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SOME FOURIER-TYPE OPERATORS FOR FUNCTIONS ON UNBOUNDED INTERVALS

G. MASTROIANNI and I. NOTARANGELO

Dipartimento di Matematica e Informatica, Università degli Studi della Basilicata,
Via dell'Ateneo Lucano 10, I-85100 Potenza, Italy
e-mails: mastroianni.csafta@unibas.it, incoronata.notarangelo@unibas.it

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Abstract. In order to approximate functions defined on the real line or on the real semiaxis by polynomials, we introduce some new Fourier-type operators, connected to the Fourier sums of generalized Freud or Laguerre orthonormal systems. We prove necessary and sufficient conditions for the boundedness of these operators in suitable weighted L^p-spaces, with $1 < p < \infty$. Moreover, we give error estimates in weighted L^p and uniform norms.

1. Introduction

Let us consider the case of functions defined on the real line. R. Askey and S. Wainger [1] proved that a function f can be represented by a Hermite series under very restrictive assumptions. To be more precise, denoting by $S_m(w,f)$, $w(x) = e^{-x^2}$, the mth partial Hermite sum, there exists a positive constant $\mathcal{C}$, independent of m and f such that

$$\left\| S_m(w,f)\sqrt{w} \right\|_p \leqq \mathcal{C} \left\| f\sqrt{w} \right\|_p$$

if and only if $p \in \left(\frac{4}{3}, 4\right)$.

Subsequently, B. Muckenhoupt [15] introduced the weights $u(x) = \left(1 + |x|\right)^b \sqrt{w(x)}$ and $v(x) = \left(1 + |x|\right)^B \left(1 + \log^+ |x|\right)^\eta \sqrt{w(x)}$, with $w(x) = e^{-x^2}$, and thus he proved inequalities of the type

$$\left\| S_m(w,f)u \right\|_p \leqq \mathcal{C} \|fv\|_p, \tag{1.1}$$

for $1 < p < \infty$ and $u \neq v$, under suitable assumptions on b, B, η, where $\mathcal{C}$ is a positive constant independent of m and f.

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S. W. Jha and D. S. Lubinsky [6] extended the results of B. Muckenhoupt to the case of Freud weights, namely they replaced $w(x) = e^{-x^2}$ by $w(x) = e^{-Q(x)}$ (under suitable assumptions on Q).

It is important to remark that in [15, Theorem 3] the author showed that inequality (1.1) holds with $u = v$ and $p > \frac{4}{3}$ if the norm on the left-hand side is restricted to the subset $A_m = \left\{x \in \mathbb{R} : \left| |x| - \sqrt{m} \right| \geqq \delta\sqrt{m},\ \delta \in (0,1)\right\}$. Then he proved the L^p_u convergence of $S_m(w,f)$ to f in A_m. Moreover, in [15, Theorem 4] it was proved that inequality (1.1) holds with $u = v$ and $1 < p < 4$ for all the functions vanishing in $\mathbb{R} \setminus A_m$. Therefore, denoting by χ_{A_m} the characteristic function of A_m, the inequality

$$\left\| \chi_{A_m} S_m(w, \chi_{A_m} f)\sqrt{w} \right\|_p \leqq \mathcal{C} \left\| \chi_{A_m} f \sqrt{w} \right\|_p \tag{1.2}$$

holds with $1 < p < \infty$, $w(x) = e^{-x^2}$ and $\mathcal{C}$ independent of m and f. Furthermore, by virtue of the results due to S. W. Jha and D. S. Lubinsky, inequality (1.2) has been extended to the case of Freud weights, but under weaker assumptions than the previous ones in [15].

Now, making a slight change of notation, let $A_m = \left\{ x \in \mathbb{R} : |x| \leqq \theta a_m \right\}$, with $\theta \in (0,1)$ and a_m the Mhaskar–Rahmanov–Saff number of w, and let χ_{A_m} be the related characteristic function. Recently in [14, Theorem 3.1], among other results, G. Mastroianni and P. Vértesi proved that inequality (1.2) holds also if w is a Freud weight of the form $e^{-Q(x)}$ and, moreover, the sequence $\left\{ \chi_{A_m} S_m(w, \chi_{A_m} f) \right\}_{m\in\mathbb{N}}$ converges to the function f in $L^p_{\sqrt{w}}$ with the order of best polynomial approximation.

The latter result was extended to the case of generalized Freud weights in [12]. Letting $w(x) = |x|^\alpha e^{-|x|^\beta}$, $u(x) = |x|^\gamma e^{-\frac{|x|^\beta}{2}}$, $\beta > 1$, $\alpha > -1$, $\gamma > -1/p$, $S_m(w,f)$ the mth Fourier sum of $f \in L^p_u$ in the orthonormal system $\left\{ p_m(w) \right\}_{m\in\mathbb{N}}$ and A_m defined as above, in [12] it was proved that, under suitable assumptions on α and γ, the bound

$$\left\| \chi_{A_m} S_m(w, \chi_{A_m} f) u \right\|_p \leqq \mathcal{C} \left\| \chi_{A_m} f u \right\|_p$$

holds with $p \in (1,\infty)$ and the sequence $\left\{ \chi_{A_m} S_m(w, \chi_{A_m} f) \right\}_{m\in\mathbb{N}}$ converges to the function f in $L^p_{\sqrt{w}}$ with optimal order.

However, $\left\{ \chi_{A_m} S_m(w, \chi_{A_m} f) \right\}_{m\in\mathbb{N}}$ *is not a sequence of polynomials.* Hence the operator $\chi_{A_m} S_m(w)$ does not map functions into polynomials, as it is required in several contexts. We finish this short survey of the previous results by mentioning the fact that results similar to all of the previous ones hold also for functions defined on the real semiaxis (see [1, 15, 14, 12]).

The aim of this paper is to construct a new *polynomial operator* of Fourier type having the same behaviour of $\chi_{A_m} S_m(w)$. In fact, in Section 2.1 we define this operator and we show that it is the continuous version of a Lagrange-type projector introduced and studied in [11]. In Section 2.2 we state convergence theorems and error estimates in several function spaces. The statements in Section 2 have an interesting application. In fact, by means of a simple quadratic transformation, mutatis mutandis, we can prove useful theorems concerning the orthogonal expansion of functions defined on the real semiaxis in a system of generalized Laguerre polynomials. This is done in Section 3. Finally, in Section 4 we prove our results.

We remark that we restricted ourselves to the case of functions belonging to weighted L^p-spaces, $1 < p \leqq \infty$, with the weight $u(x) = |x|^\gamma e^{-\frac{|x|^\beta}{2}}$, $\gamma > -1/p$ or $\gamma \geqq 0$ and $\beta > 1$, only for simplicity of exposition. In fact, we could have replaced the weight u by a more general weight of the form

$$\tilde{u}(x) = \prod_{k=1}^{s} |x - t_k|^{\alpha_k} e^{-\frac{|x|^\beta}{2}}, \qquad \alpha_k > 0, \quad \beta > 1,$$

where $t_k \in \mathbb{R}$ and $s \geqq 0$ are fixed. Whereas, the weight for the orthonormal system, $w(x) = |x|^\alpha e^{-|x|^\beta}$, $\alpha > -1$ and $\beta > 1$, cannot be replaced by another weight, since nothing is known in the literature concerning the properties of polynomials, orthonormal with respect to more general weights.

2. The case of generalized Freud weights

2.1. Basic facts and preliminary results. In the sequel, $\mathcal{C}$ will stand for a positive constant that could assume different values in different formulae and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ when $\mathcal{C}$ is independent of $a, b, \dots$. Furthermore if A and B are positive quantities depending on some parameters, then $A \sim B$ will mean that there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leqq \mathcal{C}$. Finally, $\lfloor a \rfloor$ will denote the largest integer smaller than or equal to $a \in \mathbb{R}^+$.

Function spaces. For $1 \leqq p < \infty$ let

$$u(x) = |x|^\gamma e^{-\frac{|x|^\beta}{2}} \tag{2.1}$$

be a generalized Freud weight with $\gamma > -1/p$ and $\beta > 1$. The space $L^p = L^p(\mathbb{R})$ is defined in the usual way equipped with the norm

$$\|f\|_{L^p} = \|f\|_p = \left(\int_{\mathbb{R}} \left| f(x) \right|^p dx \right)^{1/p}, \qquad f \in L^p.$$

We define the weighted L^p-spaces, saying $f \in L^p_u$ if and only if $fu \in L^p$, with the norm $\|f\|_{L^p_u} = \|fu\|_p$.

If $p = \infty$ and u is the above defined weight with $\gamma > 0$ and $\beta > 1$, then

$$L^\infty_u := C_u = \Big\{ f \in C^0\big(\mathbb{R} \setminus \{0\}\big) : \lim_{x\to 0 \text{ or } |x|\to\infty} f(x)u(x) = 0 \Big\},$$

while for $\gamma = 0$

$$L^\infty_u := C_u = \Big\{ f \in C^0(\mathbb{R}) : \lim_{|x|\to\infty} f(x)u(x) = 0 \Big\},$$

where $C^0(\mathbb{R})$ is the set of all continuous functions on $\mathbb{R}$. We equip the space C_u with the norm

$$\|f\|_{C_u} = \|fu\|_\infty = \sup_{x\in\mathbb{R}} \big| f(x)u(x) \big| .$$

In the sequel, for any set $A \subset \mathbb{R}$, we will denote by $\|fu\|_{L^p(A)}$, $1 \leqq p \leqq \infty$, the previous norms restricted to the set A.

Polynomial spaces and a Fourier-type operator. Let now

$$w(x) = |x|^\alpha e^{-|x|^\beta}, \tag{2.2}$$

with $\alpha > -1$ and $\beta > 1$, be another generalized Freud weight and let $\big\{p_m(w)\big\}_{m\in\mathbb{N}}$ be the corresponding sequence of orthonormal polynomials with leading coefficient $\gamma_m = \gamma_m(w) > 0$. The properties of these polynomials have been studied in [3, 4, 5].

The Mhaskar–Rahmanov–Saff (M–R–S, for short) number a_m related to the weight $\sqrt{w}$ is given by (see for instance [8])

$$a_m = a_m\big(\sqrt{w}\,\big) = \left[\frac{2^{\beta-1}\Gamma(\beta/2)^2}{\Gamma(\beta)} \right]^{1/\beta} \Big(1 + \frac{\alpha}{4m}\Big)^{1/\beta} m^{1/\beta} \sim m^{1/\beta}, \tag{2.3}$$

where Γ is the gamma function.

Denoting by $\mathbb{P}_{m+1}$ the set of all algebraic polynomials of degree at most $m+1$, with $a_m = a_m\big(\sqrt{w}\,\big)$, we introduce the following subspace of $\mathbb{P}_{m+1}$:

$$\mathcal{P}_{m+1} = \big\{ Q \in \mathbb{P}_{m+1} : Q(\pm\, a_m) = 0 \big\} .$$

Obviously, this subspace is connected with the weight w by the M–R–S number a_m.

