UNIVERSIDADE DE LISBOA
FACULDADE DE CIÊNCIAS

Nonlocal anisotropic problems with fractional type derivatives

" Documento Definitivo"

Doutoramento em Matemática
Especialidade de Análise Matemática

Catharine Wing Kwan Lo

Tese orientada por:
José Francisco Rodrigues

Documento especialmente elaborado para a obtenção do grau de doutor

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Abstract

In this thesis, for $0 < s < 1$, we consider the nonlocal operator,

$$\langle \mathcal{L}_a u, v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v(x)(u(x) - u(y)) a(x,y) \, dy \, dx, \quad \forall u, v \in H^s_0(\Omega)$$

for a singular kernel $a(x, y)$ satisfying coercivity and boundedness conditions, as well as the fractional operator

$$\langle \tilde{\mathcal{L}}_A u, v \rangle := \int_{\mathbb{R}^d} A(x) D^s u \cdot D^s v \, dx, \quad \forall u, v \in H^s_0(\Omega)$$

defined with the distributional Riesz fractional gradient D^s for a bounded, coercive and measurable matrix A. These operators provide an extension to the fractional Laplacian in two different forms of heterogeneous anisotropy. Their eigenfunctions are studied, as well as other properties including T-monotonicity for $\mathcal{L}_a$. These two operators coincide with the fractional Laplacian in the homogeneous isotropic case, and a connection is also drawn between them in the heterogeneous anisotropic case.

In the second part, these operators are introduced into various nonlocal and fractional partial differential equations, including stationary and evolutionary obstacle-type problems and some nonlinear generalisations, and Stefan-type problems. Results on existence, regularity, and asymptotic behaviour are obtained, and also the convergence to the respective problems with classical derivatives when $s \nearrow 1$, in the case of the $\tilde{\mathcal{L}}_A$ operator and the fractional Laplacian.

Keywords: Fractional and nonlocal derivatives, Fractional Dirichlet and Cauchy boundary value problems, Fractional variational inequalities

2020 Mathematics Subject Classification: 35R11, 35K35, 35R45, 35G46, 80A22

Resumo

Nesta tese, para $0 < s < 1$, consideramos o operador não local, coercivo e limitado,

$$\langle \mathcal{L}_a u, v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v(x)(u(x) - u(y)) a(x,y) \, dy \, dx, \quad \forall u, v \in H^s_0(\Omega)$$

para um kernel singular $a(x, y)$, e o operador fracionário

$$\langle \tilde{\mathcal{L}}_A u, v \rangle := \int_{\mathbb{R}^d} A(x) D^s u \cdot D^s v \, dx, \quad \forall u, v \in H^s_0(\Omega)$$

definido com o gradiente fracionário distribucional de Riesz D^s para uma matriz A com coeficientes limitados, coercivos e mensuráveis. Estes operadores fornecem duas formas diferentes de extensão heterogénea e anisotrópica do Laplaciano fracionário. Estudam-se as suas funções próprias, assim como outras propriedades tais como a T-monotonia de $\mathcal{L}_a$. Estes dois operadores coincidem com o Laplaciano fracionário no caso isotrópico homogéneo, e estabelece-se uma ligação entre eles no caso anisotrópico heterogéneo.

Na segunda parte, estes operadores são introduzidos em diversos problemas com derivadas parciais fracionárias e não locais, tais como problemas do tipo obstáculo estacionários e evolutivos, em generalizações não lineares e em problemas de tipo Stefan. Obtêm-se resultados sobre a existência, regularidade, e comportamento assintótico, e ainda a convergência para os respetivos problemas com derivadas clássicas quando $s \nearrow 1$, no caso do operador $\tilde{\mathcal{L}}_A$ e do Laplaciano fracionário.

Palavras-chave: Derivadas fracionárias e não locais, Problemas fracionários de Dirichlet e de Cauchy, Inequações variacionais fracionárias

Classificação AMS 2020: 35R11, 35K35, 35R45, 35G46, 80A22

Preface

Most parts of this thesis are already available on arXiv. [1]=[163] covers most of Chapters 2, 3 and 4. [2]=[165] will appear in Mathematics in Engineering, and covers the Stefan-type problem in Chapter 6 for the anisotropic fractional case. [3]=[164] gives the content of Chapter 7.

Publications

[1] Catharine W. K. Lo and José Francisco Rodrigues: *On a Class of Nonlocal Obstacle Type Problems Related to the Distributional Riesz Fractional Derivative.* In: *arXiv: 2101.06863* (2021). DOI: 10.48550/ARXIV.2101.06863.

[2] Catharine W. K. Lo and José Francisco Rodrigues: *On an Anisotropic Fractional Stefan-Type Problem with Dirichlet Boundary Conditions.* In: *To appear in Mathematics in Engineering* (2022). DOI: 10.48550/ARXIV.2201.07827.

[3] Catharine W. K. Lo and José Francisco Rodrigues: *Global Existence for Nonlocal Quasilinear Diffusion Systems in Non-Isotropic Non-Divergence Form.* In: *arXiv: 2206.11415* (2022). DOI: 10.48550/ARXIV.2206.11415.

Summary

In this thesis, we consider two definitions of fractional derivatives, the distributional Riesz fractional derivatives and the nonlocal derivatives, and their corresponding fractional and nonlocal operators. These operators are then introduced into various partial differential equations, and results on existence, regularity, asymptotic behaviour are obtained, as well as the convergence to the respective problems with classical derivatives when $s \nearrow 1$ in the case of the fractional operator and the fractional Laplacian.

Chapter 1 discusses the fractional D^s gradient and the nonlocal gradient $\mathcal{D}^s$ for $0 < s < 1$, which form the basis for the subsequent chapters. Some basic important properties are cited. The fractional Sobolev spaces the functions lie in are also introduced. These include the Hilbertian $H^s_0(\Omega)$ space defined for a bounded $\Omega \subset \mathbb{R}^d$ using the fractional D^s derivatives, as well as the classical Sobolev-Slobodeckij spaces $W^{s,p}_0(\Omega)$. A result on the equivalence of various norms related to the different notions of derivatives is given here

$$\|u\|^2_{H^s_0(\Omega)} = \|D^s u\|^2_{L^2(\mathbb{R}^d)} = \frac{c^2_{d,s}}{2}[u]^2_{s,\mathbb{R}^d} = \left\|(-\Delta)^{s/2}u\right\|^2_{L^2(\mathbb{R}^d)} = \frac{c^2_{d,s}}{2}\|\mathcal{D}^s u\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)},$$

corresponding to the H^s_0 norm, the L^2 norm of the Riesz fractional derivative, the $W^{s,2}$ semi-norm, the norm of the half s-fractional Laplacian, and the norm of the nonlocal derivative, respectively. Sobolev embeddings for these spaces are also considered, and the explicit dependence with respect to s of the constants for the fractional Sobolev and Poincaré inequalities are also derived.

Chapter 2 discusses the nonlocal operator $\mathcal{L}^s_a$ given by

$$\mathcal{L}^s_a u = \, P.V. \int_{\mathbb{R}^d} (u(x) - u(y))a(x,y)\,dy, \quad \forall u \in H^s_0(\Omega)$$

for the kernel $a(x,y)$ satisfying some coercivity and boundedness conditions extending the case of the fractional Laplacian. This nonlocal operator is shown to be a closed, coercive, regular (not necessarily symmetric) Dirichlet form, and so, the associated elliptic problem with non-homogeneous Dirichlet boundary conditions has a unique solution. This also allows us to apply the truncation property to obtain the new result that this operator is strictly T-monotone. Next, we give a known result on the continuous dependence of the eigenfunctions of the operator $\mathcal{L}^s_a$ on s, in the special case when $a = \frac{1}{|x-y|^{d+2s}}$ and $\mathcal{L}^s_a$ corresponds to the fractional Laplacian $(-\Delta)^s$.

Chapter 3 introduces the anisotropic fractional operator $\tilde{\mathcal{L}}^s_A$ defined by

$$\langle \tilde{\mathcal{L}}^s_A u, v\rangle := \int_{\mathbb{R}^d} A(x) D^s u \cdot D^s v \, dx, \quad \forall u, v \in H^s_0(\Omega)$$

for a bounded, coercive and measurable matrix A. It is straightforward to see that the associated Dirichlet problem has a solution, and we give some continuous dependence results for the one-parameter problem. Next, we prove a novel result on the stability of the eigenvalues and eigenfunctions for the operator $\tilde{\mathcal{L}}^s_A$ in $H^s_0(\Omega)$ with respect to s including the convergence $s \nearrow 1$. Finally, we relate this fractional operator which makes use of the distributional Riesz fractional derivatives to the nonlocal operator from the previous chapter, by deriving a well-defined kernel k_A. However, this kernel obtained has not been shown to satisfy the conditions required in Chapter 2. We give some counterexamples showing otherwise, as well as an interesting conjecture for the necessary and sufficient conditions on the matrix A such that the derived kernel k_A satisfies the conditions which are sufficient to make $(\tilde{\mathcal{L}}^s_A, H^s_0(\Omega))$ a Dirichlet form.

In Part II, both the fractional and the nonlocal operators are applied to various problems. Firstly, in Chapter 4, we consider nonlocal obstacle-type problems involving the nonlocal operator $\mathcal{L}^s_a$ considered with one obstacle, given by

$$u \geq \psi, \quad \mathcal{L}^s_a u - F \geq 0 \quad \text{and} \quad \langle \mathcal{L}^s_a u - F, u - \psi\rangle = 0,$$

as well as the similarly defined two obstacles problem and the N membranes problem. Several results are derived, such as the weak maximum principle, comparison properties, approximation by bounded penalisation, and also the Lewy-Stampacchia inequalities. This provides regularity of the solutions, including a global estimate in $L^\infty(\Omega)$, local Hölder regularity of the solutions when a is symmetric, and local regularity in fractional Sobolev spaces when $\mathcal{L}^s_a = (-\Delta)^s$ corresponds to fractional s-Laplacian obstacle-type problems.

These novel results are complemented with the extension of the Lewy-Stampacchia inequalities to the order dual of $H^s_0(\Omega)$ and some remarks on the associated s-capacity and the s-fractional obstacle problem. A convergence result is also obtained when the obstacle problem is defined with the fractional operator $\tilde{\mathcal{L}}^s_A$, showing the convergence of the fractional obstacle problem to the classical obstacle problem as $s \nearrow 1$. This is a property intrinsic to the fractional operator, resulting from its definition, which is currently not known for the nonlocal general operator.

Next, in Chapter 5, we extend the results of the previous chapter, by considering the novel nonlocal nonlinear g-Laplacian $\bar{\mathcal{L}}^s_g$ defined as

$$\langle \bar{\mathcal{L}}^s_g u, v\rangle = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g\left(x, y, \frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|x-y|^s}\frac{v(x)-v(y)}{|x-y|^s}\frac{dx\,dy}{|x-y|^d}$$

for a nonlinear kernel $g(x, y, r)$ satisfying some conditions compatible with $H^s_0(\Omega)$, and consider the corresponding elliptic and parabolic obstacle-type problems. In the first part of this chapter, we show the existence and uniqueness results as well as the Lewy-Stampacchia inequalities for the nonlocal nonlinear elliptic obstacle-type problems, namely the one obstacle problem, the two obstacles problem and the N membranes problem. This generalises Chapter 2 to the nonlinear case, and we similarly obtain local regularity of the solutions in the homogeneous case when $g(x, y, r) = g(r)$. Next, we show that in the homogeneous case, the one obstacle problem defined with the nonlocal operator $\bar{\mathcal{L}}^s_g$ converges to the solution of the classical nonlinear elliptic one obstacle corresponding to $s = 1$, concluding the analysis of the elliptic problem. Then, we extend this study to the evolutionary problem, obtaining similarly the existence and uniqueness results, and the Lewy-Stampacchia inequalities for all three obstacle-type problems, as well as local regularity when $\bar{\mathcal{L}}^s_g = \mathcal{L}^s_a$ is the linear nonlocal operator. Finally, we show that these problems converge to the stationary ones.

In Chapter 6, we first consider the non-homogeneous fractional Stefan-type problems with non-zero Dirichlet boundary conditions, given by

$$\frac{\partial}{\partial t}[\beta(\vartheta)] + \tilde{\mathcal{L}}^s_A\vartheta = f \quad \text{ in }]0, T[\times\Omega,$$

for a bounded domain Ω in the finite time interval $[0, T]$, where β is the usual maximal monotone graph for phase transitions. A unique solution is shown to exist. Several properties of this solution are derived, including the convergence as $s \nearrow 1$ as we have previously considered for the fractional obstacle problem, as well as the asymptotic behaviour as $T \to \infty$. Finally, we derive the relationship between the two-phase Stefan-type problem and the one-phase Stefan-type problem by considering the convergence of their solutions as one phase disappears.

These results are similarly obtained when the Stefan-type problems are defined using the nonlocal operator $\mathcal{L}^s_a$. In this case, the operator is further shown to be T-accretive in $L^2(\Omega)$, thereby obtaining a maximum principle for solutions to the nonlocal Stefan-type problems that is not previously obtained for fractional Stefan-type problems. Furthermore, we can extend the property to conclude that $\mathcal{L}^s_a$ is T-accretive for any $L^p(\Omega)$ for all $1 \leq p < 2$. These results only apply to the nonlocal operator, and have not been previously shown anywhere, to the best of our knowledge. A unique mild solution is thereby obtained for $f \in L^1(]0, T[\times\Omega)$.

Last but not least, in Chapter 7, the class of operators considered in Chapter 2 and 3 are considered in a quasilinear diffusion system, for $\boldsymbol{u} = (u^1, \dots, u^m)(t, x)$,

$$\begin{cases} \boldsymbol{u}' + \Pi(t, x, \boldsymbol{u}, \Sigma\boldsymbol{u})\mathbb{A}\boldsymbol{u} = \boldsymbol{f}(t, x, \boldsymbol{u}, \Sigma\boldsymbol{u}) & \text{in }]0, T[\times\Omega,\\ \boldsymbol{u} = \boldsymbol{0} & \text{in }]0, T[\times\Omega^c,\\ \boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0(\cdot) & \text{in } \Omega \end{cases}$$

for an open set $\Omega \subset \mathbb{R}^d$ and any $T \in]0, \infty[$, and $\boldsymbol{u}_0 \in \mathbf{H}^s_0(\Omega) := [H^s_0(\Omega)]^m$, where $\Sigma\boldsymbol{u}(t, x) \in \mathbb{R}^q$ for $0 < q \leq m \times d$ represents nonlocal derivatives or fractional derivatives with order σ with $\sigma < 2s$ for all $0 < s \leq 1$, including the classical gradient and derivatives of order greater than 1. We show global existence results for various quasilinear diffusion systems in non-divergence form, for different linear operators

$\mathbb{A}$, including local elliptic systems, anisotropic fractional equations and systems, and anisotropic nonlocal operators, of the following type

$$(\mathbb{A}\boldsymbol{u})^i = -\sum_{\alpha,\beta,j} \partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta u^j), \quad \mathbb{A}u = -D^s(A(x)D^s u), \quad \text{and} \quad (\mathbb{A}\boldsymbol{u})^i = \int_{\mathbb{R}^d} A_{ij}(x,y) \frac{u^j(x) - u^j(y)}{|x-y|^{d+2s}}\, dy,$$

for coercive, invertible matrices Π and suitable vectorial functions $\boldsymbol{f}$.

Sumário

Nesta tese, consideramos duas definições de derivadas fracionárias, as derivadas fracionárias distribucionais de Riesz e as derivadas não locais, e os seus correspondentes operadores fracionários e operadores não locais. Estes operadores são introduzidos em diversos problemas com derivadas parciais fracionárias e não locais, obtendo-se resultados sobre a existência, regularidade, comportamento assintótico, e ainda a convergência para os respetivos problemas com derivadas clássicas quando $s \nearrow 1$ no caso do operador fracionário e do Laplaciano fracionário.

O Capítulo 1 trata o gradiente fracionário D^s e o gradiente não local $\mathcal{D}^s$ para $0 < s < 1$, que formam a base para os capítulos subsequentes. São revistas algumas propriedades básicas importantes dos espaços Sobolev fracionários, incluindo o espaço Hilbertiano $H_0^s(\Omega)$ para um domínio limitado $\Omega \subset \mathbb{R}^d$, definido usando as derivadas fracionárias D^s, assim como os clássicos espaços de Sobolev-Slobodeckij $W_0^{s,p}(\Omega)$. Damos um resultado sobre a equivalência de várias normas relacionadas com as diferentes noções das derivadas

$$\|u\|^2_{H_0^s(\Omega)} = \|D^s u\|^2_{L^2(\mathbb{R}^d)} = \frac{c_{d,s}^2}{2}[u]^2_{s,\mathbb{R}^d} = \left\|(-\Delta)^{s/2}u\right\|^2_{L^2(\mathbb{R}^d)} = \frac{c_{d,s}^2}{2}\|\mathcal{D}^s u\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)},$$

correspondendo à norma H_0^s, a norma L^2 da derivada fracionária de Riesz, a norma $W^{s,2}$ semi-norma, a norma do $s/2$-Laplaciano fracionário, e a norma da derivada não local, respetivamente. Consideramos também as inclusões de Sobolev para estes espaços, bem como obtemos a dependência explícita relativamente a s das constantes para as desigualdades fracionárias de Sobolev e Poincaré.

No Capítulo 2 consideramos o operador não local, coercivo e limitado $\mathcal{L}_a^s$ dado por

$$\mathcal{L}_a^s u = P.V.\int_{\mathbb{R}^d}(u(x)-u(y))a(x,y)\,dy, \quad \forall u \in H_0^s(\Omega)$$

para um kernel singular $a(x,y)$ estendendo o caso do Laplaciano fracionário. Demonstramos que este operador não local define uma forma de Dirichlet fechada, coerciva, regular (não necessariamente simétrica). Deste modo, o problema elíptico associado com condições de fronteira não homogéneas de Dirichlet tem uma solução única. Isto permite-nos também aplicar a propriedade de truncatura para obter o novo resultado de que este operador é estritamente T-monótono. Em seguida, damos um resultado conhecido sobre a dependência contínua das funções próprias do operador $\mathcal{L}_a^s$ em s, no caso especial em que $a = \frac{1}{|x-y|^{d+2s}}$ e $\mathcal{L}_a^s$ corresponde ao Laplaciano fracionário $(-\Delta)^s$.

O Capítulo 3 introduz o operador fracionário anisotrópico $\tilde{\mathcal{L}}_A^s$ definido por

$$\langle \tilde{\mathcal{L}}_A^s u, v\rangle := \int_{\mathbb{R}^d} A(x)D^s u \cdot D^s v\,dx, \quad \forall u, v \in H_0^s(\Omega)$$

para uma matriz A com coeficientes limitados, coercivos e mensuráveis. É fácil ver que o problema de Dirichlet associado tem uma solução, e damos alguns resultados de dependência contínua para o problema com um parâmetro. A seguir, provamos um resultado novo sobre a estabilidade dos valores próprios e das funções próprias do operador $\tilde{\mathcal{L}}_A^s$ em $H_0^s(\Omega)$ em relação a s, incluindo a convergência $s \nearrow 1$. Finalmente, relacionamos este operador fracionário, que usa as derivadas fracionárias de Riesz, com o operador não local definido no capítulo anterior, obtendo um kernel bem definido k_A. No entanto, este kernel não satisfaz as condições exigidas no Capítulo 2. Damos alguns contra-exemplos mostrando o contrário, assim como uma conjetura para as condições necessárias e suficientes na matriz A, de modo que o kernel derivado k_A satisfaça as condições suficientes para fazer $(\tilde{\mathcal{L}}_A^s, H_0^s(\Omega))$ uma forma de Dirichlet.

Na Parte II, aplicam-se tanto os operadores fracionários como os não locais a diversos problemas. Em primeiro lugar, no Capítulo 4, consideramos problemas do tipo obstáculo envolvendo o operador não local $\mathcal{L}_a^s$ com um obstáculo, dado por

$$u \geq \psi, \quad \mathcal{L}_a^s u - F \geq 0 \quad \text{e} \quad \langle \mathcal{L}_a^s u - F, u - \psi\rangle = 0,$$

assim como o problema dos dois obstáculos definidos de forma semelhante e o problema das N membranas. Obtêm-se vários resultados, tais como o princípio do máximo fraco, propriedades de comparação, aproximação por penalização limitada, e também as desigualdades de Lewy-Stampacchia. Isto fornece regularidade

das soluções, incluindo uma estimativa global em $L^\infty(\Omega)$, regularidade local Hölderiana das soluções quando a é simétrico, e regularidade local em espaços de Sobolev fracionários quando $\mathcal{L}_a^s = (-\Delta)^s$ corresponde a problemas fracionários do tipo obstáculo com o s-Laplaciano. Complementam-se estes novos resultados com a extensão das desigualdades de Lewy-Stampacchia ao dual de order de $H_0^s(\Omega)$ e algumas observações sobre a capacidade associada ao problema do obstáculo s-fracionário. Também se obtém um resultado de convergência quando o problema do obstáculo é definido com o operador $\tilde{\mathcal{L}}_A^s$, mostrando a convergência do problema fracionário para o clássico problema do obstáculo quando $s \nearrow 1$. Esta é uma propriedade intrínseca do operador fracionário $\tilde{\mathcal{L}}_A^s$, que não é ainda conhecida para o operador não local $\mathcal{L}_a^s$.

No Capítulo 5, estendemos os resultados do capítulo anterior, considerando os novos g-Laplacianos $\bar{\mathcal{L}}_g^s$ não lineares e não locais definidos por

$$\langle \bar{\mathcal{L}}_g^s u, v\rangle = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g\left(x, y, \frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|x-y|^s}\frac{v(x)-v(y)}{|x-y|^s}\frac{dx\,dy}{|x-y|^d}$$

para um kernel não linear $g(x,y,r)$ satisfazendo algumas condições de compatibilidade com $H_0^s(\Omega)$, e consideramos os correspondentes problemas elípticos e parabólicos com obstáculos. Na primeira parte deste capítulo, mostramos os resultados de existência e unicidade, assim como as desigualdades de Lewy-Stampacchia para os problemas não locais e não lineares do tipo obstáculo, especificamente o problema de um obstáculo, o problema dos dois obstáculos e o problema das N membranas. Isto generaliza o Capítulo 2 para o caso não linear e, de modo análogo, obtemos a regularidade local das soluções, no caso homogéneo, quando $g(x,y,r) = g(r)$. A seguir, mostramos que no caso homogéneo, o problema de um obstáculo definido com o operador não local $\bar{\mathcal{L}}_g^s$ converge para a solução do problema clássico não linear do obstáculo correspondente a $s = 1$, concluindo a análise do problema elíptico. Seguidamente, estendemos este estudo ao problema evolutivo, obtendo de forma semelhante os resultados de existência e unicidade, as desigualdades de Lewy-Stampacchia para os três problemas do tipo obstáculo, assim como a regularidade local quando $\bar{\mathcal{L}}_g^s = \mathcal{L}_a^s$ é o operador linear não local. Finalmente, mostramos que estes problemas convergem para os respetivos problemas estacionários.

No Capítulo 6, são considerados os problemas não homogéneos do tipo Stefan fracionário com condições de fronteira não homogéneas de Dirichlet,

$$\frac{\partial}{\partial t}[\beta(\vartheta)] + \tilde{\mathcal{L}}_A^s\vartheta = f \qquad \text{em }]0,T[\times\Omega,$$

para um domínio limitado Ω no intervalo de tempo finito $[0,T]$, onde β denota o gráfico máximal monótono usual para as transições de fase. Obtém-se a existência de uma solução única. Obtêm-se também várias propriedades desta solução, incluindo a convergência quando $s \nearrow 1$, de uma forma semelhante ao que foi considerado anteriormente para o problema do obstáculo fracionário, assim como o comportamento assintótico quando $T \to \infty$. Finalmente, obtemos a relação entre o problema de tipo Stefan com duas fases e o problema de tipo Stefan com uma fase, considerando a convergência das respetivas soluções quando uma das fases desaparece.

Obtêm-se resultados semelhantes para os problemas do tipo Stefan utilizando o operador não local $\mathcal{L}_a^s$. Neste caso, o operador sendo T-acretivo em $L^2(\Omega)$, permite obter um princípio de comparação de soluções para os problemas de tipo Stefan não local que não foi possível obter para os problemas de tipo Stefan fracionário. Além disso, podemos estender a propriedade de $\mathcal{L}_a^s$ ser T-accretivo em qualquer $L^p(\Omega)$ para todos os $1 \leq p < 2$. Estes novos resultados só se aplicam ao operador não local e permitem obter assim uma solução suave única para $f \in L^1(]0,T[\times\Omega)$.

Por último, no Capítulo 7, os operadores considerados nos Capítulos 2 e 3 são retomados num sistema de difusão quasilinear para $\boldsymbol{u} = (u^1, \ldots, u^m)(t,x)$,

$$\begin{cases} \boldsymbol{u}' + \Pi(t,x,\boldsymbol{u},\Sigma\boldsymbol{u})\mathbb{A}\boldsymbol{u} = \boldsymbol{f}(t,x,\boldsymbol{u},\Sigma\boldsymbol{u}) & \text{em }]0,T[\times\Omega,\\ \boldsymbol{u} = \boldsymbol{0} & \text{em }]0,T[\times\Omega^c,\\ \boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0(\cdot) & \text{em } \Omega \end{cases}$$

num conjunto aberto $\Omega \subset \mathbb{R}^d$, com $T \in]0,\infty]$ e $\boldsymbol{u}_0 \in \mathbf{H}_0^s(\Omega) := [H_0^s(\Omega)]^m$, onde $\Sigma\boldsymbol{u}(t,x) \in \mathbb{R}^q$ para $0 < q \leq m \times d$ representa derivadas não locais ou derivadas fracionárias com ordem σ com $\sigma < 2s$ para todos os $0 < s \leq 1$, incluindo o gradiente clássico e derivadas de ordem superior a 1. Mostramos resultados de existência global para vários sistemas de difusão quasilineares em forma não-divergente, para diferentes operadores lineares $\mathbb{A}$, incluindo sistemas elípticos locais, equações e sistemas anisotrópicas fracionárias, e operadores anisotrópicos não locais, do seguinte tipo

$$(\mathbb{A}\boldsymbol{u})^i = -\sum_{\alpha,\beta,j} \partial_\alpha (A_{ij}^{\alpha\beta} \partial_\beta u^j), \quad \mathbb{A}u = -D^s(A(x)D^s u), \quad \text{e} \quad (\mathbb{A}\boldsymbol{u})^i = \int_{\mathbb{R}^d} A_{ij}(x,y) \frac{u^j(x) - u^j(y)}{|x-y|^{d+2s}} \, dy,$$

para matrizes coercivas e invertíveis Π e funções vectoriais adequadas $\boldsymbol{f}$.

Contents

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List of Abbreviations, Acronyms and Symbols

Symbols

$:=$	Definition of a left-hand-side by a right-hand-side expression
$[\cdot]_V$	Semi-norm on a Banach space V
$\langle \cdot, \cdot \rangle$	Bilinear pairing of spaces in duality
$\mapsto$	Mapping of elements into other ones
$\nearrow, \searrow$	Convergence on $\mathbb{R}$ from the left, right
$\|\cdot\|_V$	Norm on a Banach space V
$\to, \rightharpoonup$	Strong, weak convergence

Constants and Standard Functions

$\cdot'$	Time derivative
$\cdot^+, \cdot^-$	Positive, negative parts of a function
$\cdot^{-1}, \cdot^T$	Inverse, transpose
$\cdot^{sym}, \cdot^{anti}$	Symmetric, anti-symmetric parts of a function/map/matrix
$\chi_\epsilon(x,z)$	Characteristic function of the set $\{\|z - x\| > \epsilon\}$ for $\epsilon > 0$
lim	Limit
ω_n	Volume of the n-dimensional unit ball
$\sup, \inf$	Supremum, infimum
$\vee, \wedge$	Maximum, minimum
$c_{d,s}$	Constant of the s-fractional gradient
$m\{\cdot\}$	Lebesgue measure of a set
p'	Hölder conjugate of p
p^*	Sobolev conjugate of p
$p^\#$	Exponent in the trace operator $u \mapsto u\|_{\partial\Omega} : W^{s,p}(\Omega) \to L^{p^\#}(\partial\Omega)$
$sign, sign^+$	Signum, Heaviside/signum$^+$ function
$supp(u)$	Support of a function u, i.e. the closure of $\{x \in \Omega : u(x) \neq 0\}$

Operators

$(-\Delta)^s$	Fractional Laplacian
$-\Delta$	Classical Laplacian
$\frac{\partial}{\partial x_i}$ or ∂_i	Classical gradient in x_i
$\mathbb{L}$	Classical vectorial elliptic operator
$\mathbb{L}_A$	Nonlocal vectorial operator
$\mathcal{D}^s, \mathcal{D}_s$	Nonlocal gradient, divergence
$\mathcal{L}_a$ or $\mathcal{L}_a^s$	Nonlocal operator
$\mathcal{R}_j$	Riesz transform
∇	Classical gradient
$\partial\Phi$	Subdifferential of a convex functional Φ
$\tilde{\mathcal{L}}_a$	Fractional operator
$D^s, D^s\cdot$	Distributional Riesz fractional gradient, divergence

Sets and Spaces

$\bar{\Omega}$	Closure of Ω
$\mathbb{N}, \mathbb{R}$	Natural numbers, real numbers
$\mathbb{R}^+, \mathbb{R}^+_0$	Positive, non-negative real numbers
$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathcal{L}(V_1, V_2)$	Space of linear continuous mappings $A : V_1 \to V_2$ with norm $\sup_{\|v\|_{V_1} \leq 1} \|Av\|_{V_2}$
$\partial\Omega$	Boundary of the set Ω
$C^k(\bar{\Omega})$	Space of functions whose derivatives up to order k are all continuous on $\bar{\Omega}$
$C^{0,1}(\Omega)$	Space of Lipschitz functions on Ω
$C^\infty_c(\Omega)$	Space of smooth functions with compact support in Ω
$D(A)$	Domain of the mapping A
$H^1(0,T;V)$	Sobolev space of functions that are $H^1(0,T)$ in time and lie in V in space
$H^s(\Omega)$	Fractional Hilbertian Sobolev space
$H^s_0(\Omega)$	Completion of $C^\infty_c(\Omega)$ with respect to the norm of $H^s(\mathbb{R}^d)$
$H^{-s}(\Omega)$	Dual of $H^s_0(\Omega)$
$L^2(0,T;V)$	Sobolev space of functions that are $L^2(0,T)$ in time and lie in V in space
$L^p(\Omega)$	Lebesgue space of p-integrable functions on Ω, with norm $\|u\|^p_{L^p(\Omega)} = \int_\Omega \|u\|^p$
V^*	Topological dual space of V
V^m or $\boldsymbol{V}$	m-dimensional Banach space V
$V_{loc}(\Omega)$	The set of functions v on Ω whose restrictions to $\mathcal{O}$ lie in $V(\mathcal{O})$ for any open $\mathcal{O}$ such that $\bar{\mathcal{O}} \subset \Omega$
$W^{k,p}(\Omega)$	Sobolev spaces with order $k \in \mathbb{N}$
$W^{s,p}(\Omega)$	Sobolev-Slobodeckij spaces with order $s \in]0,1[$, with semi-norm $[\cdot]_{s,\Omega}$

Part I

Introduction to the Nonlocal and Fractional Operators

1 Preliminaries

1.1 Definitions

In this thesis, we consider the Riesz fractional derivative and the nonlocal gradient, which are the basis of generalisations of the well-known fractional Laplacian in two different ways. The motivation for considering these two types of derivatives is that, not only can they be used in multi-dimensional situations, but they are also not subject to the problems of commonly used fractional gradient operators. For instance, the commonly considered Riemann-Liouville derivative is not 0 when applied on non-zero constant functions, with its kernel functions being of the form $(x-a)^{s-1}$. The other commonly used derivative is the Caputo derivative, which requires at least one derivative to exist. This means that the function is required to be at least H^1, and not just H^s.

1.1.1 The Distributional Riesz Fractional Gradient

We consider first the *distributional Riesz fractional gradient* D^s of order $s \in]0,1[$: for $u \in L^p(\mathbb{R}^d)$, $p \in]1,\infty[$, we set

$$D_j^s u = \frac{\partial^s u}{\partial x_j^s} = \frac{\partial}{\partial x_j}(I_{1-s} * u), \quad 0 < s < 1, \quad j = 1, \dots, d, \tag{1.1}$$

where $\frac{\partial}{\partial x_j}$ is taken in the distributional sense, for every $v \in C_c^\infty(\mathbb{R}^d)$,

$$\left\langle \frac{\partial^s u}{\partial x_j^s}, v \right\rangle = -\left\langle (I_{1-s} * u), \frac{\partial v}{\partial x_j} \right\rangle = -\int_{\mathbb{R}^d} (I_{1-s} * u) \frac{\partial v}{\partial x_j} \, dx,$$

with I_s denoting the Riesz potential of order s, $0 < s < 1$:

$$(I_s * u)(x) = \gamma_{d,s} \int_{\mathbb{R}^d} \frac{u(y)}{|x-y|^{d-s}} \, dy, \quad \text{where} \quad \gamma_{d,s} = \frac{\Gamma\left(\frac{d-s}{2}\right)}{2^s \pi^{\frac{d}{2}} \Gamma\left(\frac{s}{2}\right)}. \tag{1.2}$$

Conversely, by Theorem 1.12 of [213], every $u \in C_c^\infty(\Omega)$ can be expressed as

$$u = I_s * \sum_{j=1}^{d} \mathcal{R}_j \frac{\partial^s u}{\partial x_j^s}, \tag{1.3}$$

where $\mathcal{R}_j$ is the Riesz transform, which we recall, is given by

$$\mathcal{R}_j f(x) := \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \lim_{\epsilon \to 0} \int_{\{|y|>\epsilon\}} \frac{y_j}{|y|^{d+1}} f(x-y) \, dy, \quad j = 1, \dots, d.$$

We can also write the s-gradient (D^s) and the s-divergence ($D^s\cdot$) for sufficiently regular functions u and vectors ϕ ([74, 213, 214, 216]) in integral form, respectively, by

$$D^s u(x) := c_{d,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \frac{z u(x+z)}{|z|^{d+s+1}} \chi_\epsilon(0,z) \, dz = c_{d,s} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+s}} \frac{x-y}{|x-y|} \, dy \tag{1.4}$$

and

$$D^s \cdot \phi(x) := c_{d,s} \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \frac{z \cdot \phi(x+z)}{|z|^{d+s+1}} \chi_\epsilon(0,z) \, dz = c_{d,s} \int_{\mathbb{R}^d} \frac{\phi(x)-\phi(y)}{|x-y|^{d+s}} \cdot \frac{x-y}{|x-y|} \, dy \tag{1.5}$$

for the constant

$$c_{d,s} = 2^s \pi^{-\frac{d}{2}} \frac{\Gamma\left(\frac{d+s+1}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)},$$

where $\chi_\epsilon(x,z)$ is the characteristic function of the set $\{(x,z) : |z-x| > \epsilon\}$ for $\epsilon > 0$. The second form is obtained from the first form by making use of the property

$$\int_{\mathbb{R}^d} \frac{z}{|z|^{d+s+1}} \chi_\epsilon(0,z)\, dz = 0, \quad \forall \epsilon > 0.$$

This property also guarantees that $D^s c = 0$ for all $c \in \mathbb{R}$. Sometimes, for brevity, we just replace χ_ϵ and $\lim_{\epsilon\to 0}$ by the term *P.V.* signifying the principal value of the integral.

The above fractional operators are dual, in the sense that

$$\int_{\mathbb{R}^d} u(D^s \cdot \phi)\, dx = -\int_{\mathbb{R}^d} \phi \cdot (D^s u)\, dx, \tag{1.6}$$

by Lemma 2.5 of Comi-Stefanie [74], for $u \in C_c^\infty(\mathbb{R}^d)$, $\phi \in [C_c^\infty(\mathbb{R}^d)]^d$, and by density, for functions in $H^s(\mathbb{R}^d)$.

These fractional operators were first considered by Silhavy [216], who observed that they have the advantage of satisfying basic physical invariance requirements, and developed a fractional vector calculus for such operators. These operators are later further developed by Shieh, Spector, Comi, Stefani, and many others (see for instance, [213, 214], [206], [74, 73],[31]. Indeed, this definition of the fractional derivative does not depend on the chosen basis, and is translationally invariant, rotationally invariant, and homogeneous of degree s under isotropic scaling. Moreover, the fractional derivatives are well-defined and continuous operators in the Schwartz space. On the other hand, since the Riesz kernel is an approximation to the identity as $1-s \to 0$, the s-derivatives approach the classical derivatives as $s \nearrow 1$, i.e.

$$D^s u \to Du,$$

as it was observed via Fourier transform in the proof of Lemma 1.5 , for $u \in H^1(\mathbb{R}^d)$ and $u \in H_0^1(\Omega)$, as well as in Rodrigues-Santos [193], Comi-Stefani [73] and Bellido et al [31].

