# Conditions at infinity for the inhomogeneous filtration equation

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**Abstract**

We investigate existence and uniqueness of solutions to the filtration equation with an inhomogeneous density in $\mathbb{R}^N$ $(N \geq 3)$, approaching at infinity a given continuous datum of Dirichlet type.

## 1 Introduction

We provide sufficient conditions for existence and uniqueness of *bounded* solutions to the following nonlinear Cauchy problem (given $T > 0$):

$$\begin{cases} \rho\, \partial_t u = \Delta\big[G(u)\big] & \text{in } I\!R^N \times (0,T] =: S_T \\ u = u_0 & \text{in } I\!R^N \times \{0\}\,, \end{cases} \tag{1.1}$$

where $\rho = \rho(x)$ does not depend on $t$. Concerning the density $\rho$, the initial condition $u_0$ and the nonlinearity $G$ we shall mostly assume the following:

$$(H_0) \qquad \begin{cases} (i) & \rho \in C(I\!R^N),\ \rho > 0\,; \\ (ii) & G \in C^1(I\!R),\ G(0) = 0,\ G'(s) > 0 \text{ for any } s \in I\!R \setminus \{0\}, \\ & G' \text{ decreasing in } (-\delta, 0) \text{ and increasing in } (0, \delta) \\ & \text{if } G'(0) = 0 \ (\delta > 0), \text{ for some } \delta > 0; \\ (iii) & u_0 \in L^\infty(I\!R^N) \cap C(I\!R^N). \end{cases}$$

A typical choice for the function $G$ is $G(u) = |u|^{m-1}u$ for some $m \geq 1$. In this case, for $m > 1$, the differential equation in problem (1.1) becomes the *inhomogeneous porous media equation*, which arises in various situations of physical interest. We quote, without any claim of generality, the papers [13], [14], [6], [21], [4], [5], [22], [23], [16]–[20], [11], [8], [9] as references on this topic, and the recent monograph [24] as a general reference on the porous media equation.

As it is well-known, if assumption $(H_0)$ is satisfied, then there exists a bounded solution of problem (1.1) (see, *e.g.*, [14], [7], [21]). Moreover, if $N = 1$ or $N = 2$, and $\rho \in L^\infty(I\!R^N)$, then the solution to problem (1.1) is unique (see [10]).

When $N \geq 3$, we can have uniqueness or nonuniqueness of bounded solutions to problem (1.1), in dependence of the behavior *at infinity* of the density $\rho$. In fact, given $R > 0$, set $B_R := \{x \in I\!R^N : |x| < R\}$ and $B_R^c := I\!R^N \setminus B_R$. Suppose that $\rho$ does not decay too fast at

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infinity, in the sense that there exist $\widehat{R} > 0$ and $\underline{\rho} \in C([\widehat{R}, \infty))$ such that $\rho(x) \geq \underline{\rho}(|x|) > 0$ for all $x \in B^c_{\widehat{R}}$, with $\int_{\widehat{R}}^{\infty} \eta \underline{\rho}(\eta)\, d\eta = \infty$. Then problem (1.1) admits *at most* one bounded solution (see [16], [20]). A natural choice for $\underline{\rho}$ above is $\underline{\rho}(\eta) := \eta^{-\alpha}$ $(\eta \in [\widehat{R}, \infty))$ for some $\alpha \in (-\infty, 2]$ and $\widehat{R} > 0$.

On the contrary if $\rho$ decays sufficiently fast at infinity, in the sense that $\Gamma * \rho \in L^\infty(I\!R^N)$, where $\Gamma$ is the fundamental solution of the Laplace equation in $I\!R^N$, then nonuniqueness prevails (see [16], [20] and also [12] for the linear case, namely $G(u) = u$). To be specific, for any function $A \in \mathrm{Lip}\,[0, T]$ with $A(0) = 0$ there exists a solution $u$ of problem (1.1) such that

$$\lim_{R\to\infty} \frac{1}{|\partial B_R|} \int_{\partial B_R} \left|U(x,t) - A(t)\right| d\sigma \,=\, 0 \tag{1.2}$$

uniformly with respect to $t \in [0, T]$, where

$$U(x,t) := \int_0^t G(u(x,\tau))\, d\tau \quad \forall (x,t) \in S_T\,. \tag{1.3}$$

The condition $\Gamma * \rho \in L^\infty(I\!R^N)$ can be replaced by the following stronger (but more explicit) condition: there exists $\widehat{R} > 0$ and $\overline{\rho} \in C([\widehat{R}, \infty))$ such that $\rho(x) \leq \overline{\rho}(|x|)$ for all $x \in B^c_{\widehat{R}}$, with $\int_{\widehat{R}}^{\infty} \eta \overline{\rho}(\eta)\, d\eta \,<\, \infty$. Then, instead of (1.2), we can impose that

$$\lim_{|x|\to\infty} \left|U(x,t) - A(t)\right| \,=\, 0 \tag{1.4}$$

uniformly with respect to $t \in [0, T]$ (which clearly implies (1.2)), with $U$ defined in (1.3). A natural choice for $\overline{\rho}$ as above is $\overline{\rho}(\eta) := \eta^{-\alpha}$ $(\eta \in [\widehat{R}, \infty))$ for some $\alpha \in (2, \infty]$ and $\widehat{R} > 0$.

Observe that equalities (1.2) and (1.4) can be also regarded as nonhomogeneous Dirichlet conditions at infinity in a suitable integral sense. From this point of view, it seems natural to study whether imposing conditions at infinity *in a pointwise sense* that resembles more closely the usual Dirichlet boundary conditions restores existence and uniqueness of solutions. In fact, up to now, it was only known that there exists at most one solution $u \in L^\infty(S_T)$ to problem (1.1) satisfying condition (1.2) or (1.4) either when $G(u) = u$ (see the important results obtained, in such linear case, in [12]) or when $u_0 \geq 0$ and $A \equiv 0$ (see [7]). However, the methods used to obtain the mentioned uniqueness results did not work for general $G$ and $A$.

In this paper we shall then address existence and uniqueness of bounded solutions to problem (1.1) satisfying at infinity suitable nonhomogeneous Dirichlet conditions in a pointwise sense. More precisely, at first we shall prove that if $\rho$ decays sufficiently fast at infinity, the diffusion is non-degenerate in an appropriate sense, $u_0 \in C(I\!R^N)$ and $\lim_{|x|\to\infty} u_0(x)$ exists and is finite then for any $a \in C([0, T])$ with

$$a(0) = \lim_{|x|\to\infty} u_0(x) \tag{1.5}$$

there exists a bounded solution $u$ to problem (1.1) satisfying

$$\lim_{|x|\to\infty} u(x,t) = a(t) \quad \text{uniformly for } t \in [0, T] \tag{1.6}$$

(see Theorem 2.2). Furthermore, we can remove the assumption of nondegeneracy of the diffusion for suitable classes of initial data $u_0$ and conditions at infinity $a$. Indeed if $a_0 := \lim_{|x|\to\infty} u_0(x)$ exists and is finite and $(H_0)$ holds true, then there exists a bounded solution to problem (1.1) such that

$$\lim_{|x|\to\infty} u(x,t) = a_0 \quad \text{uniformly for } t \in [0, T] \tag{1.7}$$

(see Theorem 2.3 and Remark 2.4).

Moreover, if $(H_0)$ holds true and $\rho$ decays sufficiently fast at infinity, then there exists a bounded solution $u$ to problem (1.1) satisfying (1.6) for any $a \in C([0,T])$ with $a > 0$ in $[0,T]$, provided $u_0$ complies with (1.5) (see Theorem 2.5).