Moreover, for any function $f \in L^p_u$, $1 \leqq p \leqq \infty$, we denote by

$$E_m(f)_{u,p} = \inf_{P \in \mathbb{P}_m} \big\| (f-P)u \big\|_p$$

the error of best polynomial approximation of f by polynomials of $\mathbb{P}_m$ and by

$$\widetilde{E}_{m+1}(f)_{u,p} = \inf_{Q \in \mathcal{P}_{m+1}} \big\| (f-Q)u \big\|_p$$

the error of best approximation of f by means of polynomials of $\mathcal{P}_{m+1}$.

In the sequel, with $M < m/2$, we will consider the subspace

$$\mathbb{P}_M \cap \mathcal{P}_{m+1} = \big\{ Q \in \mathbb{P}_M : Q(\pm a_m) = 0 \big\}.$$

The next lemma is crucial for our aims.

LEMMA 2.1. *Let $1 \leqq p \leqq \infty$, let u and w be the weights given by* (2.1) *and* (2.2)*, with $\beta > 1$, $\alpha > -1$ and $\gamma > -1/p$ (for $p < \infty$) or $\gamma \geqq 0$ (for $p = \infty$). For any $f \in L^p_u$ and uniformly with respect to the parameters α and γ, the inequality*

$$\inf_{Q \in \mathbb{P}_M \cap \mathcal{P}_{m+1}} \big\| (f-Q)u \big\|_p \leqq \mathcal{C}\big\{ E_M(f)_{u,p} + e^{-Am} \|fu\|_p \big\} \tag{2.4}$$

holds for $M = \Big\lfloor \frac{m}{2(\delta+1)^\beta} \Big\rfloor$, $\delta > 0$ fixed, with A and $\mathcal{C}$ positive constants independent of f and m.

We remark that, by Lemma 2.1, we have

$$\widetilde{E}_{m+1}(f)_{u,p} \leqq \mathcal{C}\big\{ E_M(f)_{u,p} + e^{-Am} \|fu\|_p \big\}. \tag{2.5}$$

with $M = \Big\lfloor \frac{m}{2(\delta+1)^\beta} \Big\rfloor$, $\delta > 0$, and with A and $\mathcal{C}$ independent of f and m. This ensures the density of $\bigcup_m \mathcal{P}_{m+1}$ in L^p_u, $1 \leqq p \leqq \infty$, and it estimates the error of best approximation by elements of $\mathcal{P}_{m+1}$ in terms of the error by polynomials of degree at most M.

If f belongs to L^p_u, with u the weight in (2.1), then its mth Fourier sum $S_m(w,f)$ is defined in the usual way as

$$S_m(w,f,x) = \sum_{k=0}^{m-1} c_k(w,f) p_k(w,x) = \int_{\mathbb{R}} K_m(w,x,t) f(t) w(t)\, dt,$$

where $c_k(w,f) = \int_{\mathbb{R}} p_k(w,t) f(t) w(t)\, dt$ is the kth Fourier coefficient of f in the system $\big\{ p_m(w) \big\}_{m \in \mathbb{N}}$ and

$$K_m(w,x,t) = \sum_{k=0}^{m-1} p_k(w,x) p_k(w,t) \tag{2.6}$$

$$= \frac{\gamma_{m-1}}{\gamma_m} \frac{p_m(w,x) p_{m-1}(w,t) - p_{m-1}(w,x) p_m(w,t)}{x-t}$$

is the Christoffel–Darboux kernel.

Now we introduce a new Fourier-type operator. Given $f \in L^p_u$ and $\theta \in (0,1)$, with a_m the M–R–S number related to the weight w, we set $f_\theta = \chi_\theta f$, denoting by χ_θ the characteristic function of the interval $[-\theta a_m, \theta a_m]$. Then we define the operator

(2.7)
$$S^*_m(w,f,x) = S^*_m(w,f_\theta,x) = \left(a_m^2 - x^2\right) S_m\left(w, \frac{f_\theta}{a_m^2 - \cdot^2}, x\right), \quad f \in L^p_u.$$

Since $\frac{f_\theta}{a_m^2 - \cdot^2} \in L^p_u$, this Fourier-type operator is well-defined.

By (2.6) we have

$$S^*_m(w,f,x) = S^*_m(w,f_\theta,x) = \left(a_m^2 - x^2\right) \frac{\gamma_{m-1}}{\gamma_m}$$
$$\times \int_{-\infty}^{\infty} \frac{p_m(w,x)p_{m-1}(w,t) - p_{m-1}(w,x)p_m(w,t)}{x-t} \, \frac{f_\theta(t)}{a_m^2 - t^2} w(t)\,dt.$$

Then, using a Gaussian rule (with $f \in C^0(\mathbb{R})$), we obtain the following "truncated" Lagrange polynomial, based on the zeros x_k of $p_m(w)$ and the additional points $\pm a_m$:

$$L^*_{m+2}(w,f,x) = \sum_{|x_k| \leqq \theta a_m} \ell^*_k(w,x) f(x_k), \qquad f \in C^0(\mathbb{R}),$$

where

$$\ell^*_k(w,x) = \frac{p_m(w,x)}{p'_m(w,x_k)(x-x_k)} \, \frac{\left(a_m^2 - x^2\right)}{\left(a_m^2 - x_k^2\right)}.$$

The interpolation process based on the zeros of Freud polynomials plus two additional points was introduced by J. Szabados in [19], while the operator $L^*_{m+2}(w)$, which is the discrete version of $S^*_m(w)$, was introduced and studied in [11].

We also remark that the operator $\widetilde{S}_m(w)$, where

$$\widetilde{S}_m(w,f,x) = \left(a_m^2 - x^2\right) S_m\left(w, \frac{f}{a_m^2 - \cdot^2}, x\right),$$

could seem more natural than $S^*_m(w)$, since $\widetilde{S}_m(w,Q) = Q$ for any $Q \in \mathcal{P}_{m+1}$. However, the Fourier coefficients of the function $f(t)/(a_m^2 - t^2)$, given by

$$c_k\left(\frac{f}{a_m^2 - \cdot^2}\right) = \int_{\mathbb{R}} \frac{f(t)}{a_m^2 - t^2} p_k(w,t) w(t)\,dt,$$

are finite only under proper assumptions on the function f. This is one of the reasons why we have defined the operator $S_m^*(w)$ replacing the function f by its "finite section" f_θ. Obviously $S_m^*(w)$ *maps* L_u^p *into* $\mathcal{P}_{m+1}$*, but it is not a projector.*

The following lemma shows a property of $S_m^*(w)$, which will be useful in the sequel.

LEMMA 2.2. *Let* w *and* u *be the weights given by* (2.2) *and* (2.1). *For any* $Q(x) = \left(a_m^2 - x^2\right) P_{M-2}(x) \in \mathbb{P}_M \cap \mathcal{P}_{m+1}$*, with* $a_m = a_m\left(\sqrt{w}\right)$ *and* $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right)^\beta \frac{m}{2} \right\rfloor$, $\theta \in (0,1)$*, we have*

$$S_m^*(w, Q, x) = S_m^*(w, Q_\theta, x) = Q(x) + \Gamma_{m+1}(x),$$

where $\Gamma_{m+1} \in \mathcal{P}_{m+1}$ *with*

$$\|\Gamma_{m+1} u\|_p \leqq \mathcal{C} e^{-Am} \|Qu\|_p, \qquad 1 \leqq p \leqq \infty, \tag{2.8}$$

uniformly with respect to the parameters α *and* γ *of the weights, with* $\mathcal{C}$ *and* A *positive constants independent of* m *and* Q.

2.2. Main results. Now we are able to state boundedness and convergence theorems of the operator $S_m^*(w)$.

THEOREM 2.3. *Let* $1 < p < \infty$, $\theta \in (0,1)$ *and* w, u *be the weights in* (2.2) *and* (2.1)*, with parameters* $\alpha > -1$, $\beta > 1$ *and* $\gamma > -1/p$*. Then, for any* $f \in L_u^p$*, the inequality*

$$\left\| S_m^*(w, f_\theta) u \right\|_p \leqq \mathcal{C} \|f_\theta u\|_p, \qquad \mathcal{C} \neq \mathcal{C}(m, f), \tag{2.9}$$

holds if and only if

$$-\frac{1}{p} < \gamma - \frac{\alpha}{2} < 1 - \frac{1}{p}, \quad \gamma - \alpha < 1 - \frac{1}{p}. \tag{2.10}$$

Moreover, conditions (2.10) *imply*

$$\left\| \left[f - S_m^*(w, f_\theta) \right] u \right\|_p \leqq \mathcal{C} \left\{ E_M(f)_{u,p} + e^{-Am} \|fu\|_p \right\} \tag{2.11}$$

with $M = \left\lfloor \left(\frac{\theta}{\theta+1}\right)^\beta \frac{m}{2} \right\rfloor$, $\mathcal{C}$ *and* A *positive constants independent of* m *and* f.

The following corollary is an immediate consequence of Theorem 2.3 so we will omit its proof.

COROLLARY 2.4. *Let* $1 < p < \infty$, $\theta \in (0,1)$ *and* w, u *be the weights in* (2.2) *and* (2.1), *with* $\alpha > -1$, $\beta > 1$ *and* $\gamma \geqq 0$. *Let* $\sigma(x) = \big(1+|x|\big)^{\nu} u(x)$. *If*

$$-\frac{1}{p} < \gamma - \frac{\alpha}{2} < 1 - \frac{1}{p}, \quad \gamma - \alpha < 1 - \frac{1}{p}, \quad \nu > \frac{1}{p},$$

then, for any $f \in C_\sigma$ *and for* $\theta \in (0,1)$, *we get*

$$\big\| S_m^*(w, f_\theta) u \big\|_p \leqq \mathcal{C} \|f_\theta \sigma\|_\infty, \tag{2.12}$$

and

$$\big\| \big[f - S_m^*(w, f_\theta) \big] u \big\|_p \leqq \mathcal{C} \big\{ E_M(f)_{\sigma,\infty} + e^{-Am} \|f\sigma\|_\infty \big\} \tag{2.13}$$

with $M = \Big\lfloor \big(\frac{\theta}{\theta+1}\big)^{\beta} \frac{m}{2} \Big\rfloor$, $\mathcal{C}$ *and* A *positive constants independent of* m *and* f.