Moreover, as it was shown in [213] and [216], D^s has nice properties for $u \in C_c^\infty(\mathbb{R}^d)$, namely it coincides with the fractional Laplacian as follows:

$$(-\Delta)^s u = -D^s \cdot D^s u,$$

where, for $0 < s < 1$,

$$(-\Delta)^s u(x) = c_{d,s}^2 \lim_{\epsilon\to 0} \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}} \chi_\epsilon(x,y)\, dy = -\frac{1}{2} c_{d,s}^2 \int_{\mathbb{R}^d} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2s}}\, dy. \tag{1.7}$$

Observe that for the s-gradient (D^s) and the s-divergence ($D^s\cdot$) in (1.4)–(1.5), we need to consider the Cauchy principal value (P.V.) in the first expressions, but not in the second ones. This is because for the second expressions, we can estimate the integrand, as in [74], by separating the integrals into the parts $\{|y-x| \le 1\}$ and $\{|y-x| > 1\}$. Then the first integral can be controlled by $\omega_{d-1}\, Lip(u) \int_0^1 r^{-s}\, dr$, while the second integral can be controlled by $2\omega_d \|u\|_{L^\infty(\mathbb{R}^d)} \int_1^{+\infty} r^{-(1+s)}\, dr$, where $Lip(u)$ is the Lipschitz constant for the function u and ω_d is the spherical measure $\omega_{d-1} = \int_{\{|x|=1\}} d\sigma$. Therefore, the second expressions in (1.4)–(1.5) are well-defined for all Lipschitz functions u with compact support, in particular for $u \in C_c^\infty(\mathbb{R}^d)$.

Similarly, the fractional Laplacian for smooth u, by Lemma 3.2 of [95], has two representations, with the second one being Lebesgue integrable by using a second order Taylor expansion. As a matter of fact, the s-divergence, s-gradient and s-Laplacian are linear operators from C^∞ functions with compact support into C^∞ functions that are rapidly decreasing at ∞ and are in $L^p(\mathbb{R}^d)$ for any $p \in [1,\infty]$.

Observe that for $u, v \in C_c^\infty(\mathbb{R}^d)$, by using the duality between s-divergence and s-gradient as in (1.6) and (1.7), we have

$$\begin{aligned}
&2\int_{\mathbb{R}^d} D^s u \cdot D^s v = 2\int_{\mathbb{R}^d} v(-\Delta)^s u \\
&= c_{d,s}^2 \left[\int_{\mathbb{R}^d} \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} v(y)\frac{u(y)-u(x)}{|x-y|^{d+2s}}\chi_\epsilon(x,y)\,dx\,dy + \int_{\mathbb{R}^d} \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} v(x)\frac{u(x)-u(y)}{|x-y|^{d+2s}}\chi_\epsilon(x,y)\,dy\,dx\right] \\
&= c_{d,s}^2 \left[\lim_{\epsilon\to 0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(y)\frac{u(y)-u(x)}{|x-y|^{d+2s}}\chi_\epsilon(x,y)\,dx\,dy + \lim_{\epsilon\to 0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(x)\frac{u(x)-u(y)}{|x-y|^{d+2s}}\chi_\epsilon(x,y)\,dx\,dy\right] \\
&= c_{d,s}^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}}\,dx\,dy,
\end{aligned} \tag{1.8}$$

where we have used the above definitions together with the Lebesgue and Fubini theorems.

1.1.2 The Nonlocal Gradient

The second type of fractional derivatives we consider, is the *nonlocal gradient* $\mathcal{D}^s$ and *nonlocal divergence* $\mathcal{D}_s$ (as used in [109], see also [80, 100, 101, 129]) by

$$\mathcal{D}^s u(x,y) := \frac{u(x)-u(y)}{|x-y|^{\frac{d}{2}+s}}, \quad \mathcal{D}_s\phi(x) := \ P.V.\int_{\mathbb{R}^d} \frac{\phi(x,y)-\phi(y,x)}{|x-y|^{\frac{d}{2}+s}}\,dy \tag{1.9}$$

for $u \in H^s(\mathbb{R}^d)$ and $\phi \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$. Sometimes, this nonlocal gradient is also known as the unweighted gradient, in contrast to the weighted derivative (1.4) which are convolutions with a singular weight.

Observe that if $u \in H^s(\mathbb{R}^d)$, $\mathcal{D}^s u \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, so it is well-defined. Furthermore, as with the Riesz fractional gradients, $\mathcal{D}^s$ also coincides with the fractional Laplacian as follows:

$$(-\Delta)^s u = \frac{1}{2}c_{d,s}^2 \mathcal{D}_s(\mathcal{D}^s u)$$

for $0 < s < 1$, and the nonlocal operators are dual in the following sense (see Definition 3.2 of [101] or Theorem 3.1 of [109]), provided $\mathcal{D}_s\phi \in L^2(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} u(\mathcal{D}_s\phi)\,dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\mathcal{D}^s u)\phi\,dx\,dy. \tag{1.10}$$

1.2 Fractional Sobolev Spaces

The fractional Sobolev spaces $H^s(\mathbb{R}^d)$ for all real positive s are defined by

$$H^s(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : \{\xi \mapsto (1+|\xi|^2)^{s/2}\hat{u}(\xi)\} \in L^2(\mathbb{R}^d)\}, \tag{1.11}$$

with norm

$$\|u\|_{H^s(\mathbb{R}^d)} = \left\|(1+|\xi|^2)^{s/2}\hat{u}\right\|_{L^2(\mathbb{R}^d)},$$

and its dual space

$$H^{-s}(\mathbb{R}^d) := \{\xi \in S'(\mathbb{R}^d) : \{1+|\xi|^{-s}\hat{\xi}\} \in L^2(\mathbb{R}^d)\}, \tag{1.12}$$

where S is the Schwartz space and S' the dual, and $\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}u(x)\,dx$ is the Fourier transform of u. For $0 < s < 1$, this norm is well known to be equivalent to

$$\|u\|_{H^s(\mathbb{R}^d)}^2 = \|u\|_{L^2(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,dx\,dy =: \|u\|_{L^2(\mathbb{R}^d)}^2 + [u]_{H^s(\mathbb{R}^d)}^2. \tag{1.13}$$

On the other hand, as it was shown in [213] and (1.8), the $H^s(\mathbb{R}^d)$-norm given by (1.13) is in fact equal to

$$\|u\|_{H^s(\mathbb{R}^d)}^2 = \|u\|_{L^2(\mathbb{R}^d)}^2 + \frac{2}{c_{d,s}^2}\int_{\mathbb{R}^d} |D^s u|^2 = \|u\|_{L^2(\mathbb{R}^d)}^2 + \frac{2}{c_{d,s}^2}\|D^s u\|_{L^2(\mathbb{R}^d)}^2. \tag{1.14}$$

For other equivalent spaces, see Section 2 of [213]. Then, if Ω has Lipschitz boundary, hence satisfying the extension property, $H^s(\Omega)$ coincides with the space of restrictions to Ω of the elements of $H^s(\mathbb{R}^d)$ as in [162] and [93], with norm

$$\|u\|_{H^s(\Omega)} = \inf_{U=u \text{ a.e. } \Omega} \|U\|_{H^s(\mathbb{R}^d)} . \tag{1.15}$$

Next, we define the subspace $H_0^s(\Omega)$ to be the usual fractional Sobolev space, for $0 < s \le 1$, given by the closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$ for general open sets $\Omega \subset \mathbb{R}^d$, as in [162], and $H^{-s}(\Omega)$ its dual. Since $C_c^\infty(\Omega)$ is dense in $H^s(\Omega)$ if and only if $s \le \frac{1}{2}$, in this case, $H_0^s(\Omega) = H^s(\Omega)$. Otherwise, if $s > \frac{1}{2}$, $H_0^s(\Omega)$ is strictly contained in $H^s(\Omega)$. On the other hand, as in [93], for bounded sets with Lipschitz boundary, $\mathcal{O} \subset \mathbb{R}^d$, $C_c^\infty(\bar{\mathcal{O}})$ is dense in $H^s(\mathcal{O})$ for all $s \ge 0$.

This can be further extended for $s > 1$, by an abuse of notation, by defining $H_0^s(\Omega)$ to be the space

$$H_0^s(\Omega) := \{u \in H^s(\mathbb{R}^d) : \text{ supp } u \subset \bar{\Omega}\}.$$

It should also be noted (see Theorem 1.11.5 of [162]) that for Ω bounded with C^∞ boundary, and $s > \frac{1}{2}$,

$$u \in H_0^s(\Omega) \quad \iff \quad u \in H^s(\Omega) \text{ and } \frac{\partial^j u}{\partial \nu^j} = 0, \quad \forall 0 \le j < s - \frac{1}{2},$$

where $\frac{\partial^j u}{\partial \nu^j}$ is the normal j-th order derivative of u on $\partial\Omega$, oriented towards the interior of Ω.

Next, we recall the fractional Sobolev-Slobodeckij spaces $W_0^{s,p}(\Omega)$ and its norm

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : [u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dx dy \right)^{\frac{1}{p}} < +\infty \right\},$$

for $0 < s < 1$, with the natural norm

$$\|u\|^p_{W^{s,p}(\Omega)} := \|u\|^p_{L^p(\Omega)} + [u]^p_{W^{s,p}(\Omega)}.$$

In the case of $p = 2$ and the set Ω considered being all of $\mathbb{R}^d$, $W^{s,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$, we write the seminorm as

$$[\cdot]_{W^{s,2}(\mathbb{R}^d)} = [\cdot]_{s,\mathbb{R}^d}.$$

Owing to (1.8), we have the following equality of norms

$$\|D^s u\|^2_{L^2(\mathbb{R}^d)} = \frac{c^2_{d,s}}{2} [u]^2_{s,\mathbb{R}^d} = \left\| (-\Delta)^{s/2} u \right\|^2_{L^2(\mathbb{R}^d)} = \frac{c^2_{d,s}}{2} \|\mathcal{D}^s u\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} , \tag{1.16}$$

for $(-\Delta)^{s/2}$ as defined in [95], where u is extended by 0 in $\mathbb{R}^d \backslash \Omega$, so that this extension is also in $H^s(\mathbb{R}^d)$. Therefore, our definition of $H_0^s(\Omega)$ above is equivalent to $W_0^{s,2}(\Omega)$, which is given by the closure in $W^{s,2}(\mathbb{R}^d)$ of all smooth functions having a compact support contained in Ω. Furthermore, by the Sobolev-Poincaré inequality (see Theorem 1.7 of [213] and Lemma 1.3 below), we may consider the space $H_0^s(\Omega)$ with the following equivalent norms,

$$\|u\|^2_{H_0^s(\Omega)} := \|D^s u\|^2_{L^2(\mathbb{R}^d)} = \frac{c^2_{d,s}}{2} [u]^2_{s,\mathbb{R}^d} := \frac{c^2_{d,s}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x-y|^{d+2s}} \, dx \, dy. \tag{1.17}$$

We can subsequently denote the dual space of $H_0^s(\Omega)$ by $H^{-s}(\Omega)$ for $0 < s \le 1$. Then, by the Sobolev-Poincaré inequalities, we have the embeddings

$$H_0^s(\Omega) \hookrightarrow L^q(\Omega), \qquad L^{2^\#}(\Omega) \hookrightarrow H^{-s}(\Omega) = (H_0^s(\Omega))'$$

for $1 \le q \le 2^*$, where $2^* = \frac{2d}{d-2s}$ and $2^\# = \frac{2d}{d+2s}$ when $s < \frac{d}{2}$, and if $d = 1$, $2^* = q$ for any finite q and $2^\# = q' = \frac{q}{q-1}$ when $s = \frac{1}{2}$ and $2^* = \infty$ and $2^\# = 1$ when $s > \frac{1}{2}$. We recall that those embeddings are compact for $1 \le q < 2^*$ (see for example, Theorem 4.54 of [93]). In the whole thesis, we use $2^\#$ to indicate this number that depends on $d \ge 1$ and $0 < s \le 1$.

Using the norm given in (1.11), we can also equivalently define the $H_0^s(\Omega)$ with norm

$$\|u\|_{H_0^s(\Omega)}^2 := \iint_{D_\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\, dx\, dy, \tag{1.18}$$

where $D_\Omega = \mathbb{R}^d \times \mathbb{R}^d \backslash (\Omega^c \times \Omega^c)$. The pair $(H_0^s(\Omega), \|\cdot\|_{H_0^s(\Omega)})$ yields a Hilbert space (see for instance Lemma 7 of [211] for more details). Note that this norm is different from the classical Sobolev-Slobodeckij norm, where the integral is only taken over $\Omega \times \Omega$. This norm is important since we are commonly conducting the integration over all $\mathbb{R}^d$ for nonlocal and fractional operators, rather than just in Ω.

Naturally, we can extend this definition to all $H_0^{s,p}(\Omega)$, defined by the closure of all functions in $C_c^\infty(\Omega)$ with respect to the norm $\|D^s(\cdot)\|_{L^p(\mathbb{R}^d)}$.

We next recall the following useful inequalities.

Lemma 1.1 (Sobolev-Poincaré inequality, Theorem 1.8 of [213])**.** *Let $s \in]0,1[$ such that $s < \frac{d}{2}$. Then there exists a constant $C_S = C(d,s) > 0$ such that*

$$\|u\|_{L^{2^*}(\Omega)} \leq C_S \|D^s u\|_{L^2(\mathbb{R}^d)}$$

for all $u \in H_0^s(\Omega)$ or $u \in H_0^{s,2}(\Omega)$, where $2^* = \frac{2d}{d-2s} > 0$. Here C_S is called the Sobolev constant, and is of the form

$$C_S = C(d)\left(\frac{\omega_{d-1}}{s} + \left(\frac{\omega_{d-1}}{d-2s}\right)^{1/2}\right).$$

Furthermore, for general p, we have

$$\|u\|_{L^{p^*}(\Omega)} \leq C_S \|D^s u\|_{L^p(\Omega)},$$

where $p^* = \frac{pd}{d-sp}$, $d > sp$.

Remark 1.2. *Observe that for $s > 0$, the order of convergence as $s \to 0$ is $\mathcal{O}(1/s)$.*

From this lemma, we have, by the L^p inclusions in bounded domains, the following:

Lemma 1.3 (Fractional Poincaré inequality, Theorem 2.9 of [31])**.** *Let $s \in]0,1[$. Then there exists a constant $C_P = C(d,\Omega)/s > 0$, called the Poincaré constant, such that*

$$\|u\|_{L^2(\Omega)} \leq C_P \|D^s u\|_{L^2(\mathbb{R}^d)}$$

for all $u \in H_0^s(\Omega)$.

Remark 1.4. *From the previous Lemma 1.1, the order of convergence as $s \to 0$ for C_P is also given by $\mathcal{O}(1/\sigma)$, which coincides with that given in Theorem 2.9 of [31].*

Here, we provide a proof of Lemma 1.1 to give a better understanding of the constants, following [57].

Proof of Lemma 1.1. First recall the Hardy-Littlewood maximal function Mf of $f \in C_c^\infty(\mathbb{R}^d)$ defined as

$$(Mf)(x) := \sup_{x\in B} \frac{1}{|B|}\int_B |f|.$$

It is obvious that

$$\|Mf\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^\infty(\mathbb{R}^d)}.$$

Also, it is well-known that (see Theorem I.1 page 5 of [223] or Theorem 5.6 of [57])

$$m\{(Mf)(x) > \alpha\} \leq \frac{C(d)}{\alpha}\int_{\mathbb{R}^d} |f|.$$

By the interpolation theorem for (p,p) (see Theorem I.5 of [223] or Theorem 4.19 of [57]), we have

$$\|Mf\|_{L^p(\mathbb{R}^d)} \le \frac{pC}{p-1}\|f\|_{L^p(\mathbb{R}^d)}$$

where $C = C(d)$ depends only on the dimension d.

With these preliminaries, we can derive the Hardy-Littlewood-Sobolev inequality, as in Theorem 8.8 of [57]. Then

$$\frac{1}{\gamma_{d,s}}|I_s * f(x)| = \left(\int_{B_R(x)} \frac{|f(y)|}{|x-y|^{d-s}}\,dy + \int_{\mathbb{R}^d\setminus B_R(x)} \frac{|f(y)|}{|x-y|^{d-s}}\,dy\right) =: I(x) + II(x).$$

By majorisation,

$$I(x) \le (Mf)(x)\int_{B_R(x)} \frac{1}{|x-y|^{d-s}}\,dy \le (Mf)(x)\omega_{d-1}\int_0^R r^{s-d}r^{d-1}\,dr = (Mf)(x)\frac{R^s}{s}\omega_{d-1},$$

where ω_d is the spherical measure on the d-sphere. For the integral $II(x)$, writing $1/p' = 1-1/p$, we use Hölder's inequality to obtain

$$\begin{aligned}
II(x) \le& \|f\|_{L^p(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d\setminus B_R(x)} \frac{1}{|x-y|^{p'(d-s)}}\,dy\right)^{1/p'}\\
=& \|f\|_{L^p(\mathbb{R}^d)}\left(\omega_{d-1}\int_{r>R} r^{(s-d)p'}r^{d-1}\,dr\right)^{1/p'}\\
=& \|f\|_{L^p(\mathbb{R}^d)}\left(-\omega_{d-1}\frac{R^{(s-d)p'+d}}{(s-d)p'+d}\right)^{1/p'}\\
=& \|f\|_{L^p(\mathbb{R}^d)}\,\omega_{d-1}^{1/p'}\frac{R^{s-d/p}}{\left(\frac{d-sp}{p-1}\right)^{1/p'}},
\end{aligned}$$

where we require $s < d/p$ for the integral to converge. Choosing $R = (\|f\|_{L^p(\mathbb{R}^d)}/(Mf)(x))^{p/d}$ gives

$$\frac{1}{\gamma_{d,s}}|I_s * f(x)| \le \left(\frac{\omega_{d-1}}{s} + \left(\frac{\omega_{d-1}(p-1)}{d-sp}\right)^{1/p'}\right)(Mf)(x)^{1-sp/d}\|f\|^{sp/d}_{L^p(\mathbb{R}^d)}.$$

Raising this inequality to the power $pd/(d-sp) = p^*$ and integrating then gives

$$\begin{aligned}
\left\|\frac{1}{\gamma_{d,s}}I_s * f(x)\right\|_{L^{p^*}(\mathbb{R}^d)} &\le \left(\frac{\omega_{d-1}}{s} + \left(\frac{\omega_{d-1}(p-1)}{d-sp}\right)^{1/p'}\right)\|f\|^{sp/d}_{L^p(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}|(Mf)(x)|^p\right)^{1/p^*}\\
&= \left(\frac{\omega_{d-1}}{s} + \left(\frac{\omega_{d-1}(p-1)}{d-sp}\right)^{1/p'}\right)\|f\|^{sp/d}_{L^p(\mathbb{R}^d)}\|(Mf)(x)\|^{p/p^*}_{L^p(\mathbb{R}^d)}\\
&\le \left(\frac{\omega_{d-1}}{s} + \left(\frac{\omega_{d-1}(p-1)}{d-sp}\right)^{1/p'}\right)\left(\frac{p}{p-1}C(d)\right)^{1-sp/d}\|f\|^{1-sp/d}_{L^p(\mathbb{R}^d)}\|f\|^{sp/d}_{L^p(\mathbb{R}^d)}\\
&\le \left(\frac{\omega_{d-1}}{s} + \left(\frac{\omega_{d-1}(p-1)}{d-sp}\right)^{1/p'}\right)\left(1 \vee \frac{p}{p-1}C(d)\right)\|f\|_{L^p(\mathbb{R}^d)} =: C_S\|f\|_{L^p(\mathbb{R}^d)}
\end{aligned}$$

using the assumption that $sp < d$ and the interpolation theorem for Mf, for the Sobolev constant C_s given by

$$C_S = C(p,d)\left(\frac{\omega_{d-1}}{s} + \left(\frac{\omega_{d-1}(p-1)}{d-sp}\right)^{1/p'}\right).$$

The remaining follows with $p = 2$ as in the proof of Theorem 1.8 in [213], by noting that $u = I_s * g$, where $g = \sum_{j=1}^{d} \mathcal{R}_j \frac{\partial^s u}{\partial x_j^s}$ as in (1.3). □

Finally, we end with a continuous dependence property of the Riesz fractional derivatives as s varies.

Lemma 1.5. *For $u \in H_0^{s'}(\Omega)$, $D^s u$ is continuous in $[L^2(\mathbb{R}^d)]^d$ as s varies in $[\sigma, s']$ for $0 < \sigma < s' \leq 1$. As a consequence, we have the following estimate: for $\sigma \leq s \leq 1$,*

$$\|D^\sigma u\|_{L^2(\mathbb{R}^d)} \leq c_\sigma \|D^s u\|_{L^2(\mathbb{R}^d)}, \tag{1.19}$$

for any $u \in H_0^s(\Omega)$, where the constant c_σ is independent of s.

Proof. Consider first $u \in C_c^\infty(\Omega)$. Recall that the Fourier transform of $D^s u$ is given, by Theorem 1.4 of [213],

$$\widehat{D^s u} = (2\pi)^s i\xi|\xi|^{-1+s}\hat{u},$$

where $\hat{u}$ is the Fourier transform of u extended by 0 outside Ω. Since $u \in H_0^{s'}(\Omega)$, the mapping $s \mapsto (2\pi)^s i\xi|\xi|^{-1+s}\hat{u}$ is continuous in $[L^2(\mathbb{R}^d)]^d$ as s varies in $[\sigma, s']$. Therefore, conducting the inverse Fourier transform, we have $\lim_{s\to t}\left\|D^s u - D^t u\right\|_{L^2(\mathbb{R}^d)} = 0$ for $t \in [\sigma, s']$ for $u \in C_c^\infty(\Omega)$. Extending this by density to all $u \in H_0^{s'}(\Omega)$, we have the continuity result on s. Finally, the estimate (1.19) follows similarly to Proposition 2.7 of [31] using Lemma 1.3. □

2 The Nonlocal Operator

2.1 Definition and Motivation

The nonlocal (not necessarily symmetric) operator $\mathcal{L}_a : H_0^s(\Omega) \to H^{-s}(\Omega)$ is defined in the duality sense for $u \in H_0^s(\Omega)$ by

$$\mathcal{E}_a(u,v) = \langle \mathcal{L}_a u, v\rangle = P.V. \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}(x)(\tilde{u}(x) - \tilde{u}(y))a(x,y)\,dy\,dx \quad \forall v \in H_0^s(\Omega), \tag{2.1}$$

with $\tilde{u}$ and $\tilde{v}$ being the extension of $u, v \in H_0^s(\Omega)$ by zero outside the Lipschitz domain Ω. Here $\sim$ denotes the zero extension of the function outside Ω, but we will drop all $\sim$'s from now on. We assume that the measurable kernel function $a(x,y) : \mathbb{R}^d \times \mathbb{R}^d \backslash D \to \mathbb{R}_0^+$ for the diagonal $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ satisfies

$$a_* \leq \hat{a} := a(x,y)|x-y|^{d+2s} \leq a^* \quad \forall x,y \in \mathbb{R}^d, x \neq y \tag{2.2}$$

and

$$\sup_{x\in\mathbb{R}^d} \int_{\{a^{sym}(x,y)\neq 0\}} \frac{[a^{anti}(x,y)]^2}{a^{sym}(x,y)}\,dy \leq Z < \infty \tag{2.3}$$

for some $a_*, a^* > 0$ and $Z \geq 1$, where $a^{sym}(x,y) = \frac{1}{2}[a(x,y) + a(y,x)]$ and $a^{anti}(x,y) = \frac{1}{2}[a(x,y) - a(y,x)]$ are the symmetric and anti-symmetric parts of $a(x,y)$ respectively. Also, for $u \in H_0^s(\Omega)$, we can extend it by 0 outside Ω to obtain a function in $H^s(\mathbb{R}^d)$. The operator $\mathcal{L}_a$ corresponds to the class of uniformly irreducible random walks that admit a cycle decomposition with bounded range, bounded length of cycles, and bounded jump rates [94].

Remark 2.1. *Supposing $a(x,y)$ satisfies (2.2), we have, by switching x and y,*

$$a_* \leq a(y,x)|y-x|^{d+2s} \leq a^*.$$

Taking the sum of this with (2.2), we have that (2.2) is satisfied by $a^{sym}(x,y)$, i.e.

$$a_* \leq a^{sym}(x,y)|x-y|^{d+2s} \leq a^* \quad \forall x,y \in \mathbb{R}^d, x \neq y.$$

If $\hat{a}$ is symmetric, making use of nonlocal gradient and nonlocal divergence, we can write

$$\mathcal{L}_{a^{sym}} u = \frac{1}{2}\mathcal{D}_s(\hat{a}^{sym}\mathcal{D}^s u).$$

Observe that the fractional Laplacian is defined, for all $u \in H_0^s(\Omega)$, by

$$(-\Delta)^s u(x) = c_{d,s}^2 \lim_{\epsilon\to 0} \int_{\mathbb{R}^d\backslash B_\epsilon(x)} \frac{\tilde{u}(x) - \tilde{u}(y)}{|x-y|^{d+2s}}\,dy, \tag{2.4}$$

which corresponds to the kernel

$$c_{d,s}^2 |x-y|^{-d-2s}, \tag{2.5}$$

where $c_{d,s} = 2^s \pi^{-\frac{d}{2}} \frac{\Gamma\left(\frac{d+s+1}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}$. Furthermore, by Fubini's theorem, since $a(x,y)$ satisfies the condition (2.2), we have

$$\begin{aligned}
\langle \mathcal{L}_a u, u\rangle &= P.V. \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{u}(x)(\tilde{u}(x) - \tilde{u}(y))a(x,y)\,dy\,dx \\
&= \frac{1}{2}P.V. \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{u}(x)(\tilde{u}(x) - \tilde{u}(y))a(x,y)\,dy\,dx + \frac{1}{2}P.V. \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{u}(y)(\tilde{u}(y) - \tilde{u}(x))a(y,x)\,dy\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\tilde{u}(x) - \tilde{u}(y)|^2 a^{sym}(x,y)\,dy\,dx,
\end{aligned} \tag{2.6}$$

so we have the bound, applying (2.2) to $a^{sym}(x,y)$ as discussed in Remark 2.1,

$$\frac{a_*}{c_{d,s}^2}\left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^d)}^2 \leq \langle \mathcal{L}_a u, u\rangle \leq \frac{a^*}{c_{d,s}^2}\left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^d)}^2 \tag{2.7}$$

since it is well-known (see, for example, Proposition 3.6 of [95]) that

$$\left\|(-\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^d)}^2 = \frac{1}{2}c_{d,s}^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,dx\,dy = \int_{\mathbb{R}^d} u(-\Delta^s u) \tag{2.8}$$

by Fourier transform. Moreover, taking u, v such that $u \equiv v \equiv 0$ in Ω^c, we have (see for example, Equation (1.5) of [199])

$$\langle \mathcal{L}_{\mathbb{I}} u, v\rangle = \int_\Omega v(-\Delta)^s u = \int_{\mathbb{R}^d}(-\Delta)^{s/2}u(-\Delta)^{s/2}v. \tag{2.9}$$

Also,

$$\frac{1}{2}a_*[u]_{s,\mathbb{R}^d}^2 \leq \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\hat{a}|\mathcal{D}^s u|^2\,dx\,dy = \langle \mathcal{L}_a u, u\rangle = \langle \mathcal{L}_{a^{sym}}u,u\rangle = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a^{sym}|\mathcal{D}^s u|^2\,dx\,dy \leq \frac{1}{2}a^*[u]_{s,\mathbb{R}^d}^2. \tag{2.10}$$

It is known that equations defined with such nonlocal operators arise naturally when we consider stochastic processes with jumps, particularly in Lévy processes, which have a large number of applications, including in peridynamics [217], diffusion processes [58] and finance [76]. By considering the nonlocal operator in place of the classical Laplacian, we are able to extend the concept of Brownian motion, to include paths which are merely stochastically continuous [196]. As such, this nonlocal operator is related to classical studies in stochastics for Dirichlet forms.

2.2 The Nonlocal Bilinear Form as a Coercive Dirichlet Form

Our first main result is to show that the nonlocal bilinear form together with its domain, $(\mathcal{E}_a, H_0^s(\Omega))$, defined as

$$\mathcal{E}_a(u,v) := P.V.\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tilde{v}(x)(\tilde{u}(x)-\tilde{u}(y))a(x,y)\,dy\,dx, \tag{2.11}$$

where $\tilde{u}, \tilde{v}$ are the zero extensions of $u, v \in H_0^s(\Omega)$ to Ω^c and $a : \mathbb{R}^d\times\mathbb{R}^d \to [0,\infty[, d \geq 1$ is a not necessarily symmetric kernel satisfying (2.2)–(2.3), is in fact a regular (not necessarily symmetric) Dirichlet form. This will also imply that the nonlocal bilinear form is also strictly T-monotone and will give us many properties, including Harnack's inequality (see for example [59]), Hölder regularity of solutions of equations involving this bilinear form (see for instance [138, 139], or [140]), and other results in stochastic processes (as given in [122]).

We will begin with a remark on the symmetric case.

Proposition 2.2. *For given $u, v \in H_0^s(\Omega)$ and $a : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty[$ symmetric, we have*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tilde{v}(x)(\tilde{u}(x)-\tilde{u}(y))a(x,y)\chi_\varepsilon(x,y)\,dy\,dx = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))a(x,y)\chi_\varepsilon(x,y)\,dy\,dx \tag{2.12}$$

assuming that the integrands are summable for each fixed $\varepsilon > 0$, where we have set $\chi_\varepsilon(x,y)$ as the characteristic function of the set $\{|x-y| > \varepsilon\}$ for $\varepsilon > 0$.

Proof. We first show the result for $u, v \in C_c^\infty(\Omega)$, and extend it by density to $u, v \in H_0^s(\Omega)$. The integral term

$$J := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tilde{v}(x)(\tilde{u}(x)-\tilde{u}(y))a(x,y)\chi_\varepsilon(x,y)\,dy\,dx$$

can also be written in the form, using the symmetry of a,

$$J = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tilde{v}(y)(\tilde{u}(y)-\tilde{u}(x))a(x,y)\chi_\varepsilon(x,y)\,dx\,dy$$

Then, by Fubini's theorem,

$$J = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}(y)(\tilde{u}(x) - \tilde{u}(y))a(x,y)\chi_\varepsilon(x,y)\,dy\,dx.$$

Taking the sum of this and the first equation above, we obtain the result

$$2J = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\tilde{v}(x) - \tilde{v}(y))(\tilde{u}(x) - \tilde{u}(y))a(x,y)\chi_\varepsilon(x,y)\,dy\,dx.$$

□

Remark 2.3. *The bilinear form (2.11) in the symmetric case with assumption (2.2) is a coercive and well-defined quadratic form for $u \in H^s(\mathbb{R}^d)$. Indeed, we have*

$$\begin{aligned}
a_* \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} &\frac{(u(x) - u(y))^2}{|x-y|^{d+2s}}\,dx\,dy \\
&\leq \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(x) - u(y))^2 a(x,y)\chi_\varepsilon(x,y)\,dx\,dy \\
&\leq \lim_{\varepsilon\to 0} a^* \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x-y|^{d+2s}}\chi_\varepsilon(x,y)\,dx\,dy = a^*[u]^2_{s,\mathbb{R}^d}.
\end{aligned}$$

Remark 2.4. *If $\hat{a}$ is symmetric, making use of the nonlocal derivative $\mathcal{D}^s$, we can write*

$$\langle \mathcal{L}_{a^{sym}} u, v\rangle = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\hat{a}^{sym}\mathcal{D}^s u)\mathcal{D}^s v\,dx\,dy.$$

Now, we have our first main theorem on the anisotropic non-symmetric nonlocal bilinear form.

Theorem 2.5. *$(\mathcal{E}_a, H^s_0(\Omega))$ with $a(x,y)$ satisfying (2.2) and (2.3) is a closed, regular Dirichlet form in $L^2(\Omega)$, which is also bounded and coercive.*

Proof. The bilinear form $\mathcal{E}_a : H^s_0(\Omega)\times H^s_0(\Omega) \to \mathbb{R}$ is bounded in the non-symmetric case, following the ideas of [207],

$$\begin{aligned}
\mathcal{E}_a(u,v) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}(x)(\tilde{u}(x) - \tilde{u}(y))\left(a^{sym}(x,y) + a^{anti}(x,y)\right)\chi_\varepsilon(x,y)\,dy\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\tilde{v}(x) - \tilde{v}(y))(\tilde{u}(x) - \tilde{u}(y))a^{sym}(x,y)\,dy\,dx \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}(x)(\tilde{u}(x) - \tilde{u}(y))a^{anti}(x,y)\chi_\varepsilon(x,y)\,dy\,dx \\
&\leq \frac{1}{2}a^*\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\tilde{v}(x) - \tilde{v}(y)||\tilde{u}(x) - \tilde{u}(y)||x-y|^{-d-2s}\,dy\,dx \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}(x)(\tilde{u}(x) - \tilde{u}(y))[a^{sym}(x,y)]^{\frac{1}{2}}a^{anti}(x,y)[a^{sym}(x,y)]^{-\frac{1}{2}}\chi_\varepsilon(x,y)\,dy\,dx \\
&\leq a^*\left(\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\left|\tilde{v}(x) - \tilde{v}(y)\right|^2}{|x-y|^{d+2s}}\,dx\,dy\right)^{1/2}\left(\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\left|\tilde{u}(x) - \tilde{u}(y)\right|^2}{|x-y|^{d+2s}}\,dy\,dx\right)^{1/2} \\
&\quad + \left(\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\tilde{u}(x) - \tilde{u}(y)\right|^2 a^{sym}(x,y)\,dy\,dx\right)^{1/2} \\
&\qquad \times\left(2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\tilde{v}(x)\right|^2\frac{\left|a^{anti}(x,y)\right|^2}{|a^{sym}(x,y)|}\chi_\varepsilon(x,y)\,dy\,dx\right)^{1/2} \\
&\leq \frac{a^*}{c^2_{d,s}}\|D^s u\|_{L^2(\mathbb{R}^d)}\|D^s v\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

$$
\begin{aligned}
&+ \left(\frac{1}{2} a^* \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left|\tilde{u}(x) - \tilde{u}(y)\right|^2}{|x-y|^{d+2s}} \, dy \, dx \right)^{1/2} \\
&\quad \times \left(2 \int_{\mathbb{R}^d} \left|\tilde{v}(x)\right|^2 \left[\int_{\mathbb{R}^d} \frac{\left|a^{anti}(x,y)\right|^2}{|a^{sym}(x,y)|} \chi_\varepsilon(x,y) \, dy \right] \, dx \right)^{1/2} \\
&\leq \frac{a^*}{c_{d,s}^2} \|D^s u\|_{L^2(\mathbb{R}^d)} \|D^s v\|_{L^2(\mathbb{R}^d)} + \frac{(a^*)^{\frac{1}{2}}}{c_{d,s}} \|D^s u\|_{L^2(\mathbb{R}^d)} \left(2 \int_{\mathbb{R}^d} \left|\tilde{v}(x)\right|^2 Z \, dx \right)^{1/2} \\
&\leq \frac{a^*}{c_{d,s}^2} \|D^s u\|_{L^2(\mathbb{R}^d)} \|D^s v\|_{L^2(\mathbb{R}^d)} + \frac{(2a^* Z)^{\frac{1}{2}}}{c_{d,s}} \|D^s u\|_{L^2(\mathbb{R}^d)} \|v\|_{L^2(\Omega)} \\
&\leq \left(\frac{a^*}{c_{d,s}^2} + \frac{(2a^* Z)^{\frac{1}{2}}}{c_{d,s} C_P} \right) \|D^s u\|_{L^2(\mathbb{R}^d)} \|D^s v\|_{L^2(\mathbb{R}^d)}
\end{aligned}
$$

by Lemma 1.3, and coercive

$$
\mathcal{E}_a(u,u) \geq \frac{1}{2} a_* \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\tilde{u}(x) - \tilde{u}(y))^2 |x-y|^{-d-2s} \, dy \, dx = \frac{a_*}{c_{d,s}^2} \|D^s u\|_{L^2(\mathbb{R}^d)}^2 = \frac{a_*}{c_{d,s}^2} \|u\|_{H_0^s(\Omega)}^2
$$

by making use of an argument similar to Proposition 2.2.

It is well-known that $H_0^s(\Omega)$ is complete with respect to its norm, as in (1.17), therefore $\mathcal{E}_a$ is *closed*. Furthermore, it can be shown that $(\mathcal{E}_a, H_0^s(\Omega))$ is *regular*, i.e. $H_0^s(\Omega) \cap C_c(\Omega)$ is dense in $H_0^s(\Omega)$ in the $H_0^s(\Omega)$-norm, and dense in $C_c(\Omega)$ with uniform norm. The first density result follows from the density of compactly supported smooth functions in the space $H_0^s(\Omega)$. The second density result follows since $C_c^\infty(\Omega) \subset H_0^s(\Omega)$ and $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$ with uniform norm, by considering the mollification of any $C_c(\Omega)$ function.

Next, recall that a coercive closed (not necessarily symmetric) bilinear form $\mathcal{E}$ on $L^2(\Omega)$ is a *Dirichlet form* ([167] Proposition I.4.7 and equation (4.7) pages 34–35) if and only if the following property holds: For each $\varepsilon > 0$, there exists a real function $\phi_\varepsilon(t), t \in \mathbb{R}$, such that

$$
\phi_\varepsilon(t) = t, \quad \forall t \in [0,1] \quad -\varepsilon \leq \phi_\varepsilon(t) \leq 1 + \varepsilon, \quad \forall t \in \mathbb{R}, \quad 0 \leq \phi_\varepsilon(t') - \phi_\varepsilon(t) \leq t' - t \text{ whenever } t < t' \tag{2.13}
$$

$$
u \in H_0^s(\Omega) \implies \phi_\varepsilon(u) \in H_0^s(\Omega), \quad \begin{cases} \liminf_{\varepsilon \to 0} \mathcal{E}(\phi_\varepsilon(u), u - \phi_\varepsilon(u)) \geq 0, \\ \liminf_{\varepsilon \to 0} \mathcal{E}(u - \phi_\varepsilon(u), \phi_\varepsilon(u)) \geq 0. \end{cases} \tag{2.14}
$$

A classic example of ϕ_ε is the mollification of a cut-off function (see [122] Example 1.2.1). Specifically, consider a mollifier such as

$$
j(x) = \begin{cases} \gamma e^{-\frac{1}{1-|x|^2}} & \text{for } |x| < 1 \\ 0 & \text{otherwise,} \end{cases}
$$

where γ is a positive constant such that

$$
\int_{|x| \leq 1} j(x) \, dx = 1.
$$

Set $j_\delta(x) = \delta^{-d} j(\delta^{-1} x)$ for $\delta > 0$. For any $\varepsilon > 0$, consider the function $\psi_\varepsilon(t) = ((-\varepsilon) \vee t) \wedge (1 + \varepsilon)$ on $\mathbb{R}$ (refer to (2.15) for the notations $\vee$ and $\wedge$) and set $\phi_\varepsilon(t) = j_\delta * \psi_\varepsilon(t)$ for $0 < \delta < \varepsilon$. Then our choice of ϕ_ε satisfies (2.13). Furthermore, it satisfies $\phi_\varepsilon(t) = 1 + \varepsilon$ for $t \in [1 + 2\varepsilon, \infty[$ and $\phi_\varepsilon(t) = -\varepsilon$ for $t \in (-\infty, -2\varepsilon]$, and $|\phi_\varepsilon(t)| \leq |t|$ with $t\phi_\varepsilon(t) \geq 0$.

