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RELATIONS BETWEEN MULTIDIMENSIONAL INTERVAL -VALUED VARIATIONAL PROBLEMS AND VARIATIONAL INEQUALITIES

Anurag Jayswal and Ayushi Baranwal

In this paper, we introduce a new class of variational inequality with its weak and split forms to obtain an LU-optimal solution to the multi-dimensional interval-valued variational problem, which is a wider class of interval-valued programming problem in operations research. Using the concept of (strict) LU-convexity over the involved interval-valued functionals, we establish equivalence relationships between the solutions of variational inequalities and the (strong) LU-optimal solutions of the multi-dimensional interval-valued variational problem. In addition, some applications are constructed to illustrate the established results.

Keywords: LU-convexity, LU-optimal solution, multi-dimensional inter-valued variational problem, variational inequality

Classification: 26B25, 26D10, 49J40, 90C30

1. INTRODUCTION

Interval-valued optimization is a developing area of operations research that plays a vital role in addressing the uncertainty in optimization problems. Since it allows to represent and deal with the problems which involve numerically inaccurate data varying in a range. For instance, it is not possible sometimes to know accurately the factors and variables involved in many industries and economic processes. These problems can be efficiently described by modeling them as interval-valued optimization problems, and one can adequately predict the optimal solutions. Moore [12, 13] introduced the concept of interval analysis to solve optimization problems with interval-valued objective functions. Recently, Li et al. [10] classified the division problems involving uncertainty of claims where each claimant's claim can vary within a closed interval into division problems under interval uncertainty and solved the problem under study. For more insights in this area readers are suggested to see the articles [1, 2, 8, 20] and references therein.

Further, it has been observed that variational inequalities are very useful instruments for solving the optimization problems. In 1966, Hartman and Stampacchia [5] first introduced the concept of a variational inequality and proved the existence and uniqueness of its solutions. Roubicek [14] solved a nonsmooth optimization problem governed by variational inequalities with linear constraints using Clark's generalized gradient and

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proposed a subgradient algorithm. Since then, this area has gained vast attention, and many authors have efficiently investigated the association between variational inequalities and optimization problems (see [3, 7, 11, 19] and their references). Recently, an intense study has been done to establish a specific alliance between the solution of the variational inequality and interval-valued optimization problems. Zhang et al. [21] derived the relationships between the solutions of interval-valued vector optimization problems and vector variational inequalities under the assumption of LU-convexity. In [15], Ruiz et al. investigated the correlations between generalized Stampacchia variational inequalities and a class of nonsmooth interval-valued programming problems.

On the other hand, numerous studies have been done on multi-dimensional variational problems that emerged from the calculus of variations. The association between optimization and variational calculus was first introduced by Hanson [4]. Afterward, many researchers began to show an interest in this field. For instance, Jha et al. [9] investigated some results for a class of interval-valued variational problem via its associated modified problems and saddle point criteria under the assumption of generalized convexity. In [16, 18], Treanţă considered a class of multi-dimensional interval-valued variational control problem and constructed several alliances between its KT-pseudoinvex point, optimal solution, and a saddle-point of the interval-valued Lagrange functional. Very recently, Treanţă [17] has given the optimality conditions for a class of multi-dimensional interval-valued variational problems and obtain the optimal solution.

Motivated by the research works mentioned above, we consider a multi-dimensional interval-valued variational problem and introduced a class of (weak and split) variational inequalities involving the multiple-integral functionals. We establish various equivalence relations between the (strong) LU-optimal solutions of the considered multi-dimensional interval-valued variational problems and solutions of the variational inequalities by imposing the assumption of LU-convexity over the involved functionals. The novelty element in this paper is that we introduce the variational inequality associated with the multi-dimensional interval-valued variational problem, which deals with the uncertainty given in a range of intervals. Further, some illustrative examples are formulated to give a better insight of the results. The application given means to minimize the production cost of a production firm and find an optimal output function for constructed industrial problem.

The management of this article is as follows: section 2 contains some basic notations, definitions, and formulation of the problem. In section 3, we establish several correlations between the considered multi-dimensional interval-valued variational problem and variational inequalities. Finally, section 4 concludes the paper.

2. PROBLEM FORMULATION AND PRELIMINARIES

In this section we consider some notations and basic concepts that will assist in framing the problem and presenting the main results.

⇒ R^m and R^n are the Euclidean spaces of dimensions m and n, respectively.

⇒ $\Theta = \Theta_{t_0,t_1} \subset R^m$ is a hyperparallelepiped, fixed by the diagonally opposite points $t_0 = (t_0^\alpha)$, $t_1 = (t_1^\alpha)$, $\alpha = \overline{1,m}$ and the point $t = (t^\alpha) \in \Theta$, $\alpha = \overline{1,m}$.

⇒ $\mathrm{d}\omega = \mathrm{d}t^1 \mathrm{d}t^2 \ldots \mathrm{d}t^m$ denote the volume element in $\Theta \subset R^m$.

⇒ D_α be the total differential operator.

⇒ $V \subset R^n$ be the space of piecewise smooth state function $v(t) : \Theta \mapsto R^n$ and its point $v(t) = (v^i(t)) \in R^n,\ i = \overline{1, n}$. $\frac{\partial v(t)}{\partial t^\alpha} = v_\alpha(t)$ denote the partial derivative of $v(t)$ with respect to $t^\alpha,\ \alpha = \overline{1, m}$.

