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"Mixing without randomness"

Paolo Ghirardato and Daniele Pennesi

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Mixing without randomness

Paolo Ghirardato[§] Daniele Pennesi[¶]

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Abstract

We provide a framework for identifying utility midpoints from preferences, by assuming that payoffs are vectors and that preferences are additively separable. This construction of utility midpoints allows us to define mixtures of acts in a purely subjective fashion, without making any assumptions as to the decision maker's reaction to the uncertainty that may be present. Indeed, we do not require any (subjective or objective) uncertainty to obtain the identification. We show that this framework makes it possible to provide a simple and fully subjective characterization of the second-order subjective expected utility model, and that it allows a clear distinction of such model from subjective expected utility.

1 Introduction

The pervasiveness of the Anscombe-Aumann (1963) setting for decision models is due to the presence of an objective "mixture-space" structure on the choice set, which makes it easy to formulate axioms with a direct mathematical counterpart and to invoke standard results from functional analysis. On the other hand, the existence of such objective mixture structure limits the scope of preferences that can be formally captured within an Anscombe-Aumann setting. This tension gives rise to a challenge: to identify a mixture-space structure which does not impose strong restrictions on preferences and on individuals' reactions to the uncertainty present in the choice problem.

In this paper, we develop a *subjective* mixture-space structure which is totally independent of the uncertainty of the problem, only imposing restrictions on preferences which are standard in economic models. We assume that payoffs are vector-valued and that preferences are additively separable over vector coordinates —as would be the case, say, if payoffs were (finite-horizon) consumption streams and the decision maker discounted future utilities geometrically or hyperbolically. In this context, we develop a procedure for identifying "utility midpoints" from preferences:

[§]DESOMAS and Collegio Carlo Alberto, University of Torino, email: *paolo.ghirardato@unito.it*
[¶]DESOMAS, University of Torino, email: *daniele.pennesi@unito.it*

for any two vectors a and b, we show how to find a vector c, the utility of which is halfway between the utility of a and the utility of b. Utility midpoints are then used recursively to super-impose a mixture-space structure on the set of payoffs, which is clearly subjective (and in general different from any vector-space structure that may exist on consequences).

The assumption of additive separability plays a dual role in this paper. On one hand, it provides us with a testable method for identifying utility midpoints from behavior (details below). On the other hand, it implies the *cardinality* of the utility function. Cardinality is crucial to our exercise as it makes the definition of utility midpoints and subjective mixtures meaningful. It is also key in the identification of risk attitude, as it is traditionally defined (i.e. the shape of the utility function). Our derivation of cardinality *via* additive separability has the additional advantage of imposing no restrictions on the decision maker's reaction to uncertainty in the decision problem.[1] As a consequence, we obtain a fully subjective axiomatization of the second-order subjective expected utility (SOSEU) model, which is otherwise hard to behaviorally distinguish from the standard subjective expected utility (SEU) model (see Strzalecki, 2011).

In brief, our analysis proceeds as follows. As mentioned, the consequences of the decision problem are elements of a product space $\prod_i X_i$; e.g., consumption streams. We begin by assuming that the decision maker's preferences over such consequences are represented by an additively separable (and cardinally unique) utility U. That is, $U(x_1,\ldots,x_n) = \sum_{i=1}^n u_i(x_i)$. Then, we introduce the notion of *preference midpoint* of two consequences. To provide intuition, consider first the case in which consequences are 2-dimensional and homogeneous and they are treated symmetrically by the decision maker: $d = (x_1, x_2) \in X \times X$ and $U(d) = u(x_1) + u(x_2)$. Consider two (constant) consequences $a = (x, x)$ and $b = (y, y)$. In this case, the vector $c = (x, y)$ provides an intuitive "mixture" of the vectors $a = (x, x)$ and $b = (y, y)$. And indeed:

$$U(c) = u(x) + u(y) = \left(\frac{1}{2}u(x) + \frac{1}{2}u(x)\right) + \left(\frac{1}{2}u(y) + \frac{1}{2}u(y)\right) = \frac{1}{2}U(a) + \frac{1}{2}U(b)$$

So $c = (x, y)$ is also a "midpoint" in terms of utility of the vectors a and b. Notice that a similar property is satisfied by the mirror-image vector $c' = (y, x)$, so that $c = (x, y) \sim (y, x) = c'$. Therefore, in the special case of homogeneous and symmetric coordinates, the utility midpoint of $a = (x, x)$

[1] Alternative methods to the construction of a cardinal utility that make such restrictions are, for example, Nau (2006); Ergin and Gul (2009); Gul (1992); Ghirardato et al. (2003); Ghirardato and Pennesi (2018). See the discussion in Section 1.1.

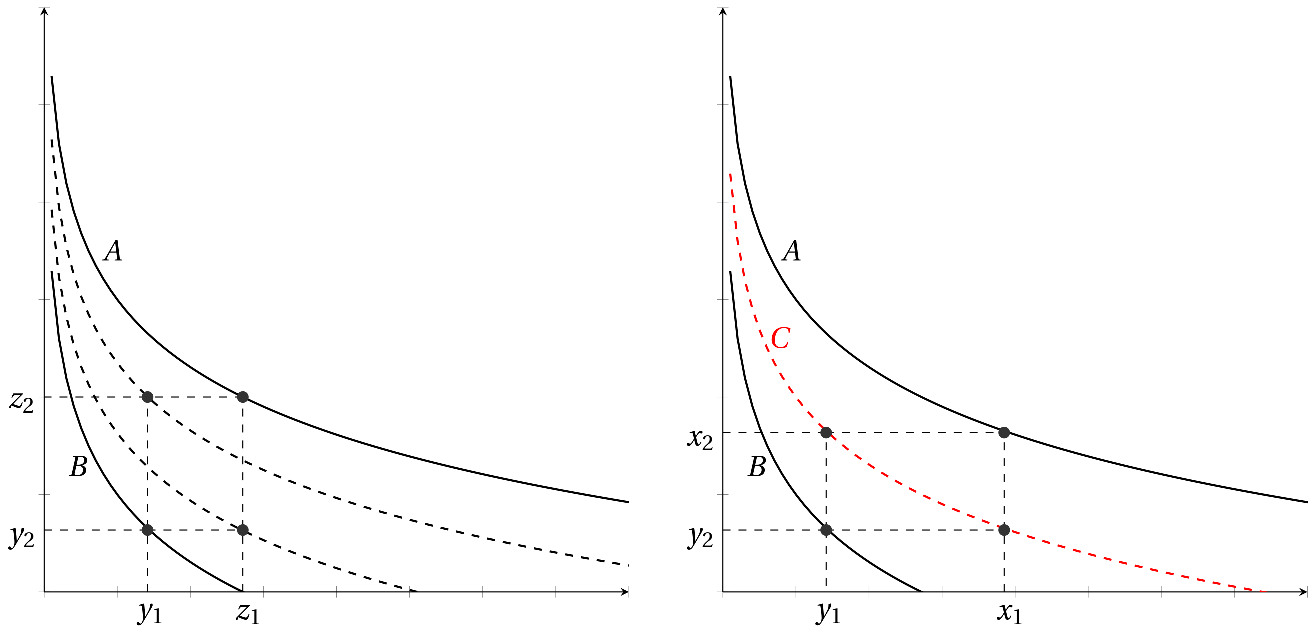

Figure 1: Looking for preference midpoints

and $b = (y, y)$ can be directly observed from preferences.

The intuition extends to the case of non-constant payoff vectors and, more generally, to the case in which consequences belong to a non-homogeneous product space $X_1 \times X_2$. Consider the left panel of Figure 1 and the points $a = (z_1, z_2)$ and $b = (y_1, y_2)$. Neither (z_1, y_2) nor (y_1, z_2) provide a "midpoint" of a and b, and indeed $(z_1, y_2) \not\sim (y_1, z_2)$. However, this does not mean that we cannot find a midpoint of a and b. We can do so by moving along the indifference curves A and B that include a and b, respectively. Consider the right panel of Figure 1, where we keep the point b fixed and find a point $a' = (x_1, x_2)$ belonging to A such that $(y_1, x_2) \sim (x_1, y_2)$ and therefore $U(y_1, x_2) = \frac{1}{2}U(y_1, y_2) + \frac{1}{2}U(x_1, x_2)$. That is, (y_1, x_2) is a midpoint of a' and b, and therefore also a midpoint of a and b. This shows that the intuition of "mixing" coordinates provided above can be extended to many[2] pairs of point in $X_1 \times X_2$.

Once midpoints are defined for a generic pair of consequences a and b, a recursive application allows us to define the $\gamma : (1-\gamma)$ mixture of a and b for any dyadic rational $\gamma \in [0, 1]$. As in Ghirardato et al. (2003), we can use this mixture notion to provide a subjective structure on the set of consequences extending it *à la* Anscombe-Aumann to define "subjective mixtures" over the set of acts (functions from states of the world to consequences).

As a first application of such subjective mixtures, we propose a purely subjective axiomatiza-

[2] As we discuss in detail in Section 3, there might be still pairs of consequences for which the "mixing" technique does not identify midpoints. In those cases, an appropriate (finite) iteration of the procedure just sketched can be used to behaviorally identify midpoints.

tion of the Monotone, Bernoullian and Archimedean (MBA) preferences of Cerreia-Vioglio et al. (2011), obtained without assuming the existence of an objective randomization device. We recall that the class of MBA preferences includes, as special cases, most of the models of choice under ambiguity (see Section 5). In our second and main application, we provide a simple axiomatic characterization of the SOSEU model, which includes Multiplier Preferences (Hansen and Sargent, 2001). In so doing, we prove that it is possible to behaviorally distinguish the SEU model from the SOSEU model, even in a fully subjective setting. Such distinction is impossible without ancillary assumptions, such as the existence of an objective randomization device (Grant et al., 2009; Cerreia-Vioglio et al., 2012) or, more generally, the existence of an unambiguous source of uncertainty (Nau, 2006; Ergin and Gul, 2009).

The paper is structured as follows: Section 2 spells out the basic assumptions on preferences, the definition of utility and preference midpoint, and it also contains our main characterization result. Section 3 provides intuition for the existence and identification of utility midpoints. Section 4 presents some examples of the identification of midpoints and extends the notion of midpoints to single dimensions. Section 5 concludes by providing the main decision-theoretic applications with the axiomatizations of MBA preferences and of the SOSEU model. The appendix contains a few additional results and all the proofs of the results in the paper.