Let us explain that in [16], generalizing arguments used in [12], the prescription of conditions (1.2) for solutions to (1.1) is made by constructing suitable *barriers* at infinity, that are sub– or supersolutions to appropriate associated linear elliptic problems. Instead, in the present case to impose at infinity Dirichlet conditions in a pointwise sense we will construct, in a neighborhood of each $t_0 \geq 0$, suitable time-dependent *barriers* at infinity, that are sub– or supersolutions to proper associated nonlinear parabolic problems.

Actually, in the existence results, hypothesis (1.5) can be removed, upon requiring that the Dirichlet condition at infinity is attained uniformly for $t \in [\tau, T]$, for any $0 < \tau < T$ (see Remark 2.6) and, in the degenerate case, a further technical condition holds.

Finally, we shall prove that the weaker condition

$$\lim_{|x|\to\infty} u(x,t) = a(t) \quad \forall t \in [0,T] \tag{1.8}$$

implies uniqueness for general $G$ satisfying $(H_0)(ii)$, bounded $\rho$ and $a \in C([0,T])$ (see Theorem 2.8). Arguments used in proving uniqueness are modeled after those in [1] (where $\rho \equiv 1, N = 1$) and [10] (where $N = 2$), for cases in which uniqueness was proved in the class of solutions not satisfying additional conditions at infinity. Although this is not our case, we use an analogous strategy, combined with the fact that solutions attain a datum at infinity in a pointwise sense. This permits to conclude.

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# 2 Existence and uniqueness results

Solutions, sub- and supersolutions to problem (1.1) are always meant in the following sense.

**Definition 2.1** *By a* solution *to problem* (1.1) *we mean a function* $u \in C(S_T) \cap L^\infty(S_T)$ *such that*

$$\begin{aligned} \int_0^\tau \int_{\Omega_1} \{\rho\, u\, \partial_t \psi + G(u)\Delta\psi\}\, dxdt = & \int_{\Omega_1} \rho \left[u(x,\tau)\psi(x,\tau) - u_0(x)\psi(x,0)\right] dx \\ & + \int_0^\tau \int_{\partial\Omega_1} G(u)\langle \nabla\psi,\, \nu\rangle\, d\sigma dt \end{aligned} \tag{2.1}$$

*for any bounded open set* $\Omega_1 \subseteq I\!R^N$ *with smooth boundary* $\partial\Omega_1$, $\tau \in (0,T]$, $\psi \in C^{2,1}(\overline{\Omega}_1 \times [0,\tau])$, $\psi \geq 0, \psi = 0$ *in* $\partial\Omega_1 \times [0,\tau]$. *Here* $\nu$ *denotes the outer normal to* $\Omega_1$ *and* $\langle\cdot,\cdot\rangle$ *the scalar product in* $I\!R^N$.

Supersolutions (subsolutions) *to* (1.1) *are defined replacing* " $=$ " *by* " $\leq$ " (" $\geq$ ", *respectively) in* (2.1).

These kind of solutions are sometimes referred to as *very weak* solutions. Observe that, according to Definition 2.1, solutions to problem (1.1) we deal with are in $S_T$.

## 2.1 Existence

In the case of *nondegenerate* nonlinearities, we have the following result.

**Theorem 2.2** *Let $N \geq 3$. Assume that $\rho \in C(\mathbb{R}^N)$, $\rho > 0$, $G \in C^1(\mathbb{R})$ with $G(0) = 0$, $G'(s) \geq \alpha_0 > 0$ for any $s \in \mathbb{R}$ and $u_0 \in C(\mathbb{R}^N)$ with $\lim_{|x|\to\infty} u_0(x)$ existing and being finite. Assume also that there exist $\widehat{R} > 0$ and $\overline{\rho} \in C([\widehat{R}, \infty))$ such that $\rho(x) \leq \overline{\rho}(|x|)$ for any $x \in B^c_{\widehat{R}}$, with $\int_{\widehat{R}}^{\infty} \eta\overline{\rho}(\eta)\, d\eta < \infty$.*

*Finally, let $a \in C([0, T])$ and suppose that*

$$a(0) = \lim_{|x|\to\infty} u_0(x) .$$

*Then there exists a solution to problem* (1.1) *such that*

$$\lim_{|x|\to\infty} u(x, t) = a(t) \quad \text{uniformly for } t \in [0, T] .$$

For appropriate classes of data and possibly *degenerate* nonlinearities of porous media type, we shall prove the following results.

**Theorem 2.3** *Let $N \geq 3$. Let assumption $(H_0)$ be satisfied. Suppose that*

$$\lim_{|x|\to\infty} u_0(x) \;=\; a_0 \tag{2.2}$$

*for some $a_0 \in \mathbb{R}$. Then there exists a solution to problem* (1.1) *such that*

$$\lim_{|x|\to\infty} u(x, t) = a_0 \quad \text{uniformly for } t \in [0, T] .$$

**Remark 2.4** *Let assumption $(H_0)$ be satisfied and suppose that $\rho$ does not decay too fast at infinity in the sense that there exist $\widehat{R} > 0$ and $\underline{\rho} \in C([\widehat{R}, \infty))$ such that $\rho(x) \geq \underline{\rho}(|x|) > 0$ for any $x \in B^c_{\widehat{R}}$, with $\int_{\widehat{R}}^{\infty} \eta\underline{\rho}(\eta)\, d\eta = \infty$. Assume also that* (2.2) *holds. Then by the uniqueness result recalled in the Introduction, and by Theorem 2.3, the* unique *solution to problem* (1.1) *necessarily satisfies*

$$\lim_{|x|\to\infty} u(x, t) = a_0 \quad \text{uniformly for } t \in [0, T] .$$

**Theorem 2.5** *Let $N \geq 3$. Let assumption $(H_0)$ be satisfied. Suppose that there exist $\widehat{R} > 0$ and $\overline{\rho} \in C([\widehat{R}, \infty))$ such that $\rho(x) \leq \overline{\rho}(|x|)$ for any $x \in B^c_{\widehat{R}}$, with $\int_{\widehat{R}}^{\infty} \eta\overline{\rho}(\eta)\, d\eta < \infty$. Let $a \in C([0, T])$, with $a(t) > 0$ for all $t \in [0, T]$. Assume also that*

$$a(0) = \lim_{|x|\to\infty} u_0(x) .$$

*Then there exists a solution to problem* (1.1) *such that*

$$\lim_{|x|\to\infty} u(x, t) = a(t) \quad \text{uniformly for } t \in [0, T] .$$

**Remark 2.6** (*i*) *In Theorem 2.2, if we* do not *assume that $a(0) = \lim_{|x|\to\infty} u_0(x)$, then the conclusion remains true, replacing the property $\lim_{|x|\to\infty} u(x, t) = a(t)$ uniformly for any $t \in [0, T]$ by the following:*

$$\textit{for any } \tau \in (0, T) , \ \lim_{|x|\to\infty} u(x, t) = a(t) \quad \text{uniformly for } t \in [\tau, T] . \tag{2.3}$$

(*ii*) *In Theorem 2.5, if we* do not *assume that $a(0) = \lim_{|x|\to\infty} u_0(x)$, then the conclusion remains true, provided we replace the property $\lim_{|x|\to\infty} u(x, t) = a(t)$ uniformly for any $t \in [0, T]$ by* (2.3) *and we also require that*

$$I := \inf_{x\in B^c_{R_0}} u_0(x) , \ S := \sup_{t\in(0,\epsilon)} a(t) , \ 2\, G(I) > G\,(S) \tag{2.4}$$

*for some $R_0, \epsilon > 0$. Clearly,* (2.4) *is technical and is needed to make our proof hold under more general assumptions. We do not know whether the result is still valid without assuming it, but notice that* (2.4) *certainly holds if $I$ is large enough compared to $S$, so that possible problems occur only if the initial datum is, in a suitable sense, small at infinity.*

See the end of Section 3 for comments on the minor changes needed in the proof of the corresponding theorems to obtain statements $(i) - (ii)$.