THEOREM 2.5. *Let* $\theta \in (0,1)$ *be fixed and* w, u *be the above defined weights with* $\alpha > -1$, $\beta > 1$ *and* $\gamma \geqq 0$. *If*

$$\max\Big\{0, \frac{\alpha}{2}\Big\} \leqq \gamma < \frac{\alpha}{2} + 1 \tag{2.14}$$

then, for any $f \in C_u$, *we have*

$$\big\| S_m^*(w, f_\theta) u \big\|_\infty \leqq \mathcal{C} (\log m) \|f_\theta u\|_\infty, \tag{2.15}$$

and

$$\big\| \big[f - S_m^*(w, f_\theta) \big] u \big\|_\infty \leqq \mathcal{C} \big\{ (\log m) E_M(f)_{u,\infty} + e^{-Am} \|f u\|_\infty \big\} \tag{2.16}$$

with $M = \Big\lfloor \big(\frac{\theta}{\theta+1}\big)^{\beta} \frac{m}{2} \Big\rfloor$, $\mathcal{C}$ *and* A *positive constants independent of* m *and* f.

As it has been mentioned, from the approximation point of view, the sequence $\big\{ S_m^*(w, f_\theta) \big\}_{m \in \mathbb{N}}$ has the same behaviour as the truncated sequence $\big\{ \chi_{A_m} S_m(w, \chi_{A_m} f) \big\}_{m \in \mathbb{N}}$ (see [14, 12, 9]), but it is a *polynomial sequence.*

3. The case of generalized Laguerre weights

3.1. Basic facts preliminary results.
Function spaces. For $1 \leqq p < \infty$ consider the generalized Laguerre weight

$$U(x) = x^{\gamma} e^{-\frac{x^{\beta}}{2}}, \qquad x \in (0,\infty), \tag{3.1}$$

with $\gamma > -1/p$ and $\beta > 1/2$. Then we write $f \in L^p_U$ if and only if $fU \in L^p[0,\infty)$.

For $p = \infty$ we set

$$L^{\infty}_U := C_U = \begin{cases} \Big\{ f \in C^0(0,\infty) : \lim\limits_{x\to 0 \text{ or } x\to\infty} f(x)U(x) = 0 \Big\}, & \gamma > 0; \\ \Big\{ f \in C^0[0,\infty) : \lim_{x\to\infty} f(x)U(x) = 0 \Big\}, & \gamma = 0. \end{cases}$$

Polynomial spaces and a Fourier-type operator. Let now

$$W(x) = x^{\alpha} e^{-x^{\beta}}, \tag{3.2}$$

with $\alpha > -1$ and $\beta > 1/2$, be another generalized Laguerre weight and let $\big\{ p_m(W) \big\}_{m\in\mathbb{N}}$ be the corresponding sequence of orthonormal polynomials with leading coefficient $\bar{\gamma}_m = \bar{\gamma}_m(W) > 0$. It is known that the Mhaskar–Rahmanov–Saff number $\bar{a}_m$ related to the weight $\sqrt{W}$ satisfies $\bar{a}_m \sim m^{1/\beta}$.

With $\bar{a}_m = \bar{a}_m\big(\sqrt{W}\,\big)$, we introduce the following subspace of polynomials of degree at most m:

$$\mathcal{P}_m = \big\{ Q \in \mathbb{P}_m : Q(\bar{a}_m) = 0 \big\}.$$

Moreover, for any function $f \in L^p_U$, $1 \leqq p \leqq \infty$, denote

$$\widetilde{E}_m(f)_{U,p} = \inf_{Q\in\mathcal{P}_m} \big\| (f - Q)U \big\|_p$$

the error of best approximation of f by means of polynomials of $\mathcal{P}_m$.

If $M < m/2$, we will consider the subspace

$$\mathbb{P}_M \cap \mathcal{P}_m = \big\{ Q \in \mathbb{P}_M : Q(\bar{a}_m) = 0 \big\}.$$

The following lemma can be proved the same way as Lemma 2.1.

Lemma 3.1. *Let $1 \leqq p \leqq \infty$, let U and W be the weights given by* (3.1) *and* (3.2), *with $\beta > 1/2$, $\alpha > -1$ and $\gamma > -1/p$ (for $p < \infty$) or $\gamma \geqq 0$ (for $p = \infty$). For any $f \in L^p_U$, uniformly with respect to the parameters α and γ, the inequality*

$$\inf_{Q \in \mathbb{P}_M \cap \mathcal{P}_m} \left\| (f-Q)U \right\|_p \leqq \mathcal{C} \left\{ E_M(f)_{U,p} + e^{-Am} \|fU\|_p \right\} \tag{3.3}$$

holds for $M = \left\lfloor \frac{m}{2(\delta+1)^\beta} \right\rfloor$, $\delta > 0$ fixed, with A and $\mathcal{C}$ positive constants independent of f and m.

By Lemma 3.1, the inequality

$$\widetilde{E}_m(f)_{U,p} \leqq \mathcal{C} \left\{ E_M(f)_{U,p} + e^{-Am} \|fU\|_p \right\} \tag{3.4}$$

holds with $M = \left\lfloor \frac{m}{2(\delta+1)^\beta} \right\rfloor$, $\delta \in (0,1)$, and with A and $\mathcal{C}$ independent of f and m. This ensures the density of $\bigcup_m \mathcal{P}_m$ in L^p_U, $1 \leqq p \leqq \infty$, in analogy with the subspaces introduced in the previous section.

If f belongs to L^p_U, with U the weight in (3.1), then its mth Fourier sum $S_m(W,f)$ is given by

$$S_m(W,f,x) = \sum_{k=0}^{m-1} c_k(W,f) p_k(W,x) = \int_{\mathbb{R}} K_m(W,x,t) f(t) W(t)\, dt,$$

where $c_k(W,f) = \int_0^\infty p_k(W,t) f(t) W(t)\, dt$ is the kth Fourier coefficient of f in the system $\left\{ p_m(W) \right\}_{m \in \mathbb{N}}$ and

$$K_m(W,x,t) = \sum_{k=0}^{m-1} p_k(W,x) p_k(W,t) \tag{3.5}$$

$$= \frac{\bar{\gamma}_{m-1}}{\bar{\gamma}_m} \frac{p_m(W,x) p_{m-1}(W,t) - p_{m-1}(W,x) p_m(W,t)}{x-t}$$

is the Christoffel–Darboux kernel.

Now we introduce the Fourier-type operator, analogous to the one in Section 2. Given $f \in L^p_U$ and $\theta \in (0,1)$, with $\bar{a}_m$ the M–R–S number related to the weight $\sqrt{W}$, we set $f_\theta = \chi_\theta f$, denoting by χ_θ the characteristic function of the interval $[0, \theta \bar{a}_m]$. Then, for $f \in L^p_U$, we define

$$S^*_m(W,f,x) = S^*_m(W,f_\theta,x) = (\bar{a}_m - x) S_m \left(W, \frac{f_\theta}{\bar{a}_m - \cdot}, x \right) \tag{3.6}$$

$$= (\bar{a}_m - x)\frac{\bar{\gamma}_{m-1}}{\bar{\gamma}_m}\int_0^\infty \frac{p_m(W,x)p_{m-1}(W,t) - p_{m-1}(W,x)p_m(W,t)}{x-t} \times \frac{f_\theta(t)}{\bar{a}_m - t}W(t)\,dt.$$

Using a Gaussian rule (with $f \in C^0[0,\infty)$), we obtain the "truncated" Lagrange polynomial, based on the zeros x_k of $p_m(W)$ and the additional point $\bar{a}_m$,

$$L^*_{m+1}(W,f,x) = \sum_{x_k \leqq \theta\bar{a}_m} \ell^*_k(W,x)f(x_k),$$

where

$$\ell^*_k(W,x) = \frac{p_m(W,x)}{p'_m(W,x_k)(x-x_k)}\,\frac{(\bar{a}_m - x)}{(\bar{a}_m - x_k)}.$$

The operator $L^*_{m+1}(W)$, which is the discrete version of $S^*_m(W)$, was studied in [7, 10].

The following lemma is the analogue of Lemma 2.2.

LEMMA 3.2. *Let W and U be the weights defined by* (3.2) *and* (3.1). *For any $Q(x) = (\bar{a}_m - x)P_{M-1}(x) \in \mathbb{P}_M \cap \mathcal{P}_m$, with $\bar{a}_m = \bar{a}_m\big(\sqrt{W}\big)$ and $M = \Big\lfloor \big(\frac{\theta}{\theta+1}\big)^\beta \frac{m}{2} \Big\rfloor$, $\theta \in (0,1)$, we have*

$$S^*_m(W,Q,x) = S^*_m(W,Q_\theta,x) = Q(x) + \Gamma_m(x),$$

where $\Gamma_m \in \mathcal{P}_m$ with

$$\|\Gamma_m U\|_p \leqq \mathcal{C}e^{-Am}\|QU\|_p, \qquad 1 \leqq p \leqq \infty, \tag{3.7}$$

uniformly with respect to the parameters α and γ of the weights, with $\mathcal{C}$ and A positive constants independent of m and Q.

3.2. Main results. The next results follow from the statements in Subsection 2.2.

THEOREM 3.3. *Let $1 < p < \infty$, $\theta \in (0,1)$ and W, U be the weights in* (3.2) *and* (3.1), *with parameters $\alpha > -1$, $\beta > 1/2$ and $\gamma > -1/p$. Then, for any $f \in L^p_U$, the inequality*

$$\big\| S^*_m(W,f_\theta)U \big\|_p \leqq \mathcal{C}\|f_\theta U\|_p, \qquad \mathcal{C} \neq \mathcal{C}(m,f), \tag{3.8}$$

holds if and only if

$$\frac{1}{4} - \frac{1}{p} < \gamma - \frac{\alpha}{2} < \frac{3}{4} - \frac{1}{p}, \quad \gamma - \alpha < 1 - \frac{1}{p}. \tag{3.9}$$

Moreover, conditions (3.9) *imply*

$$\left\| \left[f - S_m^*(W, f_\theta) \right] U \right\|_p \leqq \mathcal{C} \left\{ E_M(f)_{U,p} + e^{-Am} \|fU\|_p \right\} \tag{3.10}$$

with $M = \left\lfloor \left(\frac{\theta}{\theta+1} \right)^\beta \frac{m}{2} \right\rfloor$, $\mathcal{C}$ *and* A *positive constants independent of* m *and* f.

COROLLARY 3.4. *Let* $1 < p < \infty$, $\theta \in (0,1)$ *and* W, U *be the weights in* (3.2) *and* (3.1), *with* $\alpha > -1$, $\beta > 1/2$ *and* $\gamma \geqq 0$, *and let* $\sigma(x) = (1+x)^\nu U(x)$. *If*

$$\frac{1}{4} - \frac{1}{p} < \gamma - \frac{\alpha}{2} < \frac{3}{4} - \frac{1}{p}, \quad \gamma - \alpha < 1 - \frac{1}{p}, \quad \nu > \frac{1}{p},$$

then, for any $f \in C_\sigma$ *and for* $\theta \in (0,1)$, *we get*

$$\left\| S_m^*(W, f_\theta) U \right\|_p \leqq \mathcal{C} \|f_\theta \sigma\|_\infty, \tag{3.11}$$

and

$$\left\| \left[f - S_m^*(W, f_\theta) \right] U \right\|_p \leqq \mathcal{C} \left\{ E_M(f)_{\sigma,\infty} + e^{-Am} \|f\sigma\|_\infty \right\} \tag{3.12}$$

with $M = \left\lfloor \left(\frac{\theta}{\theta+1} \right)^\beta \frac{m}{2} \right\rfloor$, $\mathcal{C}$ *and* A *positive constants independent of* m *and* f.