⇒ let A be the set of all closed and bounded intervals in R.

⇒ $W = [w^L, w^U], Z = [z^L, z^U] \in A$, where w^L, z^L indicate the lower bound, and $w^U,\ z^U$ indicate the upper bound of W and Z, respectively.

Throughout this paper, the interval operations can be performed as follows:

- $W = Z \Rightarrow w^L = z^L$ and $w^U = z^U$,
- if $w^L = w^U = w$ then $W = [w, w] = w$,
- $W + Z = [w^L + z^L, w^U + z^U]$,
- $-W = -[w^L, w^U] = [-w^U, -w^L]$,
- $W - Z = [w^L - z^U, w^U - z^L]$,
- $u + W = [u + w^L, u + w^U],\ u \in R$,
- $uW = [uw^L, uw^U],\ u \in R,\ u \geq 0$,
- $uW = [uw^U, uw^L],\ u \in R,\ u < 0$.

⇒ Also, we use the following conventions for the intervals $W,\ Z \in A$:

- $W \preceq_{LU} Z$ iff $w^L \leq z^L$ and $w^U \leq z^U$,
- $W \prec_{LU} Z$ iff $W \preceq_{LU} Z$ and $W \neq Z$.

Definition 2.1. A functional $f : \Theta \times V \times V \mapsto A$ is said to be interval-valued functional, if it is defined by

$$\int_\Theta f(t, v(t), v_\alpha(t))\,\mathrm{d}\omega = \left[\int_\Theta f^L(t, v(t), v_\alpha(t))\,\mathrm{d}\omega,\ \int_\Theta f^U(t, v(t), v_\alpha(t))\,\mathrm{d}\omega\right],$$

where $t \in \Theta$ and $\int_\Theta f^L(t, v(t), v_\alpha(t))\,\mathrm{d}\omega$, $\int_\Theta f^U(t, v(t), v_\alpha(t))\,\mathrm{d}\omega$ are real-valued functionals satisfying the condition $\int_\Theta f^L(t, v(t), v_\alpha(t))\,\mathrm{d}\omega \leq \int_\Theta f^U(t, v(t), v_\alpha(t))\omega$.

Note: From now onward, we assume $f : \Theta \times V \times V \mapsto A$ be an interval-valued functional.

Definition 2.2. (Jayswal and Preeti [6]) Let $V \subset R^n$ be a convex set and $\psi : \Theta \times V \times V \mapsto R$ be the real valued functional. Then the functional $\int_\Theta \psi(t, v(t), v_\alpha(t))\,\mathrm{d}\omega$ is said to be (strictly) convex at $v^0 \in V$, if

$$\int_\Theta \psi(t, v(t), v_\alpha(t))\,\mathrm{d}\omega - \int_\Theta \psi(t, v^0(t), v^0_\alpha(t))\,\mathrm{d}\omega \geq (>) \int_\Theta \frac{\partial \psi}{\partial v}(t, v^0(t), v^0_\alpha(t))(v - v^0)\,\mathrm{d}\omega + \int_\Theta \frac{\partial \psi}{\partial v_\alpha}(t, v^0(t), v^0_\alpha(t)) D_\alpha(v - v^0)\,\mathrm{d}\omega,\ \forall v \in V.$$

Remark 2.3. If the above inequality holds for any $v^0, v \in V$, then the functional $\int_\Theta \psi(t, v(t), v_\alpha(t))\,\mathrm{d}\omega$ is said to be (strictly) convex on V.

Definition 2.4. If both the real valued functionals $\int_\Theta f^L(t, v(t), v_\alpha(t))\,\mathrm{d}\omega$ and $\int_\Theta f^U(t, v(t), v_\alpha(t))\,\mathrm{d}\omega$ are convex at $v^0 \in V$, then the interval-valued functional $\int_\Theta f(t, v(t), v_\alpha(t))\,\mathrm{d}\omega = [\int_\Theta f^L(t, v(t), v_\alpha(t))\,\mathrm{d}\omega,\ \int_\Theta f^U(t, v(t), v_\alpha(t))\,\mathrm{d}\omega]$ is known as LU-convex at $v^0 \in V$.

Definition 2.5. If the two real valued functionals $\int_\Theta f^L(t, v(t), v_\alpha(t))\,\mathrm{d}\omega$ and $\int_\Theta f^U(t, v(t), v_\alpha(t))\,\mathrm{d}\omega$ are convex and at least one of them is strictly convex at $v^0 \in V$, then the interval-valued functional $\int_\Theta f(t, v(t), v_\alpha(t))\,\mathrm{d}\omega = [\int_\Theta f^L(t, v(t), v_\alpha(t))\,\mathrm{d}\omega, \int_\Theta f^U(t, v(t), v_\alpha(t))\,\mathrm{d}\omega]$ is known as strictly LU-convex at $v^0 \in V$.