**Remark 2.7** *Note that the hypotheses made in Theorem 2.5 allow to assume as initial data functions $u_0$ which may be nonpositive in some compact subset $K \subset I\!R^N$.*

## 2.2 Uniqueness

We shall prove the following uniqueness result in the general case of possibly degenerate nonlinearities.

**Theorem 2.8** *Let $N \geq 3$. Let assumption $(H_0)$ be satisfied, and suppose that $a \in L^\infty(0,T)$, $\rho \in L^\infty(I\!R^N)$. Then there exists* at most *one solution to problem* (1.1) *such that*

$$\lim_{|x|\to\infty} u(x,t) = a(t) \quad \text{for almost every } t \in (0,T)\,.$$

From Theorems 2.3 and 2.8 we deduce the following.

**Corollary 2.9** *Let $N \geq 3$. Let assumption $(H_0)$ be satisfied, and suppose that $\rho \in L^\infty(I\!R^N)$. Then there exists a unique solution to problem* (1.1) *such that*

$$\lim_{|x|\to\infty} u(x,t) = a_0 \quad \text{uniformly for } t \in [0,T].$$

**Remark 2.10** *When $(H_0)$ is satisfied, $\rho$ belongs to $L^\infty(I\!R^N)$ and fulfils the assumptions appearing in Remark 2.4, then the conclusion of Corollary 2.9 is in agreement with such Remark.*

As a consequence of Theorems 2.5 and 2.8 we get

**Corollary 2.11** *Let $N \geq 3$. Let the assumptions of Theorem 2.5 be satisfied, and suppose that $\rho \in L^\infty(I\!R^N)$. Then there exists a unique solution to problem* (1.1) *such that*

$$\lim_{|x|\to\infty} u(x,t) \;=\; a(t) \quad \text{uniformly for } t \in [0,T]\,.$$

Finally, in the case of *nondegenerate* nonlinearities, from Theorems 2.2 and 2.8 we obtain the following.

**Corollary 2.12** *Let $N \geq 3$. Let the assumptions of Theorem 2.2 be satisfied, and suppose that $\rho \in L^\infty(I\!R^N)$. Then there exists a unique solution to problem* (1.1) *such that*

$$\lim_{|x|\to\infty} u(x,t) \;=\; a(t) \quad \text{uniformly for } t \in [0,T]\,.$$

# 3 Existence: proofs

In view of the assumptions on $\rho$ made in the hypotheses of Theorem 2.2 or 2.5, there exists a function $V = V(|x|) \in C^2(B^c_{\widehat{R}})$ such that

$$\Delta V \;\leq\; -\rho \quad \text{in } B^c_{\widehat{R}}\,, \tag{3.1}$$

$$V(|x|) \;>\; 0 \quad \forall x \in B^c_{\widehat{R}}\,,$$

$$|x| \mapsto V(|x|) \text{ is nonincreasing},$$
$$\lim_{|x|\to\infty} V(x) = 0; \tag{3.2}$$

here $\widehat{R} > 0$ can be assumed to be equal to the one that appears in the hypotheses of Theorem 2.2 or 2.5.

In some of the forthcoming proofs we shall make use of the function $G^{-1}$, whose domain $D$ need not coincide with $\mathbb{R}$. As we are dealing with bounded data $u_0$ (and, by the maximum principle, with bounded solutions), this makes no problem since one can modify the definition of $G(x)$ for $|x|$ large so that such a function is a bijection from $\mathbb{R}$ to itself, without changing the evolution of $u_0$.

Hereafter, for any $j \in I\!N$, $\zeta_j$ will always be a function having the following properties: $\zeta_j \in C_c^\infty(B_j)$ with $0 \le \zeta_j \le 1$ and $\zeta_j \equiv 1$ in $B_{j/2}$.

*Proof of Theorem 2.2.* Since $a \in C([0,T])$ and $G \in C^1(I\!R)$ is increasing, for any $t_0 \in [0,T]$, $\sigma > 0$ there exists $\delta = \delta(\sigma) > 0$ such that

$$G^{-1}\big[G(a(t_0)) - \sigma\big] \le a(t) \le G^{-1}\big[G(a(t_0)) + \sigma\big] \quad \forall t \in [\underline{t}_\delta, \overline{t}_\delta], \tag{3.3}$$

where $\underline{t}_\delta := \max\{t_0 - \delta, 0\}$ and $\overline{t}_\delta := \min\{t_0 + \delta, T\}$. Moreover, in view of the assumptions on $u_0$, for any $\sigma > 0$ there exists $R = R(\sigma) > \widehat{R}$ such that

$$G^{-1}\big[G(a(0)) - \sigma\big] \le u_0(x) \le G^{-1}\big[G(a(0)) + \sigma\big] \quad \forall x \in B_R^c. \tag{3.4}$$

For any $j \in I\!N$, let $u_j \in C\big(\overline{B}_j \times [0,T]\big)$ be the unique solution (see, *e.g.*, [15]) to the problem

$$\begin{cases} \rho\, \partial_t u = \Delta\big[G(u)\big] & \text{in } B_j \times (0,T) \\ u = a(t) & \text{on } \partial B_j \times (0,T) \\ u = u_{0,j} & \text{in } \overline{B}_j \times \{0\}, \end{cases} \tag{3.5}$$

where

$$u_{0,j} := \zeta_j\, u_0 + (1 - \zeta_j)\, a(0) \quad \text{in } \overline{B}_j.$$

By comparison principles,

$$|u_j| \le K := \max\{\|u_0\|_\infty, \|a\|_\infty\} \quad \text{in } B_j \times (0,T). \tag{3.6}$$

It is a matter of usual compactness arguments (see, *e.g.*, [15]) to show that there exists a subsequence $\{u_{j_k}\} \subseteq \{u_j\}$ which converges, as $k \to \infty$, *locally uniformly* in $I\!R^N \times (0,T)$ to a solution $u$ to problem (1.1).