Moreover, $\phi_\varepsilon(t)$ is infinitely differentiable, so for any $u \in C_c^\infty(\Omega)$, $\phi_\varepsilon(u) \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $H_0^s(\Omega)$, we can extend by density any results obtained, thereby obtaining $\phi_\varepsilon(u) \in H_0^s(\Omega)$ for all $u \in H_0^s(\Omega)$.

With this ϕ_ε, recalling that $a \geq 0$ (as in section II.2(d) of [167]), by Fatou's lemma,

$$
\liminf_{\varepsilon \to 0} \mathcal{E}_a(\phi_\varepsilon(u), u - \phi_\varepsilon(u))
$$

$$\geq P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \liminf_{\varepsilon\to 0} [u(x) - \widetilde{\phi_\varepsilon(u(x))}][\widetilde{\phi_\varepsilon(u(x))} - \widetilde{\phi_\varepsilon(u(y))}] a(x,y)\, dy\, dx$$

$$\geq P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \liminf_{\varepsilon\to 0} [u(x) - \widetilde{\phi_\varepsilon(u(x))}][-\widetilde{\phi_\varepsilon(u(y))}] a(x,y)\, dy\, dx \qquad (\text{since } \phi_\varepsilon(u)[u-\phi_\varepsilon(u)] \geq 0)$$

$$\geq P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \liminf_{\varepsilon\to 0} \Big\{ -[u(x) - \widetilde{\phi_\varepsilon(u(x))}](-\varepsilon)\chi_{\{u(x)<0\}} + 0\chi_{\{0\leq u(x)\leq 1\}}$$
$$+ (-\varepsilon)(-1-\varepsilon)\chi_{\{u(x)>1\}} \Big\} a(x,y)\, dy\, dx$$
$$= 0$$

since $u - \phi_\varepsilon(u) \leq 0$ for $u \leq 0$, $\phi_\varepsilon(t) = t$ for $t \in [0,1]$, and $u - \phi_\varepsilon(u) \geq u - 1 - \varepsilon \geq -\varepsilon$ for $u \geq 1$ respectively. Similarly, taking $\liminf_{\varepsilon\to 0}$ in

$$\mathcal{E}_a(u - \phi_\varepsilon(u), \phi_\varepsilon(u)) = P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{\phi_\varepsilon(u(x))} \Big\{ [u(x) - \widetilde{\phi_\varepsilon(u(x))}] - [u(y) - \widetilde{\phi_\varepsilon(u(y))}] \Big\} a(x,y)\, dx\, dy,$$

we conclude that $\mathcal{E}_a$ is a Dirichlet form.

□

From this theorem, we obtain that $\mathcal{E}_a$ possesses the property of unit contraction. Indeed, by Proposition 4.3 and Theorem 4.4 of [167], we have the following corollary.

Corollary 2.6. *For the regular Dirichlet form $\mathcal{E}_a$ the following properties hold:*

(a) the unit contraction acts on $\mathcal{E}_a$, i.e. $v := (0 \vee u) \wedge 1$ satisfies $\mathcal{E}_a(v,v) \leq \mathcal{E}_a(u,u)$;

(b) the normal contraction acts on $\mathcal{E}_a$, i.e. suppose v satisfies

$$|v(x) - v(y)| \leq |u(x) - u(y)|, \quad x, y \in \mathbb{R}^d$$
$$|v(x)| \leq |u(x)|, \quad x \in \mathbb{R}^d,$$

then $v \in H_0^s(\Omega)$ and $\mathcal{E}_a(v,v) \leq \mathcal{E}_a(u,u)$.

This result in fact follows from the fact that $\mathcal{E}_a$ is a Dirichlet form, and holds even if it is not regular.

As in [108] or Corollary 2.4 of [109], we have the existence of a unique solution to the nonlocal Dirichlet elliptic problem by the Lax-Milgram theorem.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Suppose that $f \in H^{-s}(\Omega)$ and a is measurable and satisfies* (2.2) *and* (2.3)*. Then there exists a unique $u \in H_0^s(\Omega)$ such that*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} (u(x) - u(y))v(x)a(x,y)\, dx\, dy = \int_\Omega fv\, dx$$

for every $v \in H_0^s(\Omega)$.

2.3 The Strict T-monotonicity of $\mathcal{E}_a$

We introduce the positive and negative parts of v

$$v^+ \equiv v \vee 0 \quad \text{and} \quad v^- \equiv -v \vee 0 = -(v \wedge 0),$$

and we have the Jordan decomposition of v given by

$$v = v^+ - v^- \quad \text{and} \quad |v| \equiv v \vee (-v) = v^+ + v^-$$

and the useful identities

$$\begin{aligned} u \vee v &= u + (v-u)^+ = v + (u-v)^+, \\ u \wedge v &= u - (u-v)^+ = v - (v-u)^+. \end{aligned} \tag{2.15}$$

It is well-known that such operations are closed in $H_0^s(\Omega)$ for $0 < s \leq 1$.

Since the assumptions on the kernel $a(x,y)$ imply, in particular, that it is non-negative, we can easily prove the following important property.

Theorem 2.8. *$\mathcal{E}_a$ is also strictly T-monotone in the following sense: $\mathcal{L}_a : H^s_0(\Omega) \to H^{-s}(\Omega)$ defined by*

$$\langle \mathcal{L}_a u, v\rangle = \mathcal{E}_a(u, v), \tag{2.16}$$

satisfies

$$\langle \mathcal{L}_a v, v^+\rangle > 0 \quad \forall v \in H^s_0(\Omega) \text{ such that } v^+ \neq 0.$$

Proof. $\mathcal{L}_a$ is strictly T-monotone because

$$\begin{aligned}\mathcal{E}_a(v^-, v^+) &= P.V. \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}^+(x)(\tilde{v}^-(x) - \tilde{v}^-(y))a(x,y)\,dx\,dy \\ &= -\,P.V. \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}^+(x)\tilde{v}^-(y)a(x,y)\,dx\,dy \\ &\le 0,\end{aligned}$$

since $v^+(x)v^-(x) = 0$ as v^+ and v^- cannot both be nonzero at the same point x, and $v^+, v^-, a \ge 0$. Therefore, since $a(x,y)|x-y|^{d+2s} \ge a_*$ with $a_* > 0$,

$$\begin{aligned}\langle \mathcal{L}_a v, v^+\rangle &= \mathcal{E}_a(v, v^+) \\ &= \mathcal{E}_a\left(v^+, v^+\right) - \mathcal{E}_a\left(v^-, v^+\right) \\ &\ge \frac{a_*}{c^2_{d,s}} \int_{\mathbb{R}^d} |D^s v^+|^2\end{aligned}$$

which is strictly greater than 0 if $v^+ \neq 0$. □

2.4 Dependence of Eigenfunctions of $(-\Delta)^s$ on $0 < s \le 1$

In this section, we consider the special case when $a(x,y) = \frac{1}{|x-y|^{d+2s}}$ and $\mathcal{L}^s_a$ corresponds to the fractional Laplacian $(-\Delta)^s$. We state a result regarding the continuity of the eigenfunctions of $(-\Delta)^s$ with respect to the parameter s, $0 < s \le 1$, as given in Theorem 1.2 of [51] (see also Theorem 5.1 of [111] and Theorem 4.1 of [116]).

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary. Then the eigenvalue problem*

$$(-\Delta)^s u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \Omega^c$$

admits eigenvalues $\lambda^s_m(\Omega)$ with corresponding eigenfunctions u^s_m, such that for any $m \in \mathbb{N}$,

$$\lim_{s\nearrow 1}(1-s)\lambda^s_m(\Omega) = c(d)\lambda^1_m(\Omega)$$

for a constant $c(d)$ depending only on the dimension d, and if $\|u^s_m\|_{L^p(\Omega)} = 1$, there exists a sequence $\{u^{s_k}_m\}_{k\in\mathbb{N}} \subset \{u^s_m\}_{s\in]0,1[}$ such that

$$\lim_{k\to\infty}[u^{s_k}_m - u^1_m]_{W^{t,q}(\mathbb{R}^d)} = 0, \quad \text{for every } 2 \le q < \infty \text{ and every } 0 < t < \frac{2}{q},$$

where u^1_m is the eigenfunction of

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

corresponding to the eigenvalue λ^1_m such that $\left\|u^1_m\right\|_{L^p(\Omega)} = 1$.

Remark 2.10. *While it is possible to obtain the eigenvalues and eigenfunctions of $\mathcal{L}^s_a$ with a general symmetric $a(x,y)$ by considering the Rayleigh quotient (see, for instance, [118] and [173]), an explicit form of the limit operator is not yet known. Recently, there is a result in Theorem 8.1 of [70] giving a general compactness result on the convergence of eigenvalues as $s \nearrow 1$, provided we know the limit $\mathcal{L}^1_a$ in the sense of resolvents. This assumption is known to be satisfied for the restricted and the spectral fractional Laplacian, but remains open in the case of the general operator $\mathcal{L}^s_a$. This is a limitation of the nonlocal operator $\mathcal{L}_a$, which, we will see in Section 3.3, does not apply to the fractional operator $\tilde{\mathcal{L}}^s_A$ since the fractional s-gradient D^s is defined for $0 < s \le 1$.*

3 The Fractional Operator

3.1 Definition and Motivation

The fractional operator $\tilde{\mathcal{L}}_A^s$ is defined with the fractional derivatives D^s by the continuous fractional bilinear form

$$\langle \tilde{\mathcal{L}}_A^s u, v\rangle := \int_{\mathbb{R}^d} A(x) D^s u \cdot D^s v \, dx, \quad \forall u, v \in H_0^s(\Omega) \tag{3.1}$$

for a matrix A with coefficients bounded and measurable such that

$$a_*|z|^2 \leq A(x)z \cdot z \text{ and } A(x)z \cdot z^* \leq a^*|z||z^*| \tag{3.2}$$

for some $a_*, a^* > 0$ for all $x \in \mathbb{R}^d$ and all $z, z^* \in \mathbb{R}^d$, so that the integral is well defined. This operator extends the nonlocality of the fractional Laplacian to include anisotropy, which may be useful for situations such as anomalous fractional diffusion [171]. Furthermore, unlike the nonlocal operator, this fractional operator is different from pseudodifferential operators, since it involves multiplication with an anisotropic matrix A, which gives a convolution under Fourier transform. This means that $\tilde{\mathcal{L}}_A^s$ does not correspond to a symbol.

In particular, this means that the problems considered with the nonlocal and the fractional operators are in general not equivalent, except in the isotropic homogeneous case (for more details, see Section 3.4). Furthermore, in higher dimensions, the Riesz fractional s-gradient, as proposed in [216] and discussed in Section 1.1.1, is an appropriate fractional operator maintaining translational and rotational invariance, as well as homogeneity of degree s under isotropic scaling, and so the $\tilde{\mathcal{L}}_A^s$ operator gives a natural and appropriate anisotropic generalisation of the fractional Laplacian.

Next, we give the existence and uniqueness result for the elliptic problem associated to the fractional operator $\tilde{\mathcal{L}}_A$, which generalises the result in Theorem 1.13 of [213] to non-symmetric A and for less regular source function f. This result follows directly from an application of the Lax-Milgram theorem, by the boundedness and ellipticity of the matrix A.

Theorem 3.1 (Existence of a Unique Solution to the Fractional Dirichlet Elliptic Problem). *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Suppose that $f \in L^{2^\#}(\Omega)$, $g \in H^s(\mathbb{R}^d)$ and $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is bounded and measurable such that*

$$a_*|z|^2 \leq A(x)z \cdot z \text{ and } A(x)z \cdot z^* \leq a^*|z||z^*| \tag{3.2}$$

for some $a_*, a^* > 0$ for all $x \in \mathbb{R}^d$ and all $z, z^* \in \mathbb{R}^d$. Then there exists a unique $u \in H^s(\mathbb{R}^d)$ such that $u = g$ in Ω^c and

$$\langle \tilde{\mathcal{L}}_A u, v\rangle = \int_{\mathbb{R}^d} A(x) D^s u \cdot D^s v \, dx = \int_\Omega f v \, dx$$

for every $v \in H_0^s(\Omega)$.

For the fractional operator $\tilde{\mathcal{L}}_A^s$, we are unable to obtain strict T-monotonicity. However, such a limitation does not prevent us from applying the associated s-vectorial calculus in the study of energy processes in nonlocal elasticity or peridynamics (see, for instance, [99]). In fact, the coordinate invariance of this anisotropic operator makes it more suitable in higher dimensions, thereby providing it with many physical applications, including in hyperelasticity and peridynamics (see, for instance, [30], [31] and Section 3.5 of [208]), as well as in diffusion and phase transition problems (see, for instance, [99] and Chapter 6).

Yet, it remains the question of necessary conditions on the matrix A such that we can obtain a T-monotonicity result and consequently comparison principles. It is known that such results exist for the fractional Laplacian, when A is given by the identity matrix. Therefore, we propose the following open problem:

Open Problem. *What are the conditions on the matrix $A(x)$ so that the associated fractional operator $\tilde{\mathcal{L}}_A^s$ is T-monotone?*

3.2 The One-Parameter Fractional Dirichlet Problem

In order to consider the homogeneous time-dependent problem, as in the Stefan problem in Chapter 6, we consider the one-parameter fractional Dirichlet problem, which is a consequence of the continuous dependence on Dirichlet data. The function $g = g(t)$ is constructed for every fixed $t \in J$, for the interval $J = [0, T]$ for all $T < \infty$, (using Theorem 1.13 of [213] as stated above in Theorem 3.1) by solving

$$\int_{\mathbb{R}^d} AD^s g(t) \cdot D^s v = 0 \quad \forall v \in H_0^s(\Omega) \tag{3.3}$$

with the Dirichlet boundary condition given by

$$g(t) = \tilde{g}(t) \text{ in } \Omega^c, \tag{3.4}$$

with $\tilde{g}(t)$ defined on $H^s(\mathbb{R}^d)$.

Theorem 3.2. *Suppose $\tilde{g} \in BV(0, T; H^s(\mathbb{R}^d))$ or $H^k(0, T; H^s(\mathbb{R}^d))$ for $k = 1, 2$. Then, for the Dirichlet problem* (3.3)–(3.4)*, g has the same time regularity as $\tilde{g}$.*

Indeed, consider $u = g - \tilde{g}$. Then u satisfies $u(t) = 0$ in Ω^c and

$$\int_{\mathbb{R}^d} AD^s u(t) \cdot D^s v = -\int_{\mathbb{R}^d} AD^s \tilde{g}(t) \cdot D^s v =: \langle \tilde{\mathcal{L}}_A \tilde{g}(t), v \rangle \quad \forall v \in H_0^s(\Omega) \tag{3.5}$$

Since $\tilde{\mathcal{L}}_A : H^s(\mathbb{R}^d) \to H^{-s}(\Omega)$ with $\tilde{g}(t) \in H^s(\mathbb{R}^d)$, $\tilde{\mathcal{L}}_A \tilde{g}(t)$ is a linear functional in $H^{-s}(\Omega)$. By the coercivity and boundedness of $\tilde{\mathcal{L}}_A$, there exists a unique solution $u(t) \in H_0^s(\Omega)$ satisfying (3.5) for almost every $t \in J$ by the Lax-Milgram theorem. By the uniqueness of $u(t)$, there exists a unique $g(t) := u(t) + \tilde{g}(t) \in H^s(\mathbb{R}^d)$ satisfying (3.3) for almost every $t \in J$. It is clear that $g \in L^2(0, T; H^s(\mathbb{R}^d))$ if $\tilde{g} \in L^2(0, T; H^s(\mathbb{R}^d))$.

Furthermore, by linearity of $\tilde{\mathcal{L}}_A$, considering two time slices $\{t\} \times \Omega$ and $\{\tau\} \times \Omega$, we have, taking the test function to be $u(t) - u(\tau)$,

$$\begin{aligned} a_* \left\| u(t) - u(\tau) \right\|_{H_0^s(\Omega)}^2 &\leq \int_{\mathbb{R}^d} AD^s u(t) \cdot D^s(u(t) - u(\tau)) - \int_{\mathbb{R}^d} AD^s u(\tau) \cdot D^s(u(t) - u(\tau)) \\ &= -\int_{\mathbb{R}^d} AD^s \tilde{g}(t) \cdot D^s(u(t) - u(\tau)) + \int_{\mathbb{R}^d} AD^s \tilde{g}(\tau) \cdot D^s(u(t) - u(\tau)) \\ &\leq a^* \left\| \tilde{g}(t) - \tilde{g}(\tau) \right\|_{H^s(\mathbb{R}^d)} \left\| u(t) - u(\tau) \right\|_{H_0^s(\Omega)}, \end{aligned} \tag{3.6}$$

so taking the sum of all time steps in $[t_i, t_{i-1}] \subset [0, T]$, $u \in BV(0, T; H_0^s(\Omega))$ if $\tilde{g} \in BV(0, T; H^s(\mathbb{R}^d))$, and consequently $g = u + \tilde{g} \in BV(0, T; H^s(\mathbb{R}^d))$.

Also, from (3.6), we have the continuity of $u(t)$ in time for $t \in J$. Therefore, $u \in C(J; H_0^s(\Omega))$ if $\tilde{g}(t)$ is continuous for $t \in J$. Furthermore, we consider the problem

$$\int_{\mathbb{R}^d} AD^s w(t) \cdot D^s v = -\int_{\mathbb{R}^d} AD^s \frac{\partial \tilde{g}}{\partial t}(t) \cdot D^s v = \left\langle \tilde{\mathcal{L}}_A \frac{\partial \tilde{g}}{\partial t}(t), v \right\rangle \quad \forall v \in H_0^s(\Omega) \tag{3.7}$$

when $\frac{\partial \tilde{g}}{\partial t} \in H^s(\mathbb{R}^d)$, and we can once again apply the argument above to obtain a unique solution $w \in H_0^s(\Omega)$ for almost every $t \in J$. It remains to show that

$$w(t) = \frac{\partial u}{\partial t}(t) \quad \text{a.e. } t \text{ in } H_0^s(\Omega).$$

But, as in (3.6), we have, using (3.5) and (3.7) and taking the test function to be $\frac{u(t)-u(t+h)}{h} - w(t)$,

$$
\begin{aligned}
&a_* \left\| \frac{u(t)-u(t+h)}{h} - w(t) \right\|^2_{H^s_0(\Omega)} \\
&\leq \int_{\mathbb{R}^d} AD^s \frac{u(t)-u(t+h)}{h} \cdot D^s \left(\frac{u(t)-u(t+h)}{h} - w(t) \right) - \int_{\mathbb{R}^d} AD^s w(t) \cdot D^s \left(\frac{u(t)-u(t+h)}{h} - w(t) \right) \\
&= -\int_{\mathbb{R}^d} AD^s \frac{\tilde{g}(t)-\tilde{g}(t+h)}{h} \cdot D^s \left(\frac{u(t)-u(t+h)}{h} - w(t) \right) + \int_{\mathbb{R}^d} AD^s \frac{\partial \tilde{g}}{\partial t}(t) \cdot D^s \left(\frac{u(t)-u(t+h)}{h} - w(t) \right) \\
&\leq a^* \left\| \frac{\tilde{g}(t)-\tilde{g}(t+h)}{h} - \frac{\partial \tilde{g}}{\partial t}(t) \right\|_{H^s(\mathbb{R}^d)} \left\| \frac{u(t)-u(t+h)}{h} - w(t) \right\|_{H^s_0(\Omega)} .
\end{aligned}
\tag{3.8}
$$

But recall that by definition (see, for instance, Chapter 23.5 of [244]),

$$
\frac{\tilde{g}(t)-\tilde{g}(t+h)}{h} \to \frac{\partial \tilde{g}}{\partial t}(t) \quad \text{in } H^s(\mathbb{R}^d) \text{ as } h \to 0.
$$

Therefore, for any $\epsilon > 0$, take a small enough $h > 0$ such that $\left\| \frac{\tilde{g}(t)-\tilde{g}(t+h)}{h} - \frac{\partial \tilde{g}}{\partial t}(t) \right\|_{H^s(\mathbb{R}^d)} < \epsilon$, then $\left\| \frac{u(t)-u(t+h)}{h} - w(t) \right\|_{H^s_0(\Omega)} < \frac{a^*\epsilon}{a_*}$. Since ϵ is arbitrary,

$$
w(t) = \lim_{h\to 0} \frac{u(t)-u(t+h)}{h} \quad \text{a.e. } t \text{ in } H^s_0(\Omega),
$$

and the limit of the difference quotient is, by definition, $\frac{\partial u}{\partial t}$. Therefore, $\frac{\partial g}{\partial t} = w(t) + \frac{\partial \tilde{g}}{\partial t}(t)$, and we have that g has the same regularity as $\tilde{g}$ in $H^1(0,T;H^s(\mathbb{R}^d))$. Repeating this argument again by taking a second time derivative, we have the same result for g if $\tilde{g} \in H^2(0,T;H^s(\mathbb{R}^d))$.

Analogously, for $\tilde{g} \in W^{2,1}(0,T;L^2(\mathbb{R}^d)) \cap L^2(0,T;H^s(\mathbb{R}^d))$ for $T \in]0,\infty]$, g is first constructed from $\tilde{g} \in H^2(0,T;H^s(\mathbb{R}^d))$, and then extended by density to obtain also $g \in W^{2,1}(0,T;L^2(\mathbb{R}^d)) \cap L^2(0,T;H^s(\mathbb{R}^d))$.

3.3 Dependence of Eigenfunctions of $\tilde{\mathcal{L}}^s_A$ on $0 < s \leq 1$

Here we show the continuity of the eigenfunctions of $\tilde{\mathcal{L}}^s_A$ with respect to the parameter s, $0 < s \leq 1$.

Recalling the compact embeddings $H^1_0(\Omega) \hookrightarrow H^s_0(\Omega) \hookrightarrow H^\sigma_0(\Omega) \hookrightarrow L^2(\Omega)$ for the bounded open set $\Omega \subset \mathbb{R}^d$, with Lipschitz boundary, where $0 < \sigma < s < 1$, consider the operator $T^s : L^2(\Omega) \to H^s_0(\Omega) \hookrightarrow L^2(\Omega)$, which depends on s, defined by $u^s = T^s(h) \in H^s_0(\Omega)$ corresponding to the homogeneous Dirichlet condition:

$$
u^s \in H^s_0(\Omega) : \quad \langle \tilde{\mathcal{L}}^s_A u^s, v \rangle = \int_{\mathbb{R}^d} AD^s u^s \cdot D^s v = \int_\Omega hv, \quad \forall v \in H^s_0(\Omega). \tag{3.9}
$$

Then, by the fractional Poincaré inequality Lemma 1.3 with Poincaré constant c_P/s, we have

$$
\|u^s\|^2_{L^2(\Omega)} \leq \frac{c_P^2}{s^2} \|D^s u^s\|^2_{L^2(\mathbb{R}^d)^d} \leq \frac{c_P^2}{s^2 a_*} \langle \tilde{\mathcal{L}}^s_A u^s, u^s \rangle \leq \frac{c_P^2}{s^2 a_*} \int_\Omega h u^s \leq \frac{c_P^2}{s^2 a_*} \|h\|_{L^2(\Omega)} \|u^s\|_{L^2(\Omega)} . \tag{3.10}
$$

Therefore, for $\sigma < s$,

$$
\|T^s\| = \sup_{\|h\|_{L^2(\Omega)} \leq 1} \left\| T^s(h) \right\|_{L^2(\Omega)} = \sup_{h \in L^2(\Omega)} \frac{\|u^s\|_{L^2(\Omega)}}{\|h\|_{L^2(\Omega)}} \leq \frac{c_P^2}{s^2 a_*} \leq \frac{c_P^2}{\sigma^2 a_*}.
$$

By the estimate (3.10), for $\sigma \leq s \to r \leq 1$, u^s converges strongly to some u^* in $L^2(\Omega)$. From (3.10), $\|D^s u^s\|_{L^2(\mathbb{R}^d)^d} \leq C$ for some constant C independent of s. Therefore,

$$
D^s u^s \underset{s \to r}{\rightharpoonup} \zeta \quad \text{in } L^2(\mathbb{R}^d)^d\text{-weak}
$$

for some ζ.

Now, for all $\Phi \in C_c^\infty(\mathbb{R}^d)^d$, for $s \to r$

$$D^s \cdot \Phi \to D^r \cdot \Phi \quad \text{in } L^2(\mathbb{R}^d)^d,$$

therefore

$$\int_{\mathbb{R}^d} D^s u^s \cdot \Phi = -\int_{\mathbb{R}^d} u^s (D^s \cdot \Phi) \xrightarrow[s\to r]{} -\int_{\mathbb{R}^d} u^* (D^r \cdot \Phi).$$

But by the a priori estimate on $D^s u^s$,

$$\left| \int_{\mathbb{R}^d} D^s u^s \cdot \Phi \right| \leq C \|\Phi\|_{L^2(\mathbb{R}^d)^d},$$

which implies that

$$\left| \int_{\mathbb{R}^d} u^* (D^r \cdot \Phi) \right| \leq C \|\Phi\|_{L^2(\mathbb{R}^d)^d} \quad \forall \Phi \in C_c^\infty(\mathbb{R}^d)^d.$$

This means that $D^r u^* \in L^2(\mathbb{R}^d)^d$, and since Ω has a Lipschitz boundary, $u^* \in H_0^r(\Omega)$.

Furthermore, since $D^s \cdot \Phi \to D^r \cdot \Phi$ strongly in $L^2(\mathbb{R}^d)^d$ as $s \to r$, so

$$\int_{\mathbb{R}^d} D^s (u^s - u^*) \cdot \Phi = -\int_{\mathbb{R}^d} (u^s - u^*)(D^s \cdot \Phi) \to 0 \quad \forall \Phi \in C_c^\infty(\mathbb{R}^d)^d,$$

therefore

$$\zeta = w - \lim_{s \to r} D^s u^s = D^r u^* \in L^2(\mathbb{R}^d)^d.$$

Taking test functions $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\mathbb{R}^d} A D^r u^* \cdot D^r \varphi = \lim_{s \to r} \int_{\mathbb{R}^d} A D^s u^s \cdot D^s \varphi = \lim_{s \to r} \int_\Omega h\varphi = \int_\Omega h\varphi \quad \forall \varphi \in C_c^\infty(\Omega).$$

Extending this by density to all test functions $v \in H_0^r(\Omega)$, by the uniqueness of the solution to the homogeneous Dirichlet boundary problem (3.9) with $s = r \leq 1$, we have that $u^* = u^r$. Therefore, for every $h \in L^2(\Omega)$, $T^s(h)$ converges to $T^r(h)$ in $L^2(\Omega)$ as $s \to r$.

Theorem 3.3. *Let $0 < \sigma \leq s, r \leq 1$. For the sequence of operators $T^s : L^2(\Omega) \to L^2(\Omega)$ given above, T^s converges to T^r strongly in the operator norm as $s \to r$.*

Proof. We first claim that, for each fixed s, it is possible to find an h^s in the unit ball of $L^2(\Omega)$ achieving the supremum, i.e.

$$\sup_{\|h\|_{L^2(\Omega)} \leq 1} \left\| T^s(h) - T^r(h) \right\|_{L^2(\Omega)} = \left\| T^s(h^s) - T^r(h^s) \right\|_{L^2(\Omega)}.$$

Indeed, for any maximising sequence $\{h_m\}_m$, we can extract a subsequence which converges weakly to some h^s which also belongs to the unit ball of $L^2(\Omega)$. Since the embedding from $L^2(\Omega)$ into $H^{-\sigma}(\Omega) \subset H^{-s}(\Omega) \cap H^{-r}(\Omega)$ is compact, and since T^s and T^r can also be considered continuous operators from $H^{-s}(\Omega)$ into $H_0^s(\Omega)$ and from $H^{-r}(\Omega)$ into $H_0^r(\Omega)$, respectively, both operators are also completely-continuous operators in $L^2(\Omega)$, and so taking m to infinity we have the conclusion.

Having obtained the sequence $\{h^s\}_s$, since each is the weak limit of a uniformly bounded sequence, there exists h in the unit ball of $L^2(\Omega)$ such that h^s converge weakly in $L^2(\Omega)$ and strongly in $H^{-\sigma}(\Omega)$ to h. Then, by Lemma 1.5 below, for $\sigma \leq s$, we have $\|u\|_{H_0^\sigma(\Omega)} \leq c_\sigma \|u\|_{H_0^s(\Omega)}$ for $u \in H_0^s(\Omega)$ and consequently

$$\|h\|_{H^{-s}(\Omega)} = \sup_{u \in H_0^s(\Omega)} \frac{\langle h, u \rangle}{\|u\|_{H_0^s(\Omega)}} \leq c_\sigma \|h\|_{H^{-\sigma}(\Omega)}.$$

As in (3.10), if $u = T^s(f)$ with $f \in H^{-s}(\Omega)$, we obtain

$$a_* \|u\|^2_{H_0^s(\Omega)} = a_* \|D^s u\|^2_{L^2(\mathbb{R}^d)^d} \leq \langle \tilde{\mathcal{L}}_A^s u, u \rangle = \int_\Omega f u \leq \|f\|_{H^{-s}(\Omega)} \|u\|_{H_0^s(\Omega)}, \quad \forall f \in H^{-s}(\Omega),$$

and then

$$\|T^s\|_s = \sup_{f\in H^{-s}(\Omega)} \frac{\|u\|_{H_0^s(\Omega)}}{\|f\|_{H^{-s}(\Omega)}} \leq \frac{1}{a_*}$$

for the operator norm $\|\cdot\|_s$ as an operator from $H^{-s}(\Omega)$ to $H^s(\Omega)$. Therefore, it follows that

$$\|T^s\|_\sigma = \sup_{f\in H^{-\sigma}(\Omega)} \frac{\left\|T^s(f)\right\|_{H_0^\sigma(\Omega)}}{\|f\|_{H^{-\sigma}(\Omega)}} \leq c_\sigma^2 \sup_{f\in H^{-s}(\Omega)} \frac{\left\|T^s(f)\right\|_{H_0^s(\Omega)}}{\|f\|_{H^{-s}(\Omega)}} = c_\sigma^2\|T^s\|_s \leq \frac{c_\sigma^2}{a_*}.$$

Similarly, we have

$$\|T^r\|_\sigma \leq \frac{c_\sigma^2}{a_*}.$$

Since $T^s(h)$ converges to $T^r(h)$ in $L^2(\Omega)$ for every $h \in L^2(\Omega)$, for any $\epsilon > 0$, we can pick a $\delta > 0$ such that, for $|s-r| \leq \delta$, we have

$$\|h^s - h\|_{H^{-\sigma}(\Omega)} \leq \frac{\epsilon a_*}{4c_\sigma^2} \quad \text{and } \left\|T^s(h) - T^r(h)\right\|_{L^2(\Omega)} \leq \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned}
\sup_{\|f\|_{L^2(\Omega)}\leq 1} \left\|T^s(f) - T^r(f)\right\|_{L^2(\Omega)} &= \left\|T^s(h^s) - T^r(h^s)\right\|_{L^2(\Omega)} \\
&\leq \left\|T^s(h) - T^r(h)\right\|_{L^2(\Omega)} + \left\|T^s(h^s-h) - T^r(h^s-h)\right\|_{L^2(\Omega)} \\
&\leq \frac{\epsilon}{2} + \left(\|T^s\|_\sigma + \|T^r\|_\sigma\right)\|h^s - h\|_{H^{-\sigma}(\Omega)} \\
&\leq \frac{\epsilon}{2} + \frac{2c_\sigma^2}{a_*}\frac{\epsilon a_*}{4c_\sigma^2} = \epsilon.
\end{aligned}$$

□

As a corollary, by Theorem 2.3.1 of [132], we have

Corollary 3.4. *For the operators T^s, T^r as given in the previous theorem, let $\lambda_k^s = \lambda_k^s(T^s)$ and $\lambda_k^r = \lambda_k^r(T^r)$be the k-th eigenvalues of T^s and of T^r respectively for s and for r, $0 < \sigma \leq s, r \leq 1$. Then,*

$$|\lambda_k^s - \lambda_k^r| \leq \|T^s - T^r\| := \sup_{\|f\|_{L^2(\Omega)}\leq 1} \left\|(T^s - T^r)(f)\right\|.$$

In particular, the map $[\sigma, 1] \ni s \mapsto \lambda_k^s \in]0, \infty[$ is continuous.

For each eigenvalue λ_k^s, let h_k^s be the associated eigenvector of T^s such that $T^s(h_k^s) = \lambda_k^s h_k^s$. Setting $u_k^s := T^s(h_k^s)$, we have $u_k^s = T^s(h_k^s) = \lambda_k^s h_k^s = \lambda_k^s \tilde{\mathcal{L}}_A^s u_k^s$, so $1/\lambda_k^s$ is the eigenvalue of $\tilde{\mathcal{L}}_A^s$ with associated eigenvector u_k^s.

Corollary 3.5. *Let u_k^s be the corresponding eigenfunctions of $1/\lambda_k^s$ for the operator $\tilde{\mathcal{L}}_A^s$ for $s \in [\sigma, r]$, $0 < \sigma < r \leq 1$. Then, the maps $[\sigma, 1] \ni s \mapsto u_k^s \in L^2(\Omega)$ and $]\sigma, 1] \ni r \mapsto u_k^r \in H_0^\sigma(\Omega)$ are also continuous.*

Proof. Since λ_k^s converges, so does $1/\lambda_k^s$. Therefore,

$$a_*\|D^s u_k^s\|^2_{L^2(\mathbb{R}^d)^d} \leq \langle \tilde{\mathcal{L}}_A^s u_k^s, u_k^s\rangle = \frac{1}{\lambda_k^s}\|u_k^s\|^2_{L^2(\Omega)}.$$

Normalising by $\left\|u_k^s\right\|_{L^2(\Omega)} = 1$, the convergence of the eigenvalues gives

$$a_*\|D^s u_k^s\|^2_{L^2(\mathbb{R}^d)^d} \leq \left(\frac{1}{\lambda_k^s} - \frac{1}{\lambda_k^r}\right) + \frac{1}{\lambda_k^r} \leq 1 + \frac{1}{\lambda_k^r}$$

for $|r-s|$ sufficiently small and for $r \le 1$ and k fixed. This means that the $H_0^s(\Omega)$ norm of u_k^s is bounded, so by compactness, there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ with $s_n \to r$ such that the corresponding sequence of eigenfunctions $\{u_k^{s_n}\}_{n\in\mathbb{N}}$ converges weakly in $H_0^\sigma(\Omega)$ and strongly in $L^2(\Omega)$ to some u_k^* for each k. This u_k^* corresponds to a $h_k^* = \frac{1}{\lambda_k^r} u_k^*$ which is the limit of h_k^s, where h_k^s satisfies $T^s(h_k^s) = \lambda_k^s h_k^s$. Since $\lambda_k^s \to \lambda_k^r$, $h_k^s = \frac{1}{\lambda_k^s} u_k^s$ converges to $h_k^* = \frac{1}{\lambda_k^r} u_k^*$ strongly in $L^2(\Omega)$ as $s \to r$, and by the convergence of the operator norm $T^s \to T^r$,

$$T^s(h_k^s) \to T^r(h_k^*) \quad \text{and} \quad \lambda_k^s \to \lambda_k^r \quad \text{as } s \to r.$$

Now, by the definition, the image of T^r lies in $H_0^r(\Omega)$, so $u_k^* = \lambda_k^r h_k^* = T^r(h_k^*) \in H_0^r(\Omega)$. Consequently, $h_k^* = h_k^r$, so $u_k^* = u_k^r$. Therefore, for every fixed k and r, u_k^s converges strongly to u_k^r in $L^2(\Omega)$ as $s \to r$, with $\left\|u_k^r\right\|_{L^2(\Omega)} = 1$, which yields the continuity of the map $[\sigma, 1] \ni s \mapsto u_k^s \in L^2(\Omega)$. Since $r > \sigma$, by the compactness of the inclusion $H_0^{\sigma'}(\Omega) \hookrightarrow H_0^\sigma(\Omega)$ for all $\sigma' > \sigma$, we also have the continuity of the map $]\sigma, 1] \ni r \mapsto u_k^r \in H_0^\sigma(\Omega)$.