Now, considering the above mathematical tools, we formulate the following multi-dimensional interval-valued variational problem (VP) as:

$$\text{(VP)} \quad \min_{v(\cdot)} \int_\Theta f(t, v(t), v_\alpha(t))\,\mathrm{d}\omega = \left[\int_\Theta f^L(t, v(t), v_\alpha(t))\,\mathrm{d}\omega,\ \int_\Theta f^U(t, v(t), v_\alpha(t))\,\mathrm{d}\omega \right]$$

subject to

$$G_\beta(t, v(t), v_\alpha(t)) \leq 0, \quad \beta \in Q = \overline{1, q}, \tag{1}$$

$$v(t_0) = v_0,\ v(t_1) = v_1, \tag{2}$$

where $t \in \Theta,\ f : \Theta \times V \times V \mapsto A$ is an interval-valued functional and the functionals $f^L,\ f^U,\ G_\beta : \Theta \times V \times V \mapsto R,\ \beta \in Q = \overline{1, q}$ are continuously differentiable.

We denote the set of feasible solutions to (VP) as

$$S = \{v \in V\ |G_\beta(t, v(t), v_\alpha(t)) \leq 0,\ v(t_0) = v_0,\ v(t_1) = v_1\}.$$

Throughout the paper we consider the S is convex subset of V.

For the convenience of presentation let us utilize the following notions: $v = v(t)$, $\pi = (t, v(t), v_\alpha(t))$, $\pi^0 = (t, v^0(t), v^0_\alpha(t))$, $f_v = \frac{\partial f}{\partial v}$, $f_{v_\alpha} = \frac{\partial f}{\partial v_\alpha}$, $f^L_v = \frac{\partial f^L}{\partial v}$, $f^L_{v_\alpha} = \frac{\partial f^L}{\partial v_\alpha}$, $f^U_v = \frac{\partial f^U}{\partial v}$, $f^U_{v_\alpha} = \frac{\partial f^U}{\partial v_\alpha}$.

Definition 2.6. A point $v^0 \in S$ is said to be an (strong) LU-optimal solution to (VP) if there exists no other point $v \in S$, such that

$$\int_\Theta f(\pi)\,\mathrm{d}\omega\ (\preceq_{LU})\ \prec_{LU} \int_\Theta f(\pi^0)\,\mathrm{d}\omega.$$

Now, to obtain the optimality for the problem (VP) and prove main results, we construct the following variational inequalities:

(i) Find $v^0 \in S$ such that there exist no other $v \in S$, satisfying the following variational inequality

$$\textbf{(VI)} \int_\Theta \Big\{ f_v^L(\pi^0) + f_v^U(\pi^0) \Big\} (v - v^0)\, \mathrm{d}\omega + \int_\Theta \Big\{ f_{v_\alpha}^L(\pi^0) + f_{v_\alpha}^U(\pi^0) \Big\} D_\alpha(v - v^0)\, \mathrm{d}\omega \leq 0.$$

(ii) Find $v^0 \in S$ such that there exist no other $v \in S$, satisfying the following weak variational inequality

$$\textbf{(WVI)} \int_\Theta \Big\{ f_v^L(\pi^0) + f_v^U(\pi^0) \Big\} (v - v^0)\, \mathrm{d}\omega + \int_\Theta \Big\{ f_{v_\alpha}^L(\pi^0) + f_{v_\alpha}^U(\pi^0) \Big\} D_\alpha(v - v^0)\, \mathrm{d}\omega < 0.$$

(iii) Find $v^0 \in S$ such that for all $v \in S$, the following split variational inequalities

$$\textbf{(SVI)} \int_\Theta f_v^L(\pi^0)(v - v^0)\, \mathrm{d}\omega + \int_\Theta f_{v_\alpha}^L(\pi^0) D_\alpha(v - v^0)\, \mathrm{d}\omega > 0,$$
$$\int_\Theta f_v^U(\pi^0)(v - v^0)\, \mathrm{d}\omega + \int_\Theta f_{v_\alpha}^U(\pi^0) D_\alpha(v - v^0)\, \mathrm{d}\omega > 0,$$

hold.

Next example will verify that the above mentioned variational inequality (VI) is solvable at a given point.

Example 2.7. Let $\Theta = [0,4] \times [0,4]$, $V = R_+$, $A \subset R$ and the interval-valued functional $f : \Theta \times V \times V \mapsto A$ as:

$$\int_\Theta f(\pi)\, \mathrm{d}\omega = \Big[\int_\Theta f^L(\pi)\, \mathrm{d}\omega, \int_\Theta f^U(\pi)\, \mathrm{d}\omega \Big] = \Big[\int_\Theta (e^{v^2} - 1)\, \mathrm{d}\omega, \int_\Theta (2v + 4)\, \mathrm{d}\omega \Big],$$

where $f^L,\ f^U : \Theta \times V \times V \mapsto R$ are continuously differentiable. Further, we can easily observe that $v^0 = 0$ is a solution for the associated variational inequality (VI), as for all other $v \in V$, we have

$$\int_\Theta \Big\{ f_v^L(\pi^0) + f_v^U(\pi^0) \Big\} (v - v^0)\, \mathrm{d}\omega + \int_\Theta \Big\{ f_{v_\alpha}^L(\pi^0) + f_{v_\alpha}^U(\pi^0) \Big\} D_\alpha(v - v^0)\, \mathrm{d}\omega$$
$$= \int_\Theta \Big\{ 2v^0 e^{(v^0)^2} + 2 \Big\} (v - v^0)\, \mathrm{d}\omega \nleq 0.$$

3. RELATIONSHIP BETWEEN THE MULTI-DIMENSIONAL INTERVAL-VALUED VARIATIONAL PROBLEM (VP) AND VARIATIONAL INEQUALITIES

In this section, we establish some equivalence results among the (strong) LU-optimal solutions of the multi-dimensional interval-valued variational problem (VP) and solutions of the (weak, split) variational inequalities introduced in the previous section.