Hence, it remains to prove that

$$\lim_{|x|\to\infty} u(x,t) = a(t) \quad \text{uniformly for } t \in [0,T].$$

To this end, let $t_0 \in [0,T]$. Define

$$\underline{w}(x,t) := G^{-1}\big[ -\underline{M} V(x) - \sigma + G(a(t_0)) - \underline{\lambda}(t - t_0)^2\big] \quad \forall (x,t) \in B^c_{\widehat{R}} \times (\underline{t}_\delta, \overline{t}_\delta),$$

where $\underline{M} > 0$ and $\underline{\lambda} > 0$ are constants to be chosen later. By the assumptions and (3.1),

$$\rho(x)\partial_t \underline{w} - \Delta[G(\underline{w})] = -\rho(x)\frac{2\underline{\lambda}(t - t_0)}{G'(w)} + \underline{M}\Delta V \le \rho(x)\left(\frac{2\lambda\delta}{\alpha_0} - \underline{M}\right) \le 0 \tag{3.7}$$
$$\text{in } B^c_{\widehat{R}} \times (\underline{t}_\delta, \overline{t}_\delta)$$

providing that

$$\underline{M} \geq \frac{2\underline{\lambda}\delta}{\alpha_0}\,. \tag{3.8}$$

For any $j \in I\!N$, $j > R$, let

$$N_{R,j} := B_j \setminus \overline{B}_R\,,$$

$R$ being as in (3.4). We have

$$\underline{w}(x,t) \leq -K \quad \forall (x,t) \in \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta) \tag{3.9}$$

provided

$$\underline{M} \geq \frac{G(\|a\|_\infty) - G(-K)}{V(R)}\,. \tag{3.10}$$

Furthermore,

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \sigma\big] \quad \forall (x,t) \in \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta). \tag{3.11}$$

When $\underline{t}_\delta = 0$ there holds

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \sigma\big] \quad \forall (x,t) \in \overline{N}_{R,j} \times \{\underline{t}_\delta\}\,, \tag{3.12}$$

whereas when $\underline{t}_\delta > 0$ we have

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \underline{\lambda}\delta^2\big] \leq -K \quad \forall (x,t) \in \overline{N}_{R,j} \times \{\underline{t}_\delta\} \tag{3.13}$$

provided

$$\underline{\lambda} \geq \frac{G(\|a\|_\infty) - G(-K)}{\delta^2}\,. \tag{3.14}$$

Suppose that conditions (3.8), (3.10) and (3.14) are satisfied. Hence, from (3.7) and (3.9)–(3.13) we infer that $\underline{w}$ is a subsolution to the problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, -K & \text{on } \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{on } \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,= -K & \text{in } \overline{N}_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.15}$$

when $\underline{t}_\delta > 0$, whereas it is a subsolution to the problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, -K & \text{on } \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{on } \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,= G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{in } \overline{N}_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.16}$$

when $\underline{t}_\delta = 0$.

On the other hand, (3.3), (3.4) and (3.6) show that the boundary data for the solutions to (3.5) and (3.15), (3.16) are correctly ordered on each part of the parabolic boundary of $N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta)$. In particular, we deduce that $u_j$ is a supersolution to problem (3.15) when $\underline{t}_\delta > 0$, while it is a supersolution to problem (3.16) when $\underline{t}_\delta = 0$. Therefore, by comparison principles,

$$\underline{w} \leq u_j \quad \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta)\,. \tag{3.17}$$

Now let us define

$$\overline{w}(x,t) := G^{-1}\big[\overline{M}V(x) + \sigma + G(a(t_0)) + \overline{\lambda}(t-t_0)^2\big] \quad \forall (x,t) \in B^c_{\widehat{R}} \times (\underline{t}_\delta, \overline{t}_\delta)\,,$$

with

$$\overline{M} \geq \max\left\{\frac{2\overline{\lambda}\delta}{\alpha_0}, \frac{G(K) - G(-\|a\|_\infty)}{V(R)}\right\},$$

and

$$\overline{\lambda} \geq \frac{G(K) - G(-\|a\|_\infty)}{\delta^2}\,.$$

By arguments analogous to those used above, we can infer that $\overline{w}$ is a supersolution to the problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta) \\ u = K & \text{on } \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta) \\ u = G^{-1}\big[G(a(t_0)) + \sigma\big] & \text{on } \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta) \\ u = K & \text{in } \overline{N}_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.18}$$

when $\underline{t}_\delta > 0$, whereas it is a supersolution to the problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta) \\ u = K & \text{on } \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta) \\ u = G^{-1}\big[G(a(t_0)) + \sigma\big] & \text{on } \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta) \\ u = G^{-1}\big[G(a(t_0)) + \sigma\big] & \text{in } \overline{N}_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.19}$$

when $\underline{t}_\delta = 0$.

As before, from (3.3), (3.4) and (3.6) we deduce that $u_j$ is a subsolution to problem (3.18) when $\underline{t}_\delta > 0$, while it is a subsolution to problem (3.19) when $\underline{t}_\delta = 0$. By comparison principles,

$$u_j \leq \overline{w} \quad \text{in } \; N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta)\,. \tag{3.20}$$

From (3.17) and (3.20) with $j = j_k$, sending $k \to \infty$, we then obtain

$$\underline{w} \leq u \leq \overline{w} \quad \text{in } \; B^c_{\widehat{R}} \times (\underline{t}_\delta, \overline{t}_\delta)\,. \tag{3.21}$$

By (3.21) and (3.2) we get that for $|x|$ large enough, *independently of* $t_0 \in [0,T]$, there holds

$$G^{-1}\big[G(a(t_0)) - 2\sigma\big] \leq u(x,t_0) \leq G^{-1}\big[G(a(t_0)) + 2\sigma\big]\,.$$

In order to complete the proof one just lets $\sigma \to 0^+$. □

*Proof of Theorem 2.3.* As in the proof of the previous result note that, thanks to (2.2), for any $\sigma > 0$ there exists $R = R(\sigma) > 0$ such that

$$G^{-1}\big[G(a_0) - \sigma\big] \leq u_0(x) \leq G^{-1}\big[G(a_0) + \sigma\big] \quad \forall x \in B^c_R\,.$$

In view of assumption $(H_0)$, by standard results (see, *e.g.*, [1]), for any $j \in \mathbb{N}$ there exists a unique solution $u_j$ to the problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } B_j \times (0,T) \\ u = a_0 & \text{on } \partial B_j \times (0,T) \\ u = u_{0,j} & \text{in } \overline{B}_j \times \{0\}\,, \end{cases}$$

where

$$u_{0,j} := \zeta_j u_0 + (1-\zeta_j)\, a_0 \quad \text{in } \overline{B}_j\,.$$

Note that, by the results of [3], $u_j \in C\big(\overline{B}_j \times [0,T]\big)$. By comparison principles,

$$|u_j| \le K := \max\{\|u_0\|_\infty, |a_0|\} \quad \text{in } B_j \times (0,T)\,.$$

By usual compactness techniques (one can use [2, Lemma 5.2] and a diagonal argument), there exists a subsequence $\{u_{j_k}\} \subseteq \{u_j\}$ which converges, as $k \to \infty$, locally uniformly in $I\!R^N \times (0,T)$ to a solution $u$ to problem (1.1).

Let

$$\Gamma(x) \equiv \Gamma(|x|) := |x|^{2-N} \quad \forall x \in I\!R^N \setminus \{0\}\,.$$

Clearly,

$$\begin{aligned} \Delta\Gamma \,&=\, 0 \quad \text{in } I\!R^N \setminus \{0\}\,, \\ \Gamma \,&>\, 0 \quad \text{in } I\!R^N \setminus \{0\}\,, \\ \lim_{|x|\to\infty} \Gamma(|x|) \,&=\, 0\,. \end{aligned} \tag{3.22}$$

Define

$$\overline{W}(x) := G^{-1}\big[\overline{M}\,\Gamma(x) + \sigma + G(a_0)\big] \quad \forall x \in I\!R^N \setminus \{0\}\,,$$

where

$$\overline{M} \ge \frac{G(K) - G(a_0)}{\Gamma(R)}\,. \tag{3.23}$$

Then

$$\Delta[G(\overline{W})] \,=\, 0 \quad \text{in } I\!R^N \setminus \{0\}\,. \tag{3.24}$$

