig. 4 The graph G with set of vertices $\mathcal{O}$

Table 3 Parent sets and minimum parent sets

X_i	$\mathcal{P}(X_i)$	$P_{\mathrm{m}}(X_i)$
X_1	$\{\emptyset, \{X_2\}, \{X_3\}, \{X_2, X_3\}\}$	$\emptyset$
X_2	$\{\{X_1\}, \{X_1, X_3\}\}$	$\{X_1\}$
X_3	$\{\{X_2\}, \{X_1, X_2\}\}$	$\{X_2\}$



(a) $P(X_1) = \emptyset$. **(b)** $P(X_1) = \{X_2\}$. **(c)** $P(X_1) = \{X_3\}$. **(d)** $P(X_1) = \{X_2, X_3\}$.

Fig. 5 Graphs $G_P(X_1)$'s

Remark that although we can get different $G_P(X_1)$ depending on fixed $y \in \mathcal{O}_{\mathbf{V}\setminus(\{X_1\}\cup\emptyset)}$, they are all isomorphic.

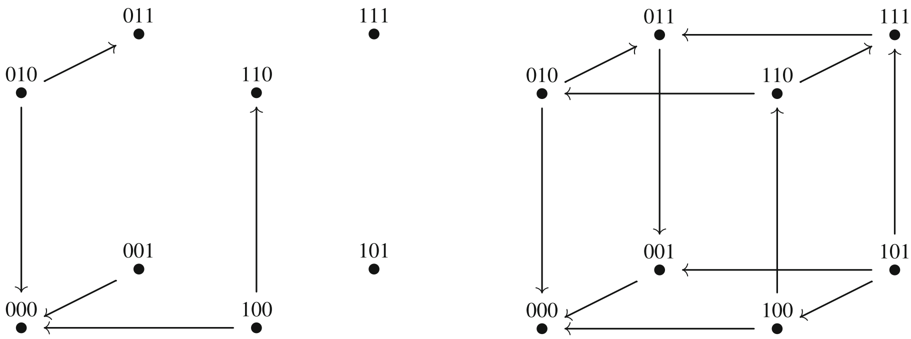
Similarly in the case $P(X_1) = \{X_2\}$ (resp. $\{X_3\}$), by fixing $y = 0 \in \mathcal{O}_{\mathbf{V}\setminus(\{X_1\}\cup\{X_2\})}$ (resp. $y = 0 \in \mathcal{O}_{\mathbf{V}\setminus(\{X_1\}\cup\{X_3\})}$), we get the graph $G_P(X_1)$ represented in Fig. 5b (resp. Fig. 5c). Finally the graph $G_P(X_1)$ for $P(X_1) = \{X_2, X_3\}$ is represented in Fig. 5d.

Comparing the graphs in Fig. 5, we observe that graphs in Fig. 5a, c and d can be obtained from the graph in Fig. 5a associated to the minimum parent set $P_m(X_1) = \emptyset$ by isomorphism, i.e. considering the union of the copies of $G_P(X_1)$ obtained when y varies opportunely in $\mathcal{O}_{\mathbf{V}\setminus(\{X_1\}\cup\emptyset)} = \{00, 01, 10, 11\}$. This suggests that it is enough to describe the graphs $G_P(X_i)$ for $P(X_i) = P_m(X_i)$ the minimum parent set. The graphs in Fig. 6a and b represent, respectively, the graphs $G_P(X_2)$, $G_P(X_3)$ in the case $P(X_i) = P_m(X_i)$, $i = 2, 3$ minimum parent sets.

Notice that graphs in Figs. 5 and 6 are subgraphs of the graph of preferences G with edges between vertices which only differ by the value of exactly one variable $X_i \in \mathbf{V}$. This is a direct consequence of CP-net definition (see Sect. 2.3) in which the conditional preference table $CPT(X_i)$ is defined using the total order among alternatives which only differ by the value of the variable X_i. In particular, fixed a parent set $P(X_i)$ for each variable X_i it is

(a) The graph $G_P(X_2), P(X_2) = P_m(X_2) = \{X_1\}$. **(b)** The graph $G_P(X_2), P(X_3) = P_m(X_3) = \{X_2\}$.