Theorem 3.1. Let $v^0 \in S$ be a solution to the variational inequality (VI). If the interval-valued functional $\int_\Theta f(\pi)\,\mathrm{d}\omega$ is LU-convex at v^0, then v^0 is also an LU-optimal solution to the problem (VP).

Proof. Let $v^0 \in S$ solves the variational inequality (VI). We proceed by the contradiction and assume that v^0 is not an LU-optimal solution to (VP), then there exist a point $v \in S$, such that

$$\int_\Theta f(\pi)\,\mathrm{d}\omega \prec_{LU} \int_\Theta f(\pi^0)\,\mathrm{d}\omega.$$

Therefore, one of the following inequality hold

$$\int_\Theta f^L(\pi)\,\mathrm{d}\omega < \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \quad \text{and} \quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega \leq \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega, \tag{3}$$

$$\text{or} \quad \int_\Theta f^L(\pi)\,\mathrm{d}\omega \leq \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \quad \text{and} \quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega < \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega, \tag{4}$$

$$\text{or} \quad \int_\Theta f^L(\pi)\,\mathrm{d}\omega < \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \quad \text{and} \quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega < \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega. \tag{5}$$

Since the functional $\int_\Theta f(\pi)\,\mathrm{d}\omega$ is LU-convex at $v^0 \in S$, it follows that

$$\int_\Theta f^L(\pi)\,\mathrm{d}\omega - \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \geq \int_\Theta \{f^L_v(\pi^0)(v-v^0) + f^L_{v_\alpha}(\pi^0) D_\alpha(v-v^0)\}\,\mathrm{d}\omega,$$

$$\text{and} \quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega - \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega \geq \int_\Theta \{f^U_v(\pi^0)(v-v^0) + f^U_{v_\alpha}(\pi^0) D_\alpha(v-v^0)\}\,\mathrm{d}\omega, \quad \forall\, v \in S.$$

In the virtue of the above inequalities, the inequalities (3)-(5) yield

$$\int_\Theta \{f^L_v(\pi^0) + f^U_v(\pi^0)\}(v-v^0)\,\mathrm{d}\omega + \{f^L_{v_\alpha}(\pi^0) + f^U_{v_\alpha}(\pi^0)\} D_\alpha(v-v^0)\,\mathrm{d}\omega \leq 0,$$

which contradicts that v^0 is a solution of the variational inequality (VI). This completes the proof. □

Now, we present an application which validates above theorem.

Example 3.2. A production firm of a company produces some goods and the company wants to minimize the production cost. Production firm has a range of total production cost as an interval-valued functional given by

$$\Big[\int_\Theta f^L(\pi)\,\mathrm{d}t^1\mathrm{d}t^2, \int_\Theta f^U(\pi)\,\mathrm{d}t^1\mathrm{d}t^2\Big] = \Big[\int_\Theta (v^2+1)\,\mathrm{d}t^1\mathrm{d}t^2, \int_\Theta \{(e^{2v}+5v-1)^2+v+1\}\,\mathrm{d}t^1\mathrm{d}t^2\Big],$$

where $v \in V = R$ is the output function and time $t \in \Theta \subset R^2$ with Θ fixed by the diagonally opposite points $t_0 := (t_0^1, t_0^2) = (0,0),\ t_1 := (t_1^1, t_1^2) = (1,1) \in R^2$. The production cost should be minimized subject to the constraint $v^2 - 16 \leq 0$, the endpoints conditions are $v(0,0) = 0$ and $v(1,1) = 4$. The firm have to find suitable output function of time which minimizes production cost. This problem can be mathematically modeled as the following multi-dimensional interval-valued variational problem:

$$\text{(VP1)} \quad \min_{v(\cdot)} \int_\Theta f(\pi)\,\mathrm{d}t^1\mathrm{d}t^2 = \Big[\int_\Theta f^L(\pi)\,\mathrm{d}t^1\mathrm{d}t^2, \int_\Theta f^U(\pi)\,\mathrm{d}t^1\mathrm{d}t^2\Big]$$
$$= \Big[\int_\Theta (v^2+1)\,\mathrm{d}t^1\,\mathrm{d}t^2, \int_\Theta \{(e^{2v}+5v-1)^2+v+1\}\,\mathrm{d}t^1\mathrm{d}t^2\Big]$$

subject to

$$v^2 - 16 \leq 0,$$
$$v(0,0) = 0, \quad v(1,1) = 4,$$

The set of all feasible solutions to the problem (VP1) is denoted by

$$S = \{v \in V : -4 \leq v \leq 4,\ v(0,0) = 0,\ v(1,1) = 4\}.$$

It can be seen that both the functionals $\int_\Theta f^L(\pi)\,\mathrm{d}t^1\mathrm{d}t^2,\ \int_\Theta f^U(\pi)\,\mathrm{d}t^1\mathrm{d}t^2$ are convex at $v^0 = 2(t^1+t^2),\ t^1 = t^2 = 0$. Thus, the interval-valued functional is LU-convex at $v^0 = 0$.