In view of (3.23) there holds

$$\overline{W}(x) \ge K \quad \forall x \in \partial B_R\,. \tag{3.25}$$

Furthermore, we have

$$\overline{W}(x) \ge a_0 \quad \forall x \in \partial B_j \tag{3.26}$$

and

$$\overline{W}(x) \ge G^{-1}\big[G(a_0) + \sigma\big] \quad \forall x \in \overline{N}_{R,j}\,. \tag{3.27}$$

From (3.24)–(3.27) it follows that $\overline{W}$ is a supersolution to the problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (0,T) \\ u \,=\, K & \text{on } \partial B_R \times (0,T) \\ u \,=\, a_0 & \text{on } \partial B_j \times (0,T) \\ u \,= G^{-1}\big[G(a_0) + \sigma\big] & \text{in } \overline{N}_{R,j} \times \{0\}\,. \end{cases} \tag{3.28}$$

On the other hand, $u_j$ is a subsolution to problem (3.28). Hence, by comparison principles,

$$u_j \le \overline{W} \quad \text{in } N_{R,j} \times (0,T)\,. \tag{3.29}$$

Now let us define

$$\underline{W}(x) := G^{-1}\big[-\underline{M}\Gamma(x) - \sigma + G(a_0)\big] \quad \forall x \in I\!R^N \setminus \{0\}\,,$$

where

$$\underline{M} \ge \frac{G(a_0) - G(-K)}{\Gamma(R)}\,.$$

By arguments similar to those used above we can infer that $\underline{W}$ is a subsolution to the problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j}\times(0,T) \\ u \,=\, -K & \text{on } \partial B_R\times(0,T) \\ u \,=\, a_0 & \text{on } \partial B_j\times(0,T) \\ u \,= G^{-1}\big[G(a_0)-\sigma\big] & \text{in } \overline{N}_{R,j}\times\{0\}\,. \end{cases} \tag{3.30}$$

On the other side, $u_j$ is a supersolution to problem (3.30). Hence, by comparison principles,

$$\underline{W}\le u_j \quad \text{in } N_{R,j}\times(0,T)\,. \tag{3.31}$$

From (3.29) and (3.31) with $j=j_k$, sending $k\to\infty$, we obtain

$$\underline{W}\le u\le \overline{W} \quad \text{in } B_R^c\times(0,T)\,. \tag{3.32}$$

Letting $|x|\to\infty$ in (3.32), from (3.22) we have that for $|x|$ large enough, *independently of* $t\in[0,T]$, there holds

$$G^{-1}\big[G(a_0)-2\sigma\big]\le u(x,t)\le G^{-1}\big[G(a_0)+2\sigma\big]\,.$$

The proof is completed by letting $\sigma\to 0^+$. □

In order to prove Theorem 2.5 we need some intermediate results.

**Lemma 3.1** *Let $N\ge 3$. For any $\alpha, R, M>0$ there exists a subsolution $\underline{u}_0$ to the equation $-\Delta[G(u)]=0$ in $I\!R^N$ which is bounded, continuous, radial, nondecreasing as a function of $|x|$, satisfies $\lim_{|x|\to+\infty}\underline{u}_0(x)=\alpha$ and is equal to $-M$ in $B_R$.*

*Proof.* Define

$$\widetilde{U}_0(x):=G(\alpha)-\frac{\beta}{|x|}\quad \forall x\in B_\varepsilon^c\,,$$

where $0<\varepsilon<\gamma:=\frac{\beta}{G(\alpha)-G(-M)}$. It is easily seen that ($N\ge 3$)

$$-\Delta\widetilde{U}_0(x)\le 0\quad \forall x\in B_\varepsilon^c\,.$$

Then $\widetilde{u}_0:=G^{-1}(\widetilde{U}_0)$ is a subsolution to $-\Delta[G(u)]=0$ in $B_\varepsilon^c$. Consider the function

$$\widehat{u}_0:=\max\big\{\widetilde{u}_0,-M\big\}\quad \text{in } B_\varepsilon^c\,.$$

Since $\widetilde{u}_0$ solves $-\Delta[G(\widetilde{u}_0)]\le 0$ in $B_\varepsilon^c$, from Kato's inequality we deduce that $\widehat{u}_0$ is a subsolution to $-\Delta[G(u)]=0$ in $I\!R^N\setminus\overline{B}_\varepsilon$. Now observe that $\widehat{u}_0=-M$ in $B_\gamma\setminus B_\varepsilon$, so that the function

$$\underline{u}_0:=\begin{cases}\widehat{u}_0 & \text{in } B_\varepsilon^c \\ -M & \text{in } B_\varepsilon\end{cases}$$

is a subsolution to $-\Delta[G(u)]=0$ in $I\!R^N$. The fact that $\underline{u}_0$ is bounded, continuous, radial, nondecreasing as a function of $|x|$ and satisfies the limit property at infinity is clear by construction. The constant condition in $B_R$ is achieved by choosing $\beta=R\,(G(\alpha)-G(-M))$. □

**Lemma 3.2** *Suppose that, besides the assumptions of Theorem 2.5, there exists a function $\underline{u}_0$ having the properties stated in Lemma 3.1 and such that, for a suitable $\varepsilon > 0$ small enough,*

$$
\begin{aligned}
u_0(x) \geq \underline{u}_0(x) \quad \forall x \in I\!R^N \,, \\
\lim_{|x|\to\infty} \underline{u}_0(|x|) = \min_{t\in[0,T]} a(t) - \varepsilon > 0\,.
\end{aligned} \tag{3.33}
$$

*Moreover assume that, for the same $\varepsilon$ given above,*

$$
2\,G\left(\min_{t\in[0,T]} a(t) - \varepsilon\right) > G(\|a\|_\infty)\,. \tag{3.34}
$$

*Then there exists a solution to problem* (1.1) *satisfying condition* (1.6).

*Proof.* First we repeat the proof of Theorem 2.2 up to the construction of the sequence $\{u_j\}$, keeping the same notation. Note that, as in Theorem 2.3, when we allow for a degenerate nonlinearity $G$, in view of hypothesis $(H_0)$ existence of solutions to problem (3.5) is due to standard results (see, *e.g.*, [1]). Again, by the results of [3], $u_j \in C\big(\overline{B}_j \times [0,T]\big)$.

Then notice that, by the assumptions on $\underline{u}_0$, (3.33), (3.34) and $(H_0)$, we can find $\beta > 0$ and $\widetilde{R} > \widehat{R}$ such that for all $R \geq \widetilde{R}$

$$
\begin{aligned}
\beta < \overline{u}_0(R)\,, \\
2G\big(\overline{u}_0(R)\big) - G(\beta) - G(\|a\|_\infty) > 0\,.
\end{aligned} \tag{3.35}
$$

Still from the assumptions on $\underline{u}_0$ we deduce that it is a subsolution to problem (3.5). By comparison principles we have

$$
\underline{u}_0(|x|) \leq u_j(x,t) \leq K \quad \forall (x,t) \in B_j \times (0,T)\,,
$$

where $K$ is as in (3.6). Hence, by the monotonicity of $\underline{u}_0$,

$$
\underline{u}_0(R) \leq u_j(x,t) \leq K \quad \forall (x,t) \in N_{R,j} \times (0,T)\,. \tag{3.36}
$$

Let

$$
\gamma := \min_{[\beta,K]} G'\,. \tag{3.37}
$$

Given $\sigma > 0$, in view of (3.2) we can fix $R = R(\sigma) > \widetilde{R}$ in (3.4) so large that in (3.3) we are allowed to set

$$
\delta = \frac{2}{\gamma} V(R)\,. \tag{3.38}
$$

Note that $\beta$ and $\gamma$ are independent of $R$ and $\delta$. Let $t_0 \in [0,T]$. Define

$$
\underline{\lambda} := \frac{G(a(t_0)) - G(\underline{u}_0(R))}{\delta^2}, \quad \underline{M} := \frac{2\underline{\lambda}\delta}{\gamma}\,. \tag{3.39}
$$

From (3.35), (3.38) and (3.39) it follows that

$$
\underline{M} = \frac{G(a(t_0)) - G(\underline{u}_0(R))}{V(R)}\,, \tag{3.40}
$$

$$
-\,\underline{M}V(R) - \sigma + G(a(t_0)) - \underline{\lambda}\delta^2 > G(\beta) \tag{3.41}
$$

for $\sigma > 0$ small enough.