THEOREM 3.5. *Let* W *and* U *be the above defined weights with* $\alpha > -1$, $\beta > 1/2$ *and* $\gamma \geqq 0$. *If*

$$\max\left\{0, \frac{\alpha}{2} + \frac{1}{4}\right\} \leqq \gamma < \frac{\alpha}{2} + \frac{3}{4} \tag{3.13}$$

then, for any $f \in C_U$ *and for* $\theta \in (0,1)$, *we have*

$$\left\| S_m^*(W, f_\theta) U \right\|_\infty \leqq \mathcal{C} (\log m) \|f_\theta U\|_\infty, \tag{3.14}$$

and

$$\left\| \left[f - S_m^*(W, f_\theta) \right] U \right\|_\infty \leqq \mathcal{C} \left\{ (\log m) E_M(f)_{U,\infty} + e^{-Am} \|fU\|_\infty \right\} \tag{3.15}$$

with $M = \left\lfloor \left(\frac{\theta}{\theta+1} \right)^\beta \frac{m}{2} \right\rfloor$, $\mathcal{C}$ *and* A *positive constants independent of* m *and* f.

As we have already emphasized in Subsection 2.2, from the approximation point of view, the sequence $\left\{ S_m^*(W, f_\theta) \right\}_{m \in \mathbb{N}}$ has the same behaviour as the truncated sequence $\left\{ \chi_\theta S_m(W, \chi_\theta f) \right\}_{m \in \mathbb{N}}$ (see [14, 12, 9]).

4. Proofs

We are going to prove only the statements in Section 2, since the theorems in Section 3 can be deduced from these ones. First of all we recall some known results, which will be used in the proofs. The following lemma collects some polynomial inequalities, which can be found in [13] (see also [18], pp. 277–343).

Lemma 4.1. *Let* $1 \leqq p \leqq \infty$ *and* $u(x) = |x|^\gamma e^{-\frac{|x|^\beta}{2}}$, *with* $\beta > 1$ *and* $\gamma > -\frac{1}{p}$ *if* $1 \leqq p < \infty$, $\gamma \geqq 0$ *if* $p = \infty$. *Moreover let* $a_m(u)$ *be the M–R–S number related to* u. *Then for any polynomial* $P \in \mathbb{P}_m$, *we have*

$$\|P'u\|_p \leqq \mathcal{C}\,\frac{m}{a_m(u)}\|Pu\|_p \tag{4.1}$$

and

$$\|Pu\|_q \leqq \mathcal{C}\left(\frac{m}{a_m(u)}\right)^{\frac{1}{p}-\frac{1}{q}}\|Pu\|_p, \qquad 1 \leqq p < q \leqq \infty, \tag{4.2}$$

where $\mathcal{C}$ *is a constant independent of* m *and* P *in both cases.*

Moreover, for any fixed $d > 0$, *there exists* $\mathcal{C} = \mathcal{C}(d) \neq \mathcal{C}(m,P)$ *such that*

$$\|Pu\|_p \leqq \mathcal{C}\|Pu\|_{L^p(\mathcal{A}_m(u))}, \tag{4.3}$$

where $\mathcal{A}_m(u) = \big[-a_m(u), -d\frac{a_m(u)}{m}\big] \cup \big[d\frac{a_m(u)}{m}, a_m(u)\big]$. *Finally, for all* $\delta > 0$, *the inequality*

$$\|Pu\|_{L^p(\mathcal{B}_m(u))} \leqq \mathcal{C}\,e^{-Am}\|Pu\|_p, \tag{4.4}$$

holds with $\mathcal{B}_m(u) = \big\{x \in \mathbb{R} : |x| > (1+\delta)a_m(u)\big\}$, $\mathcal{C}$ *and* A *positive constants independent of* m *and* P.

The Mhaskar–Rahmanov–Saff number related to the weight u is defined by (see for instance [8])

$$a_m(u) = \left[\frac{2^{\beta-1}\Gamma(\beta/2)^2}{\Gamma(\beta)}\right]^{1/\beta}\Big(1+\frac{\gamma}{2m}\Big)^{1/\beta} m^{1/\beta} \sim m^{1/\beta}, \tag{4.5}$$

where Γ is the gamma function. By (2.3) and (4.5), we have $a_m(u) \sim a_m\big(\sqrt{w}\,\big)$, where $w(x) = |x|^\alpha e^{-|x|^\beta}$, $\alpha > -1$ and $\beta > 1$.

The following inequalities deal with the behaviour of the orthonormal system $\{p_m(w)\}_{m\in\mathbb{N}}$. In [3] it has been proved that

$$\sup_{x\in[-a_m,a_m]}\left|p_m(w,x)e^{-\frac{|x|^\beta}{2}}\left(|x|+\frac{a_m}{m}\right)^{\alpha/2}\sqrt[4]{a_m^2-x^2}\right|\sim 1 \tag{4.6}$$

and

$$\left|p_m(w,x)\right|\sqrt{w(x)}\leqq\frac{\mathcal{C}}{\sqrt[4]{a_m^2-x^2+m^{-2/3}a_m^2}},\quad x\in\mathcal{A}_m\left(\sqrt{w}\right), \tag{4.7}$$

where $a_m=a_m\left(\sqrt{w}\right)$ and

$$\mathcal{A}_m\left(\sqrt{w}\right)=\left[-a_m\left(\sqrt{w}\right),-d\frac{a_m\left(\sqrt{w}\right)}{m}\right]\cup\left[d\frac{a_m\left(\sqrt{w}\right)}{m},a_m\left(\sqrt{w}\right)\right],$$

$d>0$.

In the sequel we will also need the following proposition, proved in [11].

PROPOSITION 4.2. *Let w be the weight in* (2.2), *$\{p_m(w)\}_{m\in\mathbb{N}}$ be its corresponding orthonormal system and $a_m=a_m\left(\sqrt{w}\right)$. Then, for any $G\in L^p$, $1\leqq p<\infty$, we have*

$$\left(\int_{\mathbb{R}}\left|G(x)p_m(w,x)\sqrt{w(x)}\sqrt[4]{a_m^2-x^2}\right|^p dx\right)^{1/p}\geqq\mathcal{C}\left(\int_{-a_m}^{a_m}\left|G(x)\right|^p dx\right)^{1/p} \tag{4.8}$$

with $\mathcal{C}\neq\mathcal{C}(G,m,w)$.

For each set $A\subset\mathbb{R}$ and for $y\in\mathbb{R}$, denote by

$$\mathcal{H}_A(f;y)=\int_A\frac{f(x)}{x-y}\,dx\qquad\text{and}\qquad\mathcal{H}(f;y)=\int_{\mathbb{R}}\frac{f(x)}{x-y}\,dx$$

the Cauchy principal values of these integrals, i.e. $\mathcal{H}_A$ is the Hilbert transform extended to the subset A. If $A=[-1,1]$ and v is a weight function on the same subset, then, for any measurable function f with $fv\in L^p[-1,1]$, $1<p<\infty$, the inequality

$$\left\|\mathcal{H}_{[-1,1]}(f)v\right\|_p\leqq\mathcal{C}\|fv\|_p,\qquad\mathcal{C}\neq\mathcal{C}(f), \tag{4.9}$$

holds if and only if v is an A_p weight (see [16, 2, 17]), that is for any interval $I \subset [-1,1]$

(4.10)
$$\left(\frac{1}{|I|}\int_I v^p(x)\,dx\right)^{1/p}\left(\frac{1}{|I|}\int_I v^{-q}(x)\,dx\right)^{1/q} \leqq \mathcal{C} \neq \mathcal{C}(I), \qquad \frac{1}{p}+\frac{1}{q}=1,$$

where $|I|$ is the measure of I and $1<p<\infty$. Observe that $v(x)=|x|^\lambda$ is an A_p weight if and only if

$$-\frac{1}{p}<\lambda<1-\frac{1}{p}.$$

Finally, we say that $P_m \in \mathbb{P}_m$ is a near best polynomial approximant of $f \in L^p_u$, if

$$\left\| (f-P_m)u \right\|_p \leqq \mathcal{C}E_m(f)_{u,p}.$$

Proof of Lemma 2.1. We are going to prove (2.4) only for $1 \leqq p < \infty$, the case $p=\infty$ being simpler. Let $P_M \in \mathbb{P}_M$ be a near best polynomial approximant of $f \in L^p_u$, with $M=\left\lfloor \frac{m}{2(\delta+1)^\beta} \right\rfloor$ and $\delta>0$ to be fixed in the sequel. Set

$$Q_M(x)=\sum_{|k|\leqq\lfloor\frac{M+1}{2}\rfloor-1} \ell_k(w,x)P_M(x_k)$$

and

$$P_M(x)=\sum_{|k|\leqq\lfloor\frac{M+1}{2}\rfloor} \ell_k(w,x)P_M(x_k),$$

where

$$\ell_k(w,x)=\frac{p_{M-1}(w,x)}{p'_{M-1}(w,x_k)(x-x_k)}\frac{\left(a_m^2-x^2\right)}{\left(a_m^2-x_k^2\right)}, \qquad |k| \leqq \left\lfloor \frac{M-1}{2} \right\rfloor, \tag{4.11}$$

and

$$\ell_{\pm\lfloor\frac{M+1}{2}\rfloor}(w,x)=\frac{a_m \pm x}{2a_m}\frac{p_{M-1}(w,x)}{p_{M-1}(w,\pm a_m)}. \tag{4.12}$$

Namely, these interpolation processes are based on the zeros of $p_{M-1}(w)$, i.e. $x_{-\lfloor\frac{M-1}{2}\rfloor}<\dots<x_1<x_2<\dots<x_{\lfloor\frac{M-1}{2}\rfloor}$, plus the additional points

$x_{\pm\lfloor\frac{M+1}{2}\rfloor} := \pm a_m$, with $a_m = a_m\big(\sqrt{w}\big)$. If $M-1$ is odd we replace the zero $x_0 = 0$ by $x_0 := \frac{x_1}{2}$. Obviously $Q_M \in \mathbb{P}_M \cap \mathcal{P}_{m+1}$. We have

$$\begin{aligned}(4.13)\qquad \inf_{Q\in\mathbb{P}_M\cap\mathcal{P}_{m+1}} \big\|(f-Q)u\big\|_p &\leqq \big\|(f-Q_M)u\big\|_p \\ &\leqq \big\|(Q_M-P_M)u\big\|_p + \mathcal{C}E_M(f)_{u,p} \\ &= \Big\|\Big[\ell_{-\lfloor\frac{M+1}{2}\rfloor}(w)P_M(-a_m) + \ell_{\lfloor\frac{M+1}{2}\rfloor}(w)P_M(a_m)\Big]u\Big\|_p + \mathcal{C}E_M(f)_{u,p} \\ &\leqq \Big\|P_M(-a_m)\ell_{-\lfloor\frac{M+1}{2}\rfloor}(w)u\Big\|_p + \Big\|P_M(a_m)\ell_{\lfloor\frac{M+1}{2}\rfloor}(w)u\Big\|_p + \mathcal{C}E_M(f)_{u,p} \\ &\leqq I_1 + I_2 + \mathcal{C}E_M(f)_{u,p}.\end{aligned}$$