Since the following inequality

$$\int_\Theta \{f_v^L(\pi^0) + f_v^U(\pi^0)\}(v - v^0) + \{f_{v_\alpha}^L(\pi^0) + f_{v_\alpha}^U\}D_\alpha(v - v^0)\,\mathrm{d}t^1\mathrm{d}t^2$$
$$= \int_\Theta \{2v^0 + 4e^{4v^0} + 20v^0e^{2v^0} + 6e^{2v^0} + 50v^0 - 9\}(v - v^0)\,\mathrm{d}t^1\mathrm{d}t^2$$
$$= \int_\Theta \{2t^1 + 2t^2\}\,\mathrm{d}t^1\mathrm{d}t^2 \nleq 0, \quad \forall t = (t^1, t^2) \in \Theta,$$

holds at $v^0 = 2(t^1+t^2),\ t^1 = t^2 = 0$, it is a solution to variational inequality (VI). Now it remains to show that $v^0 = 0$ is an LU-optimal solution to the problem (VP1). It can be easily verified that the inequality

$$\int_\Theta f(\pi)\,\mathrm{d}t^1\mathrm{d}t^2 - \int_\Theta f(\pi^0)\,\mathrm{d}t^1\,t^2$$
$$= \Big[\int_\Theta (v^2 - (v^0)^2)\,\mathrm{d}t^1\mathrm{d}t^2, \int_\Theta \{(e^{2v}+5v-1)^2 + v - (e^{2v^0}+5v^0-1)^2 - v^0\}\,\mathrm{d}t^1\mathrm{d}t^2\Big]$$
$$= \Big[\int_\Theta 4(t^1+t^2)^2\mathrm{d}t^1\mathrm{d}t^2, \int_\Theta \Big\{\Big(e^{(4t^1+4t^2)} + 10(t^1+t^2) - 1\Big)^2 + 2(t^1+t^2)\Big\}\,\mathrm{d}t^1\mathrm{d}t^2\Big]$$
$$\nless_{LU} (0,0), \quad \forall\ t = (t^1, t^2) \in \Theta,\ v \neq v^0,$$

holds, which gives that to minimize the production cost the suitable output function is $v^0 = 2(t^1+t^2), t^1 = t^2 = 0$. Hence, Theorem 3.1 is verified.

Theorem 3.3. Let $v^0 \in S$ be an LU-optimal solution to (VP). If the interval-valued functional $\int_\Theta -f(\pi)\,\mathrm{d}\omega$ is strictly LU-convex at v^0, then v^0 is a solution to the variational inequality (VI).

Proof. Let $v^0 \in S$ is an LU-optimal solution to (VP), then there exists no other point $v \in S$, such that

$$\int_\Theta f(\pi)\,\mathrm{d}\omega \prec_{LU} \int_\Theta f(\pi^0)\,\mathrm{d}\omega,$$

equivalently,

$$\begin{aligned}
&\int_\Theta f^L(\pi)\,\mathrm{d}\omega < \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \quad\text{and}\quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega \le \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega,\\
\text{or}\quad &\int_\Theta f^L(\pi)\,\mathrm{d}\omega \le \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \quad\text{and}\quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega < \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega,\\
\text{or}\quad &\int_\Theta f^L(\pi)\,\mathrm{d}\omega < \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \quad\text{and}\quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega < \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega.
\end{aligned}$$

Thus, there exists no other point $v \in S$ that holds the following inequality

$$\int_\Theta \{f^L(\pi) + f^U(\pi)\}\,\mathrm{d}\omega < \int_\Theta \{f^L(\pi^0) + f^U(\pi^0)\}\,\mathrm{d}\omega. \tag{6}$$

Now, we proceed by the contradiction and assume that v^0 does not solves the variational inequality (VI), then there exists a point $v \in S$, such that

$$\int_\Theta \{f^L_v(\pi^0) + f^U_v(\pi^0)\}(v - v^0)\,\mathrm{d}\omega + \int_\Theta \{f^L_{v_\alpha}(\pi^0) + f^U_{v_\alpha}(\pi^0)\}D_\alpha(v - v^0)\,\mathrm{d}\omega \le 0. \tag{7}$$

Since the functional $\int_\Theta -f(\pi)\,\mathrm{d}\omega$ is strictly LU-convex at $v^0 \in S$, we get

$$\begin{aligned}
&\int_\Theta f^L(\pi)\,\mathrm{d}\omega - \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega < \int_\Theta \{f^L_v(\pi^0)(v - v^0) + f^L_{v_\alpha}(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,\\
&\text{and}\quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega - \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega \le \int_\Theta \{f^U_v(\pi^0)(v - v^0) + f^U_{v_\alpha}(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,\\
&\hfill \forall\, v \in S,
\end{aligned}$$

or

$$\begin{aligned}
&\int_\Theta f^L(\pi)\,\mathrm{d}\omega - \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \le \int_\Theta \{f^L_v(\pi^0)(v - v^0) + f^L_{v_\alpha}(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,\\
&\text{and}\quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega - \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega < \int_\Theta \{f^U_v(\pi^0)(v - v^0) + f^U_{v_\alpha}(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,\\
&\hfill \forall\, v \in S,
\end{aligned}$$

or

$$\int_\Theta f^L(\pi)\,\mathrm{d}\omega - \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega < \int_\Theta \{f^L_v(\pi^0)(v - v^0) + f^L_{v_\alpha}(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,$$

$$\text{and} \quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega - \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega < \int_\Theta \{f_v^U(\pi^0)(v-v^0) + f_{v_\alpha}^U(\pi^0)D_\alpha(v-v^0)\}\,\mathrm{d}\omega, \quad \forall\, v \in S.$$