1.1 Related literature

The two works that are most directly related to this paper are: Kochov (2015) and Vind and Grodal (2003). The work of Kochov (2015) shares our objective of using the separability of preferences over consequences to provide a fully subjective axiomatization of some preference models. In particular, he axiomatizes the Variational Preference model of Maccheroni et al. (2006) assuming that acts are functions from states of the world to infinite horizon consumption streams. His approach does not require the existence of an objective randomization device, but it crucially depends on the infiniteness of the time-horizon and on geometric discounting of future utilities. Our work complements and extends Kochov's in two directions: first, we derive our results assuming a finite product of possibly non-homogeneous sets. Second, we dispense with the stationarity assumption implied by geometric discounting, since we only require additive separability of preferences over consumption streams (outcomes). Another difference is that Kochov (2015) provides an extension of his model to an explicitly dynamic setting, which we do not attempt in this paper.[3]

The work of Vind and Grodal (2003) is directly related to our main theoretical contribution (Theorem 1). We adopt their notion of preference midpoint and show that it can be used to provide a foundation to the notion of utility midpoints. As we explain in detail in Section 3, preference midpoints may not always exist. In contrast, we prove the general existence of utility midpoints and we present a finite algorithm for their behavioral identification. Another difference is that the general aim of Vind and Grodal (2003) is studying conditions under which preferences have an additively separable representation: Midpoints are not defined with the objective of building a mixture-space structure.

Our construction of subjective mixtures is complementary to those that propose subjective mixture operators using bets on "special" events. For example, Gul (1992) directly assumes the existence of an event with subjective probability $\frac{1}{2}$ to identify utility midpoints for SEU preferences; Ghirardato et al. (2003) and Ghirardato and Pennesi (2018) show that utility midpoints can be identified for very general classes of preferences observing choices over bets on events that are suitably well-behaved. As the title of this paper suggests, we can identify utility midpoints even in the absence of any uncertainty.

2 Midpoints

Here we present our main definition and its behavioral characterization. We first consider the case in which the set of consequences is a *binary* product space $V = X_1 \times X_2$, but this will be seen to be without loss of generality (as long as the consequence space is finite-dimensional). We denote by $a \in V$ a generic vector in V.

More crucial is the following basic assumption on preferences, which is necessary to obtain a well-defined notion of midpoint in terms of utility in our setting.

Definition 1. *The binary relation* $\succcurlyeq$ *has a* Continuous Additively Separable (CAS) *representation if there are* $U : V \to \mathbb{R}$ *and convex-ranged* $u_i : X_i \to \mathbb{R}$ *for* $i = 1, 2$ *such that*

$$U(a) = u_1(x_1) + u_2(x_2)$$

[3] Although a dynamic extension of our analysis along the lines of Ghirardato and Pennesi (2018) is possible.

represents $\succsim$. The function U is cardinally unique; i.e., if U' also represents $\succsim$, there are $\alpha > 0$ and $\beta \in R$ such that $U' = \alpha U + \beta$.

The axioms characterizing such a representation are standard and therefore omitted (see, for example, Krantz et al., 1971). As an obvious example, suppose that V is a set of two-period consumption streams over the same final consumption space $V = X \times X$, and suppose that preferences over V are represented by discounted utility, $U(a) = u(x_1) + \delta u(x_2)$, with u convex-ranged and $\delta \in (0,1]$. This is a CAS preference.

Remark 1. As mentioned earlier, our theory can also be extended to include preferences over an n-dimensional product space $V = \prod_{i=1}^{n} X_i$, as long as the preference $\succsim$ admits an additively separable representation with at least two addends. More precisely, Theorem 1 below holds as long as there is a partition $\{I, I^c\}$ of $\{1,\dots,n\}$ such that $\succsim$ is represented by

$$U(a) = \bar{u}_I(a_I) + \bar{u}_{I^c}(a_{I^c}) \tag{1}$$

where $\bar{u}_I : \prod_{i\in I} X_i \to \mathbb{R}$ and $\bar{u}_{I^c} : \prod_{j\in I^c} X_j \to \mathbb{R}$. For example, general quasi-linear preferences $U(x_1,\dots,x_n) = u_1(x_1) + f(x_2,\dots,x_n)$, or the strongly-separable preferences[4] of Goldman and Uzawa (1964) satisfy condition (1). Obviously, this will be true *a fortiori* if $\succsim$ has a representation that is additively separable across all the n dimensions; i.e., $U(x_1,\dots,x_n) = \sum_{i=1}^{n} u_i(x_i)$.

Given $V = X_1 \times X_2$ and $\succsim$, we consider the quotient space $[V] = (X_1 \times X_2)/\sim$ and we denote by $A, B \in [V]$ indifference classes and by $U : [V] \to \mathbb{R}$ the restriction of U to the indifference sets in $[V]$. As informally explained in the introduction, we define midpoints directly in terms of pairs of indifference sets and then extend the definition to pairs in V.

Definition 2. *For any $A, B \in [V]$, $C \in [V]$ is a* utility midpoint *of A and B, denoted $C = A \odot B$, if and only if $U(C) = \frac{1}{2}U(A) + \frac{1}{2}U(B)$.*

Remark 2. The definition of utility midpoint is extended to pairs $a, b \in V$ in the obvious way: $c = a \odot b$ if and only if $C = A \odot B$ and $c \in C$, $a \in A$ and $b \in B$.

While the definition of utility midpoint is based on the representation U, we show that there is a finite preference-based procedure to identify C. This is the object of our main result:

[4]A preference over $X = \prod_{i=1}^{n} X_i$ has a strongly separable representation if there exists a partition $\{I_1, I_2, \dots, I_S\}$ with $S \geq 2$ of $\{1,2,\dots,n\}$ such that $U(a) = u_1(a^{(1)}) + \cdots + u_S(a^{(S)})$ and each u_s for $s = 1,\dots,S$ is a function of the sub-vector $a^{(s)} \in \prod_{j\in I_s} X_j$.

Theorem 1. *If $\succcurlyeq$ has a CAS representation then, for all $A, B \in [V]$ a utility midpoint $C = A \odot B$ exists and it can be elicited from preferences via a finite procedure.*

That is, given an arbitrary pair of consequences it is possible to use behavioral data to identify a utility midpoint. More precisely, given a pair of points in V and their associated indifference sets, we can identify an indifference set, each element of which yields utility equaling the average of the utilities of the points we started from.

Our construction builds on the following behavioral definition of midpoint (illustrated in Figure 2 below), introduced by Vind and Grodal (2003).

Definition 3. *For any $A, B \in [V]$, $C \in [V]$ is a* preference midpoint *of A and B, if and only if there exist $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$ such that $(x_1, y_2) \sim (y_1, x_2) \in C$.*

Preference midpoints may not exist for arbitrary $A, B \in [V]$. For instance, in Figure 3 below the indifference curves A', B', passing through the points (x_1', y_2) and (y_1', x_2) respectively, do not have a preference midpoint (Def. 3) while they do have a utility midpoint (Def. 2).[5] The main novel contribution of Theorem 1 is the development of a finite procedure that, departing from the notion of preference midpoint, allows us to identify utility midpoints for all $A, B \in [V]$ using finitely many preference statements. The following subsection outlines the main ideas of the procedure, leaving the details and formal definitions to the Appendix.

3 The identification of utility midpoints: An intuition

We start by noticing that, under the CAS assumption, whenever a preference midpoint exists, it is also a utility midpoint. Refer to Figure 2 and consider the indifference curves A, B. The indifference curve C satisfies the definition of preference midpoint, because it is indeed the case that given the points $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$, the indifference $(x_1, y_2) \sim (y_1, x_2) \in C$ holds. But, using the CAS assumption, the latter indifference is mathematically equivalent to

$$U(C) = u_1(x_1) + u_2(y_2) = u_1(y_1) + u_2(x_2) \tag{2}$$

[5] In contrast, the indifference curves a, b in Fig. 3 have a preference midpoint which, under the assumption of Theorem 1, coincides with the utility midpoint.

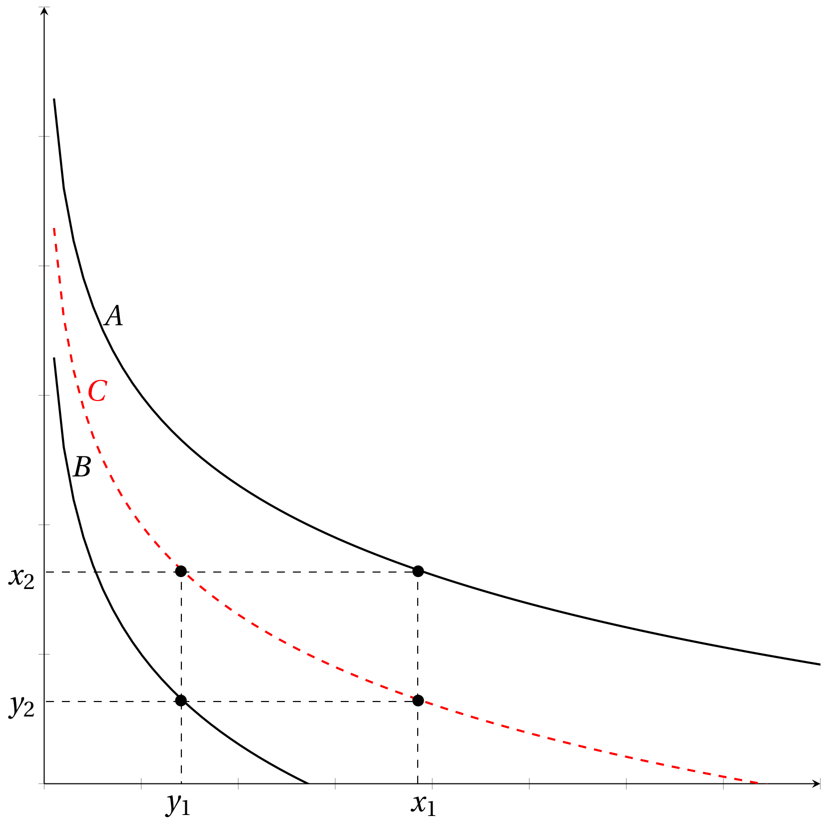

Figure 2: The preference midpoint of A and B.

which implies

$$2U(C) = u_1(x_1) + u_2(y_2) + u_1(y_1) + u_2(x_2)$$

That is,

$$U(C) = \frac{1}{2}\left[u_1(x_1) + u_2(x_2)\right] + \frac{1}{2}\left[u_1(y_1) + u_2(y_2)\right] = \frac{1}{2}U(A) + \frac{1}{2}U(B)$$

An important observation is that the previous argument only works for pairs $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$ for which we can establish the indifference $(x_1, y_2) \sim (y_1, x_2)$, which will not be the case for arbitrary choices of pairs of points in the indifference sets A and B. Our first result (Lemma 2) establishes that a preference (hence utility) midpoint of A and B always exists when A and B are intuitively "close enough."