We are going to estimate only the term I_2, the other one being similar. By the Remez-type inequality (4.3) we get

$$\begin{aligned}I_2 &\leqq \mathcal{C}\Big\|P_M(a_m)\ell_{\lfloor\frac{M+1}{2}\rfloor}(w)u\Big\|_{L^p(\mathcal{A}_M(u))} \\ &\leqq \mathcal{C}\, a_M(u)^{1/p} \sup_{|x|\geqq a_m(\sqrt{w})} \big|P_M(x)u(x)\big| \sup_{x\in\mathcal{A}_M(u)} \left|\frac{\ell_{\lfloor\frac{M+1}{2}\rfloor}(w,x)u(x)}{u(a_m)}\right|\end{aligned}$$

where $\mathcal{A}_M(u) = \big[-a_M(u), a_M(u)\big] \setminus \big[-\frac{a_M(u)}{M}, \frac{a_M(u)}{M}\big]$. Using the inequality

$$\frac{\Big|\ell_{\lfloor\frac{M+1}{2}\rfloor}(w;x)\Big|u(x)}{u(a_m)} \leqq \mathcal{C}\left|\frac{x}{a_m\big(\sqrt{w}\big)}\right|^{\gamma-\frac{\alpha}{2}} \leqq \begin{cases}\mathcal{C} & \text{if } \gamma - \dfrac{\alpha}{2} \geqq 0 \\ \mathcal{C}m^{\frac{\alpha}{2}-\gamma} & \text{if } \gamma - \dfrac{\alpha}{2} < 0\end{cases}$$

(see [11], formula (4.11)), we obtain

$$I_2 \leqq \mathcal{C}\, a_M(u)^{1/p} m^{\tau} \sup_{|x|\geqq a_m(\sqrt{w})} \big|P_M(x)u(x)\big|,$$

where $\tau = \max\big\{0, \frac{\alpha}{2}-\gamma\big\}$. Now, by the choice of M, it is easily seen that $a_m\big(\sqrt{w}\big) \geqq (1+\delta)a_M(u)$, for some $\delta > 0$ and for a sufficiently large m. Then, using inequalities (4.4) and (4.2), since $a_m\big(\sqrt{w}\big) \sim a_m(u)$, we get

$$I_2 \leqq \mathcal{C}\, e^{-Am}\|P_M u\|_p \leqq \mathcal{C}e^{-Am}\|fu\|_p.$$

From the previous estimate and (4.13), inequality (2.4) follows. □

Proof of Lemma 2.2. Let us first consider the case $1<p<\infty$. For each $Q\in\mathbb{P}_M\cap\mathcal{P}_{m+1}$, with $M=\left\lfloor\left(\frac{\theta}{\theta+1}\right)^\beta\frac{m}{2}\right\rfloor$ and $\theta\in(0,1)$ fixed, we can write

$$S_m^*(w,Q_\theta,x)=\left(a_m^2-x^2\right)S_m\left(w,\frac{Q_\theta}{a_m^2-.^2},x\right)$$

$$=Q(x)+\left(a_m^2-x^2\right)S_m\left(w,\frac{Q_\theta-Q}{a_m^2-.^2},x\right)=:Q(x)+\Gamma_{m+1}(x),$$

where $a_m=a_m\left(\sqrt{w}\right)$ and

$$\Gamma_{m+1}(x)=S_m(w,Q_\theta-Q,x)+S_m\left(w,\frac{.^2-x^2}{a_m^2-.^2}[Q_\theta-Q],x\right)\tag{4.14}$$

$$=:A_1(x)+A_2(x).$$

In order to estimate the term $A_1(x)$, we recall inequalities (18) and (19) in [12, p. 93] and Theorem 1 in [15],

$$\left\|S_m(w,f)u\right\|_p\leqq\mathcal{C}\begin{cases}\|fu\|_p, & \text{if }\ \frac{4}{3}<p<4\\ m^{1/3}\|fu\|_p, & \text{if }\ 1<p\leqq\frac{4}{3}\ \text{or}\ 4\leqq p<\infty\end{cases}$$

and since, by the choice of M, $\theta a_m\left(\sqrt{w}\right)\geqq(1+\theta)a_M(u)$ and $a_m\left(\sqrt{w}\right)\sim a_m(u)$, we can use (4.4), obtaining

$$\|A_1u\|_p\leqq\mathcal{C}m^{1/3}\left\|(Q_\theta-Q)u\right\|_p=\mathcal{C}m^{1/3}\|Qu\|_{L^p\{|x|\geqq\theta a_m(\sqrt{w})\}}\tag{4.15}$$

$$\leqq\mathcal{C}e^{-Am}\|Qu\|_p.$$

As for $A_2(x)$, by (2.6), we can write

$$A_2(x)=\frac{\gamma_{m-1}}{\gamma_m}$$

$$\times\int_{|t|>\theta a_m}\frac{(t+x)\left[p_m(w,x)p_{m-1}(w,t)-p_{m-1}(w,x)p_m(w,t)\right]Q(t)w(t)}{a_m^2-t^2}\,dt$$

and, according to the Remez-type inequality (4.3), considering the subset $\mathcal{A}_m(u)=\left[-a_m(u),a_m(u)\right]\setminus\left[-\frac{a_m(u)}{m},\frac{a_m(u)}{m}\right]$, it is sufficient to estimate only the term

$$\|\widehat{A}_2u\|_p\tag{4.16}$$

$$\leqq\frac{\gamma_{m-1}}{\gamma_m}\left(\int_{\mathcal{A}_m(u)}\left|u(x)p_m(w,x)\int_{\theta a_m}^\infty\frac{t+x}{a_m^2-t^2}p_{m-1}(w,t)Q(t)w(t)\,dt\right|^p dx\right)^{1/p}$$

$$\leqq \mathcal{C} a_m \big\| p_m(w)u \big\|_{L^p(\mathcal{A}_m(u))} \int_{\theta a_m}^{\infty} \left| \frac{p_{m-1}(w,t)Q(t)w(t)}{a_m - t} \right| dt$$

where $\frac{\gamma_{m-1}}{\gamma_m} \sim a_m = a_m\big(\sqrt{w}\big)$ (see [3]), the other ones being similar. By inequality (4.7), since $a_m\big(\sqrt{w}\big) \sim a_m(u)$, we have

$$a_m \big\| p_m(w)u \big\|_{L^p(\mathcal{A}_m(u))} \leqq \mathcal{C} a_m \frac{m^{1/6}}{\sqrt{a_m}} \big\| | \cdot |^{\gamma - \frac{\alpha}{2}} \big\|_{L^p(\mathcal{A}_m(u))} \tag{4.17}$$

$$\leqq \mathcal{C} \sqrt{a_m}\, m^{1/6} \max \left\{ a_m(u)^{\gamma - \frac{\alpha}{2} + \frac{1}{p}}, \left(\frac{a_m(u)}{m} \right)^{\gamma - \frac{\alpha}{2} + \frac{1}{p}} \right\}.$$

Concerning the last factor in (4.16), by (4.7), we have

$$\int_{\theta a_m}^{\infty} \left| \frac{p_{m-1}(w,t)Q(t)w(t)}{a_m - t} \right| dt \leqq \mathcal{C} \frac{m^{1/6}}{\sqrt{a_m}} \int_{\theta a_m}^{\infty} \left| \frac{Q(t)|t|^{\frac{\alpha}{2} - \gamma} u(t)}{a_m - t} \right| dt \tag{4.18}$$

$$\leqq \mathcal{C} m^{\frac{1}{6}}\, a_m^{\frac{\alpha}{2} - \gamma - \frac{1}{2}} \int_{\theta a_m}^{2a_m} \left| \frac{Q(t)u(t)}{a_m - t} \right| dt + \mathcal{C} \frac{m^{1/6}}{a_m^{3/2}} \int_{2a_m}^{\infty} \big| Q(t) \big| \, |t|^{\frac{\alpha}{2} - \gamma} u(t)\, dt.$$

By Hölder inequality, using the boundedness of the Hardy–Littlewood maximal function and by inequalities (4.4) and (4.1), we get

$$m^{\frac{1}{6}}\, a_m^{\frac{\alpha}{2} - \gamma - \frac{1}{2}} \int_{\theta a_m}^{2a_m} \left| \frac{Q(t)u(t)}{a_m - t} \right| dt$$

$$\leqq \mathcal{C} m^{\frac{1}{6}}\, a_m^{\frac{\alpha}{2} - \gamma - \frac{1}{2}} a_m(u)^{1 - \frac{1}{p}} \left\| \frac{Qu}{a_m - \cdot} \right\|_{L^p[\theta a_m, 2a_m]}$$

$$= \mathcal{C} m^{\frac{1}{6}}\, a_m^{\frac{\alpha}{2} - \gamma + \frac{1}{2} - \frac{1}{p}} \left(\int_{\theta a_m}^{2a_m} \left| \frac{1}{|a_m - t|} \int_t^{a_m} (Qu)'(y)\, dy \right|^p dt \right)^{1/p}$$

$$\leqq \mathcal{C} m^{\frac{1}{6}}\, a_m^{\frac{\alpha}{2} - \gamma + \frac{1}{2} - \frac{1}{p}} \big\| (Qu)' \big\|_{L^p[\theta a_m, 2a_m]}$$

$$\leqq \mathcal{C} e^{-Am} \big\| (Qu)' \big\|_p \leqq \mathcal{C} e^{-Am} \left(\frac{m}{a_m(u)} \right) \|Qu\|_p,$$

since $\theta a_m\big(\sqrt{w}\big) \geqq (1+\theta) a_M(u)$ and $a_m = a_m\big(\sqrt{w}\big) \sim a_m(u)$. Let us consider the second summand in (4.18). If $\alpha/2 - \gamma \leqq 0$, by (4.4), (4.3) and the Hölder inequality, we have

$$\frac{m^{1/6}}{a_m^{3/2}} \int_{2a_m}^{\infty} \big| Q(t) \big| \, |t|^{\frac{\alpha}{2} - \gamma} u(t)\, dt$$