From the above inequalities, we obtain

$$\begin{aligned}\int_\Theta \{f^L(\pi) + f^U(\pi)\}\,\mathrm{d}\omega &- \int_\Theta \{f^L(\pi^0) + f^U(\pi^0)\}\,\mathrm{d}\omega \\ &< \int_\Theta \{f_v^L(\pi^0) + f_v^U(\pi^0)\}(v-v^0)\,\mathrm{d}\omega + \int_\Theta \{f_{v_\alpha}^L(\pi^0) + f_{v_\alpha}^U(\pi^0)\}D_\alpha(v-v^0)\,\mathrm{d}\omega, \\ &\qquad \forall\, v \in S,\end{aligned}$$

which together with the inequality (7), yields that the following inequality

$$\int_\Theta \{f^L(\pi) + f^U(\pi)\}\,\mathrm{d}\omega < \int_\Theta \{f^L(\pi^0) + f^U(\pi^0)\}\,\mathrm{d}\omega,$$

holds for a point $v \in S$, which contradicts the inequality (6). This completes the proof. □

Theorem 3.4. Let $v^0 \in S$ solves the split variational inequality (SVI). If the interval-valued functional $\int_\Theta f(\pi)\,\mathrm{d}\omega$ is LU-convex at v^0, then v^0 is also a strong LU-optimal solution to the problem (VP).

Proof. Let $v^0 \in S$ solves the split variational inequality (SVI). We proceed by the contradiction and assume that v^0 is not a strong LU-optimal solution to (VP). Consequently, there exist a point $v \in S$, such that

$$\int_\Theta f(\pi)\,\mathrm{d}\omega \preceq_{LU} \int_\Theta f(\pi^0)\,\mathrm{d}\omega,$$

or,

$$\int_\Theta f^L(\pi)\,\mathrm{d}\omega \leq \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \quad \text{and} \quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega \leq \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega. \tag{8}$$

Since the functional $\int_\Theta f(\pi)\,\mathrm{d}\omega$ is LU-convex at $v^0 \in S$, we have

$$\begin{aligned}&\int_\Theta f^L(\pi)\,\mathrm{d}\omega - \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \geq \int_\Theta \{f_v^L(\pi^0)(v-v^0) + f_{v_\alpha}^L(\pi^0)D_\alpha(v-v^0)\}\,\mathrm{d}\omega, \\ \text{and} \quad &\int_\Theta f^U(\pi)\,\mathrm{d}\omega - \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega \geq \int_\Theta \{f_v^U(\pi^0)(v-v^0) + f_{v_\alpha}^U(\pi^0)D_\alpha(v-v^0)\}\,\mathrm{d}\omega, \\ &\qquad \forall\, v \in S.\end{aligned}$$

In agreement of the inequality (8), the above inequalities yield

$$\int_\Theta \{f_v^L(\pi^0)(v-v^0) + f_{v_\alpha}^L(\pi^0)D_\alpha(v-v^0)\}\,\mathrm{d}\omega \leq 0,$$
$$\int_\Theta \{f_v^U(\pi^0)(v-v^0) + f_{v_\alpha}^U(\pi^0)D_\alpha(v-v^0)\}\,\mathrm{d}\omega \leq 0,$$

which contradicts that v^0 is a solution to the split variational inequality (SVI). This completes the proof. □

Theorem 3.5. Let $v^0 \in S$ be a strong LU-optimal solution to the problem (VP). If the interval-valued functional $\int_\Theta -f(\pi)\,\mathrm{d}\omega$ is LU-convex at v^0, then v^0 solves the split variational inequality (SVI).

Proof. Let $v^0 \in S$ is a strong LU-optimal solution to the problem (VP). We proceed by the contradiction and assume that v^0 is not a solution of the split variational inequality (SVI), then there exists some $v \in S$, satisfying the inequalities

$$\int_\Theta \{f_v^L(\pi^0)(v - v^0) + f_{v_\alpha}^L(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega \le 0, \tag{9}$$

$$\int_\Theta \{f_v^U(\pi^0)(v - v^0) + f_{v_\alpha}^U(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega \le 0, \tag{10}$$

hold. Since the functional $\int_\Theta -f(\pi)\,\mathrm{d}\omega$ is LU-convex at $v^0 \in S$, we have

$$\int_\Theta f^L(\pi)\,\mathrm{d}\omega - \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \le \int_\Theta \{f_v^L(\pi^0)(v - v^0) + f_{v_\alpha}^L(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,$$

$$\text{and} \quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega - \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega \le \int_\Theta \{f_v^U(\pi^0)(v - v^0) + f_{v_\alpha}^U(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,$$

$$\forall\ v \in S.$$

In agreement of the inequality (9) and (10), the above inequalities yield

$$\int_\Theta f^L(\pi)\,\mathrm{d}\omega \le \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \quad \text{and} \quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega \le \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega,$$

for some $v \in S$. Equivalently, for some $v \in S$ the following inequality

$$\int_\Theta f(\pi)\,\mathrm{d}\omega \preceq_{LU} \int_\Theta f(\pi^0)\,\mathrm{d}\omega,$$

holds, which contradicts that v^0 is strong LU-optimal solution to the problem (VP). This completes the proof. □

Theorem 3.6. Let $v^0 \in S$ be an LU-optimal solution to the problem (VP). If the interval-valued functional $\int_\Theta -f(\pi)\,\mathrm{d}\omega$ is LU-convex at v^0, then v^0 solves the weak variational inequality (WVI).