To understand the notion of "closeness" (again, precise definitions are provided in the Appendix) refer to Figure 3: The indifference sets A and B in the middle of the figure are "close" because there is a pair of payoffs (z_1, z_2) such that the payoff z_1 appears in elements of both sets A and B (the intersections of the blue-dotted vertical line corresponding to z_1 with the two curves), and analogously for z_2 (in this case the intersections of the blue-dotted horizontal line corresponding to z_2 with the two curves). In contrast, the indifference sets A' and B' are not "close," as

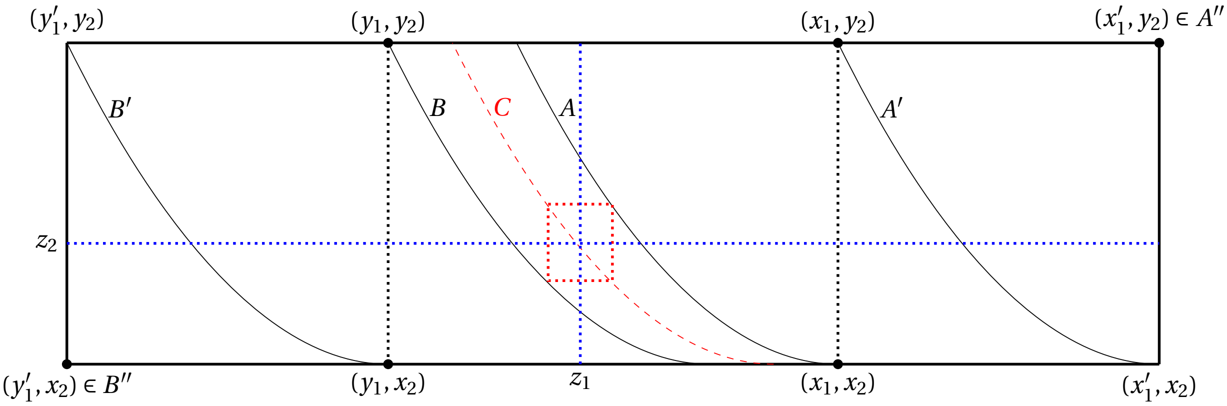

Figure 3: Identification of utility midpoints

it is impossible to find a suitable payoff z_1 (while a suitable z_2 could be found).

The second result (Lemma 3) shows that CAS preferences satisfy the so-called Diagonal property: if $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$ and C is the utility midpoint of A and B, then C is also the utility midpoint of the sets A' and B' such that $U(A') = u_1(x_1) + u_2(y_2)$ and $U(B') = u_1(y_1) + u_2(x_2)$. Referring again to Figure 3, the Diagonal property implies that the indifference set C is also the utility midpoint of A' and B', where the latter correspond to the two points $(x_1, y_2) \in A'$ and $(y_1, x_2) \in B'$ obtained from (x_1, x_2) and (y_1, y_2) by a coordinate switch. Reiterating the argument, C is also the utility midpoint of A'' and B'', derived from A' and B' by another coordinate switch.

We can now once more refer to Figure 3 to conclude the intuitive proof of Theorem 1. Consider two arbitrary indifference sets $A'', B'' \in [V]$ such that $A'' \succ B''$. It is possible to show that we can find vectors belonging to A'' and B'' that are Pareto ranked, such as $(y_1', x_2) \in B''$ and $(x_1', y_2) \in A''$. We can then use coordinate switches to construct a *finite* sequence of pairs of indifference sets $\{(A_n, B_n)\} \in [V] \times [V]$ such that eventually A_n and B_n are "close enough," and hence have a utility midpoint C. By the Diagonal property, C is also the utility midpoint of the initial A'' and B''.

4 Examples and discussion

4.1 Preference and utility midpoints: some examples

In this subsection, we provide some models of CAS preferences and show how to directly identify preference and utility midpoints for such models.

Example 1 (Two-period consumption)**.** Suppose $\succcurlyeq$ has the discounted utility representation on two-period consumption streams $V = X^2$ mentioned earlier. Then, for $A, B \in [V]$, the preference midpoint C of A, B is such that, for some $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$, $(x_1, y_2), (y_1, x_2) \in C$ and $u(x_1) + \delta u(y_2) = u(y_1) + \delta u(x_2)$. Equivalently, $\delta = \frac{u(x_1)-u(y_1)}{u(x_2)-u(y_2)}$. Geometrically, we can interpret δ as the slope of the line passing through the points $(u(x_1), u(x_2))$ and $(u(y_1), u(y_2))$. Fixing $(x_1, x_2) \in A$ and given δ and u, we can define the set M_A of all points $(y_1, y_2) \in V$ satisfying $\delta = \frac{u(x_1)-u(y_1)}{u(x_2)-u(y_2)}$. The elements of M_A can be "mixed" with $(x_1, x_2) \in A$ according to Definition 3; that is, there exists a preference midpoint of each element of M_A with (x_1, x_2). For points $(y_1, y_2) \in V$ not belonging to M_A, one can still use indifferences and/or the procedure developed in the proof of Theorem 1 to identity the utility midpoints.

Example 2 (Two-dimensional weighted attributes)**.** A simple generalization of the previous example is when $V = X_1 \times X_2$ interpreted, for example, as a product with 2 attributes. An additively separable representation of $\succcurlyeq$ is given by $U(a) = \alpha_1 u_1(x_1) + \alpha_2 u_2(x_2)$. Then, for two $A, B \in [V]$, the preference midpoint C of A, B is such that, for some $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$, $(x_1, y_2), (y_1, x_2) \in C$ and $\alpha_1 u_1(x_1) + \alpha_2 u_2(y_2) = \alpha_1 u_1(y_1) + \alpha_2 u_2(x_2)$. Equivalently $\frac{\alpha_2}{\alpha_1} = \frac{u_1(x_1)-u_1(y_1)}{u_2(x_2)-u_2(y_2)}$, and the ratio $\frac{\alpha_2}{\alpha_1}$ has a geometric interpretation analogous to that of δ in the previous example.

Example 3 (State-dependent expected utility)**.** Suppose that $V = X^S$, where S represents a finite set of states of the world and $a \in V$ is an act. A state-dependent expected utility is a CAS representation of a preference $\succcurlyeq$ defined on V: $U(a) = \sum_{s \in S} p(s) u_s(x_s)$, where $p(s) \geq 0$ and $\sum_{s \in S} p(s) = 1$. The preference midpoint C of $A, B \in [V]$ is the class of indifference of acts such that $U(C) = \frac{1}{2} U(A) + \frac{1}{2} U(B)$. In the special case of $|S| = 2$, the preference midpoint C of A, B is such that, for some $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$, $(x_1, y_2), (y_1, x_2) \in C$ and $p(s_1) u_1(x_1) + (1 - p(s_1)) u_2(y_2) = p(s_1) u_1(y_1) + (1 - p(s_1)) u_2(x_2)$. Equivalently $\frac{1-p(s_1)}{p(s_1)} = \frac{u_1(x_1)-u_1(y_1)}{u_2(x_2)-u_2(y_2)}$, and the ratio $\frac{1-p(s_1)}{p(s_1)}$ has again a geometric interpretation analogous to that of δ in the first example.

Example 4 (Social welfare under uncertainty)**.** Suppose that $V = X^n$ and $a \in V$ represents an allocation of resources to n individuals, with $\succcurlyeq$ representing the social welfare ranking. A weighted utilitarian welfare function is a CAS representation of $\succcurlyeq$: $U(a) = \sum_{i=1}^n \alpha_i u_i(x_i)$, where $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$. The preference midpoint of A and B is a set of allocations C such that $a = (x_1, x_2, \ldots, x_n) \in A$ and $b = (y_1, y_2, \ldots, y_n) \in B$ and $c = (z_1, z_2, \ldots, z_n) \sim (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n) \in C$ where $z_i = x_i, y_i$, and $z_i = x_i$ implies $\bar{z}_i = y_i$ and viceversa. A preference midpoint C exists when there exists at least one exchange of coordinates between elements of a and b that makes the new allocations indifferent. For example, take $n = 3$ and assume $U(a) = \alpha_1 u_1(x_1) + \alpha_2 u_2(x_2) + \alpha_3 u_3(x_3) = 3+2+1$ and $U(b) = \alpha_1 u_1(y_1) + \alpha_2 u_2(y_2) + \alpha_3 u_3(y_3) = 5+1+2$. Then $C = A \odot B$ where $(x_1, x_2, y_3) \sim (y_1, y_2, x_3) \in C$ and $U(c) = 7$. We notice that, when the number of coordinate n is bigger than 2, the identification of preference midpoints is easier with respect to $n = 2$.

4.2 A simple sufficient condition for the existence of preference midpoints

As we observed earlier, while utility midpoints exist for any pair A and B of indifference sets, this is not the case for preference midpoints. That is, given two arbitrary vectors $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$, a preference midpoint of (x_1, x_2) and (y_1, y_2) may not exist. Here, we introduce a sufficient condition for the existence of preference midpoints of two such vectors, which is a particular case of the informal notion of "closeness" we introduced in Section 3 (formally defined in Definition 9). The key property is given by the next condition:

Definition 4. $\succcurlyeq$ *satisfies the* Triangle condition *at* $(y_1, y_2) \in V$ *and* $(x_1, x_2) \in V$ *if and only if there exist* $z_1 \in X_1$ *and* $z_2 \in X_2$ *such that* $(y_1, z_2) \sim (z_1, y_2) \sim (x_1, x_2)$.

Figure 4 graphically illustrates the condition making it also clear that the indifference sets A and B are "close." The next proposition shows that the Triangle condition does the job.

Proposition 1. *For arbitrary* (y_1, y_2) *and* (x_1, x_2)*, if the Triangle condition holds at* (y_1, y_2) *and* (x_1, x_2)*, then there exists* $(x'_1, x'_2) \sim (x_1, x_2)$ *such that* $(y_1, x'_2) \sim (x'_1, y_2)$.

Notice that the indifference $(y_1, x'_2) \sim (x'_1, y_2)$, as in Equation (2) above, guarantees that (y'_1, x_2) is a preference midpoint of (x_1, x_2) and (y_1, y_2).[6]

While the Triangle condition is mathematically related to the notion of "closeness" employed above, an advantage of Definition 4 and Proposition 1 is that they are entirely formulated in terms of vectors, rather than indifference sets. Thus, they are in principle easier to verify and implement.