□

Remark 3.6. *For the fractional operator $\tilde{\mathcal{L}}_A^s$, it is possible to identify the limit operator $\tilde{\mathcal{L}}_A^1$, unlike the general nonlocal operator $\mathcal{L}_a^s$ which is an open problem (see also Remark 2.10). This is an advantage of the fractional operator $\tilde{\mathcal{L}}_A^s$.*

3.4 Relationship with the Nonlocal Operator

In this section, we make use the results of the nonlocal vector calculus developed by Du, Gunzburger, Lehoucq, D'Elia and coworkers in [80, 100, 101, 129] to show that this fractional bilinear form can be rewritten in the form of the nonlocal integral operator $\mathcal{L}_a$, with a measurable (not necessarily symmetric) singular kernel $k_A : \mathbb{R}^d \times \mathbb{R}^d, d \ge 1$, defined by (3.13), as

$$\mathcal{E}_{k_A}(u,v) := P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{v}(x)(\tilde{u}(x) - \tilde{u}(y)) k_A(x,y)\, dy\, dx, \tag{3.11}$$

where $\tilde{u}, \tilde{v}$ are the zero extensions of $u, v \in C_c^\infty(\Omega)$ to Ω^c. We were also motivated by the issues raised by Shieh and Spector in [214], in particular their Open Problem 1.10, which by (1.8) clearly holds when $\tilde{\mathcal{L}}_A = (-\Delta)^s$. Discussing this issue with examples, we give a counterexample in Example 3.10, showing how interesting is their Open Problem 1.10 for general strictly elliptic and bounded matrices A. Then, we conjecture in the Open Problem that the kernel k_A corresponding to $\tilde{\mathcal{L}}_A$ has the required property if and only if $\tilde{\mathcal{L}}_A$ is approximately a constant multiple of the fractional Laplacian, up to small bounded perturbations.

Theorem 3.7. *Given a matrix $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ with bounded and measurable coefficients, there exists a kernel $k_A(x,y)$ independent of u, v satisfying*

$$\int_{\mathbb{R}^d} A(x) D^s u(x) \cdot D^s v(x)\, dx = P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v(x)(u(x) - u(y)) k_A(x,y)\, dy\, dx \tag{3.12}$$

for all $u, v \in C_c^\infty(\mathbb{R}^d)$, where $k_A(x,y)$ is given by

$$k_A(x,y) = c_{d,s}^2\, P.V. \int_{\mathbb{R}^d} A(z) \frac{y-z}{|y-z|^{d+s+1}} \cdot \frac{z-x}{|z-x|^{d+s+1}}\, dz \quad \text{for } x \ne y. \tag{3.13}$$

Proof. Expanding the fractional bilinear form, we have, setting for simplicity $\int_{\mathbb{R}^d}$ as $\int$,

$$\begin{aligned}
&\int A(z) D^s u(z) \cdot D^s v(z)\, dz \\
&= c_{d,s}^2 \int A(z) \left[\int (u(y) - u(z)) \frac{y-z}{|y-z|^{d+s+1}}\, dy \right] \cdot \left[\int (v(x) - v(z)) \frac{(x-z)}{|z-x|^{d+s+1}}\, dx \right] dz \\
&= c_{d,s}^2 \iiint \lim_{\varepsilon,\delta,\eta\to 0} \left[A(z)(u(y) - u(z)) \frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot (v(x) - v(z)) \frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}} \chi_\eta(x,y) \right] dx\, dy\, dz
\end{aligned}$$

$$= c_{d,s}^2 \lim_{\varepsilon,\delta,\eta\to 0} \iiint A(z)(u(y)-u(z))\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot (v(x)-v(z))\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\,dx\,dy\,dz,$$

where $\chi_\eta(x,y)$ is the characteristic function on the set $\{|x-y|>\eta\}$ and similarly defined for χ_ε and χ_δ. The limit can be exchanged with the integral by the Fubini and Lebesgue theorems because the integrand is Lebesgue integrable. Therefore,

$$\begin{aligned}
&\int A(z)D^s u(z)\cdot D^s v(z)\,dz \\
&= c_{d,s}^2 \lim_{\varepsilon,\delta,\eta\to 0} \iiint A(z)(u(y)-u(z))\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot v(x)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\,dx\,dy\,dz \\
&\quad - c_{d,s}^2 \lim_{\varepsilon,\delta,\eta\to 0} \iiint A(z)(u(y)-u(z))\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot v(z)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\,dx\,dy\,dz \\
&= c_{d,s}^2 \lim_{\varepsilon,\delta,\eta\to 0} \iiint A(z)(u(y)-u(z))\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot v(x)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\,dx\,dy\,dz \\
&\quad - c_{d,s}^2 \lim_{\varepsilon,\delta\to 0} \iint A(z)(u(y)-u(z))\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot v(z)\left[\int \lim_{\eta\to 0}\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\,dx\right]dy\,dz \\
&= c_{d,s}^2 \lim_{\varepsilon,\delta,\eta\to 0} \iiint A(z)(u(y)-u(z))\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}}\chi_\eta(x,y) \cdot v(x)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\,dx\,dy\,dz, \qquad (3.14)
\end{aligned}$$

using the fact that $\int \frac{(x-z)\chi_\varepsilon(x,z)}{|x-z|^{d+s+1}}\,dx = 0$ for all $\varepsilon>0$. Note that we can exchange the limit in η with the integrals because the functions are Lebesgue integrable as $|x-y|\to 0$.

Adding and subtracting $u(x)$, as $u,v\in C_c^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\int A(z)D^s u(z)\cdot D^s v(z)\,dz \\
&= c_{d,s}^2 \lim_{\varepsilon,\delta,\eta\to 0} \iiint A(z)[(u(y)-u(x))+(u(x)-u(z))]\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot v(x)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\,dx\,dy\,dz \\
&= c_{d,s}^2 \lim_{\varepsilon,\delta,\eta\to 0} \iiint A(z)(u(y)-u(x))\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot v(x)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\,dx\,dy\,dz \\
&\quad + c_{d,s}^2 \lim_{\varepsilon,\delta,\eta\to 0} \int A(z)\left[\int \frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot \left[\int (u(x)-u(z))v(x)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\,dx\right]dy\right]dz \\
&= c_{d,s}^2 \lim_{\varepsilon,\delta,\eta\to 0} \iiint A(z)(u(y)-u(x))\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot v(x)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\,dx\,dy\,dz \\
&\quad + c_{d,s}^2 \lim_{\varepsilon,\delta\to 0} \int A(z)\left[\int \frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot \left[\int \lim_{\eta\to 0}(u(x)-u(z))v(x)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\,dx\right]dy\right]dz \\
&= c_{d,s}^2 \lim_{\varepsilon,\delta,\eta\to 0} \iiint A(z)(u(y)-u(x))\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot v(x)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\,dx\,dy\,dz, \qquad (3.15)
\end{aligned}$$

where we make use of the fact that $\int \frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}}\cdot f(z)\,dy = 0$ for all $\delta>0$ for any finite function $f(z)$. Once again, the limit in η can be interchanged with triple integrals, because the factor $\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}}$ is integrable for $\delta>0$. Also, the function

$$f(z) := \lim_{\eta\to 0}\int (u(x)-u(z))\chi_\eta(x,y)v(x)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\,dx$$

is a finite function of z, because the integrand has a singularity only at $x=z$ and we have introduced the characteristic function $\chi_\varepsilon(x,z)$. Furthermore, the Lipschitz continuity of u guarantees that the singularity is removable, since we have the factor $u(x)-u(z)$. This is also the reason why we used the first expression in (1.4) for the expansion of $D^s u$, rather than the second one, which will only give us a factor of $u(x)$ when we add and subtract $u(x)$. Therefore, we can take the limit $\eta\to 0$, so that it is just a function of z.

Next, we apply Fubini's theorem, since the integrand is Lebesgue integrable for fixed $\varepsilon, \delta, \eta > 0$. Therefore,

$$\int A(z)D^s u(z) \cdot D^s v(z)\, dz$$
$$= c_{d,s}^2 \lim_{\varepsilon,\delta,\eta\to 0} \iiint A(z)(u(y)-u(x))\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot v(x)\frac{(x-z)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\chi_\eta(x,y)\, dz\, dy\, dx.$$

Finally, regarding this limit as a double limit, in η and separately in ε and δ, which exists, we can consider the iterated limit in the following form

$$\int A(z)D^s u(z) \cdot D^s v(z)\, dz$$
$$= \lim_{\eta\to 0} \iint (u(x)-u(y))v(x)\left[c_{d,s}^2 \lim_{\varepsilon,\delta\to 0}\int A(z)\frac{(y-z)\chi_\delta(y,z)}{|y-z|^{d+s+1}} \cdot \frac{(z-x)\chi_\varepsilon(x,z)}{|z-x|^{d+s+1}}\, dz\right]\chi_\eta(x,y)\, dy\, dx$$

where we may interpret the term in the parentheses as the Cauchy principal value about the singularities $z = x$ and $z = y$, i.e. as a function in x, y defined for $x \neq y$, by

$$k_A(x,y) = c_{d,s}^2\, P.V. \int A(z)\frac{y-z}{|y-z|^{d+s+1}} \cdot \frac{z-x}{|z-x|^{d+s+1}}\, dz. \tag{3.13}$$

□

Remark 3.8. *Note that in general, k_A is neither translation nor rotation invariant, unlike the case for the fractional Laplacian. In particular, k_A may not have the form $j(x-y)|x-y|^{-d-2s}$. Therefore, the kernel k_A may have relevance for non-homogeneous, non-isotropic and nonlocal problems. Even in the case for A being the constant coefficient matrix when k_A is translation invariant, it may not be rotation invariant, unless if A is a constant multiple of the identity matrix.*

Remark 3.9. *Suppose that the matrix A is given by $\alpha\mathbb{I}$ for a strictly positive finite constant α and the identity matrix $\mathbb{I}$. Then, by (1.8) and Proposition 2.2, $\tilde{\mathcal{L}}_{\alpha\mathbb{I}}$ defined by (3.1) can also be defined by a symmetric bilinear form as in (3.12) with a kernel α given by*

$$\alpha(x,y) = \frac{\alpha c_{d,s}^2}{|x-y|^{d+2s}}$$

which is, up to a constant, the kernel of the fractional Laplacian and satisfies (2.2) and (2.3).

However, we observe that this representation may not be unique, and $k_{\alpha\mathbb{I}}$ may not be equal to $\alpha c_{d,s}^2|x-y|^{-d-2s}$. Indeed, consider an unbounded nonzero L^2-integrable function $h(x) : \mathbb{R}^d \to \mathbb{R}$ which integrates to 0 over $\mathbb{R}^d$ and has support outside Ω. Let $a(x,y)$ be a kernel satisfying (2.2) and (2.3) and define

$$\tilde{a}(x,y) = a(x,y) + h(x)h(y),$$

which is possible since the kernel is defined over all $(\mathbb{R}^d \times \mathbb{R}^d)\backslash\{x = y\}$ and the integrability of h means that $\mathcal{L}_{\tilde{a}}$ is well defined. Since $\int h = 0$ by the construction of h and, for any $u, v \in C_c^\infty(\Omega)$, $\int \tilde{u}h = 0$ since they have disjoint supports, we have

$$\begin{aligned}\langle \mathcal{L}_{\tilde{a}}u, v\rangle &= P.V. \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}(x)(\tilde{u}(x)-\tilde{u}(y))(a(x,y)+h(x)h(y))\, dy\, dx \\ &= P.V. \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}(x)(\tilde{u}(x)-\tilde{u}(y))a(x,y)\, dy\, dx \\ &\quad + P.V. \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{u}(x)\tilde{v}(x)h(x)h(y)\, dy\, dx - P.V. \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{u}(y)\tilde{v}(x)h(x)h(y)\, dy\, dx \\ &= P.V. \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}(x)(\tilde{u}(x)-\tilde{u}(y))a(x,y)\, dy\, dx\end{aligned}$$

$$+ P.V. \int_{\mathbb{R}^d} \tilde{u}(x)\tilde{v}(x)h(x) \left[\int_{\mathbb{R}^d} h(y)\, dy\right] dx - P.V. \int_{\mathbb{R}^d} \tilde{v}(x)h(x) \left[\int_{\mathbb{R}^d} \tilde{u}(y)h(y)\, dy\right] dx$$
$$= P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{v}(x)(\tilde{u}(x) - \tilde{u}(y))a(x,y)\, dy\, dx = \langle \mathcal{L}_a u, v \rangle$$

This example gives a class of non-uniqueness for the representation of the kernel. There may be more similar classes, and it will be interesting to know a characterisation for the equivalent class of kernels. Furthermore, even if the kernel a satisfies the conditions (2.2) *and* (2.3)*, since h may change sign and may be unbounded, our construction of the kernel $\tilde{a}$ may not satisfy the condition* (2.2)*, nor the weaker conditions*

$$\begin{cases} a_* \leq a(x,y)|x-y|^{d+2s} \leq a^*, & |x-y| \leq 1 \\ a(x,y) \leq M|x-y|^{-d-s'}, & |x-y| > 1 \end{cases} \tag{3.16}$$

for some $a_*, a^*, M, s' > 0$, as given in equation (1.11) of [214].

However, (2.2) is not satisfied for the kernel k_A for a general matrix A. Indeed, we can construct a numerical counterexample as follows.

Example 3.10. *In $d = 1$, suppose $s = 0.8$, and consider the matrix $A = \alpha(x)$ where $\alpha(x) = 0.01 + 50H(x)$ for the smooth approximation H of the characteristic function of the interval $[1, 1.5]$, such that $H(x) = 1$ in $[1, 1.5]$ and less than 0.0001 outside $[0.9, 1.6]$. Then*

$$k_A(-0.5, 0.5) = 0.01c_{1,0.8}^2 P.V. \int_{\mathbb{R}} \frac{0.5 - z}{|0.5 - z|^{2.8}} \frac{z + 0.5}{|z + 0.5|^{2.8}}\, dz + 50c_{1,0.8}^2 P.V. \int_{\mathbb{R}} H(z) \frac{0.5 - z}{|0.5 - z|^{2.8}} \frac{z + 0.5}{|z + 0.5|^{2.8}}\, dz.$$

Observe that the function

$$\kappa_{1,0.8}(-0.5, 0.5, z) := \frac{0.5 - z}{|0.5 - z|^{2.8}} \frac{z + 0.5}{|z + 0.5|^{2.8}}$$

has the shape

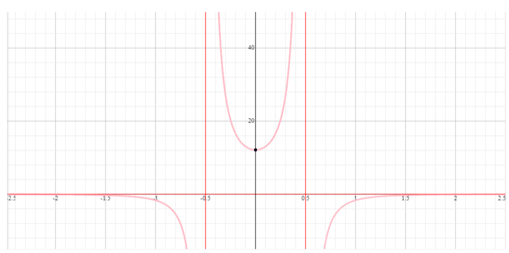

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Figure 1: Shape of an integrable but not absolutely integrable function

is integrable but not absolutely integrable, and is strictly increasing and strictly negative in the interval $[0.9, 1.6]$ with values (computed in Wolfram Alpha)

$$\kappa_{1,0.8}(-0.5, 0.5, 0.9) = -2.839, \quad \kappa_{1,0.8}(-0.5, 0.5, 1.5) = -0.287, \quad \int_{\mathbb{R}} \kappa_{1,0.8}(-0.5, 0.5, z)\, dz \approx 30 < 100.$$

Then, computing (3.13) *for*

$$\begin{aligned} k_A(-0.5, 0.5) &= 0.01c_{1,0.8}^2 P.V. \int_{\mathbb{R}} \frac{0.5 - z}{|0.5 - z|^{2.8}} \frac{z + 0.5}{|z + 0.5|^{2.8}}\, dz + 50c_{1,0.8}^2 P.V. \int_{\mathbb{R}} H(z) \frac{0.5 - z}{|0.5 - z|^{2.8}} \frac{z + 0.5}{|z + 0.5|^{2.8}}\, dz \\ &< 0.01c_{1,0.8}^2(100) + 50c_{1,0.8}^2 P.V. \int_1^{1.5} \frac{0.5 - z}{|0.5 - z|^{2.8}} \frac{z + 0.5}{|z + 0.5|^{2.8}}\, dz \\ &\quad + 50(0.0001)c_{1,0.8}^2 P.V. \int_{\mathbb{R}\setminus[0.9,1.6]} \frac{0.5 - z}{|0.5 - z|^{2.8}} \frac{z + 0.5}{|z + 0.5|^{2.8}}\, dz \end{aligned}$$

$$= c_{1,0.8}^2 + 50c_{1,0.8}^2 P.V. \int_1^{1.5} \frac{0.5-z}{|0.5-z|^{2.8}} \frac{z+0.5}{|z+0.5|^{2.8}}\, dz$$
$$+ 0.005c_{1,0.8}^2 \left(P.V. \int_{\mathbb{R}} \frac{0.5-z}{|0.5-z|^{2.8}} \frac{z+0.5}{|z+0.5|^{2.8}}\, dz - P.V. \int_{0.9}^{1.6} \frac{0.5-z}{|0.5-z|^{2.8}} \frac{z+0.5}{|z+0.5|^{2.8}}\, dz \right)$$
$$< c_{1,0.8}^2 + 50c_{1,0.8}^2 P.V. \int_1^{1.5} \kappa_{1,0.8}(-0.5, 0.5, 1.5)\, dz$$
$$+ 0.005c_{1,0.8}^2 \left(P.V. \int_{\mathbb{R}} \kappa_{1,0.8}(-0.5, 0.5, z)\, dz - P.V. \int_{0.9}^{1.6} \kappa_{1,0.8}(-0.5, 0.5, 0.9)\, dz \right)$$
$$< c_{1,0.8}^2 + 50c_{1,0.8}^2(-0.28)(0.5) + 0.005c_{1,0.8}^2(100 - (-2.84)(1.6 - 0.9)) = -5.49c_{1,0.8}^2 < 0$$

which contradicts (2.2)*. Compare with Open Problem 1.10 of [214].*

□

Theorem 3.7 and the last two remarks were inspired by the Open Problem 1.10 of [214], which asked if, given a symmetric matrix A satisfying (3.2), it is possible to find a kernel k_A satisfying (3.16) such that (3.12) holds. Complementing this open problem, we propose the following conjecture (see also Remark 3.1 of [79]):

Open Problem. *Suppose $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a bounded, measurable and strictly elliptic matrix such that*

$$a_*|z|^2 \le A(x)z \cdot z \text{ and } A(x)z \cdot z^* \le a^*|z||z^*| \tag{3.2}$$

for some $a_*, a^* > 0$ for all $x \in \mathbb{R}^d$ and all $z, z^* \in \mathbb{R}^d$. Let k_A be a corresponding kernel which is continuous outside the diagonal $x = y$ and satisfies

$$\int_{\mathbb{R}^d} A(x) D^s u(x) \cdot D^s v(x)\, dx = P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v(x)(u(x) - u(y)) k_A(x, y)\, dy\, dx$$

for all $u, v \in C_c^\infty(\mathbb{R}^d)$. Then, there exists an equivalent kernel k_A satisfying

$$a_* \le k_A(x, y)|x - y|^{d+2s} \le a^* \quad \forall x, y \in \mathbb{R}^d, x \ne y$$

for some $a_*, a^* > 0$ if and only if A is a bounded small perturbation $\tilde{\alpha}(x)$ of the identity matrix (up to a positive constant α), i.e. $A = (\alpha + \tilde{\alpha}(x))\mathbb{I}$ for some strictly positive finite constant $\alpha >> \sup_x |\tilde{\alpha}(x)|$.

Part II

Applications to Fractional and Nonlocal Problems

4 Nonlocal and the Fractional Obstacle-Type Problems

4.1 Introduction

Fractional problems with obstacle-type constraints were first considered by Silvestre as part of his thesis in 2005 in [219], which was published in 2007 in [218]. Since then, many obstacle-type problems with various nonlocal operators have been considered, extensively for the fractional Laplacian (such as in [60], [88], [177], [178] and [197]), as well as for other nonlocal operators (see, for example, [2], [14], [61], [87], [205] and [212]), which are mainly generalisations of the fractional Laplacian to nonlocal one-variable kernels $K(y)$ satisfying homogeneous and symmetry properties (see in particular, [2] and [61]).

Recall from Section 1.2 the classical fractional Sobolev space $H^s_0(\Omega)$ in a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, for $0 < s < 1$, defined as

$$H^s_0(\Omega) := \overline{C^\infty_c(\Omega)}^{\|\cdot\|_{H^s}},$$

with

$$\|u\|^2_{H^s} = \|u\|^2_{L^2(\mathbb{R}^d)} + \|D^s u\|^2_{L^2(\mathbb{R}^d)},$$

where u is extended by 0 in $\mathbb{R}^d \backslash \Omega$, so that this extension is also in $H^s(\mathbb{R}^d)$. By the Sobolev-Poincaré inequality (see Theorem 1.7 of [213] and Lemma 1.3), we may consider the space $H^s_0(\Omega)$ with the following equivalent norms

$$\|u\|^2_{H^s_0(\Omega)} := \|D^s u\|^2_{L^2(\mathbb{R}^d)} = \frac{c^2_{d,s}}{2}[u]^2_{s,\mathbb{R}^d} := \frac{c^2_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}}\,dx\,dy. \tag{1.16}$$

We can subsequently denote the dual space of $H^s_0(\Omega)$ by $H^{-s}(\Omega)$ for $0 < s \leq 1$. Then, by the Sobolev-Poincaré inequalities, we have the embeddings

$$H^s_0(\Omega) \hookrightarrow L^q(\Omega), \qquad L^{2^\#}(\Omega) \hookrightarrow H^{-s}(\Omega) = (H^s_0(\Omega))'$$

for $1 \leq q \leq 2^*$, where $2^* = \frac{2d}{d-2s}$ and $2^\# = \frac{2d}{d+2s}$ when $s < \frac{d}{2}$, and if $d = 1$, $2^* = q$ for any finite q and $2^\# = q' = \frac{q}{q-1}$ when $s = \frac{1}{2}$ and $2^* = \infty$ and $2^\# = 1$ when $s > \frac{1}{2}$. We recall that those embeddings are compact for $1 \leq q < 2^*$ (see for example, Theorem 4.54 of [93]). In this chapter, we use $2^\#$ to indicate this number that depends on $d \geq 1$ and $0 < s \leq 1$.

We consider the closed convex set

$$\mathbb{K}^s_\psi = \{v \in H^s_0(\Omega) : v \geq \psi \text{ a.e. in } \Omega\},$$

with a given obstacle ψ, such that $\mathbb{K}^s_\psi \neq \emptyset$, and the obstacle problem

$$u \in \mathbb{K}^s_\psi : \quad \langle \mathcal{L}_a u, v-u\rangle \geq \langle F, v-u\rangle \quad \forall v \in \mathbb{K}^s_\psi, \tag{4.1}$$

for F in $H^{-s}(\Omega)$. Here, the nonlocal operator $\mathcal{L}_a : H^s_0(\Omega) \to H^{-s}(\Omega)$ is a generalisation of the fractional Laplacian for a measurable, strictly positive kernel $a : \mathbb{R}^d \times \mathbb{R}^d \backslash D \to]0,\infty[$ for the diagonal $D = \{(x,x) : x \in \mathbb{R}^d\}$ satisfying (2.2) and (2.3), and is defined in the duality sense for $u, v \in H^s_0(\Omega)$, extended by zero outside Ω:

$$\langle \mathcal{L}_a u, v\rangle = P.V.\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}(x)(\tilde{u}(x)-\tilde{u}(y))a(x,y)\,dy\,dx. \tag{2.1}$$

Physically, the operator $\mathcal{L}_a$ corresponds to the class of uniformly irreducible random walks that admit a cycle decomposition with bounded range, bounded length of cycles, and bounded jump rates [94].

Recall that we have shown, in Section 2.2, that the bilinear form

$$\mathcal{E}_a(u,v) := \langle \mathcal{L}_a u, v\rangle = P.V.\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{v}(x)(\tilde{u}(x)-\tilde{u}(y))a(x,y)\,dy\,dx$$

is a (not necessarily symmetric) Dirichlet form over $H^s_0(\Omega) \times H^s_0(\Omega)$, where $\tilde{u}$ and $\tilde{v}$ are the zero extensions of u and v outside Ω respectively. This provides us with many known properties of Dirichlet forms that can be applied to the bilinear form $\mathcal{E}_a$, including the truncation property, T-monotonicity, and some regularity results.

Consequently, in Section 4.2, we make use of the comparison property to consider obstacle-type problems involving the bilinear form $\mathcal{E}_a$, thereby considering the nonlocal obstacle problem for which we derive results similar to the classical case in $H^1_0(\Omega)$ as in [221], [145] and [195], such as the weak maximum principle and comparison properties.

By considering the approximation of the obstacle problem by semilinear problems using a bounded penalisation, in Section 4.3, we give a direct proof of the Lewy-Stampacchia inequalities for the obstacle-type problems. Here we consider the non-homogeneous data $F = f \in L^{2^\#}(\Omega)$ not only for the one obstacle problem but also for the two obstacles problem and for the N membranes problem in the nonlocal framework, extending the results of [212]. In particular, we extend the estimates in energy of the difference between the approximating solutions and the solutions of the one and the two obstacle problems, which may be useful for numerical applications such as in [45] or [181]. More important is the use of the Lewy-Stampacchia inequalities that, upon restricting a to the symmetric case, allows the application of the results of [103] to obtain locally the Hölder regularity of the solutions to those three nonlocal obstacle type problems. Such regularity results are weaker than those obtained from the fractional Laplacian (such as in [198]) or other commonly considered nonlocal kernels [106], since a is in general not a constant multiple of $|x-y|^{-d-2s}$. In this special case, when $\mathcal{L}_a = (-\Delta)^s$, the one obstacle problem can be written for $\boldsymbol{f} \in [L^2(\mathbb{R}^d)]^d$

$$u \in \mathbb{K}^s_\psi(\Omega) : \int_{\mathbb{R}^d} (D^s u - \boldsymbol{f}) \cdot D^s(v-u)\, dx \geq 0 \quad \forall v \in \mathbb{K}^s_\psi(\Omega),$$

as well as for the corresponding inequalities for the two obstacles and the N membranes problems. We are then able to use the results of [37] together with the Lewy-Stampacchia inequalities to obtain locally regular solutions in the fractional Sobolev space $W^{2s,p}_{loc}(\Omega)$ for $p \geq 2^\#$ and also in $C^1(\Omega)$ for $s > 1/2$ and $p > d/(2s-1)$ when $D^s \cdot \boldsymbol{f} \in L^p(\mathbb{R}^d)$.

In Section 4.4, we further consider some properties related to the fractional s-capacity extending some classical results of Stampacchia [221]. We characterise the order dual of $H^s_0(\Omega)$ as the dual space of $L^2_{C_s}(\Omega)$, i.e. the space of quasi-continuous functions with respect to the s-capacity which are in absolute value quasi-everywhere dominated by $H^s_0(\Omega)$ functions, extending results of [24]. That dual space corresponds then to the elements of $H^{-s}(\Omega)$ that are also bounded measures, i.e. $(L^2_{C_s}(\Omega))' = H^{-s}(\Omega) \cap M(\Omega)$. Therefore, using the strict T-monotonicity of $\mathcal{L}_a$, we state the Lewy-Stampacchia inequalities in this dual space. This section ends with some new remarks on the relations of the $\mathcal{E}_a$ obstacle problem and the s-capacity.

Finally, in Section 4.5, we consider the fractional obstacle problem

$$u \in \mathbb{K}^s_\psi : \quad \langle \tilde{\mathcal{L}}_A u, v-u \rangle \geq \langle F, v-u \rangle \quad \forall v \in \mathbb{K}^s_\psi \tag{4.2}$$

where $\tilde{\mathcal{L}}_A$ is defined by

$$\langle \tilde{\mathcal{L}}^s_A u, v \rangle := \int_{\mathbb{R}^d} A(x) D^s u \cdot D^s v\, dx, \quad \forall u, v \in H^s_0(\Omega) \tag{3.1}$$

for the matrix A satisfying coercivity and boundedness properties given in (3.2). Note that the two problems (4.1) and (4.2) are in general not the same, but they coincide in the case of $(-\Delta)^s$. Making use of convergence properties of the fractional derivatives when $s \nearrow 1$ to the classical derivatives, as already observed in [193], [31] and [73], we show that its solution converges to the solution of the classical case corresponding to $s = 1$.

We start by considering the linear form for $F \in H^{-s}(\Omega)$ defined by

$$\langle F, v \rangle = \int_\Omega f_\# v + \int_{\mathbb{R}^d} \boldsymbol{f} \cdot D^s v$$

for $v \in H^s_0(\Omega)$, $\boldsymbol{f} = (f_1, \ldots, f_d) \in [L^2(\mathbb{R}^d)]^d$ and $f_\# \in L^{2^\#}(\Omega)$ where $2^\# = \frac{2d}{d+2s}$ when $s < \frac{d}{2}$, and if $d = 1$, $2^\# = q$ for any finite $q > 1$ when $s = \frac{1}{2}$ and $2^\# = 1$ when $s > \frac{1}{2}$. By the Riesz representation theorem, we have

$$\exists! \phi \in H^s_0(\Omega) : \int_{\mathbb{R}^d} D^s \phi \cdot D^s v = \langle -D^s \cdot D^s \phi, v \rangle = \langle F, v \rangle, \quad \forall v \in H^s_0(\Omega).$$

Therefore, $F \in H^{-s}(\Omega)$ may be given by $F = -D^s \cdot D^s \phi = -D^s \cdot \boldsymbol{g}$ for some $\boldsymbol{g} = (g_1, \ldots, g_d) \in [L^2(\mathbb{R}^d)]^d$ and, by the Sobolev-Poincaré inequality, it satisfies

$$\|\boldsymbol{g}\|_{[L^2(\mathbb{R}^d)]^d} = \|F\|_{H^{-s}(\Omega)} \leq C_S \|f_\#\|_{L^{2^\#}(\Omega)} + \|\boldsymbol{f}\|_{[L^2(\mathbb{R}^d)]^d}.$$

In order for

$$F \sim f_\# - D^s \cdot \boldsymbol{f}$$

to lie in the positive cone of $H^{-s}(\Omega)$, it is enough for $f_\#$ to be non-negative almost everywhere in Ω and $D^s \cdot \boldsymbol{f} \leq 0$ in the distributional sense in $\mathbb{R}^d$.

4.2 The Nonlocal Obstacle Problem and Its Properties

As a consequence of the properties of the bilinear form $\mathcal{E}_a$ defined by (2.11) in $H^s_0(\Omega)$ for $0 < s < 1$, we can derive classical properties of the fractional obstacle problems, following most of the approach of Section 4:5 in [195].

Theorem 4.1 (Obstacle Problem)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary, $f_\# \in L^{2^\#}(\Omega)$, $\boldsymbol{f} \in [L^2(\mathbb{R}^d)]^d$, and $a : \mathbb{R}^d \times \mathbb{R}^d \backslash D \to]0, \infty[$ for the diagonal $D = \{(x,x) : x \in \mathbb{R}^d\}$ be strictly elliptic and measurable satisfying* (2.2) *and* (2.3)*. Then, for every function ψ, measurable in Ω, and admissible in the sense that the closed convex set*

$$\mathbb{K}^s_\psi = \{v \in H^s_0(\Omega) : v \geq \psi \text{ a.e. in } \Omega\} \neq \emptyset,$$

there exists a unique $u \in \mathbb{K}^s_\psi$ such that

$$\mathcal{E}_a(u, v-u) \geq \int_\Omega f_\#(v-u) + \int_{\mathbb{R}^d} \boldsymbol{f} \cdot D^s(v-u) \quad \forall v \in \mathbb{K}^s_\psi. \tag{4.3}$$

Moreover, suppose $F, \hat{F}$ are given as in the beginning of this section for two different obstacle problems defined in (4.3)*, then the solution map $F \mapsto u$ is Lipschitz continuous, i.e.*

$$\|u - \hat{u}\|_{H^s_0(\Omega)} \leq \frac{c^2_{d,s}}{a_*} \left(C_S \left\| f_\# - \hat{f}_\# \right\|_{L^{2^\#}(\Omega)} + \left\| \boldsymbol{f} - \hat{\boldsymbol{f}} \right\|_{[L^2(\mathbb{R}^d)]^d} \right).$$

Proof. This is just a direct application of the Stampacchia theorem, since the bilinear form $\mathcal{E}_a : H^s_0(\Omega) \times H^s_0(\Omega) \to \mathbb{R}$ is bounded and coercive by Theorem 2.5.

For the continuous dependence on data, if $u, \hat{u}$ are the solutions corresponding to different data F and $\hat{F}$ for the obstacle problem respectively, we set $v = \hat{u}$ in the inequality for u and $v = u$ in the inequality for $\hat{u}$, and take the difference to obtain

$$\mathcal{E}_a(u - \hat{u}, u - \hat{u}) \leq \int_\Omega (f_\# - \hat{f}_\#)(u - \hat{u}) + \int_{\mathbb{R}^d} (\boldsymbol{f} - \hat{\boldsymbol{f}}) \cdot D^s(u - \hat{u}).$$

By the fractional Sobolev inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\frac{a_*}{c^2_{d,s}} \left\| D^s(u-\hat{u}) \right\|^2_{L^2(\mathbb{R}^d)} &\leq \mathcal{E}_a(u-\hat{u}, u-\hat{u}) \\
&\leq \int_\Omega (f_\# - \hat{f}_\#)(u-\hat{u}) + \int_{\mathbb{R}^d} (\boldsymbol{f} - \hat{\boldsymbol{f}}) \cdot D^s(u-\hat{u}) \\
&\leq \left\| f_\# - \hat{f}_\# \right\|_{L^{2^\#}(\Omega)} \|u-\hat{u}\|_{L^{2^*}(\Omega)} + \left\| \boldsymbol{f} - \hat{\boldsymbol{f}} \right\|_{[L^2(\mathbb{R}^d)]^d} \left\| D^s(u-\hat{u}) \right\|_{[L^2(\mathbb{R}^d)]^d} \\
&\leq \left(C_S \left\| f_\# - \hat{f}_\# \right\|_{L^{2^\#}(\Omega)} + \left\| \boldsymbol{f} - \hat{\boldsymbol{f}} \right\|_{[L^2(\mathbb{R}^d)]^d} \right) \left\| D^s(u-\hat{u}) \right\|_{[L^2(\mathbb{R}^d)]^d}
\end{aligned}$$

where C_S is the Sobolev constant of Lemma 1.1. □

Furthermore, we have the following properties of the solution as in the classical case, making use of the strict T-monotonicity of $\mathcal{E}_a$ and the fractional Poincaré inequality. See for example, Chapter IV of [145] or Section 4:5–6 of [195], where the proofs can be transferred to the nonlocal case almost in the same manner.

Proposition 4.2. *(i) (Comparison principle). Suppose u is the solution of the variational inequality* (4.3) *with data F and convex set $\mathbb{K}^s_\psi$, and $\hat{u}$ be the solution with data $\hat{F}$ and convex set $\mathbb{K}^s_{\hat{\psi}}$. If $\psi \geq \hat{\psi}$ and $F \geq \hat{F}$, then $u \geq \hat{u}$ a.e. in Ω.*

(ii) (Weak maximum principle). In the obstacle problem (4.3)*, one has*

$$u \geq 0 \ \text{a.e. in } \Omega, \ \text{if } F \geq 0; \ \text{and}$$

$$u \leq 0 \vee \sup_\Omega \psi \ \text{a.e. in } \Omega, \ \text{if } F \leq 0.$$

(iii) (Complementary problem). When $\psi \in H^s_0(\Omega)$, the variational problem (4.3) *is equivalent to the nonlinear complementary problem*

$$u \geq \psi, \quad \mathcal{L}_a u - F \geq 0 \quad \text{and} \quad \langle \mathcal{L}_a u - F, u - \psi \rangle = 0. \tag{4.4}$$

(iv) (Comparison of supersolutions).

(a) If u is the solution to the variational inequality (4.3) *and w is any supersolution, then $u \leq w$;*

(b) If v and w are two supersolutions to the variational inequality (4.3)*, then $v \wedge w$ is also a supersolution,*

where a supersolution is an element $w \in H^s_0(\Omega)$ satisfying $w \geq \psi$ and $\mathcal{L}_a w - F \geq 0$ in the sense of order dual.