$$\leqq \mathcal{C} m^{\frac{1}{6}}\, a_m^{\frac{\alpha}{2}-\gamma-\frac{3}{2}} \int_{2a_m}^{\infty} \bigl|Q(t)\bigr| u(t)\, dt \leqq \mathcal{C} m^{\frac{1}{6}}\, a_m^{\frac{\alpha}{2}-\gamma-\frac{3}{2}} e^{-Am} \|Qu\|_{L^1(\mathcal{A}_M(u))}$$

$$\leqq \mathcal{C} m^{\frac{1}{6}}\, a_m^{\frac{\alpha}{2}-\gamma-\frac{3}{2}} a_m(u)^{1-\frac{1}{p}} e^{-Am} \|Qu\|_{L^p(\mathcal{A}_M(u))} \leqq \mathcal{C} e^{-Am} \|Qu\|_p,$$

since $2a_m\bigl(\sqrt{w}\,\bigr) \geqq (1+\theta) a_m(u)$ and $a_m = a_m\bigl(\sqrt{w}\,\bigr) \sim a_m(u)$. Otherwise, if $\alpha/2 - \gamma > 0$, by (4.4), (4.3) and the Hölder inequality, we have

$$\frac{m^{1/6}}{a_m^{3/2}} \int_{2a_m}^{\infty} \bigl|Q(t)\bigr| |t|^{\frac{\alpha}{2}-\gamma} u(t)\, dt \leqq \mathcal{C} \frac{m^{1/6}}{a_m^{3/2}}\, e^{-Am} \bigl\| Q|\cdot|^{\frac{\alpha}{2}-\gamma} u \bigr\|_{L^1(\mathcal{A}_m(\sqrt{w}\,))}$$

$$\leqq \mathcal{C} \frac{m^{1/6}}{a_m^{3/2}}\, e^{-Am} a_m\bigl(\sqrt{w}\,\bigr)^{\frac{\alpha}{2}-\gamma} \|Qu\|_{L^1(\mathcal{A}_m(\sqrt{w}\,))} \leqq \mathcal{C} e^{-Am} \|Qu\|_p,$$

since $2a_m = 2a_m\bigl(\sqrt{w}\,\bigr) \geqq (1+\delta) a_m\bigl(\sqrt{w}\,\bigr)$, for some $\delta > 0$. Combining the previous estimates of the terms in (4.18) with (4.17) and (4.16), it follows that

$$\|\widehat{A}_2 u\|_p \leqq \mathcal{C} e^{-Am} \|Qu\|_p$$

whence

$$\|A_2 u\|_p \leqq \mathcal{C} e^{-Am} \|Qu\|_p. \tag{4.19}$$

Taking into account (4.14), by (4.15) and (4.19), we obtain (2.8) for $1 < p < \infty$.

Now we are going to prove (2.8) for $p = \infty$, taking into account what we have already proved for $1 < s < \infty$. Using the Nikolskii-type inequality (4.2) with $q = \infty$ and the Remez-type inequality (4.3), we get

$$\|\Gamma_{m+1} u\|_\infty \leqq \mathcal{C} \left(\frac{m}{a_m(u)}\right)^{1/s} \|\Gamma_{m+1} u\|_s \leqq \mathcal{C} \left(\frac{m}{a_m(u)}\right)^{1/s} e^{-Am} \|Qu\|_s$$

$$\leqq \mathcal{C} \left(\frac{m}{a_m(u)}\right)^{1/s} e^{-Am} \|Qu\|_{L^s(\mathcal{A}_m(u))} \leqq \mathcal{C} m^{1/s} e^{-Am} \|Qu\|_\infty$$

$$\leqq \mathcal{C} e^{-Am} \|Qu\|_\infty.$$

Finally, if $p = 1$, by the Remez-type inequality (4.3), (2.8) and the Nikolskii-type inequality (4.2), we obtain

$$\|\Gamma_{m+1} u\|_1 \leqq \mathcal{C} \|\Gamma_{m+1} u\|_{L^1(\mathcal{A}_m(u))} \leqq \mathcal{C} a_m(u) \|\Gamma_{m+1} u\|_\infty$$

$$\leqq \mathcal{C}\, a_m(u) e^{-Am} \|Qu\|_\infty \leqq \mathcal{C}\, m\, e^{-Am} \|Qu\|_1 \leqq \mathcal{C} e^{-Am} \|Qu\|_1. \quad \square$$

Proof of Theorem 2.3. First we prove that (2.10) implies (2.9). By using the Remez-type inequality (4.3), with $a_m = a_m(\sqrt{w})$, we have

$$\left\| S_m^*(w, f_\theta)u \right\|_p \leqq \mathcal{C} \left\| S_m^*(w, f_\theta)u \right\|_{L^p(\mathcal{A}_m(u))}$$

$$= \mathcal{C}\left(\int_{\mathcal{A}_m(u)} \left| \left(a_m^2 - x^2\right) u(x) \int_{-a_m(u)}^{a_m(u)} \frac{K_m(w,x,t) f_\theta(t) w(t)}{a_m^2 - t^2}\, dt \right|^p dx \right)^{1/p},$$

where $\mathcal{A}_m(u) = \left[-a_m(u), a_m(u) \right] \setminus \left[-\frac{a_m(u)}{m}, \frac{a_m(u)}{m} \right]$. Since $\frac{\gamma_{m-1}}{\gamma_m} \sim a_m$, by (2.6) and (4.7), it follows that

$$\left\| S_m^*(w, f_\theta)u \right\|_p \leqq \mathcal{C} a_m \left(\int_{\mathcal{A}_m(u)} \left| \left(a_m^2 - x^2\right) p_m(w,x) u(x) \right.\right.$$
$$\left.\left. \times \int_{-a_m(u)}^{a_m(u)} \frac{p_{m-1}(w,t) f_\theta(t) w(t)}{(a_m^2 - t^2)(x-t)}\, dt \right|^p dx \right)^{1/p}$$
$$+ \mathcal{C} a_m \left(\int_{\mathcal{A}_m(u)} \left| \left(a_m^2 - x^2\right) p_{m-1}(w,x) u(x) \right.\right.$$
$$\left.\left. \times \int_{-a_m(u)}^{a_m(u)} \frac{p_m(w,t) f_\theta(t) w(t)}{(a_m^2 - t^2)(x-t)}\, dt \right|^p dx \right)^{1/p} \tag{4.20}$$
$$\leqq \mathcal{C} a_m^{5/2} \left(\int_{\mathcal{A}_m(u)} \left| |x|^{\gamma - \frac{\alpha}{2}} \int_{-a_m(u)}^{a_m(u)} \frac{p_{m-1}(w,t) f_\theta(t) w(t)}{(a_m^2 - t^2)(x-t)}\, dt \right|^p dx \right)^{1/p}$$
$$+ \mathcal{C} a_m^{5/2} \left(\int_{\mathcal{A}_m(u)} \left| |x|^{\gamma - \frac{\alpha}{2}} \int_{-a_m(u)}^{a_m(u)} \frac{p_m(w,t) f_\theta(t) w(t)}{(a_m^2 - t^2)(x-t)}\, dt \right|^p dx \right)^{1/p} =: I_1 + I_2.$$

We are going to estimate only the term I_1, because the other one can be handled similarly.

We first consider the case $\alpha \geqq 0$. Making the changes of variables $x = a_m(u)y$ and $t = a_m(u)s$ we can reduce the integrals in I_1 to the interval $[-1, 1]$. By (2.10), the function $|\cdot|^{\gamma - \alpha/2}$ is an A_p weight, so we can use (4.9) and (4.10). Then, returning to $\left[-a_m(u), a_m(u) \right]$ we get

$$I_1 \leqq \mathcal{C} a_m^{5/2} \left(\int_{-a_m(u)}^{a_m(u)} \left| |x|^{\gamma - \frac{\alpha}{2}} \frac{p_m(w,x) f_\theta(x) w(x)}{a_m^2 - t^2} \right|^p dx \right)^{1/p}$$

$$\leqq \mathcal{C}\sqrt{a_m}\left(\int_{-\theta a_m(\sqrt{w})}^{\theta a_m(\sqrt{w})} \left||x|^{\gamma-\frac{\alpha}{2}} p_m(w,x) f_\theta(x) w(x)\right|^p dx\right)^{1/p}.$$

Being $\alpha \geqq 0$, by (4.6), we obtain $I_1 \leqq \mathcal{C}\|f_\theta u\|_p$. That means, we get (2.9) if $\alpha \geqq 0$, since an estimate, analogous to the previous one, holds for the term I_2 in (4.20).

Now let $-1 < \alpha < 0$. Concerning the term I_1 in (4.20), we can write

$$(4.21)\quad I_1 \leqq \mathcal{C}a_m^{5/2}\left(\int_{\widetilde{\mathcal{A}}_m(u)} \left||x|^{\gamma-\frac{\alpha}{2}} \int_{\widetilde{\mathcal{A}}_m(u)} \frac{p_{m-1}(w,t) f_\theta(t) w(t)}{(a_m^2-t^2)(x-t)}\, dt\right|^p dx\right)^{1/p}$$

$$+\, \mathcal{C}a_m^{5/2}\left(\int_{\mathcal{A}_m(u)} \left||x|^{\gamma-\frac{\alpha}{2}} \int_{-\frac{a_m(u)}{2m}}^{\frac{a_m(u)}{2m}} \frac{p_{m-1}(w,t) f_\theta(t) w(t)}{(a_m^2-t^2)(x-t)}\, dt\right|^p dx\right)^{1/p}$$

$$=: J_1 + J_2.$$

where $\widetilde{\mathcal{A}}_m(u) = \big[-a_m(u), a_m(u)\big] \setminus \big[-\frac{a_m(u)}{2m}, \frac{a_m(u)}{2m}\big]$. The term J_1 can be estimated as in the first part of this proof for the term I_1, using the boundedness of the Hilbert transform and inequality (4.7). Hence we get

$$(4.22)\qquad J_1 \leqq \mathcal{C}\|f_\theta u\|_p.$$

For J_2 we can write

(4.23)

$$J_2 \leqq \mathcal{C}a_m^{5/2}\left(\int_{\frac{a_m(u)}{m}\leqq|x|\leqq 1} \left||x|^{\gamma-\frac{\alpha}{2}} \int_{-\frac{a_m(u)}{2m}}^{\frac{a_m(u)}{2m}} \frac{p_{m-1}(w,t) f_\theta(t) w(t)}{(a_m^2-t^2)(x-t)}\, dt\right|^p dx\right)^{1/p}$$

$$+\, \mathcal{C}a_m^{5/2}\left(\int_{1\leqq|x|\leqq a_m(u)} \left||x|^{\gamma-\frac{\alpha}{2}} \int_{-\frac{a_m(u)}{2m}}^{\frac{a_m(u)}{2m}} \frac{p_{m-1}(w,t) f_\theta(t) w(t)}{(a_m^2-t^2)(x-t)}\, dt\right|^p dx\right)^{1/p}$$