4.3 Coordinate-wise midpoints and homogeneous coordinates

The utility midpoint operator $\odot$ allows to define coordinate-wise midpoint operators $\odot_1$ and $\odot_2$ which, to simplify notation, we define directly on X_1 and X_2, rather than on quotient spaces.

[6] More precisely, $(x'_1, y_2) \sim (y_1, x'_2) \in C$ and C is a preference midpoint of $A \ni (x_1, x_2)$ and $B \ni (y_1, y_2)$.

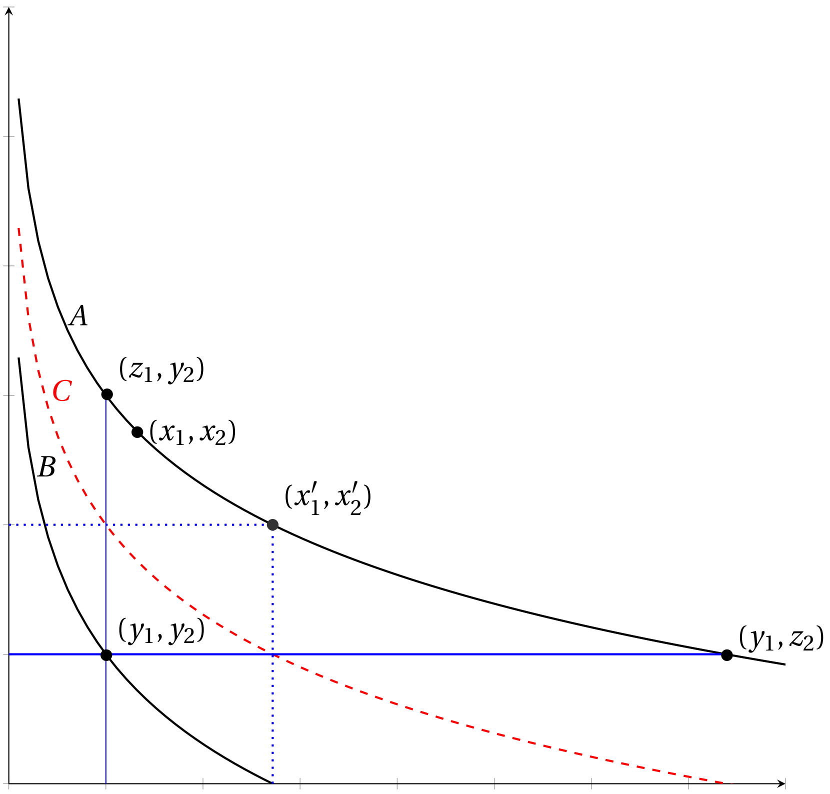

Figure 4: Triangle property at (y_1, y_2) and (x_1, x_2)

Definition 5. *For $x, y, z \in X_1$, $x \odot_1 y = z$ if and only if for some $w \in X_2$, $(x, w) \in A$, $(y, w) \in B$, $(z, w) \in C$ and $C = A \odot B$.*

The definition of $\odot_2$ is obtained symmetrically. Additive separability implies that, for $i = 1,2$, $u_i(x \odot_i y) = \frac{1}{2}u_i(x) + \frac{1}{2}u_i(y)$ for all $x, y \in X_i$. Hence, $\odot_i$ is a utility midpoint operator on X_i.

When coordinates are homogeneous $V = X^2$, coordinate mixtures can be used to compare the "shapes" of the utilities u_1, u_2. The following two preference properties can be of use:

Axiom (Coordinate-wise Ordinal Symmetry - COS)**.** *For all $x, y, z \in X$, $(x, z) \succcurlyeq (y, z)$ if and only if $(z, x) \succcurlyeq (z, y)$.*

Axiom (Coordinate-wise Cardinal Symmetry - CCS)**.** *For all $x, y, z \in X$, $(x \odot_1 y, z) \sim (x \odot_2 y, z)$.*

Axiom COS does not need discussion. As to CCS, we observe that, under additive separability, the choice of the coordinate in Axiom CCS is irrelevant (an equivalent axiom would be: for all $x, y, z \in X$, $(z, x \odot_2 y) \sim (z, x \odot_1 y)$). The axiom CCS is satisfied, for example, when utilities are affine w.r.t. to an objective mixture operation.

We have the following simple result:

Proposition 2. *Suppose $\succcurlyeq$ has a CAS representation on a homogeneous space $V = X^2$. Then, if $\succcurlyeq$ satisfies CCS, there exist $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, and a function $u : X \to \mathbb{R}$ such that $U(x, y) = \alpha u(x) + u(y) + \beta$ represents $\succcurlyeq$. If $\succcurlyeq$ also satisfies COS, $\alpha > 0$.*

So, if $\succcurlyeq$ satisfies CCS and COS then, the utilities u_1 and u_2 are cardinally equivalent.

5 A general framework for the separation of ambiguity and "risk" attitudes

This section contains the main application of the result developed above. Having defined the utility midpoint operator $\odot$, we use it to define a subjective mixture operation in a standard decision-theoretic setting in which the choice options are functions from a state space S, endowed with an algebra Σ, into consequences belonging to a product space $V = \prod_{i \in I} X_i$, with I finite.

Precisely, the decision maker has a binary relation $\succcurlyeq$ defined on the set $\mathscr{F}$ of all simple Σ-measurable functions $f : S \to V$. We denote by $B_0(\Sigma, \Gamma)$ the set of simple Σ-measurable functions on S with values in the interval $\Gamma \subseteq \mathbb{R}$ and we say that $I : B_0(\Sigma, \Gamma) \to \mathbb{R}$ is: *monotonic*, if $I(\phi) \geq I(\psi)$ when $\phi \geq \psi$; *continuous*, if it is sup-norm continuous; *normalized*, if $I(\gamma 1_S) = \gamma$ for all $\gamma \in \Gamma$.

We assume the following basic axioms:

Axiom (Preference Order - P)**.** *$\succcurlyeq$ is a complete, nontrivial and transitive relation on $\mathscr{F}$.*

Axiom (Monotonicity - M)**.** *If $f(s) \succcurlyeq g(s)$ for all $s \in S$, then $f \succcurlyeq g$.*

Axiom (Outcome Separability - OS)**.** *The restriction of $\succcurlyeq$ to A has a CAS representation.*

By Theorem 1 and Remark 2, the CAS assumption allows us to identify utility midpoints for every pair of consequences, $a, b \in V$. Using utility midpoints, we define the *act mixture* $1/2 f \oplus 1/2 g$ for any $f, g \in \mathscr{F}$ as follows: for any $s \in S$,

$$\left(\frac{1}{2} f \oplus \frac{1}{2} g\right)(s) \equiv f(s) \odot g(s) \tag{3}$$

We then consider iterated act mixtures such as $\frac{1}{2} f \oplus \left(\frac{1}{2} f \oplus \frac{1}{2} g\right)$, which corresponds to a $\frac{3}{4} : \frac{1}{4}$ mixture of f and g. This is then extended by standard continuity arguments (see Appendix C in Ghirardato et al. (2002)) to define $\alpha f \oplus (1 - \alpha) g$, for any $\alpha \in [0, 1]$, so that for any $s \in S$

$$U[(\alpha f \oplus (1 - \alpha) g)(s)] = \alpha U(f(s)) + (1 - \alpha) U(g(s))$$

Act mixtures allow us to state the next axiom, which is familiar from traditional Anscombe-Aumann treatments:

Axiom (Full Continuity - FC)**.** *For all* $f, g, h \in \mathscr{F}$*, the sets* $\{\alpha \in [0,1] : \alpha f \oplus (1-\alpha) g \succcurlyeq h\}$ *and* $\{\alpha \in [0,1] : h \succcurlyeq \alpha f \oplus (1-\alpha) g\}$ *are closed.*

Notice that, it follows from axioms P, M OS and FC that for every $f \in \mathscr{F}$ there is $a_f \in V$ such that $a_f \sim f$. That is, every act has a certainty equivalent. Adapting the argument in Cerreia-Vioglio et al. (2011), we show that the axioms stated so far are necessary and sufficient to obtain a MBA representation (Cerreia-Vioglio et al., 2011) for $\succcurlyeq$:[7]

Proposition 3. *Axioms P, M, OS, and FC hold if and only if there exist a CAS* $U : A \to \mathbb{R}$*, and a monotonic, continuous and normalized functional* $I : B_0(\Sigma, U(V)) \to \mathbb{R}$ *such that*

$$f \succcurlyeq g \iff I(U(f)) \geq I(U(g))$$

Notice that we do not assume Risk Independence,[8] since our U is, by definition, affine with respect to the mixture operator $\oplus$ and cardinally unique. Following (perhaps debatable) tradition, we interpret U as risk attitude and I as the description of the decision maker's reaction to the presence of uncertainty, in particular ambiguity.

The class of MBA preferences includes as special cases most of the models of ambiguity-sensitive preferences: the Maxmin EU model of Gilboa and Schmeidler (1989), the Variational Preferences model of Maccheroni et al. (2006), the Confidence Preferences model of Chateauneuf and Faro (2009), the Smooth Ambiguity model of Klibanoff et al. (2005), the Vector EU model of Siniscalchi (2009), and the Uncertainty Averse Preferences model of Cerreia-Vioglio et al. (2011).

5.1 Second-order subjective expected utility

In this section, we address the question of behaviorally distinguishing the subjective expected utility (SEU) model from the second-order SEU model of Grant et al. (2009), which includes the popular Multiplier Preferences (Hansen and Sargent, 2001) as a special case.

[7] We omit the uniqueness statement, which is analogous to that in Cerreia-Vioglio et al. (2011).

[8] In the presence of an objective randomization device $+$ on V, $\succcurlyeq$ satisfies Risk Independence if, for all $a, b, c \in V$ and all $\gamma \in (0,1)$, $a \succcurlyeq b$ implies $\gamma a + (1-\gamma) c \succcurlyeq \gamma b + (1-\gamma) c$.

As noted in Strzalecki (2011), in a pure Savage-style setting SOSEU is observationally equivalent to SEU. Hence, without a method to cardinally identify the Bernoulli utility U the two models cannot be told apart. Two tools have been employed to solve this identification problem. The first is the assumption of the availability of an objective randomization devices, so that consequences belong to an objective mixture space. The second is the assumption of the existence of multiple sources of uncertainty, in one of which the decision maker has SEU preferences (that is, an unambiguous source).

We now show that when the consequence space is multidimensional and preferences satisfy the OS axiom, the notion of utility midpoint introduced in this paper provides a third tool.