Now define

$$
\underline{w}(x,t) := G^{-1}\big[-\underline{M}V(x) - \sigma + G(a(t_0)) - \underline{\lambda}(t-t_0)^2\big] \quad \forall (x,t) \in B^c_{\widehat{R}} \times (\underline{t}_\delta, \overline{t}_\delta)\,.
$$

Since $|x| \mapsto V(|x|)$ is nonincreasing, by (3.41)

$$\underline{w}(x,t) \geq \beta \quad \forall (x,t) \in N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta)\,. \tag{3.42}$$

Also, from $(H_0)(ii)$, (3.1), (3.42), (3.37) and (3.39)

$$\rho(x)\partial_t \underline{w} - \Delta[G(\underline{w})] = -\rho(x)\frac{2\underline{\lambda}(t-t_0)}{G'(\underline{w})} + \underline{M}\Delta V \leq \rho(x)\left(\frac{2\underline{\lambda}\delta}{\gamma} - \underline{M}\right) = 0 \tag{3.43}$$

$$\text{in } B^c_{\widehat{R}} \times (\underline{t}_\delta, \overline{t}_\delta)\,.$$

By (3.40),

$$\underline{w}(x,t) \leq \underline{u}_0(R) \quad \forall (x,t) \in \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta)\,. \tag{3.44}$$

Furthermore,

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \sigma\big] \quad \forall (x,t) \in \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta). \tag{3.45}$$

When $\underline{t}_\delta = 0$ there holds

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \sigma\big] \quad \forall (x,t) \in \overline{N}_{R,j} \times \{\underline{t}_\delta\}\,, \tag{3.46}$$

whereas when $\underline{t}_\delta > 0$ we have

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \lambda\delta^2\big] = \underline{u}_0(R) \quad \forall (x,t) \in \overline{N}_{R,j} \times \{\underline{t}_\delta\}\,; \tag{3.47}$$

here (3.39) has been used.

From (3.43)–(3.47) we infer that $\underline{w}$ is a subsolution to the problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, \underline{u}_0(R) & \text{on } \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{on } \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, \underline{u}_0(R) & \text{in } \overline{N}_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.48}$$

when $\underline{t}_\delta > 0$, whereas it is a subsolution to the problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, \underline{u}_0(R) & \text{on } \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{on } \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{in } \overline{N}_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.49}$$

when $\underline{t}_\delta = 0$.

On the other hand, from (3.3), (3.4) (which, recall, holds true as a consequence of (1.5)) and (3.36) we easily deduce that $u_j$ is a supersolution to problem (3.48) when $\underline{t}_\delta > 0$, while it is a supersolution to problem (3.49) when $\underline{t}_\delta = 0$. Hence, by comparison principles,

$$\underline{w} \leq u_j \quad \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta)\,.$$

Finally, let us define

$$\overline{w}(x,t) := G^{-1}\big[\overline{M}V(x) + \sigma + G(a(t_0)) + \overline{\lambda}(t-t_0)^2\big] \quad \forall (x,t) \in B^c_{\widehat{R}} \times (\underline{t}_\delta, \overline{t}_\delta)\,.$$

By construction,

$$\overline{w} \geq \min_{t\in[0,T]} a(t) \quad \text{in } B^c_{\widehat{R}} \times (\underline{t}_\delta, \overline{t}_\delta)\,.$$

Choose

$$\overline{M} \geq \max\left\{\frac{2\overline{\lambda}\delta}{\min_{t\in[0,T]} G'(a(t))}, \frac{G(K)-G(-\|a\|_\infty)}{V(R)}\right\},$$

and

$$\overline{\lambda} \geq \frac{G(K)-G(-\|a\|_\infty)}{\delta^2}\,.$$

Thanks to arguments analogous to those used above, we can infer that $\overline{w}$ is a supersolution to problem (3.18) when $\underline{t}_\delta > 0$, whereas it is a supersolution to problem (3.19) when $\underline{t}_\delta = 0$. On the other hand, from (3.3), (3.4) and (3.6) we easily deduce that $u_j$ is a subsolution to problem (3.18) when $\underline{t}_\delta > 0$, while it is a subsolution to problem (3.19) when $\underline{t}_\delta = 0$. As in the proof of Theorem 2.3, by means of a compactness argument which makes use of [2, Lemma 5.2] and a diagonal procedure we deduce that there exists a subsequence $\{u_{j_k}\} \subseteq \{u_j\}$ which converges, as $k \to \infty$, locally uniformly in $I\!R^N \times (0,T)$ to a solution $u$ to problem (1.1). We then conclude arguing as in the final part of the proof of Theorem 2.2. □

*Proof of Theorem 2.5 .* First consider a datum $a(t)$ at infinity such that, for some $\varepsilon > 0$, (3.34) holds and $\min_{t\in[0,T]} a(t) - \varepsilon > 0$. Consider then the function $\underline{u}_0$ given in Lemma 3.1 with the choices $\alpha = \min_{t\in[0,T]} a(t) - \varepsilon$, $R$ great enough so that $u_0(x) \geq \min_{t\in[0,T]} a(t) - \varepsilon$ for all $x \in B_R^c$ and $M = \max(0, -\inf_{x\in I\!R^N} u_0(x))$. Clearly, under these assumptions, $u_0(x) \geq \underline{u}_0(x)$ for all $x \in I\!R^N$. Therefore the assertion of Lemma 3.2 holds true and the theorem is proved for such $a(\cdot)$.

If there exists no $\varepsilon > 0$ such that $a(t)$ fulfils (3.34) in the time interval $[0,T]$, we can always find $\varepsilon, \tau > 0$ small enough such that

$$2\,G\left(\min_{s\in[t,(t+\tau)\wedge T]} a(s) - \varepsilon\right) > G\left(\max_{s\in[t,(t+\tau)\wedge T]} a(s)\right) \quad \forall t \in [0,T)\,.$$

This is a consequence of the uniform continuity of $G(a(t))$ and of its strict positivity in $[0,T]$. Hence we get existence in the time interval $[0,\tau]$. Repeating this procedure starting from $t = \tau$ we get existence in the time interval $[\tau, 2\tau \wedge T]$ with initial datum $u(\tau)$ and hence, by Definition 2.1, existence in the time interval $[0, 2\tau\wedge T]$. A finite number of iterations yields the claim. □

*On Remark 2.6.* Note that $(i)$ follows from the same proof of Theorem 2.2, taking $t_0 > 0$, and $0 < \delta < t_0$ in (3.3). As for $(ii)$, it is enough to observe that (2.4) permits to repeat the proof of Theorem 2.5, up to choosing $R > R_0$ and $\tau \leq \epsilon$.