Proof. Let $v^0 \in S$ is LU-optimal solution to (VP). Then, there exist no other $v \in S$, such that

$$\int_\Theta f(\pi)\,\mathrm{d}\omega \prec_{LU} \int_\Theta f(\pi^0)\,\mathrm{d}\omega,$$

equivalently,

$$\int_\Theta f^L(\pi)\,\mathrm{d}\omega < \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \quad \text{and} \quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega \le \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega,$$

$$\text{or} \quad \int_\Theta f^L(\pi)\,\mathrm{d}\omega \leq \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \quad \text{and} \quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega < \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega,$$
$$\text{or} \quad \int_\Theta f^L(\pi)\,\mathrm{d}\omega < \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \quad \text{and} \quad \int_\Theta f^U(\pi)\,\mathrm{d}\omega < \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega.$$

Thus, there exists no other point $v \in S$, satisfying

$$\int_\Theta \{f^L(\pi) + f^U(\pi)\}\,\mathrm{d}\omega < \int_\Theta \{f^L(\pi^0) + f^U(\pi^0)\}\,\mathrm{d}\omega. \tag{11}$$

Now, we proceed by the contradiction and assume that v^0 is not a solution to the weak variational inequality (WVI), then there exist a point $v \in S$, such that

$$\int_\Theta \{f_v^L(\pi^0) + f_v^U(\pi^0)\}(v - v^0)\,\mathrm{d}\omega + \int_\Theta \{f_{v_\alpha}^L(\pi^0) + f_{v_\alpha}^U(\pi^0)\}D_\alpha(v - v^0)\,\mathrm{d}\omega < 0. \tag{12}$$

Since the functional $\int_\Theta -f(\pi)\,\mathrm{d}\omega$ be LU-convex at $v^0 \in S$, we get

$$\int_\Theta f^L(\pi)\,\mathrm{d}\omega - \int_\Theta f^L(\pi^0)\,\mathrm{d}\omega \leq \int_\Theta \{f_v^L(\pi^0)(v - v^0) + f_{v_\alpha}^L(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,$$

and

$$\int_\Theta f^U(\pi)\,\mathrm{d}\omega - \int_\Theta f^U(\pi^0)\,\mathrm{d}\omega \leq \int_\Theta \{f_v^U(\pi^0)(v - v^0) + f_{v_\alpha}^U(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega, \quad \forall\, v \in S.$$

From the above inequalities, we obtain

$$\begin{aligned}\int_\Theta \{f^L(\pi) + f^U(\pi)\}\,\mathrm{d}\omega &- \int_\Theta \{f^L(\pi^0) + f^U(\pi^0)\}\,\mathrm{d}\omega \\ &\leq \int_\Theta \{f_v^L(\pi^0) + f_v^U(\pi^0)\}(v - v^0)\,\mathrm{d}\omega + \int_\Theta \{f_{v_\alpha}^L(\pi^0) + f_{v_\alpha}^U(\pi^0)\}D_\alpha(v - v^0)\,\mathrm{d}\omega, \\ &\qquad \forall\, v \in S,\end{aligned}$$

which together with the inequality (12), yields the following inequality

$$\int_\Theta \{f^L(\pi) + f^U(\pi)\}\,\mathrm{d}\omega < \int_\Theta \{f^L(\pi^0) + f^U(\pi^0)\}\,\mathrm{d}\omega,$$

holds for a point $v \in S$, which contradicts the inequality (11). This completes the proof. □

Theorem 3.7. Let $v^0 \in S$ solves the weak variational inequality (WVI). If the interval-valued functional $\int_\Theta f(\pi)\,\mathrm{d}\omega$ is strictly LU-convex at v^0, then v^0 is an LU-optimal solution to the problem (VP).

P r o o f. Let $v^0 \in S$ is a solution of variational inequality (WVI), then there exist no other $v \in S$, such that

$$\int_\Theta \{f_v^L(\pi^0) + f_v^U(\pi^0)\}(v - v^0)\,\mathrm{d}\omega + \int_\Theta \{f_{v_\alpha}^L(\pi^0) + f_{v_\alpha}^U(\pi^0)\}D_\alpha(v - v^0)\,\mathrm{d}\omega < 0. \tag{13}$$