Definition 6. *A binary relation $\succsim$ has a* SOSEU representation *if there exist a continuous $U : V \to \mathbb{R}$, a continuous and strictly monotone $\phi : U(V) \to \mathbb{R}$ and a $p \in \Delta(S)$, such that $\succsim$ is represented by $J : \mathscr{F} \to \mathbb{R}$ defined by:*

$$J(f) = \int_S \left(\phi \circ U\right)(f)\,dp$$

An important special case of SOSEU is the Exponential SOSEU representation defined for $\theta \in (-\infty, 0) \cup (0, \infty]$:

$$\phi_\theta(U) = \begin{cases} -\exp\left(-\frac{U}{\theta}\right) & \theta \in (-\infty,0) \cup (0,\infty) \\ U & \theta = \infty \end{cases}$$

The Exponential SOSEU representation includes as a special case the popular Multiplier Preferences of Hansen and Sargent (2001), corresponding to $\theta \in (0, \infty]$.

As shown by Grant et al. (2009), a basic axiom characterizing SOSEU is Savage's Sure-Thing-Principle (P2), restated below. For $f, g \in \mathscr{F}$ and $E \in \Sigma$, $f_E g$ denotes the act equal to f in E and equal to g in E^c:

Axiom (P2)**.** *For all $E \in \Sigma$ and acts $f, g, h, h' \in \mathscr{F}$, if $f_E h \succsim g_E h$ then $f_E h' \succsim g_E h'$.*

The following lemma provides a characterization of a state-dependent version of SOSEU. We define a state $s \in S$ to be *null* if for all $f \in \mathscr{F}$ and $a, b \in V$, $a_s f \sim b_s f$.

Lemma 1. *Suppose there are at least 3 non-null states. A binary relation $\succsim$ satisfies axioms P, M, OS, FC, and P2 if and only if there exists a non-constant $U : V \to \mathbb{R}$ with range $U(V)$ and continuous* *weakly increasing functions $\phi_s : U(V) \to \mathbb{R}$, such that $\succsim$ is represented by:*

$$J(f) = \sum_{s \in S} \phi_s(U(f(s)))$$

If $\{\phi'_s\}_{s\in S}$ also represent $\succsim$, there are $\alpha > 0$ and $\beta_s \in \mathbb{R}$ such that $\phi_s = \alpha\phi'_s + \beta_s$.

Grant et al. (2009) provide a similar representation result. The main difference is that they require Risk Independence, a state separability assumption which presumes the presence of objective mixtures. Instead, we only rely on vector-valued consequences and the CAS assumption.

To obtain the full-fledged SOSEU representation, we impose a further condition which implies the independence of the evaluation of consequences and states. Leveraging on our previous results, we can do so without imposing restrictions on the function ϕ. The functional $J(f) = \sum_{s\in S} \phi_s(U(f(s)))$ is an additively separable representation of $\succsim$ over a homogeneous product space (as assumed in Section 4.3). Therefore, since the state space S is finite and each act $f \in \mathscr{F}$ is a vector in $V^{|S|}$, by Theorem 1 and Proposition 2, there exists a midpoint operator $\otimes$ on $[\mathscr{F}]$. The operator $\otimes$ associates an indifference class $H \in [\mathscr{F}]$ to any pair of indifference classes $F, G \in [\mathscr{F}]$, so that $J(H) = 1/2J(F) + 1/2J(G)$ whenever $H = F \otimes G$. Given $\otimes$, we can define, for each $s \in S$, an operator $\otimes_s$ on the space V as follows:

Definition 7. *For any $a, b, c \in V$, $c = a \otimes_s b$ if and only if for some $h' \in \mathscr{F}$, $a_s h' \in F$, $b_s h' \in G$ and $c_s h' \in H$ such that $F \otimes G = H$.*

The construction (and interpretation) of $\otimes_s$ is the same of the operators $\odot_i$ defined in Section 4.3, with the indexes $i \in I$ replaced by the states $s \in S$ and the utilities u_i replaced by $\phi_s \circ U$. Since, U is cardinally unique by the OS axiom, we can separately identify the state-dependent functions ϕ_s (without cardinally identifying U, we could only cardinally identify $\phi_s \circ U$).[9]

Remark 3. It is important to understand that the operators $\odot$ and $\otimes_s$, while both defined on V, are not necessarily isomorphic. For an example, consider the case $V = \mathbb{R}_+ \times \mathbb{R}_+$ and $U(x, y) = x + y$. Given two vectors $a = (x, x)$ and $b = (y, y)$, their midpoint $c = a \odot b$ is such that $U(c) = x + y$. However, if for example $\phi_s = \ln$, $c' = a \otimes_s b$ is such that $U(c') = e^{\frac{1}{2}\ln(2x) + \frac{1}{2}\ln(2y)} \neq U(c)$. The DM finds a different midpoint for a and b when the alternative payoff (as described by the act h' in

[9] To clarify further, without the OS axiom —hence without cardinally identifying U— the uniqueness part of Lemma 1 would say that if $\{(\phi_s \circ U)'\}_{s\in S}$ also represent $\succsim$, there are $\alpha > 0$, $\beta_s \in \mathbb{R}$ such that $\phi_s \circ U = a(\phi_s \circ U)' + \beta_s$.

Definition 7) is uncertain than when they are received with certainty. (Notice that this is still true even if $\phi_s = \ln$ for all $s \in S$; that is, if the $\otimes_s$ operator is state-independent.)

Remark 4. The mixture operator $\otimes$ can be extended to mix acts $f, g \in \mathscr{F}$, as we did in Remark 2. This allows us to state the the independence axiom with respect to $\otimes$: for all $f, g \in \mathscr{F}$, $f \sim g$ implies $f \otimes h \sim g \otimes h$, for all $h \in \mathscr{F}$. It is simple to prove that SOSEU satisfies such axiom, since J is affine with respect to $\otimes$: $J(f \otimes g) = \frac{1}{2}J(f) + \frac{1}{2}J(g)$. This shows that $\otimes$ is "weaker" than the objective mixture $+$ of Anscombe-Aumann, since imposing the independence axiom with respect to the objective mixture $+$ implies that the preference satisfies SEU.

We can now adapt Axiom CCS from Section 4.3 to the present setting:

Axiom (State-wise Cardinal Symmetry - SCS)**.** *For all $a, b \in V$ and $f \in \mathscr{F}$, $(a \otimes_s b)_s f \sim (a \otimes_{s'} b)_s f$ for all $s, s' \in S$.*

The act $(a \otimes_s b)_s f$ is equal to f in all states different from $s \in S$ and equal to $a \otimes_s b$ in state s. If it is indifferent to $(a \otimes_{s'} b)_s f$, it means that the state-dependent utility midpoint of a, b in state s is indifferent to the state-dependent utility midpoint of a, b in state s'. By additive separability and the definition of $\otimes_s$, $1/2\phi_s(U(a)) + 1/2\phi_s(U(b)) = \phi_s(U(a \otimes_s b)) = \phi_s(U(a \otimes_{s'} b))$, for all $a, b \in V$, implying that $a \otimes_{s'} b$ is the also the midpoint of a, b in state s. Since this holds for all states, midpoints are *state-independent*.

The main result of this section follows:

Theorem 2. *Suppose there are at least 3 non-null states. A binary relation $\succsim$ satisfies axioms P, M, OS, FC, P2 and SCS if and only if $\succsim$ has a SOSEU representation. That is, there exist a CAS utility $U : V \to \mathbb{R}$, a continuous and strictly monotone $\phi : U(V) \to \mathbb{R}$ and a $p \in \Delta(S)$, such that $\succsim$ is represented by $J : \mathscr{F} \to \mathbb{R}$ given by:*

$$J(f) = \int_S \left(\phi \circ U\right)(f)\, dp$$

If (U', ϕ', p') also represents $\succsim$, there are $\alpha, \kappa > 0$, $\beta, \zeta \in \mathbb{R}$ such that $p = p'$, $U' = \alpha U + \beta$ and $\phi'(\alpha r + \beta) = \kappa\phi(r) + \zeta$ for all $r \in U(V)$.

Remark 5. Following the discussion in Remark 3, it can be seen that imposing $a \odot b \sim a \otimes_s b$ for all $a, b \in V$, on top of the axioms of Theorem 2, implies that $\phi = \phi_s = \alpha \mathrm{id} + \beta$, for some $\alpha > 0$ and

$\beta \in \mathbb{R}$. That is, the preference is SEU. Thus, we see that SEU intuitively corresponds to the case in which the utility midpoint of a and b under certainty, $a \odot b$, is indifferent to the utility midpoint under uncertainty $a \otimes_s b$ for every $s \in S$ (thus implying Axiom SCS).

With respect to the existing literature, Theorem 2 has two advantages: first, it does not require the existence of an objective randomization device or multiple sources of uncertainty. Second, it does not constrain the ambiguity attitude of the decision maker, since ϕ is a general monotone function. Existing characterizations, for example Grant et al. (2009), only obtain a SOSEU representation with concave ϕ (cf. Corollary 1 below).

The concavity of ϕ can be obtained in our setting by introducing a preference for ambiguity hedging that mimics the Uncertainty Aversion axiom of Gilboa and Schmeidler (1989). Recall the definition of $\oplus$ provided in equation (3). We can then formulate:

Axiom (Ambiguity Hedging - AH)**.** *If $f, g \in \mathscr{F}$ and $\alpha \in (0,1)$, $f \sim g$ implies $\alpha f \oplus (1-\alpha) g \succcurlyeq f$.*

Corollary 1. *Suppose there are at least 3 non-null states. A binary relation $\succcurlyeq$ satisfies axioms P, M, OS, FC, P2, SCS and AH if and only if $\succcurlyeq$ has a SOSEU representation with concave ϕ.*

To see this, notice that Axiom AH implies quasiconcavity of $I(U \circ f) = \sum_s p(s)\phi(U(f(s)))$. By the result of Debreu and Koopmans (1982), quasiconcavity implies that all $\phi_s = p(s)\phi$, with at most one exception, are concave. Since they are all equal (up to a positive constant), the result follows. This result is analogous to Theorem 3 in Grant et al. (2009), who assume Uncertainty Aversion (i.e., a version of AH with objective act mixtures) and a condition called "Translation Invariance at Certainty" (TIC).[10]

Act mixtures can also be employed to characterize the Exponential SOSEU model, using a subjective version of the Weak Certainty Independence axiom of Maccheroni et al. (2006):

Axiom (Weak Certainty Independence - WCI)**.** *For all $f, g \in \mathscr{F}$, $a, b \in V$, and $\alpha \in (0,1)$,*

$$\alpha f \oplus (1-\alpha) a \succcurlyeq \alpha g \oplus (1-\alpha) a \;\;\Rightarrow\;\; \alpha f \oplus (1-\alpha) b \succcurlyeq \alpha g \oplus (1-\alpha) b$$

[10]Translation Invariance at Certainty holds if: $\succcurlyeq$ is locally EU at a with respect to p^a and locally EU at b with respect to p^b, then $p^a = p^b$. Where, $\succcurlyeq$ is locally EU at $a \in A$ w.r.t. $p \in \Delta(S)$ if, for all $f \in \mathscr{F}$ and $\mathbb{E}_{p^a}[U(f)] > U(a)$, there exists an $\bar{\alpha} \in (0,1]$ such that, for all $\alpha \in (0,\bar{\alpha}]$, $\alpha f \oplus (1-\alpha) a \succ a$ and $U(a) > E_{p^a}[U(f)]$ implies there exists $\bar{\alpha} \in (0,1]$ such that, for all $\alpha \in (0,\bar{\alpha}]$, $a \succ \alpha f \oplus (1-\alpha) a$. TIC is weaker than SCS, since it does not imply the state-independence of ϕ_s without ancillary assumptions.