$$=: J_2' + J_2''.$$

Let us consider J_2''. Since $|x-t| \geqq |x| - |t| \geqq \frac{|x|}{2}$ and using Hölder inequality, we obtain

$$J_2'' \leqq \mathcal{C}\sqrt{a_m} \int_{-\frac{a_m(u)}{2m}}^{\frac{a_m(u)}{2m}} \left|p_{m-1}(w,t) f_\theta(t) w(t)\right| dt$$

$$\times \left(\int_{1\leqq|x|\leqq a_m(u)} |x|^{(\gamma-\frac{\alpha}{2}-1)p}\, dx\right)^{1/p}$$

$$\leqq \mathcal{C}\sqrt{a_m}\|f_\theta u\|_p \left[\int_{-\frac{a_m(u)}{2m}}^{\frac{a_m(u)}{2m}} \left|p_{m-1}(w,t)|t|^{\alpha-\gamma} e^{-\frac{|t|^\beta}{2}}\right|^q dt\right]^{\frac{1}{q}}$$

$$\times \left[\int_{1\leqq|x|\leqq a_m(u)} |x|^{\left(\gamma-\frac{\alpha}{2}-1\right)p} \, dx\right]^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{q}=1$. From (4.6), since $a_m = a_m\left(\sqrt{w}\right) \sim a_m(u)$, it follows that

$$\left|p_{m-1}(w,t)\right| \left(\frac{a_m(u)}{m}\right)^{\alpha/2} e^{-\frac{|t|^\beta}{2}} \leqq \frac{\mathcal{C}}{\sqrt{a_m(u)}}, \qquad |t| \leqq \frac{a_m(u)}{2m}, \tag{4.24}$$

so we get

$$J_2'' \leqq \mathcal{C}\|f_\theta u\|_p \left(\frac{a_m(u)}{m}\right)^{-\alpha/2} \left[\int_{-\frac{a_m(u)}{2m}}^{\frac{a_m(u)}{2m}} |t|^{(\alpha-\gamma)q} \, dt\right]^{\frac{1}{q}} \tag{4.25}$$

$$\times \left[\int_{1\leqq|x|\leqq a_m(u)} |x|^{\left(\gamma-\frac{\alpha}{2}-1\right)p} \, dx\right]^{\frac{1}{p}} \leqq \frac{\mathcal{C}\|f_\theta u\|_p}{m^{\alpha/2-\gamma+1/q}} \leqq \mathcal{C}\|f_\theta u\|_p,$$

taking into account (2.10). Consider now the term J_2'. Since $\alpha < 0$, we have

$$J_2' \leqq \mathcal{C} a_m^{5/2} \Bigg(\int_{\frac{a_m(u)}{m}\leqq|x|\leqq 1} \Bigg| |x|^{\gamma-\frac{\alpha}{2}} \left[\frac{a_m(u)}{m}\right]^{-\frac{\alpha}{2}}$$

$$\times \int_{-\frac{a_m(u)}{2m}}^{\frac{a_m(u)}{2m}} \left[\frac{a_m(u)}{m}\right]^{\frac{\alpha}{2}} \frac{p_{m-1}(w,t) f_\theta(t) w(t)}{(a_m^2-t^2)(x-t)} \, dt \Bigg|^p dx \Bigg)^{\frac{1}{p}}$$

$$\leqq \mathcal{C} a_m^{5/2} \Bigg(\int_{\frac{a_m(u)}{m}\leqq|x|\leqq 1} \Bigg| |x|^{\gamma-\alpha} \int_{-\frac{a_m(u)}{2m}}^{\frac{a_m(u)}{2m}} \left[\frac{a_m(u)}{m}\right]^{\alpha/2}$$

$$\times \frac{p_{m-1}(w,t) f_\theta(t) w(t)}{(a_m^2-t^2)(x-t)} \, dt \Bigg|^p dx \Bigg)^{1/p}$$

$$\leqq \mathcal{C} a_m^{5/2} \left(\int_{-1}^{1} \left| |x|^{\gamma-\alpha} \int_{-1}^{1} \left[\frac{a_m(u)}{m}\right]^{\alpha/2} \frac{\chi_m(t) p_{m-1}(w,t) f_\theta(t) w(t)}{(a_m^2-t^2)(x-t)} \, dt \right|^p dx \right)^{1/p},$$

where χ_m is the characteristic function of the interval $\left[-\frac{a_m(u)}{2m}, \frac{a_m(u)}{2m}\right]$. Since, by (2.10), the function $|x|^{\gamma-\alpha}$ is an A_p weight, we can use (4.24) and (4.9), obtaining

(4.26)

$$J_2' \leqq \mathcal{C} a_m^{5/2} \left(\int_{-1}^{1} \left| |x|^{\gamma-\alpha} \left[\frac{a_m(u)}{m} \right]^{\alpha/2} \frac{\chi_m(x) p_{m-1}(w,x) f_\theta(x) w(x)}{a_m^2 - x^2} \right|^p dx \right)^{1/p}$$
$$\leqq \mathcal{C} \sqrt{a_m} \left(\int_{-\frac{a_m(u)}{2m}}^{\frac{a_m(u)}{2m}} \left| |x|^{\gamma-\alpha} \left[\frac{a_m(u)}{m} \right]^{\alpha/2} p_{m-1}(w,x) f_\theta(x) w(x) \right|^p dx \right)^{1/p}$$
$$\leqq \mathcal{C} \|f_\theta u\|_p,$$

recalling that $a_m = a_m(\sqrt{w}) \sim a_m(u)$. Combining (4.26) and (4.25) in (4.23), we get

$$J_2 \leqq \mathcal{C} \|f_\theta u\|_p.$$

From the previous inequality, (4.21) and (4.22), our claim follows for $\alpha < 0$.

Now let us prove that (2.9) implies (2.10). To this aim we emphasize the dependence on m of the quantities involved in (2.9), rewriting the inequality as

$$\left\| \left(a_m^2 - x^2\right) S_m \left(w, \frac{f_{\theta,m}}{a_m^2 - .^2}, x \right) u(x) \right\|_p \leqq \mathcal{C} \|f_{\theta,m} u\|_p, \tag{4.27}$$

where $f_{\theta,m} = \chi_{\theta,m} f$ and $\chi_{\theta,m}$ is the characteristic function of $[-\theta a_m, \theta a_m]$, $\theta \in (0,1)$ is fixed and $a_m = a_m(\sqrt{w})$. From (4.27) it follows that

(4.28)

$$\left\| \left(a_m^2 - x^2\right) S_{m+1} \left(w, \frac{f_{\theta,m}}{a_m^2 - .^2}, x \right) u(x) \right\|_p$$
$$\leqq \left\| (a_{m+1}^2 - x^2) S_{m+1} \left(w, \frac{F_{\delta,m+1}}{a_{m+1}^2 - .^2}, x \right) u(x) \right\|_p$$
$$\leqq \mathcal{C} \|F_{\delta,m+1} u\|_p \leqq \mathcal{C} \|f_{\theta,m} u\|_p,$$

where

$$F_{\delta,m+1}(t) = \frac{\chi_{\theta,m}(t)}{\chi_{\delta,m+1}(t)} \left(\frac{a_{m+1}^2 - t^2}{a_m^2 - t^2} \right) f(t),$$

$\delta \in (0,1)$. So $\left|F_{\delta,m+1}(t)\right| \leqq \mathcal{C} \left| f_{\theta,m}(t)\right|$ for $t \in [-\delta a_{m+1}, \delta a_{m+1}]$ and $[-\delta a_{m+1}, \delta a_{m+1}] \subset [-\theta a_m, \theta a_m]$.

Combining (4.27) and (4.28) we get

$$\left\| \left(a_m^2 - x^2 \right) \left[S_{m+1}\left(w, \frac{f_{\theta,m}}{a_m^2 - \cdot^2}, x \right) - S_{m+1}\left(w, \frac{f_{\theta,m}}{a_m^2 - \cdot^2}, x \right) \right] u(x) \right\|_p \leqq \mathcal{C} \| f_{\theta,m} u \|_p,$$

i.e.

$$\| (a_m^2 - \cdot^2) p_m(w) u \|_p \left| \int_{-\theta a_m}^{\theta a_m} \frac{f_{\theta,m}(t)}{a_m^2 - t^2} p_m(w,t) w(t)\, dt \right| \leqq \mathcal{C} \| f_{\theta,m} u \|_p .$$

It follows that

$$\| (a_m^2 - \cdot^2) p_m(w) u \|_p \left| \int_{-\theta a_m}^{\theta a_m} \frac{f_{\theta,m}(t) u(t)}{\| f_{\theta,m} u \|_p} \frac{p_m(w,t)}{(a_m^2 - t^2)} \frac{w(t)}{u(t)}\, dt \right| \leqq \mathcal{C}. \tag{4.29}$$

Moreover, denoting by F the functional defined, for any $g \in L^p$, by

$$F(g) = \int_{-\theta a_m}^{\theta a_m} g(t) \frac{p_m(w,t)}{(a_m^2 - t^2)} \frac{w(t)}{u(t)}\, dt,$$

it is well-known that

$$\| F \|_{L^p \to \mathbb{R}} = \sup_{\| g \|_p = 1} \left| \int_{-\theta a_m}^{\theta a_m} g(t) \frac{p_m(w,t)}{(a_m^2 - t^2)} \frac{w(t)}{u(t)}\, dt \right| = \left\| \frac{p_m(w)}{(a_m^2 - \cdot^2)} \frac{w}{u} \right\|_{L^q[-\theta a_m, \theta a_m]},$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Therefore, by (4.29), we get

$$\| (a_m^2 - \cdot^2) p_m(w) u \|_p \left\| \frac{p_m(w)}{(a_m^2 - \cdot^2)} \frac{w}{u} \right\|_{L^q[-\theta a_m, \theta a_m]} \leqq \mathcal{C},$$

which implies $u \in L^p[-1,1]$ and $\frac{w}{u} \in L^q[-1,1]$, i.e. $\gamma > -1/p$ and $\gamma - \alpha < 1 - 1/p$. Then, using Proposition 4.2 for both of the norms on the left-hand side of

$$\frac{1}{a_m^{5/4}} \| (a_m^2 - \cdot^2) p_m(w) u \|_p \left\| p_m(w) \sqrt[4]{a_m^2 - \cdot^2} \frac{w}{u} \right\|_{L^q[-\theta a_m, \theta a_m]} \leqq \mathcal{C}$$

with $G=(a_m^2-\cdot^2)^{3/4}\frac{u}{\sqrt{w}}$ and $G=\frac{\sqrt{w}}{u}$, respectively, we obtain

$$\frac{1}{a_m}\big\||\cdot|^{\gamma-\frac{\alpha}{2}}\big\|_{L^p[-1,1]}\big\||\cdot|^{\frac{\alpha}{2}-\gamma}\big\|_{L^q[-1,1]}\leqq\mathcal{C}$$

which implies $-\frac{1}{p}<\gamma-\frac{\alpha}{2}<1-\frac{1}{p}$.