Now, we proceed by the contradiction and assume that v^0 is not an LU-optimal solution to the variational problem (VP), then there exist $v \in S$, satisfying the inequality

$$\int_{\Theta} f(\pi)\,\mathrm{d}\omega \prec_{LU} \int_{\Theta} f(\pi^0)\,\mathrm{d}\omega,$$

equivalently,

$$\int_{\Theta} f^L(\pi)\,\mathrm{d}\omega < \int_{\Theta} f^L(\pi^0)\,\mathrm{d}\omega \text{ and } \int_{\Theta} f^U(\pi)\,\mathrm{d}\omega \leq \int_{\Theta} f^U(\pi^0)\,\mathrm{d}\omega, \tag{14}$$

$$\text{or } \int_{\Theta} f^L(\pi)\,\mathrm{d}\omega \leq \int_{\Theta} f^L(\pi^0)\,\mathrm{d}\omega \text{ and } \int_{\Theta} f^U(\pi)\,\mathrm{d}\omega < \int_{\Theta} f^U(\pi^0)\,\mathrm{d}\omega, \tag{15}$$

$$\text{or } \int_{\Theta} f^L(\pi)\,\mathrm{d}\omega < \int_{\Theta} f^L(\pi^0)\,\mathrm{d}\omega \text{ and } \int_{\Theta} f^U(\pi)\,\mathrm{d}\omega < \int_{\Theta} f^U(\pi^0)\,\mathrm{d}\omega. \tag{16}$$

From strict LU-convexity of the functional $\int_{\Theta} f(\pi)\,\mathrm{d}\omega$, we get

$$\int_{\Theta} f^L(\pi)\,\mathrm{d}\omega - \int_{\Theta} f^L(\pi^0)\,\mathrm{d}\omega > \int_{\Theta} \{f_v^L(\pi^0)(v - v^0) + f_{v_\alpha}^L(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,$$

and

$$\int_{\Theta} f^U(\pi)\,\mathrm{d}\omega - \int_{\Theta} f^U(\pi^0)\,\mathrm{d}\omega \geq \int_{\Theta} \{f_v^U(\pi^0)(v - v^0) + f_{v_\alpha}^U(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,\ \forall v \in S,$$

$$\text{or } \int_{\Theta} f^L(\pi)\,\mathrm{d}\omega - \int_{\Theta} f^L(\pi^0)\,\mathrm{d}\omega \geq \int_{\Theta} \{f_v^L(\pi^0)(v - v^0) + f_{v_\alpha}^L(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,$$

and

$$\int_{\Theta} f^U(\pi)\,\mathrm{d}\omega - \int_{\Theta} f^U(\pi^0)\,\mathrm{d}\omega > \int_{\Theta} \{f_v^U(\pi^0)(v - v^0) + f_{v_\alpha}^U(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,\ \forall v \in S,$$

$$\text{or } \int_{\Theta} f^L(\pi)\,\mathrm{d}\omega - \int_{\Theta} f^L(\pi^0)\,\mathrm{d}\omega > \int_{\Theta} \{f_v^L(\pi^0)(v - v^0) + f_{v_\alpha}^L(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,$$

and

$$\int_{\Theta} f^U(\pi)\,\mathrm{d}\omega - \int_{\Theta} f^U(\pi^0)\,\mathrm{d}\omega > \int_{\Theta} \{f_v^U(\pi^0)(v - v^0) + f_{v_\alpha}^U(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega,\ \forall v \in S.$$

On combining the above three inequalities with the inequalities (14) – (16), respectively, we obtain

$$\int_{\Theta} \{f_v^L(\pi^0)(v - v^0) + f_{v_\alpha}^L(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega < 0$$
$$\text{and } \int_{\Theta} \{f_v^U(\pi^0)(v - v^0) + f_{v_\alpha}^U(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega \leq 0,$$

$$\int_{\Theta} \{f_v^L(\pi^0)(v - v^0) + f_{v_\alpha}^L(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega \leq 0$$
$$\text{and } \int_{\Theta} \{f_v^U(\pi^0)(v - v^0) + f_{v_\alpha}^U(\pi^0)D_\alpha(v - v^0)\}\,\mathrm{d}\omega < 0,$$

$$\int_{\Theta}\{f_v^L(\pi^0)(v-v^0)+f_{v_\alpha}^L(\pi^0)D_\alpha(v-v^0)\}\,\mathrm{d}\omega<0$$
$$\text{and }\int_{\Theta}\{f_v^U(\pi^0)(v-v^0)+f_{v_\alpha}^U(\pi^0)D_\alpha(v-v^0)\}\,\mathrm{d}\omega<0.$$

From the above inequalities, it follows that

$$\int_{\Theta}\{f_v^L(\pi^0)+f_v^U(\pi^0)\}(v-v^0)\,\mathrm{d}\omega+\int_{\Theta}\{f_{v_\alpha}^L(\pi^0)+f_{v_\alpha}^U(\pi^0)\}D_\alpha(v-v^0)\,\mathrm{d}\omega<0,$$

holds, for some $v \in S$, which contradicts the inequality (13). This completes the proof. □

4. CONCLUSIONS

In this paper, we have shown various equivalence relations between solutions of (weak, split) variational inequalities and (strong) LU-optimal solutions to a class of multi-dimensional interval-valued variational problem involving multiple integral functional. The concept of a convex set and the LU-convexity of the involved functionals played an essential role in establishing the main results. Some examples demonstrated how the suggested methods can be used, how well they work and applicable.

It would be interesting to derive analogous results for various classes of interval-valued problems emerging in many areas of operations research for instance non-convex interval-valued variational problems, multi-dimensional interval-valued variational control problems, etc.

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