Corollary 2. *Suppose there are at least 3 non-null states. A binary relation $\succsim$ satisfies axioms P, M, OS, FC, P2, SCS and WCI if and only if $\succsim$ has an Exponential SOSEU representation.*

The argument is similar to that given by Strzalecki (2011). However, SCS and WCI do not characterize Multiplier Preferences. The reason for this is that our derivation of SOSEU does not entail restrictions on the curvature of ϕ. Indeed, to obtain the characterization of Multiplier Preferences, we need to add axiom AH to the assumptions of Corollary 2:

Corollary 3. *Suppose there are at least 3 non-null states. A binary relation $\succsim$ satisfies axioms P, M, OS, FC, P2, SCS, WCI and AH if and only if $\succsim$ has a Multiplier Preference representation.*

Figure 5 summarizes the above results with a graphical illustration of the relations between axioms and representations within the SOSEU model.

We conclude this section with two observations on the relation of our results with previous characterizations of SOSEU.

Remark 6. Our characterization of SOSEU can be compared to Nau (2006) and Ergin and Gul (2009), who identify ambiguity attitude by using multiple sources of uncertainty in lieu of an objective randomization device. In their setting, the states of the world are pairs $(s_1, s_2) \in S_1 \times S_2$ and acts are functions from the states to consequences $f : S_1 \times S_2 \to X$. By assuming $|S_2| < \infty$ one can interpret one such act as a "compound act": a map from S_1 to a homogeneous product space with finitely many coordinates $V = X^{S_2}$. A SEU evaluation for acts that depend only on the source of uncertainty S_2 is a CAS representation of the restriction of $\succsim$ to X^{S_2}: $U(f(s_1)) = \sum_{s_2 \in S_2} u(f(s_1, s_2))q(s_2)$. Then, the axioms of Theorem 2 can be adapted to obtain the following representation of $\succsim$:

$$J(f) = \int_{S_1} \phi\left(\sum_{s_2 \in S_2} u(f(s_1, s_2))q(s_2)\right) dp(s_1)$$

Our result actually requires a weaker condition than SEU over $V = X^{S_2}$. Indeed, it would be sufficient to have a CAS representation (not necessarily EU) of $\succsim$ on $V = X^{S_2}$: $U(f(s_1)) = \sum_{s_2 \in S_2} u_{s_2}(f(s_1, s_2))$. Such U allows us to define $\odot$ and $\oplus_s$ and to reproduce the proof of Theorem 2 to obtain:

$$J(f) = \int_{S_1} \phi\left(\sum_{s_2 \in S_2} u_{s_2}(f(s_1, s_2))\right) dp(s_1)$$

Remark 7. Another axiomatization of SOSEU without restrictions on the function ϕ is that provided by Cerreia-Vioglio et al. (2012). They assume a rich state space as in Savage (1954) and

$$
\begin{tikzcd}
 & \text{State-dependent Second-Order Utility (Lemma 4)} \arrow[d, "+\text{ SCS}"] & \\
 & \text{SOSEU (Th. 2)} \arrow[dl, "+\text{ AH}"'] \arrow[dr, "+\text{ WCI}"] & \\
\text{SOSEU with concave } \phi \text{ (Cor. 1)} \arrow[dr, "+\text{ WCI}"'] & & \text{Exponential SOSEU (Cor. 2)} \arrow[dl, "+\text{ AH}"] \\
 & \text{Multiplier Preferences (Cor. 3)} &
\end{tikzcd}
$$

Figure 5: Axioms and SOSEU

the existence of an objective randomization device. Our technique can be applied to substitute the presence of objective randomization with the assumption that the set of consequences is a product space $V : \prod_i X_i$. The advantage of this extension is that we do not need to assume Risk Independence, as the utility function is automatically affine with respect to $\oplus$. We can prove that a binary relation $\succcurlyeq$ satisfies axioms P1-P6 of Savage, OS and FC if and only if there exist a CAS $U : V \to \mathbb{R}$, a continuous increasing $\phi : U(V) \to \mathbb{R}$, and a non-atomic and finitely additive $p \in \Delta(S)$ such that $\succcurlyeq$ is represented by $J : \mathscr{F} \to \mathbb{R}$ defined by:

$$J(f) = \int_S (\phi \circ U)(f) dp$$

Appendix

A Comparative risk attitude

In this appendix, we consider the case of homogeneous coordinates $V = X^2$ and provide a simple generalization of the arguments in Section 4.3, using coordinate-wise midpoints to study coordinate-dependent risk attitudes.

Assuming that $\succcurlyeq$ has a CAS representation over V, we define two derived binary relations $\succcurlyeq_i$, $i = 1, 2$ on X by $x \succcurlyeq_i y$ if and only if $(x, z) \succcurlyeq (y, z)$ for some $z \in X$. The next definition introduces the behavioral characterization of coordinate-wise comparative risk aversion.[11]

Definition 8. $\succcurlyeq$ *is* more risk averse about coordinate 1 than about coordinate 2 *if* $(x \odot_2 y, z) \succcurlyeq (x \odot_1 y, z)$*, for some* $z \in X$.

The following result shows the simple characterization of this comparative notion:

Proposition 4. *If* $\succcurlyeq$ *has a CAS representation over* V*, then* $\succcurlyeq$ *is more risk averse about coordinate* 1 *than about coordinate* 2 *if and only if there exists a concave function* $\phi : u_2(X) \to \mathbb{R}$ *such that* $u_1 = \phi \circ u_2$.

Proof. Suppose that $u_1 = \phi \circ u_2$ for some concave function ϕ, then

$$\begin{aligned} u_1\left(x \odot_2 y\right) &= \phi \circ u_2\left(x \odot_2 y\right) \\ &= \phi\left(\frac{1}{2}u_2(x) + \frac{1}{2}u_2(y)\right) \\ &\geq \frac{1}{2}\phi(u_2(x)) + \frac{1}{2}\phi(u_2(y) \\ &= \frac{1}{2}u_1(x) + \frac{1}{2}u_1(y) = u_1(x \odot_1 y) \end{aligned}$$

where the inequality follows from concavity of ϕ. Hence, $(x \odot_2 y, z) \succcurlyeq (x \odot_1 y, z)$ for some (all) $z \in X$.

For the opposite implication, notice that $u_1(x \odot_2 y) = u_1\left(u_2^{-1}\left(\frac{1}{2}u_2(x) + \frac{1}{2}u_2(y)\right)\right)$, hence, by

[11] For brevity, in what follows we discuss the case in which the DM is more risk averse about the first coordinate than the second, but symmetric results can be proved in the symmetric case in which the DM is is more risk averse about the second coordinate than the first.

defining $\phi = u_1 \circ u_2^{-1}$ we have:

$$\begin{aligned}
\phi\left(\frac{1}{2}u_2(x) + \frac{1}{2}u_2(y)\right) =& \phi \circ u_2\left(x \odot_2 y\right) \\
=& u_1\left(x \odot_2 y\right) \\
\geq& u_1(x \odot_1 y) \\
=& \frac{1}{2}u_1(x) + \frac{1}{2}u_1(y) \\
=& \frac{1}{2}\phi(u_2(x)) + \frac{1}{2}\phi(u_2(y))
\end{aligned}$$

where the inequality follows from Definition 8. By continuity of ϕ, standard results imply that ϕ is concave. $\square$

The previous result can be applied to behaviorally characterize state-dependent risk aversion. Consider a state-dependent representation of a preference $\succcurlyeq$ defined over acts $a \in V = X^S$, where S is a set of states of the world:

$$U(a) = \sum_{s \in S} u_s(a_s).$$

As the intuition suggests, $\succcurlyeq$ is more risk averse in state s than in state s' if and only if $u_s = \phi \circ u_{s'}$, for some concave function ϕ. By Proposition 4, the latter condition is equivalent to assuming that $\succcurlyeq$ satisfies Definition 8. The following example illustrates:

Example 5. Suppose $X = [a, b]$ for some $0 < a < b$. If $U(x, y) = \ln(x) + y$, $x \odot_1 y = \sqrt{xy}$ and $x \odot_2 y = \frac{1}{2}x + \frac{1}{2}y$. By a well-known result about the relation between the arithmetic and the geometric means, U is more risk averse in state 1 than in state 2 and $\phi = \ln$.

B Proofs

B.1 Proof of Theorem 1

The proof of the theorem builds on several lemmas.