(v) The solution u of the obstacle problem (4.3) *is the unique function in $H^s_0(\Omega)$, such that,*

$$u = \min\{w \in H^s_0(\Omega) : \mathcal{L}_a w - F \geq 0, w \geq \psi \text{ in } \Omega\}.$$

(vi) (L^∞ estimates) The following estimate holds:

$$\|u - \hat{u}\|_{L^\infty(\Omega)} \leq \|\psi - \hat{\psi}\|_{L^\infty(\Omega)}.$$

Proof of (i). Since $u, \hat{u}$ are the solutions to the variational inequalities

$$\langle \mathcal{L}_a u - F, v - u \rangle \geq 0 \quad \forall v \in \mathbb{K}^s_\psi$$

$$\langle \mathcal{L}_a \hat{u} - \hat{F}, v - \hat{u} \rangle \geq 0 \quad \forall v \in \mathbb{K}^s_{\hat{\psi}}$$

respectively, we take $v = u \vee \hat{u} = u + (\hat{u} - u)^+ \in \mathbb{K}^s_\psi$ and $v = u \wedge \hat{u} = \hat{u} - (\hat{u} - u)^+ \in \mathbb{K}^s_{\hat{\psi}}$ because $(\hat{u} - u)^+ \in H^s_0(\Omega)$. Summing the two equations then gives

$$\langle \mathcal{L}_a(\hat{u} - u), (\hat{u} - u)^+ \rangle + \langle F - \hat{F}, (\hat{u} - u)^+ \rangle \leq 0.$$

Since $F - \hat{F} \geq 0$, using Theorem 2.8 that $\mathcal{E}_a$ is strictly T-monotone, we have $u \geq \hat{u}$. □

Proof of (ii). Take $v = u \vee 0 \in \mathbb{K}^s_\psi$ in (4.1). Then, by the strict T-monotonicity of $\mathcal{E}_a$, we have

$$0 \geq -\langle F, (0-u)^+ \rangle \geq -\langle \mathcal{L}_a u, (0-u)^+ \rangle = \langle \mathcal{L}_a(-u), (-u)^+ \rangle \geq \frac{a_*}{c^2_{d,s}} \int_{\mathbb{R}^d} |D^s(-u)^+|^2 \geq \frac{a_*}{c^2_{d,s} C_P^2} \int_\Omega |(-u)^+|^2 \geq 0$$

by the fractional Poincaré inequality 1.3. Therefore, $(-u)^+ = 0$ a.e. in Ω, and we have the first result. Analogously, choosing $v = u \wedge (0 \vee \sup_\Omega \psi) < +\infty$ gives the second result. □

Proof of (iii). Suppose u solves the variational inequality (4.1). Letting $v = u + w \geq \psi$ for arbitrary $w \geq 0$, we have

$$\langle \mathcal{L}_a u - F, w \rangle \geq 0 \quad \forall w,$$

so

$$\mathcal{L}_a u - F \geq 0.$$

Also, setting $v = \psi$ and $v = 2u - \psi$, we obtain

$$0 \leq \langle \mathcal{L}_a u - F, \psi - u \rangle = -\langle \mathcal{L}_a u - F, u - \psi \rangle = -\langle \mathcal{L}_a u - F, (2u - \psi) - u \rangle \leq 0,$$

so

$$\langle \mathcal{L}_a u - F, \psi - u \rangle = 0.$$

Conversely, if u solves the nonlinear complementary problem (4.4), then for arbitrary $v \geq \psi$, we have

$$\langle \mathcal{L}_a u - F, v - u \rangle = \langle \mathcal{L}_a u - F, v - \psi \rangle + \langle \mathcal{L}_a u - F, \psi - u \rangle \geq 0.$$

This is because $\mathcal{L}_a u - F \geq 0$ and $v - \psi \geq 0$, so the first term is greater than or equal to 0, while the second term is equal to 0. □

Proof of (iv) and (v). (a) Since both $u \geq \psi$ and $w \geq \psi$, taking $v = u \wedge w = u - (u - w)^+$ in the variational inequality (4.1), v satisfies $v \geq \psi$ and

$$\langle \mathcal{L}_a u - F, (u - w)^+ \rangle \leq 0.$$

At the same time, since $(u - w)^+ \geq 0$ and w is a supersolution, we have

$$\langle \mathcal{L}_a w - F, (u - w)^+ \rangle \geq 0,$$

hence

$$\langle \mathcal{L}_a u - \mathcal{L}_a w, (u - w)^+ \rangle \leq 0.$$

By the strict monotonicity of $\mathcal{L}_a$ given in Theorem 2.8, $(u - w)^+ = 0$, i.e. $u \leq w$.

(b) Consider, using the Lions-Stampacchia theorem, the unique solution $z \in H^s_0(\Omega)$ to the complementary problem

$$z \geq v \wedge w, \quad \mathcal{L}_a z - F \geq 0 \quad \text{and} \quad \langle \mathcal{L}_a z - F, z - v \wedge w \rangle = 0.$$

Since v and w are still supersolutions to this problem, from part (i), it follows that $z \leq v$ and $z \leq w$, so $z \leq v \wedge w$. Therefore, $z = v \wedge w$, and since this gives $z \geq \psi$, z is also a supersolution to the nonlinear complementary problem (4.4).

Applying this result to the Dirichlet fractional obstacle problem (4.3), a supersolution $w \in H^s_0(\Omega)$ to the operator $\mathcal{L}_a - F$ in Ω is such that $w \geq \psi$ in Ω. Since the solution u to (4.3) is itself a supersolution, analogously to part (a) of the (iv), one has the result for (v). □

Proof of (vi). Given that $u, \hat{u}$ are the solutions to the variational inequalities

$$\langle \mathcal{L}_a u - F, v - u \rangle \geq 0 \quad \forall v \in \mathbb{K}^s_\psi$$

$$\langle \mathcal{L}_a \hat{u} - F, v - \hat{u} \rangle \geq 0 \quad \forall v \in \mathbb{K}^s_{\hat{\psi}}$$

respectively, setting $\ell = \left\| \psi - \hat{\psi} \right\|_{L^\infty(\Omega)} < \infty$, we take $v = u + (\hat{u} - u - \ell)^+ \in \mathbb{K}^s_\psi$ and $v = \hat{u} - (\hat{u} - u - \ell)^+ \in \mathbb{K}^s_{\hat{\psi}}$. Summing the two equations then gives

$$\langle \mathcal{L}_a(\hat{u} - u), (\hat{u} - u - \ell)^+ \rangle \leq \langle F - F, (\hat{u} - u - \ell)^+ \rangle = 0.$$

But by the T-monotonicity of $\mathcal{E}_a$ given in Theorem 2.8, since ℓ is a constant,

$$\langle \mathcal{L}_a(\hat{u} - u), (\hat{u} - u - \ell)^+ \rangle = \langle \mathcal{L}_a(\hat{u} - u - \ell), (\hat{u} - u - \ell)^+ \rangle \geq \frac{a_*}{c^2_{d,s}} \int_{\mathbb{R}^d} |D^s(\hat{u} - u - \ell)^+|^2 \geq 0.$$

Therefore $(\hat{u} - u)^+ = 0$ and $\hat{u} \leq u + \ell$. Repeating with u interchanged with $\hat{u}$, we also obtain $u \leq \hat{u} + \ell$. □

Similarly, as in Theorem 6.1 of Chapter 4 of [195], we can prove the following additional result for the Dirichlet form $\mathcal{E}_\lambda$ with $\lambda > 0$.

Proposition 4.3. *All the results in Proposition 4.2 hold for $\mathcal{E}_\lambda$ when $\lambda > -\lambda_1^s$ for the first eigenvalue λ_1^s of $\mathcal{L}_a$, where*

$$\mathcal{E}_\lambda(u,v) = \mathcal{E}_a(u,v) + \lambda \int_\Omega uv, \quad u,v \in H_0^s(\Omega). \tag{4.5}$$

Moreover in this case, when $\boldsymbol{f} \equiv \boldsymbol{0}$, the following maximum principle holds a.e. in Ω:

$$0 \wedge \inf_\Omega \left(\frac{f_\#}{\lambda}\right) \leq u \leq 0 \vee \sup_\Omega \psi \vee \sup_\Omega \left(\frac{f_\#}{\lambda}\right).$$

Proof. The first part of the proposition follows since $\mathcal{E}_\lambda$ remains strictly T-monotone.

In the obstacle problem (4.3), one has

$$u \geq 0 \text{ a.e. in } \Omega, \text{ if } F \geq 0; \text{ and}$$

$$u \leq 0 \vee \sup_\Omega \psi \text{ a.e. in } \Omega, \text{ if } F \leq 0.$$

For the maximum principle, we follow similarly the previous proof of Proposition 4.2(ii), this time choosing $v = u \vee m < +\infty$ and $v = u \wedge M < +\infty$ in (4.3), where $m = 0 \wedge \inf_\Omega \left(\frac{f_\#}{\lambda}\right)$ and $M = 0 \vee \sup_\Omega \psi \vee \sup_\Omega \left(\frac{f_\#}{\lambda}\right)$ denote the left and right expressions respectively. Then, by the strict T-monotonicity of $\mathcal{E}_a$, since $\lambda m \leq f_\#$, we have, for $\lambda > -\lambda_1^s$,

$$0 \geq -\langle \lambda m - \lambda u, (m-u)^+ \rangle \geq -\langle f_\# - \lambda u, (m-u)^+ \rangle \geq -\langle \mathcal{L}_a u, (m-u)^+ \rangle = \langle \mathcal{L}_a(m-u), (m-u)^+ \rangle \geq 0.$$

Therefore, $u \geq m$ a.e. in Ω, and analogously we obtain the other bound. □

Remark 4.4. *If $\Omega = \mathbb{R}^d$ and since the kernel a is defined in the whole $\mathbb{R}^d$, the domain of $\mathcal{E}_\lambda$, $D(\mathcal{E}_\lambda)$, is instead given by $H^s(\mathbb{R}^d)$, and the Dirichlet form $\mathcal{E}_\lambda$ is coercive for $\lambda > 0$.*

4.3 Lewy-Stampacchia Inequalities and Local Regularity

In this section, we take $f = f_\#$ and $\boldsymbol{f} = 0$. We give a direct proof of the Lewy-Stampacchia inequalities. This will follow much of the approach of Section 5:3.3 in [195] or Chapter IV of [145]. The Lewy-Stampacchia inequalities will allow us to apply the results of [103], [108] to obtain local Hölder regularity of the solutions when a is symmetric, and additional regularity on fractional Sobolev spaces when $\mathcal{L}_a = (-\Delta)^s$ using [37].

4.3.1 Bounded penalisation of the obstacle problem in $H_0^s(\Omega)$

Assume now that the obstacle $\psi \in H^s(\mathbb{R}^d)$, so that we may define $\mathcal{L}_a\psi \in H^{-s}(\Omega)$ by (2.1) for any test function $v \in H_0^s(\Omega)$, and ψ is such that the convex set $\mathbb{K}_\psi^s \neq \emptyset$. Consider the approximation to the obstacle problem, where the penalisation is based on any nondecreasing Lipschitz function $\theta : \mathbb{R} \to [0,1]$ such that

$$\theta \in C^{0,1}(\mathbb{R}), \quad \theta' \geq 0, \quad \theta(+\infty) = 1 \quad \text{ and } \theta(t) = 0 \text{ for } t \leq 0;$$

$$\exists C_\theta > 0 : [1 - \theta(t)]t \leq C_\theta, \quad t > 0.$$

Then, for any $\varepsilon > 0$, consider the family of functions $\theta_\varepsilon(t) = \theta\left(\frac{t}{\varepsilon}\right)$, $\quad t \in \mathbb{R}$, which converges as $\varepsilon \to 0$ to the multi-valued Heaviside graph. Examples of such sequences of functions include $\theta(t) = t/(1+t)$, $\theta(t) = (2/\pi)\arctan t$, or from any non-decreasing Lipschitz function $0 \leq \theta \leq 1$ such that $\theta(t) = 1$ for $t \geq t_* > 0$.

Assume that

$$f, (\mathcal{L}_a\psi - f)^+ \in L^{2^\#}(\Omega). \tag{4.6}$$

For $\zeta \in L^{2^\#}(\Omega)$ such that

$$\zeta \geq (\mathcal{L}_a\psi - f)^+ \text{ a.e. in } \Omega,$$

consider now the one parameter family of approximating semilinear problems in variational form

$$u_\varepsilon \in H_0^s(\Omega): \quad \mathcal{E}_a(u_\varepsilon, v) + \int_\Omega \zeta\theta_\varepsilon(u_\varepsilon - \psi)v = \int_\Omega (f+\zeta)v \quad \forall v \in H_0^s(\Omega). \tag{4.3$_\varepsilon$}$$

Arguing as in the proof of Theorem 5:3.1 of [195], we have the following theorem.

Theorem 4.5. *The unique solution u_ε of the semilinear boundary value problem (4.3$_\varepsilon$) is such that $u_\varepsilon \in \mathbb{K}_\psi^s$ for each $\varepsilon > 0$, and it defines a monotone decreasing sequence, converging as $\varepsilon \to 0$ to the solution u of the obstacle problem* (4.3) *with the error estimate*

$$\|u - u_\varepsilon\|^2_{H_0^s(\Omega)} \leq \varepsilon(C_\theta c_{d,s}^2/a_*)\|\zeta\|_{L^1(\Omega)}\,. \tag{4.7}$$

Proof. The existence and uniqueness of the solution to (4.3$_\varepsilon$) follow easily from the assumptions and the Lions-Stampacchia theorem, as in the proof of the obstacle problem in Theorem 4.1. Therefore, it remains to show that $u_\varepsilon \in \mathbb{K}_\psi^s$, i.e. $u_\varepsilon \geq \psi$. Observe that for all $v \in H_0^s(\Omega)$ such that $v \geq 0$, we have

$$\langle \mathcal{L}_a\psi - f + f, v\rangle \leq \langle (\mathcal{L}_a\psi - f)^+ + f, v\rangle \leq \int_\Omega (\zeta + f)v \tag{4.8}$$

since $H_0^s(\Omega) \cap C^0(\bar{\Omega})$ is dense in $H_0^s(\Omega)$ and $(\mathcal{L}_a\psi - f)^+ \geq \mathcal{L}_a\psi - f$ in the sense of measures. Now, taking $v = (\psi - u_\varepsilon)^+ \in H_0^s(\Omega)$ (since $\psi \in H^s(\Omega)$ and $\psi \leq 0$ on $\partial\Omega$) which satisfies $v \geq 0$ and subtracting (4.3$_\varepsilon$) from the above equation, we have

$$\begin{aligned}
\frac{a_*}{c_{d,s}^2}\left\|(\psi - u_\varepsilon)^+\right\|^2_{H_0^s(\Omega)} &= \frac{a_*}{c_{d,s}^2}\left\|D^s(\psi - u_\varepsilon)^+\right\|^2_{L^2(\Omega)} \\
&\leq \langle \mathcal{L}_a(\psi - u_\varepsilon)^+, (\psi - u_\varepsilon)^+\rangle \\
&\leq \langle \mathcal{L}_a(\psi - u_\varepsilon), (\psi - u_\varepsilon)^+\rangle \\
&= \langle \mathcal{L}_a\psi, (\psi - u_\varepsilon)^+\rangle - \langle \mathcal{L}_a u_\varepsilon, (\psi - u_\varepsilon)^+\rangle \\
&\leq \int_\Omega (\zeta + f)(\psi - u_\varepsilon)^+ + \int_\Omega \zeta\theta_\varepsilon(u_\varepsilon - \psi)(\psi - u_\varepsilon)^+ - \int_\Omega (f+\zeta)(\psi - u_\varepsilon)^+ \\
&= \int_\Omega \zeta\theta_\varepsilon(u_\varepsilon - \psi)(\psi - u_\varepsilon)^+ \\
&= 0.
\end{aligned}$$

The last equality is true because either $u_\varepsilon - \psi > 0$ which gives $(\psi - u_\varepsilon)^+ = 0$, or $u_\varepsilon - \psi \leq 0$ which gives $\theta_\varepsilon(u_\varepsilon - \psi) = 0$ by the construction of θ, thus implying $\theta_\varepsilon(u_\varepsilon - \psi)(\psi - u_\varepsilon)^+ = 0$. Therefore, $(\psi - u_\varepsilon)^+ = 0$, i.e. $u_\varepsilon \in \mathbb{K}_\psi^s$ for any $\varepsilon > 0$.

To show that $u_\varepsilon \geq u_{\hat{\varepsilon}}$ for $\varepsilon > \hat{\varepsilon} > 0$, we just apply the comparison proposition 4.2(i), since $\zeta \geq 0$ and $\theta_\varepsilon(t) \leq \theta_{\hat{\varepsilon}}(t)$.

To show convergence $u_\varepsilon \searrow u$, it is sufficient to prove the error estimate. Taking $v = w - u_\varepsilon$ in (4.3$_\varepsilon$) for arbitrary $w \in \mathbb{K}_\psi^s$, we have

$$\begin{aligned}
\mathcal{E}_a(u_\varepsilon, (w - u_\varepsilon)) &= \int_\Omega (f+\zeta)(w - u_\varepsilon) - \int_\Omega \zeta\theta_\varepsilon(u_\varepsilon - \psi)(w - u_\varepsilon) \\
&= \int_\Omega f(w - u_\varepsilon) + \int_\Omega \zeta[1 - \theta_\varepsilon(u_\varepsilon - \psi)](w - u_\varepsilon) \\
&\geq \int_\Omega f(w - u_\varepsilon) + \int_\Omega \zeta[1 - \theta_\varepsilon(u_\varepsilon - \psi)](\psi - u_\varepsilon) \\
&= \int_\Omega f(w - u_\varepsilon) - \varepsilon\int_\Omega \zeta[1 - \theta_\varepsilon(u_\varepsilon - \psi)]\frac{u_\varepsilon - \psi}{\varepsilon} \\
&\geq \int_\Omega f(w - u_\varepsilon) - \varepsilon C_\theta \int_\Omega \zeta
\end{aligned}$$

since $\zeta, 1-\theta_\varepsilon, w-\psi \geq 0$ for $w \in \mathbb{K}^s_\psi$.

Now, taking $w = u$, we obtain

$$\mathcal{E}_a(u_\varepsilon, (u - u_\varepsilon)) \geq \int_\Omega f(u - u_\varepsilon) - \varepsilon C_\theta \int_\Omega \zeta,$$

but taking $v = u_\varepsilon \in \mathbb{K}^s_\psi$ in the original obstacle problem (4.3), we have

$$\mathcal{E}_a(u, (u_\varepsilon - u)) \geq \int_\Omega f(u_\varepsilon - u).$$

Taking the difference of these two equations, we have

$$\langle \mathcal{L}_a(u_\varepsilon - u), (u_\varepsilon - u)\rangle = \mathcal{E}_a((u_\varepsilon - u), (u_\varepsilon - u)) \leq \varepsilon C_\theta \int_\Omega \zeta.$$

Using the ellipticity of a, we have

$$\varepsilon C_\theta \int_\Omega \zeta \geq \langle \mathcal{L}_a(u_\varepsilon - u), (u_\varepsilon - u)\rangle \geq \frac{a_*}{c^2_{d,s}} \left\| D^s(u_\varepsilon - u)\right\|^2_{L^2(\Omega)} = \frac{a_*}{c^2_{d,s}} \|u_\varepsilon - u\|^2_{H^s_0(\Omega)}$$

by $H^s_0(\Omega)$ equivalent norm. Therefore, we have the estimate.

Taking $\varepsilon \searrow 0$ gives the convergence $u_\varepsilon \searrow u$. □

Remark 4.6. *(i) We can formally interpret the variational equation (4.3_ε) as corresponding to the following semilinear boundary value problem:*

$$\mathcal{L}_a u_\varepsilon + \zeta\theta_\varepsilon(u_\varepsilon - \psi) = f + \zeta \text{ in } \Omega, \quad u_\varepsilon = 0 \text{ on } \mathbb{R}^d \backslash \Omega.$$

(ii) One can also consider the translated penalisation, given by

$$\bar{\theta}_\varepsilon(t) = 1 - \theta\left(-\frac{t}{\varepsilon}\right) = \bar{\theta}\left(-\frac{t}{\varepsilon}\right)$$

for $t \in \mathbb{R}$ and $\varepsilon > 0$, to approach the solution of the obstacle problem from below using monotonicity. Then we have that the unique solution $\bar{u}_\varepsilon$ of the penalised problem

$$\bar{u}_\varepsilon \in H^s_0(\Omega): \quad \mathcal{E}_a(\bar{u}_\varepsilon, v) + \int_\Omega \zeta\bar{\theta}_\varepsilon(\bar{u}_\varepsilon - \psi)v = \int_\Omega (f + \zeta)v \quad \forall v \in H^s_0(\Omega). \tag{$4.3^\sim_\varepsilon$}$$

defines a monotone increasing sequence converging to the solution u of the obstacle problem (4.3) *weakly in $H^s_0(\Omega)$.*

(iii) For special choices of the function θ, we can estimate the uniform convergence for the approximations of u by penalisation. Suppose that, in addition to the conditions on θ,

$$\theta(t) = 1 \textit{ for } t \geq 1,$$

then the approximating solution u_ε of (4.3_ε) verifies, for each $\varepsilon > 0$,

$$u_\varepsilon - \varepsilon \leq u \leq u_\varepsilon \textit{ a.e. in } \Omega.$$

If, additionally, θ verifies

$$t \leq \theta(t) \leq 1 \textit{ if } 0 \leq t \leq 1,$$

the approximating solution $\bar{u}_\varepsilon$ yields

$$\bar{u}_\varepsilon \leq u \leq u_\varepsilon \textit{ and } 0 \leq u_\varepsilon - \bar{u}_\varepsilon \leq \varepsilon \textit{ a.e. in } \Omega.$$

Proof of (ii). The fact that $\tilde{u}_\varepsilon$ is increasing in ε follows as before from the comparison proposition 4.2(i), since $\tilde{u}_\varepsilon \geq \tilde{u}_{\hat{\varepsilon}}$ and $\tilde{\theta}_{\hat{\varepsilon}}(t) \geq \tilde{\theta}_\varepsilon(t)$ for $\hat{\varepsilon} > \varepsilon > 0$.

Since $0 \leq \tilde{\theta}_\varepsilon \leq 1$, from ($4.3^\sim_\varepsilon$), we have that $\tilde{u}_\varepsilon$ is bounded in $H^s_0(\Omega)$ independently of ε. Therefore, $\tilde{u}_\varepsilon \rightharpoonup \tilde{u}$ weakly in $H^s_0(\Omega)$ and strongly in $L^2(\Omega)$ for some subsequence $\varepsilon \to 0$. To show that $\tilde{u}$ is in fact the solution u of (4.3), we apply Minty's Lemma (see, for instance, Lemma 4.2 of [195]) to ($4.3^\sim_\varepsilon$) to obtain, for any $v \in H^s_0(\Omega)$

$$\langle \mathcal{L}_a v, (v - \tilde{u}_\varepsilon)\rangle \geq \int_\Omega [f + \zeta - \zeta\tilde{\theta}_\varepsilon(\tilde{u}_\varepsilon - \psi)](v - \tilde{u}_\varepsilon).$$

Also, for $v \in \mathbb{K}^s_\psi$, $\tilde{\theta}_\varepsilon(\tilde{u}_\varepsilon - \psi) = 1$, so we have

$$\langle \mathcal{L}_a v, (v - \tilde{u}_\varepsilon)\rangle \geq \int_\Omega f(v - \tilde{u}_\varepsilon).$$

Taking the limit $\varepsilon \to 0$, we have

$$\langle \mathcal{L}_a v, (v - \tilde{u})\rangle \geq \int_\Omega f(v - \tilde{u}).$$

Therefore, it remains, by Minty's lemma, to show that $\tilde{u} \in \mathbb{K}^s_\psi$, i.e. $\tilde{u} \geq \psi$ in Ω since $\tilde{u} = \lim_\varepsilon \tilde{u}_\varepsilon$ with $\tilde{u}_\varepsilon \in H^s_0(\Omega)$. But this follows from the fact that $(\psi - \tilde{u}_\varepsilon)^+ \to 0$ strongly in $H^s_0(\Omega)$. Indeed, choosing $v = (\psi - \tilde{u}_\varepsilon)^+ \in H^s_0(\Omega)$ in ($4.3^\sim_\varepsilon$) and recalling (4.8), we have, by the equivalent norm on $H^s_0(\Omega)$,

$$\begin{aligned}
\frac{a_*}{c^2_{d,s}}\left\|(\psi - \tilde{u}_\varepsilon)^+\right\|^2_{H^s_0(\Omega)} &\leq \frac{a_*}{c^2_{d,s}}\left\|D^s(\psi - \tilde{u}_\varepsilon)^+\right\|^2_{L^2(\Omega)} \\
&\leq \langle \mathcal{L}_a(\psi - \tilde{u}_\varepsilon)^+, (\psi - \tilde{u}_\varepsilon)^+\rangle \\
&\leq \langle \mathcal{L}_a(\psi - \tilde{u}_\varepsilon), (\psi - \tilde{u}_\varepsilon)^+\rangle \\
&= \langle \mathcal{L}_a\psi, (\psi - \tilde{u}_\varepsilon)^+\rangle - \langle \mathcal{L}_a\tilde{u}_\varepsilon, (\psi - \tilde{u}_\varepsilon)^+\rangle \\
&\leq \int_\Omega (\zeta + f)(\psi - \tilde{u}_\varepsilon)^+ + \int_\Omega \zeta\tilde{\theta}_\varepsilon(\tilde{u}_\varepsilon - \psi)(\psi - \tilde{u}_\varepsilon)^+ - \int_\Omega (f + \zeta)(\psi - \tilde{u}_\varepsilon)^+ \\
&= \int_\Omega \zeta\tilde{\theta}_\varepsilon(\tilde{u}_\varepsilon - \psi)(\psi - \tilde{u}_\varepsilon)^+ \\
&\leq \int_{\Omega\cap\{\tilde{u}_\varepsilon - \psi < 0\}} \zeta\tilde{\theta}_\varepsilon(\tilde{u}_\varepsilon - \psi)(\psi - \tilde{u}_\varepsilon) \\
&\leq \varepsilon C_\theta \int_\Omega \zeta,
\end{aligned}$$

which converges to 0 as $\varepsilon \to 0$. □

Proof of (iii). Since u_ε is decreasing in ε, $u \leq u_\varepsilon$. To prove the other inequality, take $v = u + (u_\varepsilon - u - \varepsilon)^+ = u \vee (u_\varepsilon - \varepsilon) \in \mathbb{K}^s_\psi$ (because $u \in \mathbb{K}^s_\psi \implies u \geq \psi$, and $u, u_\varepsilon \in H^s_0(\Omega)$ so $u_\varepsilon - \varepsilon < 0$ on $\partial\Omega$ and thus $u \vee (u_\varepsilon - \varepsilon) = 0$ on $\partial\Omega$) in (4.3) and $v = (u_\varepsilon - u - \varepsilon)^+ \in H^s_0(\Omega)$ (because $u, u_\varepsilon \in H^s_0(\Omega)$, and $u, u_\varepsilon = 0$ on $\partial\Omega \implies u_\varepsilon - u - \varepsilon = -\varepsilon < 0$ on $\partial\Omega \implies (u_\varepsilon - u - \varepsilon)^+ = 0$ on $\partial\Omega$) in (4.3_ε). One has, as before,

$$\begin{aligned}
\frac{a_*}{c^2_{d,s}}\left\|(u_\varepsilon - u - \varepsilon)^+\right\|^2_{H^s_0(\Omega)} &\leq \langle \mathcal{L}_a(u_\varepsilon - u - \varepsilon), (u_\varepsilon - u - \varepsilon)^+\rangle \\
&= \langle \mathcal{L}_a u_\varepsilon, (u_\varepsilon - u - \varepsilon)^+\rangle - \langle \mathcal{L}_a u, (u_\varepsilon - u - \varepsilon)^+\rangle \quad \text{since } D^s\varepsilon = 0 \\
&\leq \int_\Omega (\zeta + f)(u_\varepsilon - u - \varepsilon)^+ - \int_\Omega \zeta\theta_\varepsilon(u_\varepsilon - \psi)(u_\varepsilon - u - \varepsilon)^+ - \int_\Omega f(u_\varepsilon - u - \varepsilon)^+ \\
&= \int_\Omega \zeta[1 - \theta_\varepsilon(u_\varepsilon - \psi)](u_\varepsilon - u - \varepsilon)^+ \\
&= \int_{\Omega\cap\{u_\varepsilon > u + \varepsilon\}} \zeta[1 - \theta_\varepsilon(u_\varepsilon - \psi)](u_\varepsilon - u - \varepsilon)^+ = 0
\end{aligned}$$

since $\theta_\varepsilon(t) = 1$ for $t \geq \varepsilon$.

For the other set of inequalities, from the conditions on $\tilde{\theta}_\varepsilon$, we have $\tilde{\theta}_\varepsilon(t) = 1 - \theta_\varepsilon(-t) \le \theta_\varepsilon(t+\varepsilon)$. This is because if $0 \le -t \le 1$, $\theta_\varepsilon(-t) \ge -t/\varepsilon$, so $1 - \theta_\varepsilon(-t) \le 1 + \frac{t}{\varepsilon}$, and also, $\theta_\varepsilon(t+\varepsilon) = \theta\left(\frac{t}{\varepsilon}+1\right) \ge \frac{t}{\varepsilon}+1$ since $1 - \frac{1}{\varepsilon} \le \frac{t}{\varepsilon} + 1 \le 1$; and if $-t \le 0$, $\theta_\varepsilon(-t) = 0$ and $\theta_\varepsilon(t+\varepsilon) = 1$, so $1 - \theta_\varepsilon(-t) \le \theta_\varepsilon(t+\varepsilon)$; and if $-t \ge 1$, $\theta_\varepsilon(-t) = 1$ and $\theta_\varepsilon(t+\varepsilon) \ge 0$, so $1 - \theta_\varepsilon(-t) \le \theta_\varepsilon(t+\varepsilon)$. Now, since θ is non-decreasing, this implies the inequality

$$\tilde{\theta}_\varepsilon(t) \le \theta_\varepsilon(\tau) \text{ for } \tau - t > \varepsilon.$$

Hence, setting $v = (u_\varepsilon - \tilde{u}_\varepsilon - \varepsilon)^+ \in H^s_0(\Omega)$ in (4.3_ε) and $(4.3_{\tilde{\varepsilon}})$, we have

$$\begin{aligned}
\frac{a_*}{c^2_{d,s}}\left\|(u_\varepsilon - \tilde{u}_\varepsilon - \varepsilon)^+\right\|^2_{H^s_0(\Omega)} &\le \langle \mathcal{L}_a(u_\varepsilon - \tilde{u}_\varepsilon), (u_\varepsilon - \tilde{u}_\varepsilon - \varepsilon)^+\rangle \\
&= \langle \mathcal{L}_a u_\varepsilon, (u_\varepsilon - \tilde{u}_\varepsilon - \varepsilon)^+\rangle - \langle \mathcal{L}_a \tilde{u}_\varepsilon, (u_\varepsilon - \tilde{u}_\varepsilon - \varepsilon)^+\rangle \\
&= \int_\Omega [f + \zeta - \zeta\theta_\varepsilon(u_\varepsilon - \psi)](u_\varepsilon - \tilde{u}_\varepsilon - \varepsilon)^+ \\
&\quad - \int_\Omega [f + \zeta - \zeta\tilde{\theta}_\varepsilon(u_\varepsilon - \psi)](u_\varepsilon - \tilde{u}_\varepsilon - \varepsilon)^+ \\
&= \int_\Omega \zeta[\tilde{\theta}_\varepsilon(\tilde{u}_\varepsilon - \psi) - \theta_\varepsilon(u_\varepsilon - \psi)](u_\varepsilon - \tilde{u}_\varepsilon - \varepsilon)^+ \\
&= \int_{\Omega\cap\{u_\varepsilon - \tilde{u}_\varepsilon > \varepsilon\}} \zeta[\tilde{\theta}_\varepsilon(\tilde{u}_\varepsilon - \psi) - \theta_\varepsilon(u_\varepsilon - \psi)](u_\varepsilon - \tilde{u}_\varepsilon - \varepsilon)^+ \le 0,
\end{aligned}$$

so $(u_\varepsilon - \tilde{u}_\varepsilon - \varepsilon)^+ = 0$, and $u_\varepsilon - \tilde{u}_\varepsilon \le \varepsilon$.

To complete the proof, it remains to show that $u_\varepsilon \ge \tilde{u}_\varepsilon$. But this follows immediately from the comparison proposition 4.2(i), since $\tilde{\theta}_\varepsilon(t) = 1 \ge \theta_\varepsilon(t)$ for $t \ge 0$ and $\tilde{\theta}_\varepsilon \ge \theta(t) = 0$ for $t \le 0$. □

From this theorem, we can derive the Lewy-Stampacchia inequality.

Theorem 4.7 (Lewy-Stampacchia inequality)**.** *Under the assumptions* (4.6)*, the solution u of the obstacle problem satisfies*

$$f \le \mathcal{L}_a u \le f \vee \mathcal{L}_a\psi \qquad \textit{a.e. in } \Omega.$$

In particular, $\mathcal{L}_a u \in L^{2^\#}(\Omega)$.

Proof. Choosing $\zeta = (\mathcal{L}_a\psi - f)^+$ in (4.3_ε), and making use of the property of θ that $0 \le 1 - \theta_\varepsilon \le 1$, then for any $\varepsilon > 0$ and any $v \in H^s_0(\Omega)$, $v \ge 0$, we have

$$\int_\Omega \mathcal{L}_a u_\varepsilon v = \int_\Omega [f + (\mathcal{L}_a\psi - f)^+(1 - \theta_\varepsilon)]v \le \int_\Omega [f + (\mathcal{L}_a\psi - f)^+]v$$

from the variational form, and on the other hand

$$\mathcal{L}_a u_\varepsilon = f + \zeta - \zeta\theta_\varepsilon = f + \zeta(1 - \theta_\varepsilon) \ge f$$

holds a.e. in Ω. Together, these give

$$\int_\Omega fv \le \int_\Omega \mathcal{L}_a u_\varepsilon v \le \int_\Omega [f + (\mathcal{L}_a\psi - f)^+]v = \int_\Omega [f \vee (\mathcal{L}_a\psi)]v.$$

Using (4.7), we let $\varepsilon \to 0$, so this holds for u. Since v is arbitrary, we have the result. □

Remark 4.8. *The Lewy-Stampacchia inequalities for nonlocal obstacle problems have been first obtained in [212] for a class of symmetric integrodifferential operators $\mathcal{L}_K$, with even kernels K, which are also strictly T-monotone and include the fractional Laplacian, and with f and $\mathcal{L}_K\psi \in L^\infty(\Omega)$.*

4.3.2 Multiple obstacles problem

Two obstacles problem We next consider the two obstacles problem with a Dirichlet boundary condition in a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, which consists of finding $u \in \mathbb{K}^s_{\psi,\varphi}$ such that

$$\mathcal{E}_a(u, v-u) \geq \int_\Omega f(v-u), \quad \forall v \in \mathbb{K}^s_{\psi,\varphi}, \tag{4.9}$$

where $f \in L^{2^\#}(\Omega)$ and

$$\mathbb{K}^s_{\psi,\varphi} = \{v \in H^s_0(\Omega) : \psi \leq v \leq \varphi \text{ a.e. in } \Omega\}. \tag{4.10}$$

Assume that φ and ψ are measurable and admissible obstacles in Ω such that $\mathbb{K}^s_{\psi,\varphi} \neq \emptyset$. When $\varphi, \psi \in H^s(\mathbb{R}^d)$, a sufficient condition for these two assumptions to hold is to assume $\varphi \geq \psi$ a.e. in Ω and $\varphi \geq 0 \geq \psi$ a.e. in Ω^c.

Theorem 4.9. *The two obstacles problem has a unique solution. Moreover, if $u = u(f, \varphi, \psi)$ and $\hat{u} = u(\hat{f}, \hat{\varphi}, \hat{\psi})$ are solutions in $\mathbb{K}^s_{\psi,\varphi}$ and in $\mathbb{K}^s_{\hat{\psi},\hat{\varphi}}$, respectively, of the two obstacles problem, then*

$$f \geq \hat{f}, \varphi \geq \hat{\varphi}, \psi \geq \hat{\psi} \quad \textit{implies} \quad u \geq \hat{u} \textit{ a.e. in } \Omega.$$

In addition, if $f = \hat{f}$, we have the L^∞ estimate

$$\|u - \hat{u}\|_{L^\infty(\Omega)} \leq \|\psi - \hat{\psi}\|_{L^\infty(\Omega)} + \|\varphi - \hat{\varphi}\|_{L^\infty(\Omega)}.$$

Proof. The existence and uniqueness follow, as in the previous sections, from the monotonicity, coercivity, continuity and boundedness of the operator $\mathcal{L}_a$ and the Stampacchia theorem. The comparison property follows also as previous by the T-monotonicity of $\mathcal{L}_a$.

The L^∞ estimate follows as well, as in the classical one obstacle problem. □

Corresponding to the two obstacles problem, we also have the Lewy-Stampacchia inequality.

Theorem 4.10. *The solution u of the two obstacles problem, for $f, \mathcal{L}_a\varphi, \mathcal{L}_a\psi \in L^{2^\#}(\Omega)$ such that $\varphi, \psi \in H^s(\mathbb{R}^d)$ are compatible and $\mathcal{L}_a\varphi, \mathcal{L}_a\psi$ are given by (2.1), satisfies*

$$f \wedge \mathcal{L}_a\varphi \leq \mathcal{L}_a u \leq f \vee \mathcal{L}_a\psi \quad \textit{a.e. in } \Omega, \tag{4.11}$$

and therefore $\mathcal{L}_a u \in L^{2^\#}(\Omega)$.