Fig. 6 The graphs $G_P(X_i)$ for $P(X_i) = P_m(X_i), i = 2, 3$



(a) The subgraph $H_{\mathbf{P}} = \bigcup_{i=1}^{3} G_{P_m}(X_i)$. **(b)** The Hamming one subgraph H of G

Fig. 7 The subgraph of G equivalent to the conditional preference table of Example 4.10

possible to build the subgraph

$$H_{\mathbf{P}} = \bigcup_{i=1}^{n} G_P(X_i), \quad \mathbf{P} = (P(X_1), \ldots, P(X_n))$$

of G with vertices all vertices in G and edges between vertices differing by just one entry, i.e. having the so called *Hamming distance* equal to 1.

For example, if we consider the case illustrated in Example 4.10, the subgraph $H_{\mathbf{P}} = \bigcup_{i=1}^{3} G_{P_m}(X_i)$ is the graph represented in Fig. 7a.

In particular, it is an easy remark that isomorphisms $G_{zy} \simeq G_{zy'}$, $y, y' \in \mathcal{O}_{\mathbf{V}\setminus(\{X_i\}\cup P(X_i))}$ for each value $z \in \mathcal{O}_{P(X_i)}, i = 1, \ldots, n$ allow to extend the graph $H_{\mathbf{P}}$ in an unique way to the only subgraph H of G having as vertices all vertices of G and as edges all edges of Hamming distance 1 of G, i.e. all edges which only differ for the value of one variable X_i (Example 4.10 is depicted in Fig. 7b). We will call H the *Hamming one subgraph* of G and since, obviously, the graph H is independent from the choice of the set $\mathbf{P}$, we can state the following Theorem.

Theorem 4.11 *Let $V = \{X_1, \ldots, X_n\}$ be a set of variables, $G = (\mathcal{O}, \succ)$ the associated preference graph and $\mathcal{P}(X_i), i = 1, \ldots, n$ the families of parent sets. Then all the CP-graphs $(\mathcal{G}_{\mathbf{P}}(\mathbf{V}), \{G_P(X_i)\}_{i=1,\ldots,n})$ give rise to the same subgraph H of G for any choice of $\mathbf{P} \in \prod_{i\in\{1,\ldots,n\}} \mathcal{P}(X_i)$. In particular H is the Hamming one subgraph of G.*

As a corollary of this Theorem we get that the minimal CP-graph $(\mathcal{G}_m(\mathbf{V}), \{G_m(X_i)\}_{i=1,\dots,n})$ is a good representative since in any decision process, it provides the same preference graph with the smallest amount of choices, i.e. in the fastest time. In particular the graph of variables $\mathcal{G}_m(\mathbf{V})$ uniquely defines the families of parent sets $\mathcal{P}(X_i)$ for $i = 1, \dots, n$. By definition of variable graph we have the following Proposition.

Proposition 4.12 *Let $\boldsymbol{V} = \{X_1, \dots, X_n\}$ be a set of variables and $\mathcal{G}(\boldsymbol{V})$ be an a-cyclic complete variable graph. Then $\mathcal{G}(\boldsymbol{V}) = \mathcal{G}_m(\boldsymbol{V})$ is the minimal variable graph associated to the n-tuple of minimal parent variables $\boldsymbol{P} = (P_m(X_1), \dots, P_m(X_n)) \in \prod_{i\in\{1,\dots,n\}} \mathcal{P}(X_i)$ and if $X_{i_1} \succ X_{i_2} \succ \cdots \succ X_{i_n}$ is the transitive order on the variables induced by $\mathcal{G}(\boldsymbol{V})$ then:*

1. *$\emptyset \in \mathcal{P}(X_{i_1})$;*
2. *there exists a chain of parent variables*

$$\emptyset \subset \{X_{i_1}\} \subset \{X_{i_1}, X_{i_2}\} \subset \cdots \subset \{X_{i_1}, \dots, X_{i_j}\} \cdots \subset \{X_{i_1}, \dots, X_{i_{n-1}}\} \quad (4.13)$$

   such that $\{X_{i_1}, \dots, X_{i_j}\} \in \mathcal{P}(X_{i_{j+1}})$;
3. *the parent set $\{X_{i_1}, \dots, X_{i_j}\}$ is the minimum parent set $P_m(X_{i_{j+1}})$.*

Notice that the presence or absence of an edge in the minimal variable graph $\mathcal{G}_m(\mathbf{V})$ matters as it changes the minimal parent sets and, consequently, the family of parent sets. The question whether a given graph is the minimal variable graph of a set of variable $\mathbf{V}$ deserves to be better investigated. In the next Section we will deal with the difference on CP-graphs when the variable graph is a-cyclic or contains cycles.