We begin by showing that a midpoint of $A, B \in [V]$ exists when A and B satisfy the following preference condition:

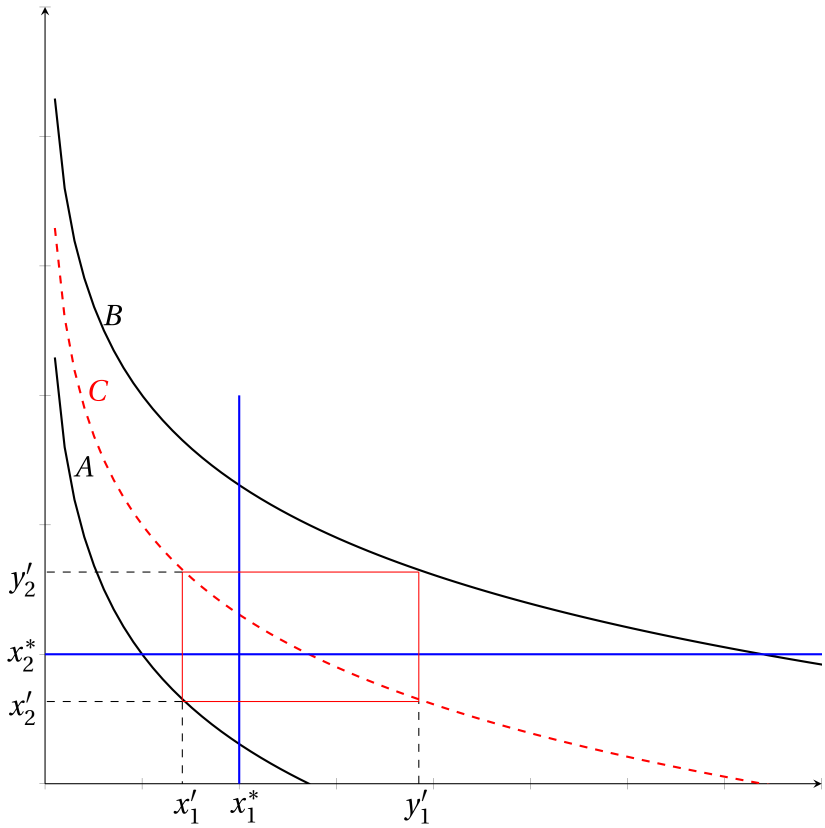

Figure 6: Crossing property and midpoints

Definition 9. *$\succcurlyeq$ satisfies* Crossing *at $A, B \in [V]$ if and only if there are $x_1^* \in X_1, x_2^* \in X_2$ such that $(x_1^*, x_2) \in A$ and $(x_1^*, y_2) \in B$ for some $x_2, y_2 \in X_2$ and $(w_1, x_2^*) \in A$ and $(z_1, x_2^*) \in B$ for some $w_1, z_1 \in X_1$ with $z_1 \succcurlyeq_1 x_1^* \succcurlyeq w_1$ and $y_2 \succcurlyeq_2 x_2^* \succcurlyeq_2 x_2$.*

Fig. 6 provides a graphical representation of the Crossing property.

Lemma 2. *Given $A, B \in [V]$, if $\succcurlyeq$ has an additively separable representation and Crossing holds at A, B, then there exist $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$ such that $(x_1, y_2) \sim (y_1, x_2)$ (hence, there exists $C \in [V]$ such that $C = A \odot B$).*

Proof. Consider directly the space of utilities and given $U(A) = k, U(B) = k'$ assume w.l.o.g that $U(B) > U(A)$. By Crossing, there are $(x_1^*, x_2) \in A$ and $(x_1^*, y_2) \in B$ for some $x_2, y_2 \in X_2$ and $(w_1, x_2^*) \in A$ and $(z_1, x_2^*) \in B$ for some $w_1, z_1 \in X_1$, such that $u_1(z_1) \geq u_1(x_1^*) \geq u_1(w_1)$ and $u_2(y_2) \geq u_2(x_2^*) \geq u_2(x_2)$. Define $a' = (x_1', x_2')$ by $u_1(x_1') = 0.5u_1(x_1^*) + 0.5u_1(w_1)$ and $u_2(x_2') = 0.5u_2(x_2^*) + 0.5u_2(x_2)$. Notice that $U(a') = U(A)$. Similarly, define $b' = (y_1', y_2')$ by $u_1(y_1') = 0.5u_1(x_1^*) + 0.5u_1(z_1)$ and $u_2(y_2') = 0.5u_2(x_2^*) + 0.5u_2(y_2)$. Notice that $U(b') = U(B)$. Both a' and b' are well-defined because of Crossing at A, B and continuity of the preference. Then, it follows that $(x_1', y_2') \sim (y_1', x_2')$, indeed:

$$\begin{aligned} u_1(x_1') + u_2(y_2') =& 0.5u_1(x_1^*) + 0.5u_1(w_1) + 0.5u_2(x_2^*) + 0.5u_2(y_2) \\ =& 0.5(u_1(x_1^*) + u_2(y_2)) + 0.5(u_1(w_1) + u_2(x_2^*)) = 0.5U(B) + 0.5U(A) \end{aligned}$$

and

$$\begin{aligned} u_1(y_1') + u_2(x_2') =& 0.5u_1(x_1^*) + 0.5u_1(z_1) + 0.5u_2(x_2^*) + 0.5u_2(x_2) \\ =& 0.5(u_1(x_1^*) + u_2(x_2)) + 0.5(u_1(z_1) + u_2(x_2^*)) = 0.5U(A) + 0.5U(B) \end{aligned}$$

□

If Crossing does not hold at A and B, we need to take additional steps in order to identify the midpoint of A and B. The following definition outlines such additional steps:

Definition 10. *An* elementary step *is:*

(SS) A smoothing swap of $a = (x_1, x_2)$ and $b = (y_1, y_2)$*: The pair* $c = (x_1, y_2)$ *and* $d = (y_1, x_2)$ *is substituted to a and b respectively (see Figure 7).*

(II) An indifference substitution of $a = (x_1, x_2)$*: The vector* $b = (x_1', x_2') \sim a$ *is substituted to* a*.*

Suppose that $a = (x_1, x_2) \in A$ Pareto dominates $b = (y_1, y_2) \in B$. A smoothing swap of a and b generates two points $c = (x_1, y_2) \in C$ and $d = (y_1, x_2) \in D$ that are "interior," in terms of preferences, to a and b; i.e., $a \succcurlyeq \{c, d\} \succcurlyeq b$. Moreover, CAS implies the following Diagonal property (see Vind and Grodal, 2003):

Lemma 3. *If* $\succcurlyeq$ *has a CAS representation,* $(x_1, x_2) \in A$ *and* $(y_1, y_2) \in B$ *and* $C = A \odot B$*, then* $C = A' \odot B'$ *where* $(x_1, y_2) \in A'$ *and* $(y_1, x_2) \in B'$.

Given $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$, if they have a utility midpoint, their midpoint is the same as that of $(x_1, y_2) \in A'$ and $(y_1, x_2) \in B'$. Therefore, if Crossing holds at a pair C, D generated by a smoothing swap of $a \in A$ and $b \in B$, then Lemma 2 implies the existence of a utility midpoint E of C and D and the Diagonal property guarantees that E is also the utility midpoint of A and B.

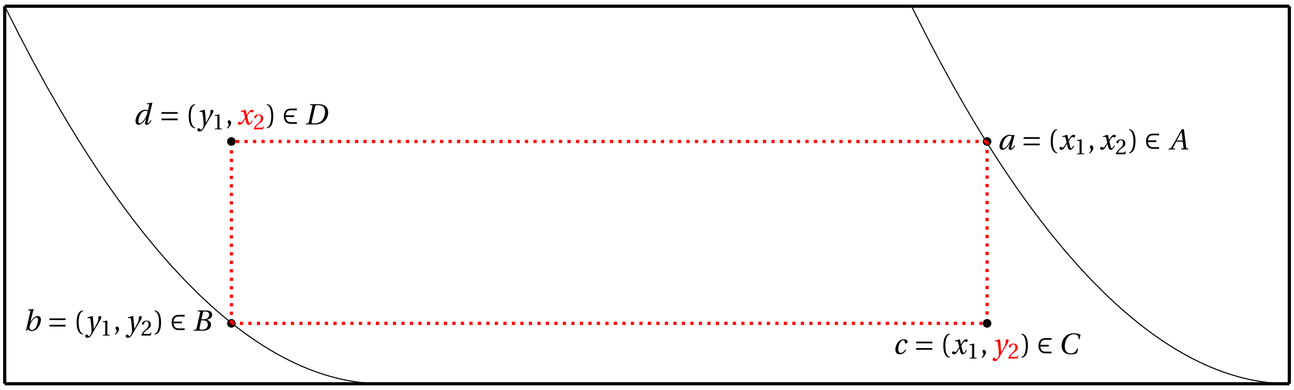

Figure 7: Smoothing swap of a and b

Lemma 4. *If, $A \succ B$ and for some $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$, $u_1(x_1) > u_1(y_1)$ and $u_2(x_2) > u_2(y_2)$ then, there exists $C = A \odot B$ which can be determined in n elementary steps. The number n is the smallest integer such that $n \geq \frac{1}{2}\frac{u_1(x_1)-u_1(y_1)}{u_2(x_2)-u_2(y_2)}$.*

Proof. If Crossing holds at A, B, by Lemma 2 there exists a midpoint. If not, consider $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$ and refer to Figure 8. Since we can always relabel the axes, it is w.l.o.g. to assume that $(x_1, y_2) \succ (y_1, x_2)$ (or equivalently $u_1(x_1) - u_1(y_1) > u_2(x_2) - u_2(y_2)$). Now apply a smoothing swap. The pairs (x_1, y_2) and (y_1, x_2) are such that $(x_1, x_2) \succ (x_1, y_2) \in A^1$ and $(y_1, y_2) \prec (y_1, x_2) \in B^1$. If Crossing holds at A^1, B^1 then, by Lemma 2 and by the Diagonal property there exists a midpoint of A and B. If not, the condition $u_1(x_1) - u_1(y_1) > u_2(x_2) - u_2(y_2)$ implies $A^1 \succ B^1$. Now, find $y_1^1 \in X_1$ such that $(y_1^1, y_2) \in B^1$. It exists by continuity and the fact that $(x_1, y_2) \in A^1 \succ B^1 \succ (y_1, y_2) \in B$. Similarly, find $x_1^1 \in X_1$ such that $(x_1^1, x_2) \in A^1$. Since Crossing does not hold at A^1, B^1, then $u_1(x_1^1) > u_1(y_1^1)$, so (y_1^1, y_2) and (x_1^1, x_2) are strictly Pareto ranked. Apply a smoothing swap to (y_1^1, y_2) and (x_1^1, x_2), to find $(x_1^1, y_2) \in B^2$ and $(y_1^1, x_2) \in A^2$. If Crossing holds at A^2, B^2 then by Lemma 2 and the Diagonal property there exists a midpoint of A^1 and B^1, by another application of the Diagonal property, there exists a midpoint of A and B. If the Crossing does not hold at A^2 and B^2, repeat the argument to find $A^3, A^4, \ldots$ and $B^3, B^4, \ldots$ as before, until Crossing holds at A^n, B^n for some n.

To find n, begin by observing that $u_1(y_1^1) - u_1(y_1) = u_1(y_1^1) + u_2(y_2) - u_1(y_1) - u_2(y_2) = U(B^1) - U(B) = u_1(y_1) + u_2(x_2) - u_1(y_1) - u_2(y_2) = u_2(x_2) - u_2(y_2)$ and $u_1(x_1) - u_1(x_1^1) = u_1(x_1) + u_2(x_2) - u_1(x_1^1) - u_2(x_2) = U(A) - U(A^1) = u_1(x_1) + u_2(x_2) - u_1(x_1) - u_2(y_2) = u_2(x_2) - u_2(y_2)$.