Proof. The proof is similar to that of the classical case $s = 1$, now for two obstacles. Consider the penalised problem given by

$$u_\varepsilon \in H^s_0(\Omega) : \quad \langle \mathcal{L}_a u_\varepsilon, v\rangle + \int_\Omega \zeta_\psi \theta_\varepsilon(u_\varepsilon - \psi)v - \int_\Omega \zeta_\varphi \theta_\varepsilon(\varphi - u_\varepsilon)v = \int_\Omega (f + \zeta_\psi - \zeta_\varphi)v \quad \forall v \in H^s_0(\Omega)$$

for

$$\zeta_\psi \geq (\mathcal{L}_a\psi - f)^+, \quad \zeta_\varphi \geq (\mathcal{L}_a\varphi - f)^-,$$

with $\theta_\varepsilon(t) = 1$ for $t \geq \varepsilon$. Then, there is a unique solution $u_\varepsilon \in H^s_0(\Omega)$ such that $\psi \leq u_\varepsilon \leq \varphi + \varepsilon$ for each $\varepsilon > 0$. Indeed, we obtain the existence and uniqueness of the solution by the Stampacchia theorem as before. To show that $\psi \leq u_\varepsilon \leq \varphi + \varepsilon$, we have

$$\langle \mathcal{L}_a\psi, v\rangle \leq \langle(\mathcal{L}_a\psi - f)^+ + f, v\rangle \leq \int_\Omega (\zeta_\psi + f)v \quad \forall v \in H^s_0(\Omega), v \geq 0, \quad \text{and}$$

$$\langle \mathcal{L}_a\varphi, v\rangle = \langle(\mathcal{L}_a\varphi - f) + f, v\rangle \geq \langle f - (\mathcal{L}_a\varphi - f)^-, v\rangle \geq \int_\Omega (f - \zeta_\varphi)v \quad \forall v \in H^s_0(\Omega), v \geq 0.$$

Taking $v = (\psi - u_\varepsilon)^+ \in H^s_0(\Omega)$ and using the strict T-monotonicity of $\mathcal{L}_a$, we have

$$\frac{a_*}{c^2_{d,s}} \left\|(\psi - u_\varepsilon)^+\right\|^2_{H^s_0(\Omega)} \leq \mathcal{E}_a(\psi, (\psi - u_\varepsilon)^+) - \mathcal{E}_a(u_\varepsilon, (\psi - u_\varepsilon)^+)$$

$$
\begin{aligned}
&\le \int_\Omega (f+\zeta_\psi)(\psi-u_\varepsilon)^+ - \int_\Omega \{f+\zeta_\psi[1-\theta_\varepsilon(u_\varepsilon-\psi)] - \zeta_\varphi[1-\theta_\varepsilon(\varphi-u_\varepsilon)]\}(\psi-u_\varepsilon)^+ \\
&= \int_\Omega \zeta_\psi \theta_\varepsilon(u_\varepsilon-\psi)(\psi-u_\varepsilon)^+ - \int_\Omega \{\zeta_\varphi[1-\theta_\varepsilon(\varphi-u_\varepsilon)]\}(\psi-u_\varepsilon)^+ \\
&\le 0
\end{aligned}
$$

because the first term is non-positive, while the factors in the second term are all non-negative. Therefore, $u_\varepsilon \ge \psi$. Similarly, taking $v = (u_\varepsilon - \varphi - \varepsilon)^+ \in H^s_0(\Omega)$ gives

$$
\begin{aligned}
\frac{a_*}{c_{d,s}^2}\left\|(u_\varepsilon-\varphi-\varepsilon)^+\right\|^2_{H^s_0(\Omega)} &= \frac{a_*}{c_{d,s}^2}\left\|D^s(u_\varepsilon-\varphi-\varepsilon)^+\right\|^2_{L^2(\mathbb{R}^d)} \\
&\le \mathcal{E}_a((u_\varepsilon-\varphi-\varepsilon)^+,(u_\varepsilon-\varphi-\varepsilon)^+) \\
&\le \mathcal{E}_a(u_\varepsilon-\varphi-\varepsilon,(u_\varepsilon-\varphi-\varepsilon)^+) \\
&\le \mathcal{E}_a(u_\varepsilon-\varphi,(u_\varepsilon-\varphi-\varepsilon)^+) \\
&= \left[\mathcal{E}_a(u_\varepsilon,(u_\varepsilon-\varphi-\varepsilon)^+) - \mathcal{E}_a(\varphi,(u_\varepsilon-\varphi-\varepsilon)^+)\right] \\
&\le \int_\Omega \{f+\zeta_\psi[1-\theta_\varepsilon(u_\varepsilon-\psi)] - \zeta_\varphi[1-\theta_\varepsilon(\varphi-u_\varepsilon)]\}(u_\varepsilon-\varphi-\varepsilon)^+ \\
&\quad - \int_\Omega (f-\zeta_\varphi)(u_\varepsilon-\varphi-\varepsilon)^+ \\
&= \int_\Omega \{\zeta_\psi[1-\theta_\varepsilon(u_\varepsilon-\psi)]\}(u_\varepsilon-\varphi-\varepsilon)^+ + \int_\Omega [\zeta_\varphi\theta_\varepsilon(\varphi-u_\varepsilon)](u_\varepsilon-\varphi-\varepsilon)^+ \\
&= \int_\Omega \zeta_\psi[1-\theta_\varepsilon(u_\varepsilon-\psi)](u_\varepsilon-\varphi-\varepsilon)^+ \\
&\le \int_\Omega \zeta_\psi[1-\theta_\varepsilon(u_\varepsilon-\psi)](u_\varepsilon-\psi-\varepsilon)^+ \text{ since } \varphi \ge \psi \\
&= 0
\end{aligned}
$$

Therefore, $u_\varepsilon \le \varphi + \varepsilon$.

Now, we can show that $u_\varepsilon \to u$, so $\psi \le u_\varepsilon \le \varphi + \varepsilon$ converges to $\psi \le u \le \varphi$. Take $v = w - u_\varepsilon$ in the penalised problem above for arbitrary $w \in \mathbb{K}^s_{\psi,\varphi}$, then

$$
\begin{aligned}
\mathcal{E}_a(u_\varepsilon, w-u_\varepsilon) &= \int_\Omega f(w-u_\varepsilon) + \int_\Omega \zeta_\psi[1-\theta_\varepsilon(u_\varepsilon-\psi)](w-u_\varepsilon) - \int_\Omega \zeta_\varphi[1-\theta_\varepsilon(\varphi-u_\varepsilon)](w-u_\varepsilon) \\
&\ge \int_\Omega f(w-u_\varepsilon) + \int_\Omega \zeta_\psi[1-\theta_\varepsilon(u_\varepsilon-\psi)](\psi-u_\varepsilon) - \int_\Omega \zeta_\varphi[1-\theta_\varepsilon(\varphi-u_\varepsilon)](\varphi-u_\varepsilon) \\
&= \int_\Omega f(w-u_\varepsilon) - \varepsilon\int_\Omega \zeta_\psi[1-\theta_\varepsilon(u_\varepsilon-\psi)]\frac{u_\varepsilon-\psi}{\varepsilon} - \varepsilon\int_\Omega \zeta_\varphi[1-\theta_\varepsilon(\varphi-u_\varepsilon)]\frac{\varphi-u_\varepsilon}{\varepsilon} \\
&\ge \int_\Omega f(w-u_\varepsilon) - \varepsilon C_\theta \int_\Omega (\zeta_\psi+\zeta_\varphi).
\end{aligned}
$$

Now, taking $w = u \in \mathbb{K}^s_{\psi,\varphi}$, we obtain

$$
\mathcal{E}_a(u_\varepsilon, u-u_\varepsilon) \ge \int_\Omega f(u-u_\varepsilon) - \varepsilon C_\theta \int_\Omega (\zeta_\psi+\zeta_\varphi),
$$

but taking $v = u_\varepsilon \in \mathbb{K}^s_\psi$ in the original obstacle problem (4.3), we have

$$
\mathcal{E}_a(u, u_\varepsilon-u) \ge \int_\Omega f(u_\varepsilon-u).
$$

Taking the difference of these two equations, by the linearity of $\mathcal{E}_a$, we have

$$
\mathcal{E}_a(u_\varepsilon-u, u_\varepsilon-u) \le \varepsilon C_\theta \int_\Omega (\zeta_\psi+\zeta_\varphi).
$$

Using the ellipticity of a, we have

$$\varepsilon C_\theta \int_\Omega (\zeta_\psi + \zeta_\varphi) \geq \mathcal{E}_a(u_\varepsilon - u, u_\varepsilon - u) \geq \frac{a_*}{c_{d,s}^2} \left\| D^s(u_\varepsilon - u) \right\|_{L^2(\mathbb{R}^d)}^2 = \frac{a_*}{c_{d,s}^2} \| u_\varepsilon - u \|_{H_0^s(\Omega)}^2 .$$

Therefore, $u_\varepsilon \to u$ in $H_0^s(\Omega)$.

Choosing $\zeta_\psi = (\mathcal{L}_a\psi - f)^+$ and $\zeta_\varphi = (\mathcal{L}_a\varphi - f)^-$, and making use of the property of θ that $0 \leq 1-\theta_\varepsilon \leq 1$, then for any $\varepsilon > 0$ and any $v \in H_0^s(\Omega)$, $v \geq 0$, we have

$$\int_\Omega (\mathcal{L}_a u_\varepsilon) v = \int_\Omega [f + (\mathcal{L}_a\psi - f)^+(1-\theta_\varepsilon) - (\mathcal{L}_a\varphi - f)^-(1-\theta_\varepsilon)]v \leq \int_\Omega [f + (\mathcal{L}_a\psi - f)^+]v$$

and

$$\int_\Omega (\mathcal{L}_a u_\varepsilon) v = \int_\Omega [f + (\mathcal{L}_a\psi - f)^+(1-\theta_\varepsilon) - (\mathcal{L}_a\varphi - f)^-(1-\theta_\varepsilon)]v \geq \int_\Omega [f - (\mathcal{L}_a\varphi - f)^-]v.$$

Therefore,

$$\int_\Omega [f \wedge (\mathcal{L}_a\varphi)]v = \int_\Omega [f - (\mathcal{L}_a\varphi - f)^-]v \leq \int_\Omega (\mathcal{L}_a u_\varepsilon)v \leq \int_\Omega [f + (\mathcal{L}_a\psi - f)^+]v = \int_\Omega [f \vee (\mathcal{L}_a\psi)]v.$$

Letting $\varepsilon \to 0$, this holds for u. Since v is arbitrary, we conclude (4.11). □

N **membranes problem** We consider now the N membranes problem, which consists of: To find $\boldsymbol{u} = (u_1, u_2, \ldots, u_N) \in \mathbb{K}_N^s$ satisfying

$$\sum_{i=1}^N \mathcal{E}_a(u_i, v_i - u_i) \geq \sum_{i=1}^N \int_\Omega f^i(v_i - u_i), \quad \forall (v_1, \ldots, v_N) \in \mathbb{K}_N^s, \tag{4.12}$$

where $\mathbb{K}_N^s$ is the convex subset of $[H_0^s(\Omega)]^N$ defined by

$$\mathbb{K}_N^s = \{(v_1, \ldots, v_N) \in [H_0^s(\Omega)]^N : v_1 \geq \cdots \geq v_N \text{ a.e. in } \Omega\} \tag{4.13}$$

and $f^i, \ldots, f^N \in L^{2^\#}(\Omega)$. As in the previous sections, the existence and uniqueness follows easily. Furthermore, the following Lewy-Stampacchia type inequality also holds.

Theorem 4.11. *The solution $\boldsymbol{u} = (u_1, \ldots u_N)$ of the N membranes problem satisfies a.e. in Ω*

$$\begin{aligned} f^1 \wedge \mathcal{L}_a u_1 &\leq f^1 \vee \cdots \vee f^N \\ f^1 \wedge f^2 \leq \mathcal{L}_a u_2 &\leq f^2 \vee \cdots \vee f^N \\ &\vdots \\ f^1 \wedge \cdots \wedge f^{N-1} \leq \mathcal{L}_a u_{N-1} &\leq f^{N-1} \vee f^N \\ f^1 \wedge \cdots \wedge f^N \leq \mathcal{L}_a u_N &\leq f^N, \end{aligned}$$

and $\mathcal{L}_a \boldsymbol{u} \in [L^{2^\#}(\Omega)]^N$.

Proof. Choosing $(v, u_2, \ldots, u_N) \in \mathbb{K}_N^s$ with $v \in \mathbb{K}_{u_2}^s$, we see that $u_1 \in \mathbb{K}_{u_2}^s$ solves (4.3) with $f = f^1$ and by Theorem 4.7

$$f^1 \leq \mathcal{L}_a u_1 \leq f^1 \vee \mathcal{L}_a u_2 \text{ a.e. in } \Omega.$$

Analogously, $u_j \in \mathbb{K}_{u_{j+1}, u_{j-1}}^s$ solves the two obstacles problem with $f = f^j$, $j = 2, 3, \ldots, N-1$, and satisfies, by (4.11),

$$f^j \wedge \mathcal{L}_a u_{j-1} \leq \mathcal{L}_a u_j \leq f^j \vee \mathcal{L}_a u_{j+1} \quad \text{ a.e. in } \Omega.$$

Finally, u_N solves the one obstacle problem with an upper obstacle $\varphi = u_{N-1}$, and so by the symmetric Lewy-Stampacchia estimates given in Theorem 4.7, we have

$$f^N \wedge \mathcal{L}_a u_{N-1} \leq \mathcal{L}_a u_N \leq f^N \quad \text{ a.e. in } \Omega.$$

The proof concludes by simple iteration (see Theorem 5.1 of [194]).

□

Remark 4.12. *The solution can also be approximated by the bounded penalisation given in [25] by*

$$\boldsymbol{u}_\varepsilon \in [H^s_0(\Omega)]^N: \quad \langle \mathcal{L}_a u^\varepsilon_i, v_i\rangle + \int_\Omega \zeta_i \theta_\varepsilon(u^\varepsilon_i - u^\varepsilon_{i+1}) v_i - \int_\Omega \zeta_{i-1}\theta_\varepsilon(u^\varepsilon_{i-1} - u^\varepsilon_i) v_i = \int_\Omega (f^i + \zeta_i - \zeta_{i-1}) v_i \quad \forall v_i \in H^s_0(\Omega)$$

for

$$\zeta_0 = \max\left\{\frac{f^1 + \cdots + f^i}{i} : i = 1, \ldots, N\right\},$$
$$\zeta_i = i\zeta_0 - (f^1 + \cdots + f^i) \quad \text{for } i = 1, \ldots, N$$

with $\theta_\varepsilon(t) = 1$ for $t \geq \varepsilon$ and $u^\varepsilon_0 = +\infty, u^\varepsilon_{N+1} = -\infty$.

As in [25], we then have the strong convergence of $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ in $[H^s_0(\Omega)]^N$, thereby giving the Lewy-Stampacchia inequalities in Theorem 4.11.

4.3.3 Local regularity of solutions

We make use of the Lewy-Stampacchia inequalities to show local regularity for the three types of nonlocal obstacle problems, but first it is useful to obtain the global boundedness of the solutions.

Let $s \in]0, 1[$. Suppose that

(a) $f, \mathcal{L}_a\psi \in L^p(\Omega)$ for some $p > \frac{d}{2s}$ for the one obstacle problem,

(b) $f \wedge \mathcal{L}_a\varphi$ and $f \vee \mathcal{L}_a\psi$ are in $L^p(\Omega)$ for some $p > \frac{d}{2s}$ for the two obstacles problem, or

(c) $f^i \in L^p(\Omega)$ for $i = 1, \ldots, N$ for some $p > \frac{d}{2s}$ for the N membranes problem.

Theorem 4.13. *Let u denote the solutions of the one obstacle problem* (4.3)*, or the two obstacles problem* (4.9)*, or $u = u_i$ for $i = 1, \ldots, N$ of the N membranes problem* (4.12)*, respectively, under the assumptions (a), (b) or (c) above. Then $g = \mathcal{L}_a u \in L^p(\Omega)$, with $p > \frac{d}{2s}$ and there exists a constant C, depending only on a_*, a^*, d, Ω, $\|u\|_{H^s_0(\Omega)}$, $\|g\|_{L^p(\Omega)}$ and s, such that*

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Proof. Assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz convex function such that $\Phi(0) = 0$, then if $u \in H^s_0(\Omega)$, we have, by repeating the proof of Proposition 4 of [154],

$$\mathcal{E}_a(\Phi(u), v) \leq \mathcal{E}_a(u, v\Phi'(u)) \quad \text{for } v \geq 0, v \in H^s_0(\Omega), \text{ weakly in } \Omega.$$

We can then repeat the proof of Theorem 13 of [154] using the Moser technique to obtain the theorem. More details are given in Section 6.8.2. Here, we focus on showing the local regularity for the solutions to the obstacle problems. ☐

Observe that in general our kernel does not satisfy the usual regularity of the kernel of the fractional Laplacian [196, 198] or other commonly considered fractional kernels [61], [106], since in general it does not satisfy the "symmetry" condition $a(x, y) = a(x, -y)$ unless a is a constant multiple of the kernel of the fractional Laplacian. However, it will still be possible to obtain local Hölder regularity on the solution with the properties of our kernel, if we assume it is symmetric, i.e. if it satisfies

$$a(x, y) = a(y, x). \tag{4.14}$$

Then, we can make use of the symmetric form as given in (2.12), and apply the results of [154].

By the Lewy-Stampacchia inequalities, as long as the upper and lower bounds are in $L^p(\Omega)$ for some $p > \frac{d}{2s}$, we can make use of the Dirichlet form nature of the bilinear form, and obtain Hölder regularity on the solutions on balls independently of the boundary conditions and of the regularity of $\partial\Omega$. Then, by Theorem 1.6 of [103], since the bilinear form satisfies (2.2) and (2.13)–(2.14), in the symmetric case, we have the weak Harnack inequality. Furthermore, as in the classical de Giorgi-Nash-Moser theory, the weak Harnack inequality implies a decay of oscillation-result and local Hölder regularity estimates for weak solutions.

Theorem 4.14 (Weak Harnack inequality)**.** *Let u denote the solutions of the one obstacle problem* (4.3)*, or the two obstacles problem* (4.9)*, or $u = u_i$ for $i = 1, \dots, N$ of the N membranes problem* (4.12)*, respectively, under the assumptions (a), (b) or (c) above. Suppose the unit ball about the origin B_1 is a subset of Ω, and a is symmetric. Then,*

$$\inf_{B_{1/4}} u \geq c \left(\fint_{B_{1/2}} u(x)^{p_0}\, dx \right)^{\frac{1}{p_0}} - \sup_{x \in B_{15/16}} \int_{\mathbb{R}^d \setminus B_1} u^-(z) d\mu_x(dz) - \|\mathcal{L}_a u\|_{L^p(B_{15/16})},$$

for $d\mu_x(dz)$ a measure depending on a as defined in [103] and [140], and the positive constants $p_0 \in]0,1[$ and c depend only on d, s, a_*, a^*.

Theorem 4.15 (Hölder regularity)**.** *Let u denote the solutions of the one obstacle problem* (4.3)*, or the two obstacles problem* (4.9)*, or $u = u_i$ for $i = 1, \dots, N$ of the N membranes problem* (4.12)*, respectively, under the assumptions (a), (b) or (c) above. Suppose $B_\rho \Subset \Omega$ is a ball of radius ρ and a is symmetric. Then, there exists $c_\rho \geq 0$ and $\beta \in]0,1[$, independent of u, such that the following Hölder estimate holds for almost every $x, y \in B_{\rho/2}$:*

$$|u(x) - u(y)| \leq c_\rho |x-y|^\beta \left(\|u\|_{L^\infty(\Omega)} + \|\mathcal{L}_a u\|_{L^p(\Omega)} \right).$$

Proof. Since the Lewy-Stampacchia inequalities in Theorems 4.7, 4.10 and 4.11 hold a.e. in $B_\rho \subset \Omega$ for $\mathcal{L}_a u$ for the one obstacle, the two obstacles problem and the N membranes problem respectively, and $\mathcal{L}_a u = g$ in B_ρ, therefore g lies in $L^p(\Omega)$ for $p > \frac{d}{2s}$, and we have the result making use of Theorem 4.14 following the classical approach. □

Remark 4.16. *(i) The local Hölder continuity of the solutions of a two membranes problem was obtained for different operators with translation invariant kernels in [61], as well as the local $C^{1,\gamma}$ regularity in the case of the fractional Laplacian as in the case of the regular obstacles of [218].*

(ii) The results in Theorems 4.14 and 4.15 can be generalised to arbitrary family of measures satisfying (2.2) *and* (2.13)–(2.14)*, as given in [68].*

(iii) Since the L^∞ bound also works in the non-symmetric case, we conjecture that the weak Harnack inequality and the Hölder continuity are also true without the symmetry assumption, but this is an open problem.

In the case where a corresponds to the kernel of the fractional Laplacian, $\mathcal{L}_a = (-\Delta)^s$, we can use Corollary 1.15 of [213] and apply local elliptic regularity of weak solutions in fractional Sobolev spaces $W^{r,p}$ associated to Dirichlet fractional Laplacian problems as in Theorem 1.4 of [37].

Theorem 4.17. *Let u denote the solutions of the one obstacle problem* (4.3) *for $f \in L^{2^\#}(\Omega)$ in the form*

$$u \in \mathbb{K}^s_\psi(\Omega) : \int_{\mathbb{R}^d} D^s u \cdot D^s(v-u)\, dx \geq \int_\Omega f(v-u) \quad \forall v \in \mathbb{K}^s_\psi,$$

or the corresponding two obstacles problem (4.9)*, or $u = u_i$ for $i = 1, \dots, N$ of the corresponding N membranes problem* (4.12)*, respectively, under the assumptions (a), (b) or (c) above, with $2^\# \leq p < \infty$ and $0 < s < 1$. Then, $(-\Delta)^s u \in L^p_{loc}(\Omega)$ and $u \in W^{2s,p}_{loc}(\Omega)$. In particular, $u \in C^1(\Omega)$ if $s > 1/2$ and $p > d/(2s-1)$, by Theorem 7.57(c) of [5].*

This theorem, which seems new, is an extension to nonlocal obstacle type problems of the well-known $W^{2,p}_{loc}(\Omega)$ regularity of solutions of the classical local obstacle problem corresponding to $s = 1$.

4.4 s-capacity and Lewy-Stampacchia Inequalities in $H^{-s}(\Omega)$

In this section, we extend the results on the Lewy-Stampacchia inequalities obtained in the previous section to data in the dual space $H^{-s}(\Omega)$. We first characterise the order dual of $H^s_0(\Omega)$, which is related to the theory of the s-capacity. This follows much of the results in the classical obstacle problem [221], [3], [195]. In [4] and [122], more general capacities are considered for general bilinear forms. Recently the fractional capacity for the Neumann problem was considered in [242]. In order to extend the results in Theorems 4.7, 4.10 and 4.11 to data in $H^{-s}(\Omega)$, we may apply the general results of [175] for the one obstacle problem and [194] for two obstacles.

4.4.1 A characterisation of the order dual $H^{-s}_{\prec}(\Omega)$ of $H^s_0(\Omega)$

Associated with any Dirichlet form, there is a *Choquet capacity.* We denote by C_s the capacity associated to the norm of $H^s_0(\Omega)$. For any compact set $K \subset \Omega$, it is defined by

$$C_s(K) = \inf \left\{ \|u\|^2_{H^s_0(\Omega)} : u \in H^s_0(\Omega), u \geq 1 \text{ a.e. in } K \right\}.$$

For an arbitrary open set $G \subset \Omega$,

$$C_s(G) = \sup \left\{ C_s(K) : K \text{ is a compact set in } G \right\}.$$

A function $u \in H^s_0(\Omega)$ is said to be *quasi-continuous* if for every $\varepsilon > 0$, there exists an open set $G \subset \Omega$ such that $C_s(G) < \varepsilon$ and $u|_{\Omega\setminus G}$ is continuous. A property is said to hold *quasi-everywhere* (q.e.) if it holds except for a set of capacity zero.

It is well-known (by [4] Proposition 6.1.2 page 156 or [122] Theorem 2.1.3 page 71) that for every $u \in H^s_0(\Omega)$, there exists a unique (up to a set of capacity 0) quasi-continuous function $\bar{u} : \Omega \to \mathbb{R}$ such that $\bar{u} = u$ a.e. on Ω. Therefore, we have the following theorem (see also Theorem 3.7 of [242]).

Theorem 4.18. *For every function $u \in H^s_0(\Omega)$, there exists a unique (up to q.e. equivalence) $\bar{u} : \Omega \to \mathbb{R}$ quasi-continuous function such that $u = \bar{u}$ a.e. in Ω.*

Thus, it makes sense to identify a function $u \in H^s_0(\Omega)$ with the class of quasi-continuous functions that are equivalent quasi-everywhere. Denote the space of such equivalent classes by $Q_s(\Omega)$. Then, for every element $u \in H^s_0(\Omega)$, there is an associated $\bar{u} \in Q_s(\Omega)$.

Define the space $L^2_{C_s}(\Omega)$ by

$$L^2_{C_s}(\Omega) = \{\phi \in Q_s(\Omega) : \exists u \in H^s_0(\Omega) : \bar{u} \geq |\phi| \text{ q.e. in } \Omega\}$$

and

$$R_{C_s}(\phi) = \inf\{\|u\|_{H^s_0(\Omega)} : u \in H^s_0(\Omega), \bar{u} \geq |\phi| \text{ q.e.}\},$$

which is a norm that makes $L^2_{C_s}(\Omega)$ a Banach space (see Proposition 1.2 of [24]). We want to show that the dual space of $L^2_{C_s}(\Omega)$ can be identified with the order dual of $H^s_0(\Omega)$, i.e.

$$[L^2_{C_s}(\Omega)]' = H^{-s}(\Omega) \cap M(\Omega) = H^{-s}_{\prec}(\Omega) = [H^{-s}(\Omega)]^+ - [H^{-s}(\Omega)]^+,$$

where $M(\Omega)$ is the set of bounded measures in Ω. Then we have the following result by Theorem 4.18 (corresponding to the classical case in [24] Proposition 1.7).

Proposition 4.19. *The injection of $H^s_0(\Omega) \cap C_c(\Omega) \hookrightarrow L^2_{C_s}(\Omega)$ is dense.*

Proof. This simply follows from Theorem 4.18, since $u \geq 0$ a.e. on Ω implies $u \geq 0$ q.e. on Ω. □

For $K \subset \Omega$, recall that one says that $u \succeq 0$ on K (or $u \geq 0$ on K in the sense of $H^s_0(\Omega)$) if there exists a sequence of Lipschitz functions $u_k \to u$ in $H^s_0(\Omega)$ such that $u_k \geq 0$ on K.

Let $K \subset \Omega$ be any compact subset. Define the nonempty closed convex set of $H^s_0(\Omega)$ by

$$\mathbb{K}^s_K = \{v \in H^s_0(\Omega) : v \succeq 1 \text{ on } K\}.$$

Consider the following variational inequality

$$u \in \mathbb{K}^s_K : \mathcal{E}_a(u, v - u) \geq 0, \quad \forall v \in \mathbb{K}^s_K. \tag{4.15}$$

This variational inequality has clearly a unique solution and consequently we can also extend to the s-fractional framework the following theorem which is due to Stampacchia [221] in the case $s = 1$.

Theorem 4.20 (Radon measure for the bilinear form $\mathcal{E}_a$)**.** *For any compact $K \subset \Omega$, the unique solution u of* (4.15)*, which is called the* (s,a)-capacitary potential *of K, is such that*

$$u = 1 \text{ on } K \text{ (in the sense of } H^s_0(\Omega))$$

$$\mu = \mathcal{L}_a u \geq 0 \text{ with } supp(\mu) \subset K.$$

Moreover, for the non-negative Radon measure μ, one has

$$C^a_s(K) = \mathcal{E}_a(u,u) = \int_\Omega d\mu = \mu(K)$$

and this number is called the (s,a)-capacity *of K with respect to $\mathcal{E}_a(\cdot,\cdot)$ (or to the operator $\mathcal{L}_a$).*

Proof. The proof follows a similar approach to the classical case ([221] Theorem 3.9 or [195] Theorem 8.1). Taking $v = u \wedge 1 = u - (u-1)^+ \in \mathbb{K}^s_K$ in (4.15), one has

$$\frac{a_*}{c^2_{d,s}} \left\|(u-1)^+\right\|^2_{H^s_0(\Omega)} \leq \mathcal{E}_a(u-1,(u-1)^+) = \mathcal{E}_a(u,(u-1)^+) \leq 0$$

since the s-grad of a constant is zero. Hence $u \preceq 1$ in Ω. But $u \in \mathbb{K}^s_K$, so $u \succeq 1$ on K. Therefore, the first result $u = 1$ on K follows.

For the second result, set $v = u + \varphi \in \mathbb{K}^s_K$ in (4.15) with an arbitrary $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$. Then, by the Riesz-Schwartz theorem (see for instance [4] Theorem 1.1.3), there exists a non-negative Radon measure μ on Ω such that

$$\langle \mathcal{L}_a u, \varphi\rangle = \mathcal{E}_a(u,\varphi) = \int_\Omega \varphi \, d\mu, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Moreover, for $x \in \Omega\backslash K$, there is a neighbourhood $O \subset \Omega\backslash K$ of x so that $u + \varphi \in \mathbb{K}^s_K$ for any $\varphi \in \mathcal{D}(O)$. Therefore,

$$\mathcal{E}_a(u,\varphi) = 0, \quad \forall \varphi \in \mathcal{D}(\Omega\backslash K)$$

which means $\mu = \mathcal{L}_a u = 0$ in $\Omega\backslash K$. Therefore, $supp(\mu) \subset K$ and the third result follows immediately. □

We observe that when a corresponds to the kernel of the fractional Laplacian, the (s,a)-capacity corresponds to the s-capacity and the *s-capacitary potential* of a compact set K is the solution of the obstacle problem (4.15) when the bilinear form is the inner product in $H^s_0(\Omega)$ and we have a simple comparison of the capacities in the following proposition.

Proposition 4.21. *For any compact subset $E \subset \Omega$,*

$$\frac{a_*}{c^2_{d,s}} C_s(E) \leq C^a_s(E) \leq \frac{{a^*}^2}{a_* c^2_{d,s}} C_s(E).$$

Proof. Let u be the (s,a)-capacitary potential of E, and $\bar{u}$ be the s-capacitary potential of E. Since $\bar{u} \succeq 1$ on E, we can choose $v = \bar{u} \in \mathbb{K}^s_E$ in (4.15) to get

$$\begin{aligned} C^a_s(E) = \mathcal{E}_a(u,u) &\leq \mathcal{E}_a(u,\bar{u}) \\ &\leq \frac{a^*}{c^2_{d,s}} \left(\int_{\mathbb{R}^d} |D^s u|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |D^s \bar{u}|^2\right)^{\frac{1}{2}} \\ &\leq \frac{a_*}{2c^2_{d,s}} \int_{\mathbb{R}^d} |D^s u|^2 + \frac{{a^*}^2}{2a_* c^2_{d,s}} \int_{\mathbb{R}^d} |D^s \bar{u}|^2 \\ &\leq \frac{1}{2}\mathcal{E}_a(u,u) + \frac{{a^*}^2}{2a_* c^2_{d,s}} C_s(E) \\ &= \frac{1}{2} C^a_s(E) + \frac{{a^*}^2}{2a_* c^2_{d,s}} C_s(E) \end{aligned}$$

by Cauchy-Schwarz inequality and the coercivity of a. Similarly, we can choose $v = u \in \mathbb{K}_E^s$ for (4.15) with $a = \frac{1}{|x-y|^{d+2s}}$ for $C_s(E)$ to get, using the coercivity of a,

$$\begin{aligned}
\frac{1}{c_{d,s}^2} C_s(E) = \mathcal{E}_a(\bar{u}, \bar{u}) \leq \mathcal{E}_a(\bar{u}, u) \\
&\leq \frac{1}{c_{d,s}^2} \left(\int_{\mathbb{R}^d} |D^s \bar{u}|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |D^s u|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2c_{d,s}^2} \int_{\mathbb{R}^d} |D^s \bar{u}|^2 + \frac{1}{2c_{d,s}^2} \int_{\mathbb{R}^d} |D^s u|^2 \\
&\leq \frac{1}{2c_{d,s}^2} C_s(E) + \frac{1}{2a_*} \mathcal{E}_a(u, u) \\
&= \frac{1}{2c_{d,s}^2} C_s(E) + \frac{1}{2a_*} C_s^a(E).
\end{aligned}$$

□

Using this definition of the Radon measure, we recall that two quasi-continuous functions which are equal (or, $\leq$) μ-a.e. on an open subset of $\mathbb{R}^d$ are also equal (or, $\leq$) q.e. on that set (see [122] Lemma 2.1.4).

Recall that a Radon measure μ is said to be of finite energy relatively to $H_0^s(\Omega)$ if its restriction to $H_0^s(\Omega) \cap C_c(\Omega)$ is continuous for the topology of $H_0^s(\Omega)$, by means of

$$\langle \mu, v \rangle = \int_\Omega v \, d\mu, \quad \forall v \in H_0^s(\Omega) \cap C_c(\Omega).$$

Such a finite energy measure can in fact be defined for any Dirichlet form $\mathcal{E}$ (see [122] Section 2.2 and Example 2.2.1 pages 87–91). We denote by $E^+(H_0^s(\Omega))$ the cone of positive finite energy measures relative to $H_0^s(\Omega)$. Then μ is of finite energy if and only if there exists $w_\mu \in H^{-s}(\Omega)$ such that

$$\langle w_\mu, v \rangle = \int_\Omega v \, d\mu \quad \forall v \in H_0^s(\Omega) \cap C_c(\Omega),$$

and $E^+(H_0^s(\Omega))$ can be identified with $[H^{-s}(\Omega)]^+$, the positive cone of $H^{-s}(\Omega) = [H_0^s(\Omega)]'$, by the mapping $\mu \mapsto w_\mu$. Moreover, whenever $\mu \in E^+(H_0^s(\Omega))$, the mapping $u \in H_0^s(\Omega) \mapsto \bar{u}$ is continuous from $H_0^s(\Omega)$ into $L^1(\mu)$ and whenever $u \in H_0^s(\Omega)$, $\int_\Omega \bar{u} \, d\mu = \langle w_\mu, v \rangle$. Note that in the particular case of the space $H_0^s(\Omega)$, the mapping $u \in H_0^s(\Omega) \mapsto \bar{u} \in L^1(\mu)$ is compact; this follows from the fact that $\int_\Omega |\bar{u}_n| \, d\mu = \langle w_\mu, |u_n| \rangle$ and that if $u_n \rightharpoonup 0$ in $H_0^s(\Omega)$ then $|u_n| \rightharpoonup 0$ in $H_0^s(\Omega)$.

Extending these results to $L^2_{C_s}(\Omega)$, we have the following result.

Proposition 4.22. *Let $\mu \in E^+(H_0^s(\Omega))$. Then $L^2_{C_s}(\Omega) \subset L^1(\mu)$ and this inclusion is continuous.*

Proof. Let $u \in L^2_{C_s}(\Omega)$. There exists $v \in H_0^s(\Omega)$ such that $\bar{v} \geq |u|$ a.e., and therefore μ-q.e.. Since $\bar{v} \in L^1(\mu)$, $u \in L^1(\mu)$.

Let (u_n) be a sequence in $L^2_{C_s}(\Omega)$ such that $R_{C_s}(u_n) \to 0$. Then there exists $(v_n) \in H_0^s(\Omega)$ such that $\bar{v}_n \geq |u_n|$ q.e., and therefore μ-q.e., and $\|v_n\|_{H_0^s(\Omega)} \to 0$. As a result, $\int_\Omega |u_n| \, d\mu \leq \int_\Omega \bar{v}_n \, d\mu = \langle w_\mu, v_n \rangle \leq \|w_\mu\|_{H^{-s}(\Omega)} \|v_n\|_{H_0^s(\Omega)} \to 0$. Therefore $u_n \to 0$ in $L^1(\mu)$. □

Having these results, we can now identify the dual space of $L^2_{C_s}(\Omega)$ with the order dual of $H_0^s(\Omega)$, as given in the following theorem.

Theorem 4.23 (Characterisation of Order Dual)**.** *The dual of $L^2_{C_s}(\Omega)$ is the space of finite energy measures $E^+(H_0^s(\Omega)) - E^+(H_0^s(\Omega))$, that is identified with the order dual $H_\prec^{-s}(\Omega)$ of $H_0^s(\Omega)$. More precisely, $L \in [L^2_{C_s}(\Omega)]'$ if and only if there is a unique μ such as $|\mu| \in E^+(H_0^s(\Omega))$ and $L(\phi) = \int_\Omega \phi \, d\mu$ for all $\phi \in L^2_{C_s}(\Omega)$. In addition, the norm of L in $[L^2_{C_s}(\Omega)]'$ is such that $\|L\| = \|w_{|\mu|}\|_{H^{-s}(\Omega)}$.*

Proof. According to Proposition 4.19, $C_c(\Omega)$ is dense in $L^2_{C_s}(\Omega)$ and moreover this injection is continuous; therefore the dual of $L^2_{C_s}(\Omega)$ is a space of measures.

Let μ be a Radon measure such that $|\mu| \in E^+(H^s_0(\Omega))$. For any $\phi \in L^2_{C_s}(\Omega)$ (ϕ is then μ integrable by Proposition 4.22), set $L(\phi) = \int_\Omega \phi\, d\mu$. For any $v \in H^s_0$ such that $\bar{v} \geq |\phi|$ quasi-everywhere, so μ-a.e., we have

$$|L(\phi)| = \left|\int_\Omega \phi\, d\mu\right| \leq \int_\Omega |\phi| d|\mu| \leq \bar{v} d|\mu| = (w_{|\mu|}, v) \leq \|w_{|\mu|}\|_{H^{-s}(\Omega)} \|v_n\|_{H^s_0(\Omega)},$$

so $|L(\phi)| \leq \|w_{|\mu|}\| R_{C_s}(\phi)$. Therefore $L \in [L^2_{C_s}(\Omega)]'$ and $\|L\| \leq \|w_{|\mu|}\|_{H^{-s}(\Omega)}$.

Conversely, suppose $L \in [L^2_{C_s}(\Omega)]'$. Let K be a compact subset of Ω and $\psi \in \mathcal{D}(\Omega)$ such that $\psi \geq 1$ in K. Whenever $\phi \in C_s(K)$ such that $\|\phi\|_\infty \leq 1$, we have $\psi \geq |\phi|$ in Ω, therefore

$$|L(\phi)| \leq \|L\| \cdot R_{C_s}(\psi).$$

We deduce that there exists a Radon measure μ in Ω such that

$$\forall \phi \in C_c(\Omega), \quad L(\phi) = \int_\Omega \phi\, d\mu.$$

In addition $|\mu| \in E^+(H^s_0)$ because, whenever $u \in H^s_0(\Omega) \cap C_c(\Omega)$,

$$\begin{aligned}\left|\int_\Omega u d|\mu|\right| &\leq \int_\Omega |u| d|\mu| \\ &= \sup\{L(\phi) : \phi \in C_c(\Omega), |\phi| \leq |u|\} \\ &\leq \sup\{\|L\| \cdot R_{C_s}(\phi) : |\phi| \leq |u|\} \\ &\leq \|L\| \cdot R_{C_s}(u) \\ &\leq \|L\| \cdot \|u\|_{H^s_0(\Omega)}\end{aligned}$$

and also $\|w_{|\mu|}\|_{H^{-s}(\Omega)} \leq \|L\|$.