# 4 Uniqueness: proofs

Let $u_1, u_2$ be any two solutions to problem (1.1). Define

$$q(x,t) := \begin{cases} \frac{G(u_1)-G(u_2)}{u_1-u_2} & \text{if } u_1(x,t) \neq u_2(x,t) \\ 0 & \text{if } u_1(x,t) = u_2(x,t) \end{cases}$$

for all $(x,t) \in S_T$. Observe that, in view of $(H_0)(ii)$, $q \geq 0$ in $S_T$ and $q \in L^\infty(S_T)$. Fix $\tau \in (0,T)$. Consider a sequence $\{q_n\} \subseteq C^\infty(S_T)$ such that for every $n \in I\!N$ there hold

$$\frac{1}{n^2} \leq q_n \leq \|q\|_{L^\infty(S_T)} + \frac{1}{n^2} \quad \text{in} \quad Q_{n,\tau} := B_n \times (0,\tau)$$

and

$$\left\|\frac{(q_n - q)}{\sqrt{q_n}}\right\|_{L^2(Q_{n,\tau})} \to 0 \quad \text{as } n \to \infty\,. \tag{4.1}$$

For any $n \in I\!N$, let $\psi_n \in C^2(\overline{Q}_{n,\tau})$ be the unique solution to the backward parabolic problem

$$\begin{cases} \rho\,\partial_t \psi_n + q_n \Delta \psi_n = 0 & \text{in } Q_{n,\tau} \\ \psi_n = 0 & \text{on } \partial B_n \times (0,\tau) \\ \psi_n = \chi & \text{in } \overline{B}_n \times \{\tau\}\,, \end{cases} \tag{4.2}$$

where $\chi \in C^\infty(I\!R^N)$, $0 \le \chi \le 1$ and *supp* $\chi \subseteq B_{n_0}$ for some fixed $n_0 \in I\!N$.

The following lemma will play a central role in the proof of Theorem 2.8.

**Lemma 4.1** *For every $n \in I\!N$ let $\psi_n \in C^2(\overline{Q}_{n,\tau})$ be the unique solution to problem* (4.2). *Then*

$$0 \le \psi_n \le 1 \quad \text{in} \quad Q_{n,\tau}\,. \tag{4.3}$$

*Furthermore, there exists a constant $C > 0$ such that for every $n > n_0$*

$$-\frac{C}{n^{N-1}} \le \langle \nabla \psi_n, \nu_n \rangle \le 0 \quad \text{on } \partial B_n \times (0,\tau), \tag{4.4}$$

*where $\nu_n = \nu_n(\sigma)$ is the outer normal at $\sigma \in \partial B_n$.*

*Proof.* First notice that $\underline{\psi} \equiv 0$ is a subsolution, while $\overline{\psi} \equiv 1$ is a supersolution to problem (4.2), so that by comparison we get (4.3). Now, since

$$\psi_n = 0 \quad \text{in } \partial B_n \times (0,\tau)$$

for all $n \in I\!N$, from (4.3) we deduce that

$$\langle \nabla \psi_n, \nu_n \rangle \le 0 \quad \text{in } \partial B_n \times (0,T)\,. \tag{4.5}$$

For every $n > n_0$ set

$$E_n := B_n \setminus B_{n_0}\,.$$

From (4.3) and the fact that $\operatorname{supp}\chi \subset B_{n_0}$ we infer that, for all $n > n_0$, the function $\psi_n$ is a *subsolution* to problem

$$\begin{cases} \rho\,\partial_t \psi_n + q_n \Delta \psi_n = 0 & \text{in } E_n \times (0,\tau) \\ \psi = 1 & \text{on } \partial B_{n_0} \times (0,\tau) \\ \psi = 0 & \text{on } \partial B_n \times (0,\tau) \\ \psi = 0 & \text{in } \overline{B}_n \times \{\tau\}\,. \end{cases} \tag{4.6}$$

Define

$$z(x) := \widehat{C}\,\frac{|x|^{2-N} - n^{2-N}}{1 - n^{2-N}} \quad \forall x \in E_n\,,$$

where $\widehat{C}$ is a positive constant to be chosen. It is easily seen that, for $\widehat{C} = \widehat{C}(n_0)$ sufficiently large, the function $z$ is a *supersolution* to problem (4.6). Furthermore,

$$\psi_n = z = 0 \quad \text{on } \partial B_n \times (0,\tau);$$

hence,

$$\langle \nabla \psi_n, \nu_n \rangle \ge \langle \nabla z, \nu_n \rangle = \frac{(2-N)\,\widehat{C}\,n^{1-N}}{1 - n^{2-N}} \quad \text{on } \partial B_n \times (0,\tau) \tag{4.7}$$

for all $n > n_0$. From (4.5) and (4.7), (4.4) follows with $C := (N-2)\widehat{C}/(1-n_0^{2-N})$. This completes the proof. $\square$

*Proof of Theorem 2.8.* Let $u_1$, $u_2$ be two bounded solutions to problem (1.1) satisfying

$$\lim_{|x|\to\infty} u_1(x,t) = \lim_{|x|\to\infty} u_2(x,t) = a(t) \quad \text{for almost every } t \in (0,T)\,.$$

Clearly, by dominated convergence, this implies that for any $\tau \in (0,T)$

$$\lim_{R\to\infty} \frac{1}{R^{N-1}} \int_0^\tau \int_{\partial B_R} \left|G\big(u_1(x,t)\big) - G\big(u_2(x,t)\big)\right| d\sigma dt = 0\,. \tag{4.8}$$

Put $w := u_1 - u_2$. By Definition 2.1,

$$\begin{aligned}\int_{\Omega_1} \rho w(x,\tau)\psi(x,\tau)dx = &\int_0^\tau \int_{\Omega_1} \big\{\rho w \psi_t + [G(u_1) - G(u_2)]\Delta\psi\big\} dxdt \\ &- \int_0^\tau \int_{\partial\Omega_1} [G(u_1) - G(u_2)]\langle\nabla\psi, \nu\rangle d\sigma dt\end{aligned} \tag{4.9}$$

for any $\tau$, $\Omega_1$ and $\psi$ as in Definition 2.1.

Moreover, multiplying the first equation in (4.2) by $\Delta\psi_n/\rho$ and integrating by parts we obtain (recall that $\rho \in L^\infty(\mathbb{R}^N)$), for any $n \in \mathbb{N}$,

$$\int_0^\tau \int_{B_n} q_n(\Delta\psi_n)^2 dxdt \leq \widetilde{C} \tag{4.10}$$

for some constant $\widetilde{C} > 0$ independent of $n$.

Taking $\Omega_1 = B_n$ and $\psi = \psi_n$ in (4.9) we get, for any $n \in \mathbb{N}$,

$$\int_{B_n} \rho w(x,\tau)\chi dx = \int_0^\tau \int_{B_n} (q - q_n) w \Delta\psi_n dxdt - \int_0^\tau \int_{\partial B_n} qw\langle\nabla\psi_n, \nu_n\rangle\, d\sigma dt. \tag{4.11}$$

We shall prove that both integrals on the right-hand side of inequality (4.11) tend to 0 as $n \to \infty$. In fact, from (4.1) and (4.10) we have:

$$\begin{aligned}&\left(\int_0^\tau \int_{B_n} (q - q_n) w \Delta\psi_n dxdt\right)^2 \\ \leq &\overline{C} \int_0^\tau \int_{B_n} \left|\frac{q - q_n}{\sqrt{q_n}}\right|^2 dxdt \int_0^\tau \int_{B_n} q_n |\Delta\psi_n|^2 dxdt \to 0 \quad \text{as } \; n \to \infty\,,\end{aligned} \tag{4.12}$$

where $\overline{C} := (\|u_1\|_\infty + \|u_2\|_\infty)^2$.