4.4 Satisfiability and B-optima

In Sect. 4.3 we proved that a CP-graph is equivalent to provide a minimum graph $H_{\mathbf{P}}$ which can be completed by isomorphisms to the hamming one subgraph $H = (\mathcal{O}, \succ_H)$ of the acyclic complete graph $G = (\mathcal{O}, \succ)$. Notice that two different complete graphs $G = (\mathcal{O}, \succ)$ and $G' = (\mathcal{O}, \succ')$ can have the same hamming one subgraph H. On the other hand any graph $G = (\mathcal{O}, \succ)$ which has H as subgraph has to contain the acyclic graph $T = (\mathcal{O}, \succ_T)$ obtained completing H by transitivity, i.e. adding to H all the edges (o, o') such that it exists an oriented path from o to o' in H. To build the graph T from $H_{\mathbf{P}}$ is equivalent to what [3] called *entailment*. The fact that T can be completed to, at least, one acyclic graph G is equivalent to their definition of *satisfiable* CP-net. We get that the following statement equivalent to Theorem 1. in [3] holds.

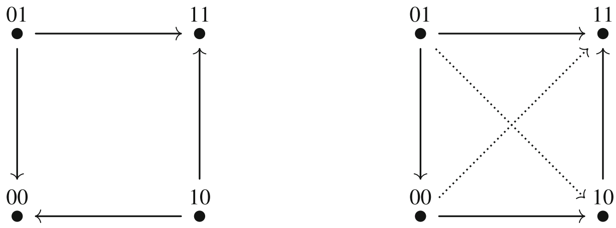
Theorem 4.14 *Every acyclic CP-net is satisfiable.*

If we denote by $\mathbf{P}_m = (P_m(X_1), \dots, P_m(X_n))$ the n-tuple of minimal parent sets, then Theorem 4.14 is equivalent to the following statement.

Theorem 4.15 *If the $\boldsymbol{P}_m$-variable graph $\mathcal{G}_m(\boldsymbol{V})$ is acyclic, then H is the hamming distance one subgraph of at least one acyclic complete graph G.*

Since the graph G from Theorem 4.15 is not unique, we can ask what we can say about the existence of a most preferred choice[5]. In order to study this problem, we introduce the notion of *B-optimum*, and we compare it with the one of local optimum defined in Sect. 3.1.

[5] Notice that if G would be unique then the most preferred choice would be the unique sink of G acyclic complete graph.



(a) The hamming one subgraph H_1 **(b)** The hamming one subgraph H_2

Fig. 8 Hamming one subgraphs. The added edges in H_2 are the dotted ones

(a) $\mathbf{P}_m$-variable graph associated to H_1 **(b)** $\mathbf{P}_m$-variable graph associated to H_2

Fig. 9 Minimal variable graphs associated to the hamming one subgraphs in Fig. 8

Definition 4.16 Let $G = (\mathcal{O}, \succ)$ be an acyclic complete graph, $H \subset G$ be its hamming one subgraph and $T = (\mathcal{O}, \succ_T)$ the acyclic extended graph of H. An element $o \in \mathcal{O}$ is called a *B-optimum* if $o' \not\succ_T o$ for any $o' \in \mathcal{O}, o' \neq o$. The set of all B-optima will be denoted by $\mathcal{B}$.

Example 4.17 Let's consider the hamming one subgraphs H_1 and H_2 depicted in Fig. 8. While the graph H_2 can be completed by transitivity by adding the dotted edges as drawn in Fig. 8b the graph H_1 cannot and hence it coincides with its acyclic extended graph T_1. It is easy to check that H_1 admits two B-optima 00 and 11 while H_2 only the B-optimum 11. It is also an easy remark that in both cases the B-optima coincide with the local optima of the graphs. This is not a coincidence as stated in the following Theorem 4.18.