As soon as you move from A and B to A^1 and B^1, one of the two conditions required by Crossing is satisfied, because we can set, for instance, $x_2 = x_2^*$, and the same x_2^* will satisfy the condition for all A^n, B^n. To verify that Crossing holds, say, at A^{n+1} and B^{n+1}, we thus only need to find a suitable

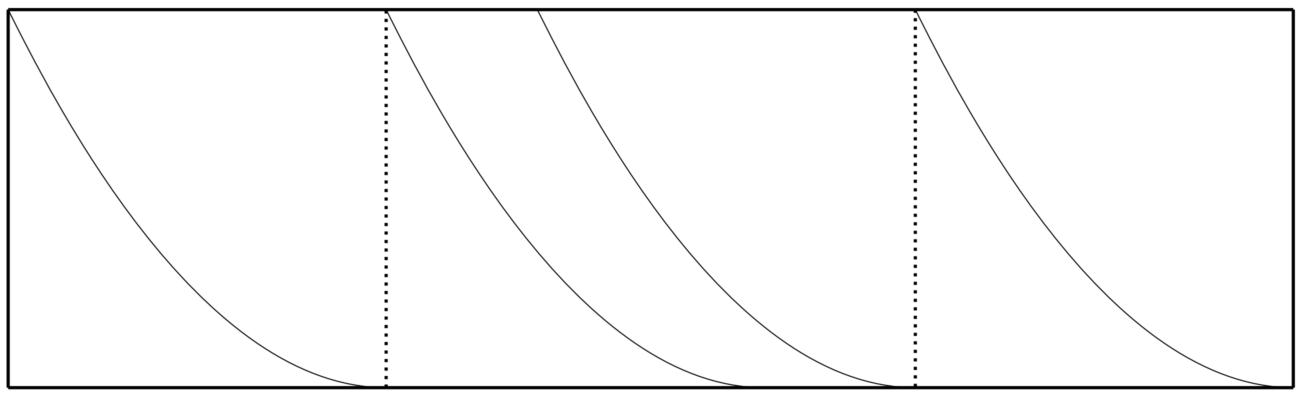

Figure 8: Elementary steps

x_1^*. A sufficient condition for that is $u_1(y_1^n) \geq u_1(x_1^n)$ with $(y_1^n, y_2) \in B^{n+1}$ and $(x_1^n, x_2) \in A^{n+1}$. But, by iteration if the previous calculations, $u_1(y_1^n) = u_1(y_1) + n(u_2(x_2) - u_2(y_2))$ and $u_1(x_1^n) = u_1(x_1) - n(u_2(x_2) - u(y_2))$. Therefore, $u_1(y_1^n) \geq u_1(x_1^n)$ holds whenever $u_1(y_1) + n(u_2(x_2) - u_2(y_2)) \geq u_1(x_1) - n(u_2(x_2) - u(y_2))$; that is, n satisfies

$$n \geq \frac{1}{2} \frac{u_1(x_1) - u_1(y_1)}{u_2(x_2) - u_2(y_2)}$$

□

Proof of Theorem 1. Given arbitrary $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$, if $\succcurlyeq$ satisfies Crossing at A adn B, a midpoint of A and B exists by Lemma 2. Suppose $\succcurlyeq$ does not satisfy Crossing at A, B, and assume w.l.o.g. that $A \succ B$. Then for arbitrary $(x_1, x_2) \in A$ and $(y_1, y_2) \in B$, there are three possible cases:

1. $u_1(x_1) > u_1(y_1)$ and $u_2(y_2) > u_2(x_2)$
2. $u_1(y_1) > u_1(x_1)$ and $u_2(x_2) > u_2(y_2)$
3. $u_1(x_1) \geq u_1(y_1)$ and $u_2(x_2) \geq u_2(y_2)$

For Case 1, we apply a smoothing swap to find $c = (x_1, y_2) \in C$ and $d = (y_1, x_2) \in D$ for some $C, D \in [V]$. $c \succ d$ and they are strictly Pareto ranked. Hence, we can apply Lemma 4 (the condition of Lemma 4 is satisfied because $A \succ B$). Therefore, there exists a midpoint E of C, D. By the Diagonal property (Lemma 3), $E = A \odot B$.

Case 2 can be treated as Case 1, up to relabeling of the axes.

Case 3 has three subcases:

a. $u_1(x_1) > u_1(y_1)$ and $u_2(x_2) = u_2(y_2)$

b. $u_1(x_1) = u(y_1)$ and $u_2(x_2) > u_2(y_2)$

c. $u_1(x_1) > u_1(y_1)$ and $u_2(x_2) > u_2(y_2)$

For case a, since $u_2(X_2)$ is an interval in $\mathbb{R}$, if $u_2(y_2) \in u_2(X_2)^\circ$ or $u_2(y_2) = \max_{w_2 \in X_2} u_2(w_2)$, we can always find $\epsilon > 0$ such that $u_1(x_1) - u_1(y_1) > \epsilon$, $u_2(y_2) - \epsilon = u_2(z_2)$, and $(y_1, y_2) \sim (z_1, z_2)$ for some $z_1 \in X_1$. By construction, $u_1(x_1) > u_1(z_1)$, because

$$\begin{aligned} u_1(x_1) >& u_1(y_1) + \epsilon \\ =& u_1(y_1) + u_2(y_2) - u_2(z_2) \\ =& u_1(z_1) + u_2(z_2) - u_2(z_2) \\ =& u_1(z_1) \end{aligned}$$

therefore, (z_1, z_2) and (x_1, x_2) are strictly Pareto ranked. We can thus apply Lemma 4 to prove existence of the midpoint of A and B.

If $u_2(y_2) = \min_{w_2 \in X_2} u_2(w_2)$, take $\epsilon > 0$ with $u_1(x_1) - u_1(y_1) > \epsilon$, define $u_2(x_2) + \epsilon = u_2(y_2) + \epsilon = u_2(z_2)$, and find $z_1 \in X_1$ such that $(z_1, z_2) \in a$. By construction, $u_1(z_1) > u_1(y_1)$, because

$$\begin{aligned} u_1(y_1) <& u_1(x_1) - \epsilon \\ =& u_1(x_1) + u_2(y_2) - u_2(z_2) \\ =& u_1(z_1) + u_2(z_2) - u_2(z_2) \\ =& u_1(z_1) \end{aligned}$$

Therefore, (z_1, z_2) strictly Pareto dominates (y_1, y_2), and Lemma 4 can by applied.

Case b is symmetric to case a. For Case c we can directly apply Lemma 4. □

B.2 Other proofs

Proof of Proposition 1. First consider the case $u_1(y_1) + u_2(y_2) = k' = U(B) < U(A) = k$. Define $f : X_1 \to \mathbb{R}$ as follows $f(x) = u_1(y_1) + k - 2u_1(x) - u_2(y_2)$. By the Triangle condition, take $x = y_1$, then

$f(y_1) = u_1(y_1) + k - 2u_1(y_1) - u_2(y_2) = k - (u_1(y_1) + u_2(y_2)) = k - k' > 0$. Again by the Triangle condition, take $x = z_1$, then $f(z_1) = u_1(y_1) + u_1(z_1) + u_2(y_2) - 2u_1(z_1) - u_2(y_2) = u_1(y_1) - u_1(z_1) < 0$, because $k' = u_1(y_1) + u_2(y_2) < u_1(z_1) + u_2(y_2) = k$. By continuity of u_1, there exists $f(x^*) = 0$. The case $u_1(y_1) + u_2(y_2) = U(B) > U(A)$ is symmetric. □

Proof of Proposition 2. By CAS and CCS we have

$$
\begin{aligned}
u_1(x \odot_1 y) =& \frac{1}{2}u_1(x) + \frac{1}{2}u_1(y) \\
=& u_1(x \odot_2 y)
\end{aligned}
$$

Hence, u_1 respects the midpoint operation induced by $\odot_2$. Axiom COS implies the ordinal equivalence of u_1 and u_2. By the uniqueness property of Fuchs (1963, Th.15 p. 183), there are $\alpha \geq 0$ and $\beta \in \mathbb{R}$ such that $u_1 = \alpha u_2 + \beta$. $\alpha \neq 0$, otherwise the additively separable representation would not be cardinally unique. □

Proof of Theorem 2. By a straightforward extension of Proposition 2 to $|S|$ dimensions, SCS and the fact that all ϕ_s are weakly increasing, imply that, for a non-null s', $\phi_{s'} = \alpha_s \phi_s + \beta_s$ for some non-null $s \in S, \alpha_s \geq 0$ and $\beta_s \in \mathbb{R}$. By assumption $\alpha_s > 0$ for at least 3 states. Therefore, $J(f) = \sum_{s\in S} \alpha_s \phi(u(f(s))) + \beta_s$ where $\phi \triangleq \phi_s$. Renormalizing by $K = \left(\sum_{s\in S} \alpha_s\right) > 0$ gives a SOSEU with $p(s) = \frac{\alpha_s}{K}$. □

Proof of Corollary 2. The argument is similar to the one of Strzalecki (2011): WCI implies translation invariance of the functional $I : B_0(\Sigma, U(V)) \to \mathbb{R}$ defined by $I(U \circ f) = \sum_{s\in S} \phi\big(U(f(s))\big)$ (i.e. $I(\xi + k) = I(\xi) + k$ for all $\xi \in B_0(\Sigma, U(V))$ and $k \in \mathbb{R}$). In turn, translation invariance forces the function ϕ to satisfy a generalized Pexider's functional equations whose solution is the exponential function $\phi(r) = \gamma e^{\alpha r} + \beta$ for $\gamma, \alpha \neq 0$ and arbitrary β (see Aczél, 1966, Cor. 1 p. 150). □

Proof of Proposition 7. The proof is straightforward. By Savage's result, there exist $W : V \to \mathbb{R}$ and a non-atomic and finitely additive $p \in \Delta(S)$ such that $J(f) = \int_S W(f)dp$ represents $\succsim$. Therefore $\succsim$ satisfies Axiom M. By P1, M and FC, for each $f \in \mathscr{F}$ there exists $a_f \in V$ such that $f \sim a_f$. By OS, there exists a cardinally unique $U : V \to \mathbb{R}$ which is additively separable and represents the restriction of $\succsim$ to V. By defining $I(U \circ f) = U(a_f)$, we obtain $f \succsim g$ if and only $I(U \circ f) \geq I(U \circ g)$. I is well-defined by M. Since both U and W represents $\succsim$ on V, there exists a monotone function

ϕ such that $W = \phi \circ U$. The proof that ϕ is continuous is identical to that in Cerreia-Vioglio et al. (2012). □

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