Moreover, by (4.4) and (4.8), for every $n > n_0$ there holds

$$\begin{aligned}\left|\int_0^\tau \int_{\partial B_n} qw\langle\nabla\psi_n, \nu_n\rangle d\sigma dt\right| &= \left|\int_0^\tau \int_{\partial B_n} \big[G(u_1) - G(u_2)\big]\langle\nabla\psi_n, \nu_n\rangle d\sigma dt\right| \\ &\leq \max_{\partial B_n} |\langle\nabla\psi_n, \nu_n\rangle| \int_0^\tau \int_{\partial B_n} \big|G(u_1) - G(u_2)\big| d\sigma dt \\ &\leq \frac{C}{n^{N-1}} \int_0^\tau \int_{\partial B_n} \big|G(u_1) - G(u_2)\big| d\sigma dt \to 0\end{aligned} \tag{4.13}$$

as $n \to \infty$. Sending $n \to \infty$ in (4.11), from (4.12) and (4.13) it follows that

$$\int_{\mathbb{R}^N} \rho(x)\chi(x)w(x,\tau)dx = 0 \tag{4.14}$$

for any $\tau \in (0,T)$ and any $\chi \in C_c^\infty(I\!R^N)$ with $0 \le \chi \le 1$.

Now fix any compact subset $K \subset I\!R^N$. Define

$$\zeta(x,\tau) := \begin{cases} 1 & \text{if } x \in K \text{ and } w(x,\tau) > 0\,, \\ \\ 0 & \text{elsewhere}\,. \end{cases}$$

Pick a sequence $\{\chi_n\} \subseteq C_c^\infty(I\!R^N)$, with $0 \le \chi_n \le 1$, such that $\chi_n(x) \to \zeta(x)$ as $n \to \infty$ for any $x \in I\!R^N$. In view of (4.14) we deduce that, for any $n \in I\!N$,

$$\int_{I\!R^N} \rho(x)\chi_n(x) w(x,\tau)\,dx = 0\,. \tag{4.15}$$

Letting $n \to \infty$ in (4.15), by dominated convergence we get

$$\int_{K\cap\{w(\cdot,\tau)>0\}} \rho(x)w(x,\tau)dx = 0\,.$$

Hence $w(x,\tau) \le 0$ for any $x \in K$. Since the compact subset $K \subset I\!R^N$ and $\tau \in (0,T)$ are arbitrary, we get

$$w \le 0 \quad \text{in } I\!R^N \times (0,T)\,,$$

that is

$$u_1 \le u_2 \quad \text{in } I\!R^N \times (0,T)\,.$$

Interchanging the role of $u_1$ and $u_2$ we obtain also the opposite inequality, and this completes the proof. □

# References


[1] D. Aronson, M.C. Crandall, L.A. Peletier, *Stabilization of solutions of a degenerate nonlinear diffusion problem*, Nonlin. Anal. TMA **6** (1982), 1001-1022.

[2] E. Di Benedetto, *Continuity of Weak Solutions to a General Porous Medium Equation,* Indiana Univ. Math. J. **32** (1983), 83–118.

[3] E. Di Benedetto, *A Boundary Modulus of Continuity for a Class of Singular Parabolic Equations,* J. Diff. Eq. **63** (1986), 418–447.

[4] J. Dolbeault, I. Gentil, A. Guillin, F.-Y. Wang, *$L^q$-functional inequalities and weighted porous media equations*, Potential Anal. **28** (2008), 35–59.

[5] J. Dolbeault, B. Nazaret, G. Savaré, *On the Bakry-Emery criterion for linear diffusions and weighted porous media equations*, Commun. Math. Sci. **6** (2008), 477–494.

[6] D. Eidus, *The Cauchy problem for the nonlinear filtration equation in an inhomogeneous medium*, J. Differential Equations **84** (1990), 309–318.

[7] D. Eidus, S. Kamin, *The filtration equation in a class of functions decreasing at infinity*, Proc. Amer. Math. Soc. **120** (1994), 825–830.

[8] G. Grillo, M. Muratori, *Sharp short and long time $L^\infty$ bounds for solutions to porous media equations with Neumann boundary conditions*, J. Differential Equations **254** (2013), 2261-2288.

[9] G. Grillo, M. Muratori, M. M. Porzio, *Porous media equations with two weights: existence, uniqueness, smoothing and decay properties of energy solutions via Poincaré inequalities*, Discr. Contin. Dynam. Systems A **33** (2013), 3599-3640.

[10] M. Guedda, D. Hilhorst, M.A. Peletier, *Disappearing interfaces in nonlinear diffusion*, Adv. Math. Sc. Appl. **7** (1997), 695–710.

[11] S. Kamin, G. Reyes, J. L. Vázquez, *Long time behavior for the inhomogeneous PME in a medium with rapidly decaying density*, Discrete Contin. Dyn. Syst. **26** (2010), 521–549.

[12] S. Kamin, M.A. Pozio, A. Tesei, *Admissible conditions for parabolic equations degenerating at infinity*, St. Petersburg Math. J. **19** (2008), 239–251.

[13] S. Kamin, P. Rosenau, *Propagation of thermal waves in an inhomogeneous medium*, Comm. Pure Appl. Math. 34 (1981), 831–852.

[14] S. Kamin, P. Rosenau, *Non-linear diffusion in a finite mass medium*, Comm. Pure Appl. Math. **35** (1982), 113–127.

[15] O.A. Ladyzhenskaya, V.A. Solonnikov, N.A. Uraltseva, "Linear and Quasilinear Equations of Parabolic Type", Nauka, Mocow (1967) (English translation: series Transl. Math. Monographs, **23** AMS, Providence, 1968).

[16] F. Punzo, *On the Cauchy problem for nonlinear parabolic equations with variable density*, J. Evol. Equations **9** (2009), 429-447.

[17] F. Punzo, *Uniqueness and nonuniqueness of bounded solutions to singular nonlinear parabolic equations,* Nonlin. Anal. TMA **70** (2009), 3020–3029.

[18] F. Punzo, *On support of solutions to singular nonlinear parabolic equations in bounded domains*, Interf. Free Bound. **13** (2011), 397–410.

[19] F. Punzo, *Uniqueness and support properties of solutions to singular quasilinear parabolic equations on surfaces of revolution*, Ann. Mat. Pura Appl. **191** (2012), 311–338.

[20] F. Punzo, *Uniqueness and Nonuniqueness of Solutions to Quasilinear Parabolic Equations with a Singular Coefficient on Weighted Riemannian Manifolds*, Asympt. Anal. **79** (2012), 273-301.

[21] G. Reyes, J. L. Vazquez, *The Cauchy problem for the inhomogeneous porous medium equation*, Netw. Heterog. Media **2** (2006), 337–351.

[22] G. Reyes, J. L. Vázquez, *The inhomogeneous PME in several space dimensions. Existence and uniqueness of finite energy solutions*, Commun. Pure Appl. Anal. 7 (2008), 1275–1294.

[23] G. Reyes, J. L. Vázquez, *Long time behavior for the inhomogeneous PME in a medium with slowly decaying density*, Commun. Pure Appl. Anal. 8 (2009), 493–508.

[24] J. L. Vázquez, *The porous medium equation. Mathematical theory*, The Clarendon Press, Oxford University Press, Oxford, 2007.