Theorem 4.18 *Let $G = (\mathcal{O}, \succ)$ be an acyclic complete graph. An element $o \in \mathcal{O}$ is a B-optimum if and only if it is a local optimum.*

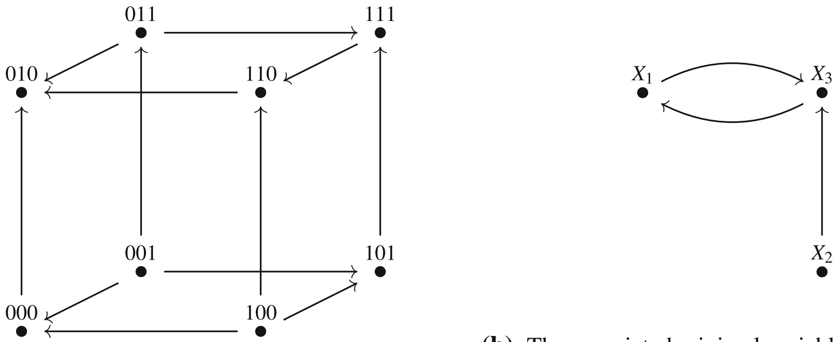
Proof An element $o \in \mathcal{O}$ is a local optimum if and only if it satisfies the condition $o \succ o'$ for any element $o' \in \mathcal{O}$ such that $d_H(o, o') = 1$. This implies that there is no path along H that can end in o hence $o' \not\succ_T o$ for any $o' \in \mathcal{O}$, that is o is a B-optimum. The vice versa is obvious. □

Finally let's consider the two minimum variable graphs depicted in Fig. 9. The two graphs represent the minimum variable graphs associated, respectively, to the graphs H_1 and H_2 depicted in Fig. 8. We notice that while the minimum graph associated to H_1 is cyclic, the one associated to H_2 is acyclic. This is related to the existence of multiple B-optima as stated in the following theorem.

Theorem 4.19 *Let V be a set of n variables and $G = (\mathcal{O}, \succ)$ an acyclic complete graph. If the minimum variable graph $\mathcal{G}_m(\mathbf{V})$ is acyclic, then the B-optimum exists and is unique.*

Proof Let's suppose, by absurd, that there are two B-optima $o_1, o_2, o_1 \neq o_2$. Then the set

$$\mathbf{W} = \{X_i \in \mathbf{V} \mid pr_{X_i}(o_1) \neq pr_{X_i}(o_2)\}$$



(b) The associated minimal variable graph.

(a) The hamming one (sub)graph

Fig. 10 The counterexample to the converse of Theorem 4.19

is not empty and fixed an element $X \in \mathbf{W}$ we can write $o_1 = x_1y_1z, o_2 = x_2y_2z$ with $x_i \in \mathcal{O}_X, y_i \in \mathcal{O}_{\mathbf{W}\setminus X}, z \in \mathcal{O}_{\mathbf{V}\setminus\mathbf{W}}$. If we define $o'_1 = x_2y_1z$ and $o'_2 = x_1y_2z$ then o'_i differs from o_i only on the entry of the variable X and hence $o_i \succ o'_i$ since, by Theorem 4.18, o_1 and o_2 are also local optima. It follows that the graphs G_{y_1z} and G_{y_2z} are not isomorphic and hence X is nor preferentially independent of $\mathbf{V} \setminus \{X\}$ nor conditionally preferentially independent of $\mathbf{W} \setminus \{X\}$ given $\mathbf{V} \setminus \mathbf{W}$, i.e. $\emptyset \notin \mathcal{P}(X)$ and $\mathbf{V} \setminus \mathbf{W} \notin \mathcal{P}(X)$. The latter implies that there is another variable $X' \in \mathbf{W}$, $X' \neq X$ such that $X' \in P_m(X)$, that is (X', X) is an edge in the subgraph $\mathcal{G}_m(\mathbf{W})$ of $\mathcal{G}_m(\mathbf{V})$.

Let's now consider the longest path $X_{i_1} \ldots X_{i_h}$ in $\mathcal{G}_m(\mathbf{W})$. By the above considerations, fixed the first variable of the path X_{i_1}, there is a variable X' such that (X', X_{i_1}) is an edge of $\mathcal{G}_m(\mathbf{W})$. Since $X_{i_1} \ldots X_{i_h}$ is the longest path, then there is an index $j \in \{1, \ldots, h\}$ such that $X' = X_{i_j}$, that is $\mathcal{G}_m(\mathbf{W})$ contains the cycle $X_{i_1} \ldots X_{i_j}$ which is an absurd as $\mathcal{G}_m(\mathbf{W})$ is a subgraph of an acyclic graph. □

Remark that the converse of the Theorem 4.19 does not hold in general. A counterexample is shown in Fig. 10: the vertex 010 is the unique B-optimum of the humming one graph in (a), but the variable graph, depicted in (b), is not acyclic.

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Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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