Finally, it follows, from the density of $C_c(\Omega)$ in $L^2_{C_s}(\Omega)$ and Proposition 4.22, that

$$\forall \phi \in L^2_{C_s}(\Omega), \quad L(\phi) = \int_\Omega \phi\, d\mu.$$

□

4.4.2 Lewy-Stampacchia inequalities in $H^{-s}_{\prec}(\Omega)$

For completeness, we state the Lewy-Stampacchia inequalities in the dual space $H^{-s}_{\prec}(\Omega)$.

Theorem 4.24. *The unique solution u to the obstacle problem (4.3) with compatible obstacle $\psi \in H^s(\mathbb{R}^d)$ and $F, \mathcal{L}_a\psi \in H^{-s}_{\prec}(\Omega)$, satisfies*

$$F \leq \mathcal{L}_a u \leq F \vee \mathcal{L}_a \psi \quad \text{in } H^{-s}_{\prec}(\Omega). \tag{4.16}$$

Proof. Since $\mathcal{L}_a$ is strictly T-monotone, this is a direct consequence of the abstract Lewy-Stampacchia inequality obtained by Mosco in [175] (see also Theorem 5:2.1 of [195]). □

We next consider the generalisations to the two obstacles problem and to the N membranes problem.

Similarly, as a direct consequence of Theorem 4.2 of [194], we may also state the Lewy-Stampacchia inequality for the two obstacles problem.

Theorem 4.25. *The solution u of the two obstacles problem*

$$u \in \mathbb{K}^s_{\varphi,\psi} : \quad \mathcal{E}_a(u, v-u) \geq \langle F, v-u\rangle \quad \forall v \in \mathbb{K}^s_{\varphi,\psi}$$

with $\mathbb{K}^s_{\varphi,\psi}$ given by (4.10) with data $F \in H^{-s}_{\prec}(\Omega)$ and $\mathcal{L}_a\psi, \mathcal{L}_a\varphi \in H^{-s}_{\prec}(\Omega)$ satisfies

$$F \wedge \mathcal{L}_a\varphi \leq \mathcal{L}_a u \leq F \vee \mathcal{L}_a \psi \quad \text{in } H^{-s}_{\prec}(\Omega). \tag{4.17}$$

Then, applying the general Lewy-Stampacchia inequalities for the one obstacle and for the two obstacles problem iteratively in the previous theorem as in the proof of Theorem 4.11, we obtain

Theorem 4.26. *The solution $\boldsymbol{u}$ of the N membranes problem*

$$\boldsymbol{u} \in \mathbb{K}_N^s : \quad \sum_{i=1}^{N} \mathcal{E}_a(u_i, v_i - u_i) \geq \sum_{i=1}^{N} \langle F^i, v_i - u_i \rangle \quad \forall (v_1, \ldots v_N) \in \mathbb{K}_N^s$$

with $\mathbb{K}_N^s$ given by (4.13) with data $F = (F^1, \ldots, F^N)$ for $F^i \in H_{\prec}^{-s}(\Omega)$ satisfies

$$\begin{aligned} F^1 \wedge \mathcal{L}_a u_1 &\leq F^1 \vee \cdots \vee F^N \\ F^1 \wedge F^2 \leq \mathcal{L}_a u_2 &\leq F^2 \vee \cdots \vee F^N \\ &\vdots \\ F^1 \wedge \cdots \wedge F^{N-1} \leq \mathcal{L}_a u_{N-1} &\leq F^{N-1} \vee F^N \\ F^1 \wedge \cdots \wedge F^N \leq \mathcal{L}_a u_N &\leq F^N \end{aligned}$$

in $H_{\prec}^{-s}(\Omega)$.

Remark 4.27. *In the symmetric case, the Lewy-Stampacchia inequalities also follow from the general results of [125]. The application to Theorem 4.26 for the N membranes problem is new.*

4.4.3 The $\mathcal{E}_a$ obstacle problem and the s-capacity

As a simple application of s-capacity, we consider the corresponding nonlocal obstacle problem extending some results of [221] and [3] (see also [195]). In this section we start by the following comparison property for the (s,a)-capacity, the proof of which is exactly as in Theorem 3.10 of [221], which states that in the case when the kernel a is symmetric the (s,a)-capacity is an increasing set function.

Proposition 4.28. *For any compact subsets $E_1 \subset E_2 \subset \Omega$,*

$$C_s^a(E_1) \leq \left(1 + \frac{M}{a_*}\right)^2 C_s^a(E_2),$$

where $M = \sup \frac{1}{2}(\mathcal{E}_a(u,v) - \mathcal{E}_a(v,u))$ for u, v such that $\|u\|_{H_0^s(\Omega)} = \|v\|_{H_0^s(\Omega)} = 1$.

Theorem 4.29. *Let ψ be an arbitrary function in $L^2_{C_s}(\Omega)$. Suppose that the closed convex set $\bar{\mathbb{K}}_\psi^s$ is such that*

$$\bar{\mathbb{K}}_\psi^s = \{v \in H_0^s(\Omega) : \bar{v} \geq \psi \text{ q.e. in } \Omega\} \neq \emptyset.$$

Then there is a unique solution to

$$u \in \bar{\mathbb{K}}_\psi^s : \quad \mathcal{E}_a(u, v-u) \geq 0, \quad \forall v \in \bar{\mathbb{K}}_\psi^s, \tag{4.18}$$

which is non-negative and such that

$$\|u\|_{H_0^s(\Omega)} \leq (a^*/a_*) \|\psi^+\|_{L^2_{C_s}(\Omega)}; \tag{4.19}$$

there is a unique measure $\mu = \mathcal{L}_a u \geq 0$, concentrated on the coincidence set $\{u = \psi\} = \{u = \psi^+\}$, verifying

$$\mathcal{E}_a(u,v) = \int_\Omega \bar{v}\, d\mu, \quad \forall v \in H_0^s(\Omega), \tag{4.20}$$

and

$$\mu(K) \leq \left(\frac{a^{*2}}{a_*^{3/2} c_{d,s}}\right) \|\psi^+\|_{L^2_{C_s}(\Omega)} \left[C_s^a(K)\right]^{1/2}, \quad \forall K \Subset \Omega, \tag{4.21}$$

in particular μ does not charge on sets of capacity zero.

Proof. By the maximum principle Proposition 4.2(ii), taking $v = u + u^-$, the solution is non-negative. Hence, the variational inequality (4.18) is equivalent to solving the variational inequality with $\bar{\mathbb{K}}^s_\psi$ replaced by $\bar{\mathbb{K}}^s_{\psi^+}$. Since $\psi^+ \in L^2_{C_s}(\Omega)$, by definition of $L^2_{C_s}(\Omega)$, $\bar{\mathbb{K}}^s_{\psi^+} \neq \emptyset$ and we can apply the Stampacchia theorem to obtain a unique non-negative solution.

Since $\mathcal{E}_a(u, v-u) \geq 0$,

$$\frac{a_*}{c^2_{d,s}}\|u\|^2_{H^s_0(\Omega)} \leq \mathcal{E}_a(u,u) \leq \mathcal{E}_a(u,v) \leq \frac{a^*}{c^2_{d,s}}\|u\|_{H^s_0(\Omega)}\|v\|_{H^s_0(\Omega)}\,,$$

we have

$$\|u\|_{H^s_0(\Omega)} \leq (a^*/a_*)\|v\|_{H^s_0(\Omega)}\,, \quad \forall v \in \bar{\mathbb{K}}^s_{\psi^+},$$

which, by the definition of the $L^2_{C_s}(\Omega)$ norm of ψ^+, gives (4.19).

The existence of a Radon measure for (4.20) follows exactly as in Theorem 4.20. Finally, recalling the definitions, it is sufficient to prove (4.21) for any compact subset $K \subset \Omega$. But this follows from

$$\mu(K) \leq \int_\Omega \bar{v}\,d\mu = \mathcal{E}_a(u,v) \leq \frac{a^*}{c^2_{d,s}}\|u\|_{H^s_0(\Omega)}\|v\|_{H^s_0(\Omega)} \leq (a^{*2}/a_*c^2_{d,s})\big\|\psi^+\big\|_{L^2_{C_s}(\Omega)}\|v\|_{H^s_0(\Omega)}\,, \quad \forall v \in \mathbb{K}^s_K.$$

Now, recall from Proposition 4.21 that we have

$$C^a_s(K) \geq \frac{a_*}{c^2_{d,s}}C_s(K) = \frac{a_*}{c^2_{d,s}}\inf_{v\in\mathbb{K}^s_K}\|v\|^2_{H^s_0(\Omega)}$$

thereby obtaining (4.21). □

Corollary 4.30. *If u and $\hat{u}$ are the solutions to* (4.18) *with non-negative compatible obstacles ψ and $\hat{\psi}$ in $L^2_{C_s}(\Omega)$ respectively, then*

$$\|u-\hat{u}\|_{H^s_0(\Omega)} \leq k\|\psi-\hat{\psi}\|^{1/2}_{L^2_{C_s}(\Omega)},$$

where

$$k = (a^*/a_*)\left[\|\psi\|_{L^2_{C_s}(\Omega)} + \|\hat{\psi}\|_{L^2_{C_s}(\Omega)}\right]^{1/2}.$$

Proof. Since $supp(\mu) \subset \{u = \psi\}$ and $supp(\hat{\mu}) \subset \{\hat{u} = \hat{\psi}\}$ (where $\mu = \mathcal{L}_a u$ and $\hat{\mu} = \mathcal{L}_a\hat{u}$), for an arbitrary $v \in \bar{\mathbb{K}}^s_{|\psi-\hat{\psi}|}$, we have

$$\begin{aligned}
\frac{a_*}{c^2_{d,s}}\|u-\hat{u}\|^2_{H^s_0(\Omega)} &\leq \mathcal{E}_a(u-\hat{u}, u-\hat{u}) \\
&= \mathcal{E}_a(u, u-\hat{u}) - \mathcal{E}_a(\hat{u}, u-\hat{u}) \\
&= \int_\Omega (u-\hat{u})\,d\mu - \int_\Omega (u-\hat{u})d\hat{\mu} \text{ by setting } \bar{v} = u-\hat{u} \text{ for } \mu,\hat{\mu} \text{ in (4.20)} \\
&\leq \int_\Omega (\psi-\hat{\psi})\,d\mu - \int_\Omega (\psi-\hat{\psi})d\hat{\mu} \\
&\leq \int_\Omega |\psi-\hat{\psi}|d(\mu+\hat{\mu}) \\
&\leq \int_\Omega \bar{v}d(\mu+\hat{\mu}) \text{ since } v \in \bar{\mathbb{K}}^s_{|\psi-\hat{\psi}|} \\
&= \int_\Omega \bar{v}\,d\mu + \int_\Omega \bar{v}d\hat{\mu} \\
&= \mathcal{E}_a(u,v) + \mathcal{E}_a(\hat{u},v) \\
&\leq \frac{a^*}{c^2_{d,s}}\left[\|u\|_{H^s_0(\Omega)} + \|\hat{u}\|_{H^s_0(\Omega)}\right]\|v\|_{H^s_0(\Omega)} \\
&\leq \frac{a^{*2}}{a_*c^2_{d,s}}\left[\|\psi\|_{L^2_{C_s}(\Omega)} + \|\hat{\psi}\|_{L^2_{C_s}(\Omega)}\right]\|v\|_{H^s_0(\Omega)} \text{ by (4.19)}
\end{aligned}$$

$$\leq \frac{a^{*2}}{a_* c_{d,s}^2} \left[\|\psi\|_{L^2_{C_s}(\Omega)} + \|\hat{\psi}\|_{L^2_{C_s}(\Omega)} \right] \|\psi - \hat{\psi}\|_{L^2_{C_s}(\Omega)}$$

since v is arbitrary in $\bar{\mathbb{K}}^s_{|\psi-\hat{\psi}|}$, by the definition of the norm of $|\psi - \hat{\psi}|$ in $L^2_{C_s}(\Omega)$. $\square$

Remark 4.31. *Further properties on the s-capacity and the regularity of the solution to the s-obstacle problem are an interesting topic to be developed. For instance, as in the classical local case $s = 1$ [3], it would be interesting to show that ψ is compatible, i.e. $\bar{\mathbb{K}}^s_\psi \neq \emptyset$, if and only if*

$$\int_0^\infty C_s(\{|\psi^+| > t\}) dt^2 < \infty.$$

4.5 The Fractional s-obstacle Problem and its Convergence as $s \nearrow 1$

Consider the fractional obstacle problem given by

$$u \in \mathbb{K}^s_\psi : \quad \int_{\mathbb{R}^d} AD^s u \cdot D^s(v-u) \geq \langle F, v-u \rangle \quad \forall v \in \mathbb{K}^s_\psi$$

for the same convex set $\mathbb{K}^s_\psi$, where $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a bounded, measurable and strictly elliptic matrix satisfying

$$a_*|z|^2 \leq A(x)z \cdot z \text{ and } A(x)z \cdot z^* \leq a^*|z||z^*|. \tag{3.2}$$

We observe that the existence of a solution follows from a direct application of the Stampacchia theorem, since the bilinear form

$$\langle \tilde{\mathcal{L}}_A u, v \rangle = \int_{\mathbb{R}^d} AD^s u \cdot D^s v$$

is bounded

$$\langle \tilde{\mathcal{L}}_A u, v \rangle \leq a^* \|u\|_{H^s_0(\Omega)} \|v\|_{H^s_0(\Omega)} \quad \forall u, v \in H^s_0(\Omega)$$

and coercive

$$\langle \tilde{\mathcal{L}}_A u, u \rangle \geq a_* \|D^s u\|^2_{L^2(\mathbb{R}^d)} = a_* \|u\|^2_{H^s_0(\Omega)}.$$

As could be expected, we have a continuous transition from the fractional obstacle problem to the classical local obstacle problem as $s \nearrow 1$ in the following sense.

Theorem 4.32. *Suppose ψ is such that $\mathbb{K}^1_\psi := \{v \in H^1_0(\Omega) : v \geq \psi \text{ a.e. in } \Omega\} \neq \emptyset$. Let $u^s \in \mathbb{K}^s_\psi$ for $0 < s < 1$ be the solution to the fractional obstacle problem, i.e.*

$$\int_{\mathbb{R}^d} AD^s u^s \cdot D^s(v - u^s) \geq \langle F, v - u^s \rangle \quad \forall v \in \mathbb{K}^s_\psi,$$

where $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a bounded, measurable and strictly elliptic matrix satisfying

$$a_*|z|^2 \leq A(x)z \cdot z \text{ and } A(x)z \cdot z^* \leq a^*|z||z^*|, \tag{3.2}$$

and $F \in H^{-\sigma}(\Omega)$. Then, there exists a unique solution $u^s \in H^s_0(\Omega)$. Furthermore, the sequence $(u^s)_s$ converges strongly to u in $H^\sigma_0(\Omega)$ as $s \nearrow 1$ for any fixed $0 < \sigma < 1$, where $u \in \mathbb{K}^1_\psi$ solves uniquely the obstacle problem for $s = 1$, i.e.

$$\int_\Omega ADu \cdot D(v-u) \geq \langle F, v-u \rangle \quad \forall v \in \mathbb{K}^1_\psi.$$

Proof. Next, we want to show the convergence of the fractional obstacle problem to the classical one. We first prove an a priori estimate for $D^s u^s$. For $v^1 \in \mathbb{K}^1_\psi = \bigcap_{\sigma \leq s < 1} \mathbb{K}^s_\psi$, by Cauchy-Schwarz inequality and Sobolev inequality,

$$a_* \|D^s u^s\|^2_{L^2(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} AD^s u^s \cdot D^s u^s$$

$$\le \int_{\mathbb{R}^d} AD^s u^s \cdot D^s v^1 - \langle F, v^1 - u^s \rangle$$
$$\le \frac{a^{*2}\epsilon}{2}\|D^s u^s\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2\epsilon}\left\|D^s v^1\right\|^2_{L^2(\mathbb{R}^d)} - \langle F, v^1\rangle + \frac{1}{2\epsilon'}\|F\|^2_{H^{-s}(\Omega)} + \frac{\epsilon'}{2}\|D^s u^s\|^2_{L^2(\mathbb{R}^d)}$$
$$\le \frac{a_*}{4}\|D^s u^s\|^2_{L^2(\mathbb{R}^d)} + \frac{a^{*2}}{a_*}\left\|D^s v^1\right\|^2_{L^2(\mathbb{R}^d)} - \langle F, v^1\rangle + \frac{c_\sigma^2}{a_*}\|F\|^2_{H^{-\sigma}(\Omega)} + \frac{a_*}{4}\|D^s u^s\|^2_{L^2(\mathbb{R}^d)}$$

by taking $\epsilon = \frac{a_*}{2a^{*2}}$ and $\epsilon' = \frac{a_*}{2}$ and c_σ^2 may be chosen independent of s for $0 < s \le 1$, as a consequence of (1.19) for the dual norms $\|\cdot\|_{H^{-s}(\Omega)}$. Therefore, we have

$$\|D^s u^s\|_{L^2(\mathbb{R}^d)} \le C,$$

where the constant $C = C(\sigma, a_*, a^*) > 0$ is independent of $s \ge \sigma$.

Also by (1.19), for $\sigma \le s < 1$, we have

$$\|D^\sigma u^s\|_{L^2(\mathbb{R}^d)} \le c_\sigma \|D^s u^s\|_{L^2(\mathbb{R}^d)} \le C, \tag{4.22}$$

for some constant C independent of s, $\sigma \le s < 1$, and we may take a sequence

$$D^s u^s \xrightarrow[s\nearrow 1]{} \zeta \quad \text{in } [L^2(\mathbb{R}^d)]^d\text{-weak}$$

for some ζ. By compactness, since u^s is also uniformly bounded in $H^\sigma_0(\Omega)$, there exists a subsequence and a limit $u \in L^2(\Omega)$ such that

$$u^s \xrightarrow[s\nearrow 1]{} u \quad \text{strongly in } L^2(\Omega).$$

Now, by Lemma 1.5, for all $\Phi \in [C_c^\infty(\Omega)]^d$, denoting by $\tilde{\Phi}$ the zero extension of Φ outside Ω,

$$D^s \cdot \tilde{\Phi} \to D \cdot \tilde{\Phi} \quad \text{in } [L^2(\mathbb{R}^d)]^d,$$

therefore

$$\int_{\mathbb{R}^d} D^s u^s \cdot \tilde{\Phi} = -\int_{\mathbb{R}^d} \tilde{u}^s (D^s \cdot \tilde{\Phi}) \xrightarrow[s\nearrow 1]{} -\int_{\mathbb{R}^d} \tilde{u}(D \cdot \tilde{\Phi}).$$

But by the a priori estimate on $D^s u^s$,

$$\left|\int_{\mathbb{R}^d} D^s u^s \cdot \tilde{\Phi}\right| \le C\|\Phi\|_{[L^2(\Omega)]^d},$$

which implies that

$$\left|\int_{\mathbb{R}^d} \tilde{u}(D \cdot \tilde{\Phi})\right| \le C\|\Phi\|_{[L^2(\Omega)]^d}.$$

This means that $D\tilde{u} \in [L^2(\mathbb{R}^d)]^d$, and since Ω has a Lipschitz boundary, $\widetilde{Du} = D\tilde{u}$, so $Du \in [L^2(\Omega)]^d$. Together with the first inequality in (1.19) which implies that $u \in \mathbb{K}^\sigma_\psi$ for any $\sigma < 1$, we have $u \in \mathbb{K}^1_\psi$.

Furthermore, by Lemma 1.5, $D^s u \to \widetilde{Du}$ strongly in $[L^2(\mathbb{R}^d)]^d$ as $s \nearrow 1$, so

$$\int_{\mathbb{R}^d} D^s(u^s - u) \cdot \tilde{\Phi} = -\int_{\mathbb{R}^d} (\widetilde{u^s - u})(D^s \cdot \Phi) \to 0,$$

therefore

$$\zeta = \lim_{s\nearrow 1} D^s u^s = \widetilde{Du}.$$

Finally, it remains to show that u satisfies the obstacle problem for $s = 1$. For any $v \in \mathbb{K}^1_\psi \subset \mathbb{K}^s_\psi$, since $D^s u^s$ are uniformly bounded, we have, up to a subsequence and using the lower-semicontinuity of $A_{sym} = \frac{1}{2}(A + A^T)$,

$$\int_\Omega ADu \cdot Dv = \int_{\mathbb{R}^d} A\widetilde{Du} \cdot \widetilde{Dv}$$

$$
\begin{aligned}
&= \lim_{s\nearrow 1}\int_{\mathbb{R}^d} AD^s u^s\cdot D^s v\\
&\geq \lim_{s\nearrow 1}\langle F, v-u^s\rangle + \liminf_{s\nearrow 1}\int_{\mathbb{R}^d} AD^s u^s\cdot D^s u^s\\
&= \langle F, v-u\rangle + \liminf_{s\nearrow 1}\int_{\mathbb{R}^d} A_{sym}D^s u^s\cdot D^s u^s\\
&\geq \langle F, v-u\rangle + \int_{\mathbb{R}^d} A_{sym}\widetilde{Du}\cdot \widetilde{Du}\\
&= \langle F, v-u\rangle + \int_{\Omega} ADu\cdot Du
\end{aligned}
$$

since $D^s u^s \rightharpoonup \widetilde{Du}$ weakly in $[L^2(\mathbb{R}^d)]^d$ and $D^s v \to \widetilde{Dv}$ strongly in $[L^2(\mathbb{R}^d)]^d$. The conclusion follows by the compactness of the inclusion of $H^\sigma_0(\Omega)$ in $H^{\sigma'}_0(\Omega)$ when $\sigma > \sigma'$.

□

Remark 4.33. *The case with $A = \mathbb{I}$ corresponds to the obstacle problem for the fractional Laplacian and was first considered by Silvestre in [219]. Indeed, from* (1.8)*, since*

$$
\int_{\mathbb{R}^d} D^s u\cdot D^s v = \frac{c^2_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}}\,dx\,dy,
$$

also holds for $u, v\in H^s_0(\Omega)$, Theorem 4.32 gives the convergence of the solution u^s to the nonlocal obstacle problem (4.3)*, which is equivalent, up to a constant, to*

$$
u^s \in \mathbb{K}^s_\psi : \int_{\mathbb{R}^d} D^s u^s\cdot D^s(v-u^s) \geq \langle F, v-u^s\rangle \quad \forall v\in \mathbb{K}^s_\psi
$$

towards the solution u of the classical problem

$$
u \in \mathbb{K}^1_\psi : \int_{\Omega} Du\cdot D(v-u) \geq \langle F, v-u\rangle \quad \forall v\in \mathbb{K}^1_\psi.
$$

Remark 4.34. *A similar convergence result as $s\nearrow 1$ for the symmetric nonlinear nonlocal operator $\bar{\mathcal{L}}^s_g$ is shown in Section 5.3, given by the Γ-convergence of the associated symmetric energy functionals. Such a result remains open not only for the general non-symmetric case, but also for the linear nonlocal operator $\mathcal{L}_a$ when the kernel $a(x, y)$ depends on $x, y \in \mathbb{R}^d$.*

5 Elliptic and Parabolic Nonlocal Nonlinear Obstacle-Type Problems

5.1 The Nonlocal Nonlinear g-Laplacian

In this chapter, we consider the nonlocal nonlinear nonhomogeneous g-Laplacian $\bar{\mathcal{L}}_g^s : H_0^s(\Omega) \to H^{-s}(\Omega)$ as defined in the homogeneous case of g independent of (x,y) in [113] (see also [77], [204], [90] and [115]) by

$$\langle \bar{\mathcal{L}}_g^s u, v\rangle = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g\left(x,y,\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|x-y|^s}\frac{v(x)-v(y)}{|x-y|^s}\frac{dx\,dy}{|x-y|^d}, \tag{5.1}$$

and consider the corresponding elliptic and parabolic obstacle-type problems. Here, $g(x,y,r) : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+ \to [g_*, g^*]$ for some $g_* > 0$ is a function symmetric in x and y, i.e. $g(x,y,r) = g(y,x,r)$, is bounded in x and y for all $r > 0$, and Lipschitz continuous in r for almost every $x, y \in \mathbb{R}^d$, such that

$$0 < a_* \le \frac{r\frac{\partial}{\partial r}g(x,y,r)}{g(x,y,r)} + 1 \le a^* \quad \text{for } r > 0 \tag{5.2}$$

for some constants $0 < a_* \le 1 \le a^*$, and $\lim_{r\to 0^+} rg(x,y,r) = 0$. Then, in particular, $\bar{\mathcal{L}}_g^s$ generalises the symmetric nonlocal operator $\mathcal{L}_a$ which corresponds to the case where $g(x,y,r) = \hat{a}(x,y) = a(x,y)|x-y|^{d+2s}$.

Remark 5.1. *A different definition of a nonlocal g-Laplacian is given in [143] and [92], where the integral is instead taken over the domain Ω.*

We first observe some properties regarding the strong monotonicity and Lipschitzness of g. Indeed,

Proposition 5.2. *For a.e. $x, y \in \mathbb{R}^d$, for all $\xi, \zeta \in \mathbb{R}^n$,*

$$(g(x,y,|\xi|)\xi - g(x,y,|\zeta|)\zeta)\cdot(\xi-\zeta) \ge a_* g_* |\xi-\zeta|^2$$

and

$$|g(x,y,|\xi|)\xi - g(x,y,|\zeta|)\zeta| \le a^* g^* |\xi-\zeta|.$$

In particular, $\lim_{r\to\infty} rg(x,y,r) = \infty$.

Proof. The proof is similar to that given in page 2 of [69], and for simplicity, we write $g(r)$ for $g(x,y,r)$. Indeed, setting $\theta_r = r\xi + (1-r)\zeta$, we have

$$\begin{aligned}
(g(|\xi|)\xi - g(|\zeta|)\zeta)\cdot(\xi-\zeta) &= \left(\int_0^1 \frac{d}{dr}(g(|\theta_r|)\theta_r)\,dr\right)\cdot(\xi-\zeta) \\
&= \left(\int_0^1 g(|\theta_r|)\frac{d\theta_r}{dr}\,dr\right)\cdot(\xi-\zeta) + \left(\int_0^1 \theta_r\frac{dg(|\theta_r|)}{dr}\,dr\right)\cdot(\xi-\zeta) \\
&= |\xi-\zeta|^2\left(\int_0^1 g(|\theta_r|)\,dr\right) + \left(\int_0^1 \theta_r\frac{\theta_r g'(|\theta_r|)}{|\theta_r|}\frac{d\theta_r}{dr}\,dr\right)\cdot(\xi-\zeta) \\
&= |\xi-\zeta|^2\left(\int_0^1 g(|\theta_r|)\,dr\right) + |\xi-\zeta|^2\left(\int_0^1 |\theta_r|g'(|\theta_r|)\,dr\right) \\
&\ge |\xi-\zeta|^2\left(\int_0^1 g(|\theta_r|)\,dr\right) + (a_*-1)|\xi-\zeta|^2\left(\int_0^1 g(|\theta_r|)\,dr\right) \quad \text{by (5.2)} \\
&= a_*|\xi-\zeta|^2\left(\int_0^1 g(|\theta_r|)\,dr\right) \ge a_* g_* |\xi-\zeta|^2
\end{aligned}$$

which is strictly positive for $\xi \neq \zeta$.

At the same time, once again by (5.2),

$$\begin{aligned}|g(|\xi|)\xi - g(|\zeta|)\zeta| &\le |\xi-\zeta|\left(\int_0^1 g(|\theta_r|)\,dr\right) + |\xi-\zeta|\left(\int_0^1 |\theta_r| g'(|\theta_r|)\,dr\right)\\ &\le |\xi-\zeta|\left(\int_0^1 g(|\theta_r|)\,dr\right) + (a^*-1)|\xi-\zeta|\left(\int_0^1 g(|\theta_r|)\,dr\right)\\ &= a^*|\xi-\zeta|\left(\int_0^1 g(|\theta_r|)\,dr\right) \le a^* g^* |\xi-\zeta|.\end{aligned}$$

□

Next, let the mapping $\bar{g} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$\bar{g}(x, y, r) := g(x, y, r)r.$$

Then, from the properties derived above, it can be seen that $\bar{g}$ satisfies the following condition:

(1) For all $x, y \in \mathbb{R}^d$, $\bar{g}(x, y, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is an odd, increasing homeomorphism from $\mathbb{R}^+$ onto $\mathbb{R}^+$, $\bar{g}(x, y, r) > 0$ when $r > 0$.

Moreover, its primitive $G : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^+$ defined for all $x, y \in \mathbb{R}^d$ and all $r \ge 0$ by

$$G(x, y, r) := \int_0^r \bar{g}(x, y, \rho)\,d\rho$$

satisfies:

(2) For all $x, y \in \mathbb{R}^d$, $G(x, y, \cdot) : [0, \infty[\to \mathbb{R}$ is an increasing function, $\lim_{r\to\infty} \bar{g}(x, y, r) = \infty$, $G(x, y, 0) = 0$ and $G(x, y, r) > 0$ whenever $r > 0$;

(3) For every $r \ge 0$, $G(\cdot, \cdot, r) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a measurable function.

(4) For the same constants $a_* < a^*$ as in (5.2),

$$0 < 1 + a_* \le \frac{r\bar{g}(x, y, r)}{G(x, y, r)} \le a^* + 1, \quad \text{a.e. } x, y \in \mathbb{R}^d, \quad r \ge 0, \tag{5.3}$$

so G satisfies the Δ_2-condition. (For a definition of the Δ_2-condition, see [5] or [151].)

This means that G is a strictly convex Young function. We write $G(r)$ for $G(x, y, r)$ if there is no confusion.

Now, denoting G^* as the conjugate Young function of G, which is defined by

$$G^*(x, y, r) = \sup_{\rho > 0}\{r\rho - G(x, y, \rho)\,\forall r \ge 0\}, \quad \text{for a.e. } x, y \in \mathbb{R}^d$$

we have the following properties (see Equations (P5) and (2.3)–(2.6) of [113], as well as Equation (G2) of [168]):

(i) For every $a, b \ge 0$, $ar \le G(b) + G^*(a)$.

(ii) There exists $q > 1$ such that $b^{2q}G(a) \le G(ab)$ for every $a > 0$ and $0 \le b \le 1$.

(iii) At the same time, for any $0 < b < 1$ and $a > 0$, $G(ab) \le bG(a)$,

(iv) while, for $a \ge 0$ and $b \ge 1$, $G(ab) \le b^{a^*+1}G(a)$.

(v) Also, for all $a, b > 0$, $G(a + b) \le 2^{a^*}(1 + a^*)(G(a) + G(b))$.

(vi) Finally, $G^*(\bar{g}(r)) = \bar{g}(r)r - G(r)$, and

(vii) $G^*(\bar{g}(r)) \le a^* G(r)$.

As a result, by the assumptions on G in (5.3), $\bar{\mathcal{L}}_g^s$ is the potential operator defined in $H_0^s(\Omega)$ with respect to the convex functional

$$\Gamma(v) := \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G\left(x, y, \frac{|v(x) - v(y)|}{|x-y|^s}\right) \frac{dx\, dy}{|x-y|^d} \tag{5.4}$$

(see, for instance, Section 20.4 of [151].)

Furthermore,

Proposition 5.3. *The operator $\bar{\mathcal{L}}_g^s$ is strictly coercive, Lipschitz, and strictly T-monotone.*

Proof. Writing $\xi = \frac{u(x)-u(y)}{|x-y|^s}$ and $\zeta = \frac{v(x)-v(y)}{|x-y|^s}$ for all $u, v \in H_0^s(\Omega)$,

$$\begin{aligned}
\langle \bar{\mathcal{L}}_g^s u - \bar{\mathcal{L}}_g^s v, u - v \rangle &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g\left(x, y, |\xi|\right) \xi \cdot (\xi - \zeta) \frac{dx\, dy}{|x-y|^d} - \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g\left(x, y, |\zeta|\right) \zeta \cdot (\xi - \zeta) \frac{dx\, dy}{|x-y|^d} \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (g(x,y,|\xi|)\xi - g(x,y,|\zeta|)\zeta) \cdot (\xi - \zeta) \frac{dx\, dy}{|x-y|^d} \\
&\ge \frac{1}{2} a_* g_* \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\xi - \zeta|^2 \frac{dx\, dy}{|x-y|^d} \\
&= \frac{1}{2} a_* g_* \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \frac{u(x) - u(y)}{|x-y|^s} - \frac{v(x) - v(y)}{|x-y|^s} \right|^2 \frac{dx\, dy}{|x-y|^d} = \frac{a_* g_*}{2 c_{d,s}^2} \|u - v\|_{H_0^s(\Omega)}^2,
\end{aligned}$$

so $\bar{\mathcal{L}}_g^s$ is strictly coercive.

Also, $\bar{\mathcal{L}}_g^s$ is Lipschitz since for all $u, v, w \in H_0^s(\Omega)$ with $\|w\|_{H_0^s(\Omega)} = 1$,

$$\begin{aligned}
|\langle \bar{\mathcal{L}}_g^s u - \bar{\mathcal{L}}_g^s v, w \rangle| &\le \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g(x,y,|\xi|)\xi - g(x,y,|\zeta|)\zeta| \frac{|w(x) - w(y)|}{|x-y|^s} \frac{dx\, dy}{|x-y|^d} \\
&\le \frac{1}{2} a^* g^* \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\xi - \zeta|}{|x-y|^{\frac{d}{2}}} \frac{|w(x) - w(y)|}{|x-y|^{s+\frac{d}{2}}} \, dx\, dy \le \frac{a^* g^*}{2 c_{d,s}^2} \|u - v\|_{H_0^s(\Omega)}.
\end{aligned}$$

The strict T-monotonicity follows in a similar way to the proof of Proposition 5.2. Once again, setting $\theta_r = r\xi + (1-r)\zeta$ and writing $w = u - v$, we have

$$\begin{aligned}
&\langle \mathcal{L}_g^s u - \mathcal{L}_g^s v, w^+ \rangle \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (w^+(x) - w^+(y)) \left[g\left(x, y, |\xi|\right) \xi - g\left(x, y, |\zeta|\right) \zeta \right] \frac{dy\, dx}{|x-y|^{d+s}} \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (w^+(x) - w^+(y)) \left[\int_0^1 g(x,y,|\theta_r|)\, dr + \int_0^1 |\theta_r| g'(x,y,|\theta_r|)\, dr \right] (\xi - \zeta) \frac{dy\, dx}{|x-y|^{d+s}}
\end{aligned}$$

Now, by (5.2),

$$a^* g^* \ge J(x,y) = \int_0^1 g(x,y,|\theta_r|)\, dr + \int_0^1 |\theta_r| g'(x,y,|\theta_r|)\, dr \ge a_* g_* > 0,$$

so we have

$$\begin{aligned}
\langle \mathcal{L}_g^s u - \mathcal{L}_g^s v, (u-v)^+ \rangle &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x,y) \frac{w^+(x) - w^-(x) - w^+(y) + w^-(y)}{|x-y|^{d+2s}} (w^+(x) - w^+(y))\, dx\, dy \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x,y) \frac{(w^+(x) - w^+(y))^2 + w^-(x) w^+(y) + w^+(x) w^-(y)}{|x-y|^{d+2s}} \, dx\, dy \\
&\ge \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x,y) \frac{(w^+(x) - w^+(y))^2}{|x-y|^{d+2s}} \, dx\, dy > 0
\end{aligned}$$

if $w^+ \ne 0$, since $w^-(x) w^+(x) = w^-(y) w^+(y) = 0$. □

Remark 5.4. *Since g is symmetric about x and y, as observed in Theorem 6.12 of [113], the operator $\bar{\mathcal{L}}_g^s$ can also be written as*

$$\bar{\mathcal{L}}_g^s u = P.V. \int_{\mathbb{R}^d} g\left(x, y, \frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|x-y|^{d+2s}}\, dy.$$

In the first part of this chapter, we show the existence and uniqueness results for the nonlocal nonlinear nonhomogeneous elliptic obstacle-type problems, namely the one obstacle problem, two obstacles problem and the N membranes problem, as well as the Lewy-Stampacchia inequalities, providing local regularity results in the homogeneous case when $g(x,y,r) = g(r)$. This generalises Chapter 2 to the nonlinear case. Next, in Section 5.3, we will show that in the homogeneous case, the one obstacle problem defined with the nonlocal operator $\bar{\mathcal{L}}_g^s$ converges to the solution of the classical nonlinear elliptic one obstacle corresponding to $s = 1$, concluding the analysis of the elliptic problem. Following, we will extend this study to the evolutionary nonhomogeneous problem in Section 5.4, obtaining similarly the existence and uniqueness results, and the Lewy-Stampacchia inequalities for all three obstacle-type problems, which gives local regularity in the case when $\bar{\mathcal{L}}_g^s = \mathcal{L}_a$ is the linear nonlocal operator. Finally, we show that these problems converge to the stationary ones in Section 5.5.

5.2 Elliptic Obstacle-Type Problems

We consider $\bar{\mathcal{L}}_g^s$ which is defined by

$$\langle \bar{\mathcal{L}}_g^s u, v\rangle = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g\left(x, y, \frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{u(x)-u(y)}{|x-y|^s}\,\frac{v(x)-v(y)}{|x-y|^s}\,\frac{dx\,dy}{|x-y|^d}, \tag{5.1}$$

to study elliptic obstacle-type problems in a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. We first introduce some notations.