Finally we prove the estimate (2.11). For any $Q\in\mathbb{P}_M\cap\mathcal{P}_{m+1}$, where $M=\left\lfloor\left(\frac{\theta}{\theta+1}\right)^{\beta}\frac{m}{2}\right\rfloor$, by the bound (2.9) and Lemma 2.2, we get

$$\big\|\big[f-S_m^*(w,f_\theta)\big]u\big\|_p\leqq\|(f-Q)u\|_p+\big\|\big[Q-S_m^*(w,Q_\theta)\big]u\big\|_p$$

$$+\big\|S_m^*(w,Q_\theta-f_\theta)u\big\|_p\leqq\mathcal{C}\big\{\big\|(f-Q)u\big\|_p+e^{-Am}\|Qu\|_p\big\}.$$

Thus, taking the infimum over all $Q\in\mathbb{P}_M\cap\mathcal{P}_{m+1}$, by Lemma 2.1, we obtain (2.11). $\square$

Proof of Theorem 2.5. By (2.7) and (2.6), we have

$$\big|S_m^*(w,f_\theta,x)\big|u(x)\sim a_m\bigg|\big(a_m^2-x^2\big)\mathcal{H}_{[-\theta a_m,\theta a_m]}$$

$$\times\left(\frac{p_m(w,x)p_{m-1}(w)-p_{m-1}(w,x)p_m(w)}{a_m^2-\cdot^2}f_\theta w,x\right)\bigg|u(x)$$

since $\frac{\gamma_{m-1}}{\gamma_m}\sim a_m=a_m\big(\sqrt{w}\big)$. According to the Remez-type inequality (4.3) we may assume $x\in\mathcal{A}_m(u)=\big[-a_m(u),a_m(u)\big]\setminus\big[-\frac{a_m(u)}{m},\frac{a_m(u)}{m}\big]$ and, for symmetry, it is sufficient to consider the case $x\in\big[\frac{a_m(u)}{m},a_m(u)\big]$. We use the decomposition

$$\big|S_m^*(w,f_\theta,x)\big|u(x)\sim \tag{4.30}$$

$$\sim a_m\bigg|\big(a_m^2-x^2\big)\bigg\{\int_{|x-t|>\frac{a_m(u)}{m}}+\int_{|x-t|\leqq\frac{a_m(u)}{m}}\bigg\}$$

$$\times\frac{K_m(w,x,t)\chi_\theta(t)f(t)w(t)}{(a_m^2-t^2)}\,dt\bigg|u(x)=:\big|I_1(x)+I_2(x)\big|\leqq\big|I_1(x)\big|+\big|I_2(x)\big|.$$

Let us consider the term $I_1(x)$. We can write

$$\big|I_1(x)\big|\leqq\mathcal{C}\|f_\theta u\|_\infty a_m\big(a_m^2-x^2\big)u(x)\int_{|x-t|>\frac{a_m(u)}{m}}\frac{\big|K_m(w,x,t)\big|\chi_\theta(t)w(t)}{(a_m^2-t^2)u(t)}\,dt.$$

Observe that, by (4.7), for $|x| \geqq a_m(u)/m$ and $|t| \leqq \theta a_m\big(\sqrt{w}\,\big)$, the inequality

$$a_m\big(\sqrt{w}\,\big)\,\frac{a_m\big(\sqrt{w}\,\big)^2 - x^2}{a_m\big(\sqrt{w}\,\big)^2 - t^2}\big|p_m(w,x)p_{m-1}(w,t)\big|\,\frac{w(t)}{u(t)}u(x) \leqq \mathcal{C}|x|^{\gamma-\frac{\alpha}{2}}|t|^{\frac{\alpha}{2}-\gamma}$$

holds. Using this inequality we get

$$\big|I_1(x)\big| \leqq \mathcal{C}\|fu\|_\infty |x|^{\gamma-\frac{\alpha}{2}}\left(\int_{-\theta a_m}^{x-\frac{a_m(u)}{m}} \frac{|t|^{\frac{\alpha}{2}-\gamma}}{x-t}\,dt + \int_{x+\frac{a_m(u)}{m}}^{\theta a_m} \frac{|t|^{\frac{\alpha}{2}-\gamma}}{t-x}\,dt\right)$$
$$\leqq \mathcal{C}\|f_\theta u\|_\infty \log m,$$

since, by (2.14), $\frac{\alpha}{2}-\gamma \in (-1,0]$.

It remains to estimate the term $I_2(x)$. We can write

$$\big|I_2(x)\big| \leqq \mathcal{C}a_m u(x)\big(a_m^2 - x^2\big)\left|\int_{-\frac{a_m(u)}{m}}^{\frac{a_m(u)}{m}} \frac{K_m(w,x,t)}{a_m^2 - t^2} f_\theta(t)w(t)\,dt\right| \tag{4.31}$$

$$\leqq \mathcal{C}\int_{-\frac{a_m(u)}{m}}^{\frac{a_m(u)}{m}} \big|R_m(x,t)f_\theta(t)u(t)\big|\,dt \leqq \mathcal{C}\|f_\theta u\|_\infty \int_{-\frac{a_m(u)}{m}}^{\frac{a_m(u)}{m}} \big|R_m(x,t)\big|\,dt,$$

where

$$\big|R_m(x,t)\big| \tag{4.32}$$

$$= a_m u(x)\big(a_m^2 - x^2\big)\left|\frac{p_m(w,x)p_{m-1}(w,t) - p_{m-1}(w,x)p_m(w,t)}{(a_m^2 - t^2)(x-t)}\right|\frac{w(t)}{u(t)}$$

$$\leqq a_m u(x)\left(\frac{a_m^2 - x^2}{a_m^2 - t^2}\right)\left|\frac{p_m(w,x) - p_m(w,t)}{x-t}\right|\big|p_{m-1}(w,t)\big|\,\frac{w(t)}{u(t)}$$

$$+ a_m u(x)\left(\frac{a_m^2 - x^2}{a_m^2 - t^2}\right)\big|p_m(w,t)\big|\left|\frac{p_{m-1}(w,t) - p_{m-1}(w,x)}{x-t}\right|\frac{w(t)}{u(t)}$$

$$=: \big|T_m(x,t)\big| + \big|V_m(x,t)\big|.$$

By the mean value theorem we have

$$\big|T_m(x,t)\big| \leqq a_m u(x)\big|p_m'(w,\xi)\big|\big|p_{m-1}(w,t)\big|\,\frac{w(t)}{u(t)}$$

with $\xi \in (x,t)$ and, since $w(x) \sim w(\xi) \sim w(t)$, $u(x) \sim u(t)$ and $a_m = a_m\big(\sqrt{w}\,\big) \sim a_m(u)$, using the inequalities (4.1) and (4.7), we get

$$\big|T_m(x,t)\big| \leqq a_m \frac{u(x)}{u(t)} \big|p'_m(w,\xi)\sqrt{w(\xi)}\,\big|\big|p_{m-1}(w,t)\sqrt{w(t)}\,\big|$$

$$\leqq \mathcal{C} a_m \frac{u(x)}{u(t)} \frac{m}{a_m} \frac{1}{\sqrt{a_m}} \frac{1}{\sqrt{a_m}} \leqq \mathcal{C}\frac{m}{a_m}$$

for $|t| \leqq a_m(u)/m$ and $x \in \big[a_m(u)/m, a_m(u)\big]$.

Analogously we can show that

$$\big|V_m(x,t)\big| \leqq \mathcal{C}\frac{m}{a_m}.$$

Therefore, by (4.31) and (4.32), it follows that $\big|I_2(x)\big| \leqq \mathcal{C}\|f_\theta u\|_\infty$. Combining the estimates for $I_1(x)$ and $I_2(x)$ in (4.30) and taking the supremum on all $x \in \mathcal{A}_m(u)$, we obtain (2.15).

In order to prove the estimate (2.16) we can proceed as for (2.11). We omit the details. □

Finally, let us consider the case of functions defined on the real semiaxis. We omit the proofs of the results in Section 3, since they can be deduced from those in Section 2. To give the ideas, we are going only to prove Theorem 3.3.

Proof of Theorem 3.3. If $W(x) = x^\alpha e^{-x^\beta}$ and $U(x) = x^\gamma e^{-\frac{x^\beta}{2}}$ are generalized Laguerre weights (see (3.2) and (3.1)), consider the generalized Freud weights $w(y) = |y|^{2\alpha+1} e^{-y^{2\beta}}$ and $u(y) = |y|^{2\gamma+1/p} e^{-\frac{y^{2\beta}}{2}}$. Then, for any $f \in L^p_U$, set $\widetilde{f}(y) = f(y^2)$, where $\widetilde{f} \in L^p_u$. If we apply the transformation $x = y^2$, $y \in \mathbb{R}$, the orthonormal polynomials and M–R–S numbers corresponding to W and w are related by $p_m(W,x) = p_{2m}(w,y)$ and $\bar{a}_m\big(\sqrt{W}\,\big) = a_{2m}\big(\sqrt{w}\,\big)^2$. Hence, the operator $S^*_m(W)$ in (3.6) can be obtained from $S^*_{2m}(w)$ in (2.7) by the quadratic transformation $x = y^2$ as follows:

$$S^*_m(W,f_\theta,x) = \big(\bar{a}_m\big(\sqrt{W}\,\big) - x\big) S_m\left(W, \frac{f_\theta}{\bar{a}_m\big(\sqrt{W}\,\big) - \cdot}, x\right) \tag{4.33}$$

$$= \big(a_{2m}\big(\sqrt{w}\,\big)^2 - y^2\big) S_{2m}\left(W, \frac{\widetilde{f}_{\sqrt{\theta}}}{a_{2m}\big(\sqrt{w}\,\big)^2 - \cdot^2}, y\right) = S^*_{2m}\big(w, \widetilde{f}_{\sqrt{\theta}}, y\big).$$

Then by Theorem 2.3 we have

$$\big\|S^*_{2m}\big(w, \widetilde{f}_{\sqrt{\theta}}\big)u\big\|_p \leqq \mathcal{C}\big\|\widetilde{f}_{\sqrt{\theta}}\, u\big\|_p, \qquad \mathcal{C} \neq \mathcal{C}(m, \widetilde{f}),$$

if and only if

$$-\frac{1}{p} < 2\gamma + \frac{1}{p} - \left(\alpha + \frac{1}{2}\right) < 1 - \frac{1}{p}, \qquad 2\gamma + \frac{1}{p} - (2\alpha + 1) < 1 - \frac{1}{p}.$$

Since $\|f_\theta U\|_p = \left\|\widetilde{f}_{\sqrt{\theta}}\, u\right\|_p$ and by (4.33), it follows that (3.8) holds if and only if the conditions (3.9) are satisfied. Similarly we can prove the error estimate (3.10). □

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