For every function $u \in L^2(\Omega)$, we assume it is extended by 0 outside Ω. Recall that for $0 < s \leq 1$, the classical fractional Sobolev space $H_0^s(\Omega)$ is given by the closure of $C_c^\infty(\Omega)$ with respect to the $\|\cdot\|_{H^s(\mathbb{R}^d)}$ norm, when Ω has Lipschitz boundary.

We want to consider the elliptic obstacle-type problems, such as the nonlinear nonlocal one obstacle problem which is given as

$$u \in K_\psi^s: \quad \langle \bar{\mathcal{L}}_g^s u, v-u\rangle \geq \int_\Omega f(v-u) \quad \forall v \in K_\psi^s, \tag{5.5}$$

for $f \in L^2(\Omega) \subset H^{-s}(\Omega)$ and a measurable obstacle function $\psi \in H^s(\mathbb{R}^d)$, which is admissible in the sense that the closed convex set

$$K_\psi^s = \{v \in H_0^s(\Omega) : v \geq \psi \text{ a.e. in } \Omega\} \neq \emptyset.$$

We first state the corresponding result for elliptic Dirichlet problems, given as Theorems 6.15 and 6.16 in [113], as well as in [114] and Theorem 2 of [115].

Theorem 5.5. *Let $0 < s < 1$ and $\Omega \subset \mathbb{R}^d$ be an open bounded domain. For $f \in L^2(\Omega)$, there exists a unique weak solution $u \in H_0^s(\Omega)$ to*

$$\bar{\mathcal{L}}_g^s u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \Omega^c,$$

which is equivalent to the infimum in $L^2(\Omega)$ of the functional $\mathcal{G}_s : L^2(\Omega) \to \bar{\mathbb{R}}$ defined by

$$\mathcal{G}_s(v) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G\left(x, y, \frac{|v(x)-v(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^d} - \int_\Omega fv \quad \forall v \in H_0^s(\Omega) \quad \text{and} \quad \mathcal{G}_s(v) = +\infty \text{ otherwise.} \tag{5.6}$$

Moreover, when $g(x,y,r) = g(r)$ is the one-parameter homogeneous kernel, assume that Ω is an open bounded set with Lipschitz boundary, and $f \in L^\infty(\Omega)$. Then there exist constants $\alpha = \alpha\left(s, d, a_*, a^*, \|f\|_{L^\infty(\Omega)}\right)$ $\in]0, s]$ and $C_1 = C_1\left(s, d, a_*, a^*, \|f\|_{L^\infty(\Omega)}, \Omega\right) > 0$ such that the weak solution $u \in H_0^s(\Omega)$ satisfies

$$u \in C^\alpha(\bar{\Omega}) \quad \text{such that } \|u\|_{C^\alpha(\bar{\Omega})} \leq C_1.$$

Remark 5.6. *Observe that in the results of Bonder et al. [113, 114, 115], the existence result in $H_0^s(\Omega)$ is given for the operator $\bar{\mathcal{L}}_g^s$ with the homogeneous one-parameter nonlinear kernel $g(r)$. However, since $\bar{\mathcal{L}}_g^s$ is coercive and Lipschitz in $H_0^s(\Omega)$, it holds for the three-parameter kernel $g(x, y, r)$.*

Remark 5.7. *The functional $\mathcal{G}_s(v)$ can also be written for all $v \in L^2(\Omega)$ as*

$$\mathcal{G}_s(v) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G\left(x, y, \frac{|v(x) - v(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^d} - \int_\Omega fv + I_{H_0^s(\Omega)}(v)$$

where $I_{H_0^s(\Omega)}(v) = 0$ if $v \in H_0^s(\Omega)$ and $+\infty$ if $v \in L^2(\Omega)\backslash H_0^s(\Omega)$.

Moreover, by the strict T-monotonicity of $\bar{\mathcal{L}}_g^s$ in $H_0^s(\Omega)$ given by Proposition 5.3, we have, in addition, the comparison property.

Proposition 5.8. *If $u, \hat{u}$ denotes the solution corresponding to f, ψ and $\hat{f}, \hat{\psi}$ respectively, then*

$$f \geq \hat{f} \quad \text{implies} \quad u \geq \hat{u} \quad \text{a.e. in } \Omega.$$

Proof. Taking $v = u \vee \hat{u} \in H_0^s(\Omega)$ for the original problem and $\hat{v} = u \wedge \hat{u} \in H_0^s(\Omega)$ for the other problem and adding, we have

$$\langle \bar{\mathcal{L}}_g^s \hat{u} - \bar{\mathcal{L}}_g^s u, (\hat{u} - u)^+\rangle + \int_\Omega (f - \hat{f})(\hat{u} - u)^+ = 0.$$

Since $f \geq \hat{f}$, the result follows by the strict T-monotonicity of $\bar{\mathcal{L}}_g^s$. □

Next, we want to show similar results for the obstacle-type problems. We begin with the one obstacle problem.

Theorem 5.9. *Let $0 < s < 1$ and $\Omega \subset \mathbb{R}^d$ be a Lipschitz bounded domain. The one obstacle problem (5.5) has a unique solution $u = u(f, \psi) \in K_\psi^s$, and is equivalent to minimizing in K_ψ^s the functional $\mathcal{F}_s : L^2(\Omega) \to \bar{\mathbb{R}}$ defined by*

$$\mathcal{F}_s(v) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G\left(x, y, \frac{|v(x) - v(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^d} - \int_\Omega fv \quad \forall v \in K_\psi^s \quad \text{and} \quad \mathcal{F}_s(v) = +\infty \text{ otherwise.} \tag{5.7}$$

Moreover, if $\hat{u}$ denotes the solution corresponding to $\hat{f}$ and $\hat{\psi}$, then

$$f \geq \hat{f}, \psi \geq \hat{\psi} \quad \text{implies} \quad u \geq \hat{u} \quad \text{a.e. in } \Omega.$$

And, when $f = \hat{f}$,

$$\|u - \hat{u}\|_{L^\infty(\Omega)} \leq \left\|\psi - \hat{\psi}\right\|_{L^\infty(\Omega)}.$$

Proof. The existence and uniqueness follow directly by Stampacchia theorem (see, for instance Theorem 4:3.1 of [195]) from the coercivity and Lipschitzness of $\bar{\mathcal{L}}_g^s$ in Proposition 5.3. The comparison property is once again standard and follows from the strict T-monotonicity of $\bar{\mathcal{L}}_g^s$ as given in Proposition 5.3. Indeed, taking $v = u \vee \hat{u} \in K_\psi^s$ for the original problem and $\hat{v} = u \wedge \hat{u} \in K_{\hat{\psi}}^s$ for the other problem and adding, we have

$$\langle \bar{\mathcal{L}}_g^s \hat{u} - \bar{\mathcal{L}}_g^s u, (\hat{u} - u)^+\rangle + \int_\Omega (f - \hat{f})(\hat{u} - u)^+ \leq 0.$$

Since $f \geq \hat{f}$ and $\bar{\mathcal{L}}_g^s$ is strictly T-monotone, $(\hat{u} - u)^+ = 0$, i.e. $u \geq \hat{u}$.

The L^∞-continuous dependence follows from similarly from taking $v = u + \left(\hat{u} - u - \left\|\psi - \hat{\psi}\right\|_{L^\infty(\Omega)}\right)^+ \in K_\psi^s$ and $\hat{v} = \hat{u} - \left(\hat{u} - u - \left\|\psi - \hat{\psi}\right\|_{L^\infty(\Omega)}\right)^+ \in K_{\hat{\psi}}^s$.

Finally, the equivalence with the minimisation problem follows similarly to the elliptic Dirichlet problem in Theorem 6.15 of [113]. For all $v \in K_\psi^s$, we have, by Property (i) and Property (vi) above,

$$\langle \bar{\mathcal{L}}_g^s u, u\rangle - \int_\Omega fu$$

$$
\begin{aligned}
&\le \langle \bar{\mathcal{L}}^s_g u, v\rangle - \int_\Omega fv \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|x-y|^s}\frac{v(x)-v(y)}{|x-y|^s}\frac{dx\,dy}{|x-y|^d} - \int_\Omega fv \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \bar{g}\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{v(x)-v(y)}{|x-y|^s}\frac{dx\,dy}{|x-y|^d} - \int_\Omega fv \\
&\le \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \bar{g}\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{|u(x)-u(y)|}{|u(x)-u(y)|}\frac{|v(x)-v(y)|}{|x-y|^s}\frac{dx\,dy}{|x-y|^d} - \int_\Omega fv \\
&\le \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G^*\left(\bar{g}\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\right) + G\left(\frac{|v(x)-v(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^d} - \int_\Omega fv \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \bar{g}\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{|u(x)-u(y)|}{|x-y|^s} - G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) + G\left(\frac{|v(x)-v(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^d} - \int_\Omega fv \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{|u(x)-u(y)|^2}{|x-y|^{2s}} - G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) + G\left(\frac{|v(x)-v(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^d} - \int_\Omega fv \\
&= \langle \bar{\mathcal{L}}^s_g u, u\rangle - \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^d} + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G\left(\frac{|v(x)-v(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^d} - \int_\Omega fv
\end{aligned}
$$

Conversely, suppose $u \in L^2(\Omega)$ such that $u \ge \psi$ a.e. in Ω is a minimum in K^s_ψ of $\mathcal{F}_s$. In particular, $\mathcal{F}_s(u) < \infty$ so $u \in H^s_0(\Omega)$. Fix $v \in K^s_\psi$, and define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) := \mathcal{F}_s(u + t(v-u)).$$

Since u is the infimum of $\mathcal{F}_s$, $\phi(0) = \inf_t \phi(t)$, so $\phi'(0) \ge 0$, which is just the the Euler inequality $\mathcal{F}'_s(u)(v-u) \ge 0$ for the obstacle problem (5.5). □

Remark 5.10. *This argument also works for $p \neq 2$ in the framework of the fractional p-Laplacian as given in [119]. Indeed, the fractional p-Laplacian defined for $1 < p < \infty$ by*

$$(-\Delta)^s_p u(x) = \textit{P.V.}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{d+sp}}\,dy$$

is T-monotone in $W^{s,p}_0(\Omega)$ by a similar proof as in Proposition 5.3.

As a result, we can similarly obtain the comparison property and L^∞ estimates for the fractional p-Laplacian, which coincides, up to a constant, with $(-\Delta)^s$ when $p = 2$.

Furthermore, as in classical cases, we have the following convergence result, and we give the proof here for completeness.

Proposition 5.11. *Suppose g satisfies (5.2) and $u_n, u \in H^s_0(\Omega)$, $u_n \rightharpoonup u$ weakly in $H^s_0(\Omega)$. If*

$$\limsup_{n\to\infty} \langle \bar{\mathcal{L}}^s_g u_n, u_n - u\rangle \le 0,$$

then

$$u_n \to u \text{ strongly in } H^s_0(\Omega), \quad \text{and} \quad \bar{\mathcal{L}}^s_g u_n \to \bar{\mathcal{L}}^s_g u \text{ strongly in } H^{-s}(\Omega).$$

Proof. Using the coercivity of $\bar{\mathcal{L}}^s_g$ from Proposition 5.3,

$$\frac{g_*}{c^2_{d,s}}\|u_n - u\|^2_{H^s_0(\Omega)} \le \langle \bar{\mathcal{L}}^s_g u_n - \bar{\mathcal{L}}^s_g u, u_n - u\rangle \le \langle \bar{\mathcal{L}}^s_g u_n, u_n - u\rangle - \langle \bar{\mathcal{L}}^s_g u, u_n - u\rangle.$$

Then, taking the limsup and using the assumptions, since $u_n \rightharpoonup u$ in $H^s_0(\Omega)$, we have the strong convergence of u_n to u in $H^s_0(\Omega)$.

Furthermore, since $\bar{\mathcal{L}}^s_g$ is a Lipschitz operator by Proposition 5.3, it follows that $\bar{\mathcal{L}}^s_g u_n \to \bar{\mathcal{L}}^s_g u$ strongly in $H^{-s}(\Omega)$. □

Moreover, using classical bounded penalisation techniques, by the strict T-monotonicity of $\bar{\mathcal{L}}^s_g$ and the assumption on g in (5.2), we can easily derive the Lewy-Stampacchia inequality, as in Section 4.3.1.

Theorem 5.12. *Suppose*

$$f, (\bar{\mathcal{L}}^s_g\psi - f)^+ \in L^2(\Omega).$$

Then, the solution u of the nonlinear one obstacle problem (5.5) *satisfies*

$$f \leq \bar{\mathcal{L}}^s_g u \leq f \vee \bar{\mathcal{L}}^s_g \psi \quad \text{a.e. in } \Omega. \tag{5.8}$$

In particular, $\bar{\mathcal{L}}^s_g u \in L^2(\Omega)$.

Proof. The proof follows as in Section 4.3.1, by considering the corresponding penalised problem

$$u_\varepsilon \in H^s_0(\Omega): \quad \langle \bar{\mathcal{L}}^s_g u_\varepsilon, v\rangle + \int_\Omega \zeta\theta_\varepsilon(u_\varepsilon - \psi)v = \int_\Omega (f+\zeta)v \quad \forall v \in H^s_0(\Omega).$$

Then the solution $u_\varepsilon \in K^s_\psi$ for each $\varepsilon > 0$, since $\bar{\mathcal{L}}^s_g$ is strictly coercive, we have the error estimate

$$\|u - u_\varepsilon\|^2_{H^s_0(\Omega)} \leq \varepsilon(C_\theta c^2_{d,s}/g_*)\|\zeta\|_{L^1(\Omega)},$$

which implies that u_ε converges strongly in $H^s_0(\Omega)$ as $\varepsilon \to 0$ to the solution u of the obstacle problem. Then, choosing $\zeta = (\bar{\mathcal{L}}^s_g\psi - f)^+$ in the penalised problem, the inequality (5.8) is also satisfied for u_ε, and since $\bar{\mathcal{L}}^s_g$ is Lipschitz, (5.8) is therefore satisfied by u at the limit $\varepsilon \to 0$. ☐

Furthermore, as in Chapter 4, we similarly have the result for the two obstacles problem and the N membranes problem.

Theorem 5.13. *Let $0 < s < 1$ and $\Omega \subset \mathbb{R}^d$ be a Lipschitz bounded domain. Suppose $f \in L^2(\Omega)$ and the measurable obstacle functions $\psi, \varphi \in H^s(\mathbb{R}^d)$ are admissible in the sense that the closed convex set*

$$K^s_{\psi,\varphi} = \{v \in H^s_0(\Omega) : \psi \leq v \leq \varphi \text{ a.e. in } \Omega\} \neq \emptyset.$$

Then, the two obstacles problem

$$u \in K^s_{\psi,\varphi}: \quad \langle \bar{\mathcal{L}}^s_g u, v-u\rangle \geq \int_\Omega f(v-u) \quad \forall v \in K^s_{\psi,\varphi}, \tag{5.9}$$

has a unique solution $u = u(f,\psi,\varphi) \in K^s_{\psi,\varphi}$, and is equivalent to minimizing in $K^s_{\psi,\varphi}$ the functional $\mathcal{F}_s$ defined similarly as in (5.7).

Moreover, if $\hat{u}$ denotes the solution corresponding to $\hat{f}$, $\hat{\psi}$ and $\hat{\varphi}$, then

$$f \geq \hat{f}, \varphi \geq \hat{\varphi}, \psi \geq \hat{\psi} \quad \text{implies} \quad u \geq \hat{u} \text{ a.e. in } \Omega,$$

and if $f = \hat{f}$, L^∞ estimate holds:

$$\|u - \hat{u}\|_{L^\infty(\Omega)} \leq \|\psi - \hat{\psi}\|_{L^\infty(\Omega)} + \|\varphi - \hat{\varphi}\|_{L^\infty(\Omega)}.$$

Assume further that

$$f, (\bar{\mathcal{L}}^s_g\psi - f)^+, (\bar{\mathcal{L}}^s_g\varphi - f)^+ \in L^2(\Omega).$$

Then, the solution u of the nonlinear two obstacles problem (5.9) *satisfies*

$$f \wedge \bar{\mathcal{L}}^s_g\varphi \leq \bar{\mathcal{L}}^s_g u \leq f \vee \bar{\mathcal{L}}^s_g\psi \quad \text{a.e. in } \Omega,$$

and so $\bar{\mathcal{L}}^s_g u \in L^2(\Omega)$.

Proof. The existence and uniqueness follow, as in the previous sections, from the coercivity, continuity and Lipschitzness of the operator $\bar{\mathcal{L}}_g^s$ in Proposition 5.3 and the Stampacchia theorem. The comparison property follows also as previous by the strict T-monotonicity of $\bar{\mathcal{L}}_g^s$.

The L^∞ estimate follows as in the one obstacle problem, while the Lewy-Stampacchia inequalities extend from the one obstacle case as in Theorem 4.10.

It remains to proof the equivalence to the minimisation problem. But the proof is almost similar to the one obstacle problem, except that in this case, we consider $u \in L^2(\Omega)$ such that $\psi \leq u \leq \varphi$ with the corresponding functional finite only when $u \in K^s_{\psi,\varphi}$. □

Finally, we consider now the N membranes problem, which consists of: To find $\boldsymbol{u} = (u_1, u_2, \ldots, u_N) \in K^s_N$ satisfying

$$\sum_{i=1}^{N} \langle \bar{\mathcal{L}}_g^s u_i, v_i - u_i \rangle \geq \sum_{i=1}^{N} \int_\Omega f^i (v_i - u_i), \quad \forall (v_1, \ldots, v_N) \in K^s_N, \tag{5.10}$$

where K^s_N is the convex subset of $[H^s_0(\Omega)]^N$ defined by

$$K^s_N = \{(v_1, \ldots, v_N) \in [H^s_0(\Omega)]^N : v_1 \geq \cdots \geq v_N \text{ a.e. in } \Omega\}$$

and $f^i, \ldots, f^N \in L^2(\Omega)$. As with the one and two obstacles problems, the existence and uniqueness follow easily, while the equivalence with the minimisation problem over the convex set K^s_N follows since the sequence is decreasing. Furthermore, the following Lewy-Stampacchia type inequality also holds.

Theorem 5.14. *The solution $\boldsymbol{u} = (u_1, \ldots u_N)$ of the N membranes problem satisfies a.e. in Ω*

$$\begin{aligned} f^1 \wedge \bar{\mathcal{L}}_g^s u_1 &\leq f^1 \vee \cdots \vee f^N \\ f^1 \wedge f^2 \leq \bar{\mathcal{L}}_g^s u_2 &\leq f^2 \vee \cdots \vee f^N \\ &\vdots \\ f^1 \wedge \cdots \wedge f^{N-1} \leq \bar{\mathcal{L}}_g^s u_{N-1} &\leq f^{N-1} \vee f^N \\ f^1 \wedge \cdots \wedge f^N \leq \bar{\mathcal{L}}_g^s u_N &\leq f^N, \end{aligned}$$

and $\bar{\mathcal{L}}_g^s \boldsymbol{u} \in [L^2(\Omega)]^N$.

Given the Lewy-Stampacchia inequalities, which applies similarly assuming $f, \bar{\mathcal{L}}_g^s\psi, \bar{\mathcal{L}}_g^s\varphi \in L^p(\Omega)$ for any $2 < p \leq \infty$, we can show local regularity for the three nonlocal obstacle-type problems, as in Section 4.3.3. Indeed, as long as the upper and lower bounds are in $L^\infty(\Omega)$, by the Lewy-Stampacchia inequalities, we can make use of the Dirichlet form nature of the quadratic form, and obtain Hölder regularity on the solutions on balls independently of the boundary conditions and of the regularity of $\partial\Omega$ applying the local elliptic regularity Theorem 5.5.

Suppose that $g(x, y, r) = g(r)$ is the one-parameter homogeneous kernel, and

(a) $f, \bar{\mathcal{L}}_g^s\psi \in L^\infty(\Omega)$ for the one obstacle problem,

(b) $f \wedge \bar{\mathcal{L}}_g^s\varphi$ and $f \vee \bar{\mathcal{L}}_g^s\psi$ are in $L^\infty(\Omega)$ for the two obstacles problem, or

(c) $f^i \in L^\infty(\Omega)$ for $i = 1, \ldots, N$ for the N membranes problem.

Theorem 5.15. *Suppose $\bar{\mathcal{L}}_g^s$ is defined with the one-parameter kernel $g(x, y, r) = g(r)$, Let u denote the solutions of the one obstacle problem* (5.5)*, or the two obstacles problem* (5.9)*, or $u = u_i$ for $i = 1, \ldots, N$ of the N membranes problem* (5.10)*, respectively, under the assumptions (a), (b) or (c) above. Suppose $B_\rho \Subset \Omega$ is a ball of radius ρ. Then, there exists $c_\rho \geq 0$ and $\alpha = \alpha\left(s, d, a_*, a^*, \|f\|_{L^\infty(\Omega)}\right) \in]0, s]$, independent of u, such that the following Hölder estimate holds for almost every $x, y \in B_{\rho/2}$:*

$$|u(x) - u(y)| \leq c_\rho |x - y|^\alpha \left(\|u\|_{L^\infty(\Omega)} + \left\| \bar{\mathcal{L}}_g^s u \right\|_{L^\infty(\Omega)} \right).$$

Consequently, $u \in C^\alpha(\Omega)$, i.e. u is locally Hölder continuous.

Proof. Since the Lewy-Stampacchia inequalities in Theorems 5.12, 5.13 and 5.14 hold a.e. in $B_\rho \subset \Omega$ for $\bar{\mathcal{L}}_g^s u$ for the one obstacle, the two obstacles problem and the N membranes problem respectively, and $\bar{\mathcal{L}}_g^s u = f$ in B_ρ, therefore f lies in $L^\infty(\Omega)$, and we have the result as in Section 4.3.3 making use of Theorem 5.5 following the classical approach. □

5.3 Convergence of the One Obstacle Problem to the Classical Problem as $s \nearrow 1$ in the Homogeneous Case

5.3.1 The Young functions G and their modulars

In this section, we consider the special case of the one-parameter nonlinear kernels $g(r)$. Given the corresponding Young functions G, we can subsequently define the modulars Φ_G and $\Phi_{s,G}$, following that of [113], given for $0 < s \leq 1$ and u extended by 0 outside Ω by

$$\Phi_G(u) := \int_{\mathbb{R}^d} G(|u(x)|)\,dx,$$

$$\Phi_{s,G}(u) := \begin{cases} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^d} & \text{if } 0 < s < 1, \\ \int_{\mathbb{R}^d} G(|\nabla u(x)|)\,dx & \text{if } s = 1. \end{cases}$$

Observe that $\Phi_{1,G}(u) = \Phi_G(|\nabla u|)$.

Remark 5.16. *Suppose we define the corresponding Orlicz and Orlicz-Sobolev spaces*

$$L^G(\mathbb{R}^d) := \left\{u : \mathbb{R}^d \to \mathbb{R}, \textit{measurable} : \Phi_G(u) < \infty\right\},$$

$$W^{s,G}(\mathbb{R}^d) := \left\{u \in L^G(\mathbb{R}^d) : \Phi_{s,G}(u) < \infty\right\}$$

with their corresponding Luxemburg norms (see, for instance, Chapter 8 of [5] or Chapter 2 of [176]), given by

$$\|u\|_G = \|u\|_{L^G(\mathbb{R}^d)} := \inf\left\{\lambda > 0 : \Phi_G\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

and

$$\|u\|_{s,G} = \|u\|_{W^{s,G}(\mathbb{R}^d)} := \|u\|_G + [u]_{s,G},$$

where

$$[u]_{s,G} := \inf\left\{\lambda > 0 : \Phi_{s,G}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

$L^G(\mathbb{R}^d)$ and $W^{s,G}(\mathbb{R}^d)$ are known to be reflexive Banach spaces. Then, by Lemma 2.1 of [77] on Page 309, for $a^* > 1 > a_* > 0$ as given in (5.2),

$$L^{a^*+1}(\mathbb{R}^d) \cap L^{a_*+1}(\mathbb{R}^d) \subset L^G(\mathbb{R}^d) \subset L^{a^*+1}(\mathbb{R}^d) + L^{a_*+1}(\mathbb{R}^d).$$

Furthermore, for bounded domains $\Omega \subset \mathbb{R}^d$,

$$L^{a^*+1}(\Omega) \subset L^G(\Omega) \subset L^{a_*+1}(\Omega).$$

Moreover, by (5.2),

$$g_* r^2 \leq G(r) = \int_0^r g(\rho)\rho\,d\rho \leq \int_0^r g^*\rho\,d\rho = g^* r^2, \tag{5.11}$$

we have

$$g_*\int_{\mathbb{R}^d}|u|^2 \leq \Phi_G(u) \leq g^*\int_{\mathbb{R}^d}|u|^2,$$

so the space $L^G(\mathbb{R}^d)$ equipped with the Luxemburg norm is equivalent to the Lebesgue space $L^2(\mathbb{R}^d)$.

Moreover, applying an argument similar to that of Theorem 3.11 of [26], by (5.2),

$$\frac{1+a_*}{r} \le \frac{\bar{g}(x,y,r)}{G(x,y,r)} \le \frac{a^*+1}{r}$$

for almost every $x, y \in \mathbb{R}^d$ *for all* $r \ge 0$*. Since* g *is Lipschitz continuous, for every* $0 < r_0 < r$*, we have*

$$\log(r^{1+a_*}) - \log(r_0^{1+a_*}) \le \int_{r_0}^{r} \frac{1+a_*}{r}\, dr \le \int_{r_0}^{r} \frac{\bar{g}(r)}{G(r)}\, dr$$
$$= \log(G(r)) - \log(G(r_0)) \le \log(r^{1+a^*}) - \log(r_0^{1+a^*}).$$

Exponentiating and integrating with respect to $x, y \in \mathbb{R}^d$ *with measure* $\frac{dx\,dy}{|x-y|^d}$*, we have, for any* $0 < s < 1$*,*

$$W^{s,a^*+1}(\mathbb{R}^d) \subset W^{s,G}(\mathbb{R}^d) \subset W^{s,a_*+1}(\mathbb{R}^d)$$

for the fractional Sobolev-Slobodeckij spaces $W^{s,p}(\mathbb{R}^d)$*.*

Furthermore, also by (5.11)*,* $\Phi_{s,G}$ *is equivalent, up to a constant, to the Gagliardo semi-norm*

$$[u]^2_{s,\mathbb{R}^d} := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}}\, dx\, dy.$$

This means that, as Banach spaces, $L^G(\mathbb{R}^d)$ *and* $W^{s,G}(\mathbb{R}^d)$*, with their corresponding Luxemburg norms, correspond to* $L^2(\mathbb{R}^d)$ *and* $H^s(\mathbb{R}^d)$ *respectively. In particular, these topologies are equivalent, and the homeomorphisms*

$$L^G(\mathbb{R}^d) \approx L^2(\mathbb{R}^d) \quad \text{and} \quad W^{s,G}(\mathbb{R}^d) \approx H^s(\mathbb{R}^d)$$

hold for any $0 < s < 1$*, with equality in the particular case when* $G(r) = r^2$*.*

Furthermore, the Rellich-type compactness results hold similarly to the classical case, as shown in Theorem 3.1 of [113]:

Proposition 5.17. *Let* $0 < s < 1$*. Then for every bounded sequence* $\{u_n\}_{n\in\mathbb{N}}$ *satisfying* $\sup_{n\in\mathbb{N}}\{\Phi_{s,G}(u_n) + \Phi_G(u_n)\} < \infty$*, there exists a limit* u *satisfying the same bound and a subsequence* $\{u_{n_k}\}_{k\in\mathbb{N}}$ *such that* $u_{n_k} \to u$ *in* $L^2_{loc}(\mathbb{R}^d)$*.*

Next, we state some estimates which show how the modular $\Phi_{s,G}$ remains bounded under regularisation and truncation, which will be useful for the convergence as $s \nearrow 1$.

Denote by $\rho \in C_c^\infty(\mathbb{R}^d)$ the standard mollifier function with $\mathrm{supp}(\rho) = B_1(0)$, such that $\rho_\varepsilon(x) = \varepsilon^{-d}\rho\left(\frac{x}{\varepsilon}\right)$ is the approximation of the identity with $\int_{\mathbb{R}^d} \rho_\varepsilon = 1$. Then, as usual, we define the regularised functions $u_\varepsilon \in L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ by

$$u_\varepsilon(x) = u * \rho_\varepsilon(x).$$

Then, as given in Lemma 2.13 of [113],

Lemma 5.18. *Suppose* $u \in L^2(\mathbb{R}^d)$ *and define* u_ε *as above. Then, for all* $\varepsilon > 0$ *and* $0 < s < 1$*,*

$$\Phi_{s,G}(u_\varepsilon) \le \Phi_{s,G}(u).$$

We also want to consider truncated functions. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \le \eta \le 1$ supported in $B_2(0)$ with $\eta = 1$ in $B_1(0)$ and $\|\nabla\eta\|_{L^\infty} \le 2$. For fixed $k \in \mathbb{N}$, define $\eta_k(x) = \eta\left(\frac{x}{k}\right)$. Then $\eta_k \in C_c^\infty(\mathbb{R}^d)$ is supported in $B_{2k}(0)$ with $|\nabla\eta_k| \le \frac{2}{k}$. Defining the truncated functions $\{u_k\}_{k\in\mathbb{N}}$, of $u \in L^2(\mathbb{R}^d)$ by $u_k = \eta_k u$, we have, following Lemma 2.14 of [113] with the constants replaced with the appropriate ones from Property (v) above,

Lemma 5.19. *Let* $u \in L^2(\mathbb{R}^d)$ *and* $\{u_k\}_{k\in\mathbb{N}}$ *be the corresponding truncated functions. Then*

$$\Phi_{s,G}(u_k) \le 2^{a^*}(1+a^*)\Phi_{s,G}(u) + 2^{2a^*+1}(1+a^*)^2\omega_{d-1}\left(\frac{1}{s} + \frac{1}{k(1-s)}\right)\Phi_G(u),$$

where $\omega_{d-1} = \int_{\{|x|=1\}} d\sigma$ *is the spherical measure of the unit ball in* $\mathbb{R}^d$*.*

5.3.2 Convergence as $s \nearrow 1$

Given the nonlinear nonlocal problem for $0 < s < 1$ as well as the classical nonlinear problem for $s = 1$, we want to consider the behaviour as $s \nearrow 1$. This follows much of the approach of [113], adopted to the case of the elliptic obstacle-type problems.

We first begin with a study of the limit function of G when $s \nearrow 1$. Define the bounded function $H : \mathbb{R}^+ \to \mathbb{R}$ as

$$H(a) := \liminf_{s\nearrow 1}(1-s)\int_0^1\int_{\mathbb{S}^{d-1}} G(a|\omega|r^{1-s})dS_\omega \frac{dr}{r}, \tag{5.12}$$

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and ω is the variable on the unit sphere. It is known that the integral converges, by Remark 2.15 of [113]. Furthermore, H is an Orlicz function (see Proposition 2.16 of [113]), and there exists a positive constant $c = c(d, q)$ depending on the dimension d and the constant q from Property (ii) such that

$$cG(r) \le H(r) \le \omega_{d-1}G(r) \quad \text{for every } w \in \mathbb{R}^d \text{ and } r > 0.$$

Correspondingly, we can define $h(r)$ such that $H(r) = \int h(r)r\,dr$ by

$$h(a) := \liminf_{s\nearrow 1}(1-s)\int_0^1\int_{\mathbb{S}^{d-1}} |\omega|^2 r^{1-2s} g(a|\omega|r^{1-s})dS_\omega\,dr. \tag{5.13}$$

Remark 5.20. *In particular, in the case of the p-Laplacian where $G(r) = r^p$ for $r \in \mathbb{R}^+$,*

$$H(r) = \frac{r^p}{p}K_{d,p}$$

for some constant $K_{d,p}$. See Example 2.17 of [113] for more details. Compare this also with the Bourgain-Brezis-Mironescu result in [47], with a similar constant.

On the other hand, if $G = G(x, y, r)$, an explicit form of the limit Young function similar to (5.12) *is not yet known, although such a limit exists, by the uniform boundedness of $g(x, y, r)$.*

Our main convergence result is then as follows

Theorem 5.21. *Suppose $\psi, \varphi \in H^1(\mathbb{R}^d)$ is such that*

$$K^1_\psi := \{v \in H^1_0(\Omega) : v \ge \psi \text{ a.e. in } \Omega\} \neq \emptyset,$$

and define $K^1_{\psi,\varphi}$ and K^1_N similarly. Let $u_s \in K^s_\psi(K^s_{\psi,\varphi}, K^s_N)$ for $0 < s < 1$ be the solution to the nonlinear nonlocal one obstacle problem

$$\langle \bar{\mathcal{L}}^s_g u, v-u\rangle \ge \int_\Omega \frac{f}{1-s}(v-u) \quad \forall v \in K^s_\psi \tag{5.5}$$

(and the corresponding two obstacles (5.9) *and N membranes* (5.10) *problems). Then, the sequence $(u_s)_s$ converges strongly to u in $L^2(\Omega)$ as $s \nearrow 1$, where $u \in K^1_\psi(K^1_{\psi,\varphi}, K^1_N)$ solves uniquely the obstacle-type problems for $s = 1$, i.e.*

$$\langle \bar{\mathcal{L}}_g u, v-u\rangle \ge \int_\Omega f(v-u) \quad \forall v \in K^1_\psi(K^1_{\psi,\varphi}, K^1_N), \tag{5.14}$$

where $\bar{\mathcal{L}}_g$ is the classical Ladyzhenskaya-Ural'tseva operator

$$\bar{\mathcal{L}}_g = -\frac{1}{2}\nabla\cdot\big(h(|\nabla u|)\nabla u\big) \tag{5.15}$$

for $h(r)$ defined by (5.13).

We will show this via a series of known lemmas, given in Sections 4 and 5 of [113], and is an extension of the result of Bourgain-Brezis-Mironescu in [47]. For the proofs, refer to [113].

Lemma 5.22. *Let $u \in H^1(\mathbb{R}^d)$. Then, for $0 < s < 1$,*

$$\Phi_{s,G}(u) \le \frac{\omega_{d-1}}{1-s}\Phi_G(|\nabla u|) + 2^{a^*+2}(1+a^*)\frac{\omega_{d-1}}{s}\Phi_G(u).$$

Furthermore, a similar result holds for $0 < s < s' < 1$. See also [8] for further embedding results.

Lemma 5.23. *Let $0 < s < s' < 1$ and $u \in L^2(\Omega)$. Then*

$$(1-s)\Phi_{s,G}(u) \le 2^{1-s}(1-s')\Phi_{s',G}(u) + 2^{a^*+2}(1+a^*)\omega_{d-1}\frac{1-s}{s}\Phi_G(u).$$

Lemma 5.24. *Suppose H is as defined above in* (5.12). *Let $u \in C_c^2(\Omega)$. Then, for every fixed $x \in \mathbb{R}^d$,*

$$\liminf_{s \nearrow 1}(1-s)\int_{\mathbb{R}^d} G\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dy}{|x-y|^d} = H(|\nabla u(x)|).$$

As a result of these lemmas, we have the following two results from [113].

Theorem 5.25. *Given an Orlicz function G, let H be as defined above in* (5.12). *Then, for $u \in L^2(\Omega)$ and $0 < s < 1$,*

$$\liminf_{s \nearrow 1}(1-s)\Phi_{s,G}(u) = \Phi_H(|\nabla u|),$$

where we recall that

$$\Phi_H(u) := \int_{\mathbb{R}^d} H(|u(x)|)\, dx.$$

As a result, by Theorem 5.17 and invoking Lemma 5.23, we have the following result for sequences of functions, given as Theorem 5.1 of [113].

Theorem 5.26. *Given an Orlicz function G, let $0 \le s_k \nearrow 1$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^2(\Omega)$ be such that*

$$\sup_{k\in\mathbb{N}}(1-s_k)\Phi_{s_k,G}(u_k) < \infty \quad \text{and} \quad \sup_{k\in\mathbb{N}}\Phi_G(u_k) < \infty.$$

Then, there exists $u \in L^2(\Omega)$ and a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ such that $u_{k_j} \to u$ in $L^2_{loc}(\mathbb{R}^d)$. Moreover, defining H as in (5.12), *we have that $u \in H_0^1(\Omega)$ and*

$$\Phi_H(|\nabla u|) \le \liminf_{k\to\infty}(1-s_k)\Phi_{s_k,G}(u_k).$$

Consequently, we can obtain the Γ-convergence of the functionals $\mathcal{F}_s$ as defined in (5.7). We first recall the definition of Γ-convergence.

Definition 5.27. *Given a metric space X with $F, F_n : X \to \bar{\mathbb{R}}$, F_j* Γ-converges *to F if for every $u \in X$:*

- *For every sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $u_j \to u$ in X,*

$$F(u) \le \liminf_{n\to\infty} F_n(u_n).$$

- *For every $u \in X$, there exists a sequence $\{u_m\}_{m\in\mathbb{N}} \subset X$ converging to u such that*

$$F(u) \ge \limsup_{m\to\infty} F_m(u_m).$$

Then, by the previous theorem, taking the metric space X to be $L^2(\Omega)$ with the functionals $\mathcal{F}_{s,f}$ defined for all $u \in L^2(\Omega)$ such that $\mathcal{F}_{s,f}(u) = +\infty$ for $u \notin K_\psi^s$ for $0 < s \le 1$, we have

Theorem 5.28. *Given the functional $\mathcal{F}_{s,f}(u) : L^2(\Omega) \to \bar{\mathbb{R}}$ as defined above by*

$$\mathcal{F}_{s,f}(v) := \frac{1}{2}\Phi_{s,G}(v) - \int_\Omega fv \quad \forall v \in K_\psi^s \quad \text{and} \quad \mathcal{F}_{s,f}(v) = +\infty \text{ otherwise}, \tag{5.7}$$

we have that

$$(1-s)\mathcal{F}_{s,f/(1-s)} \quad \Gamma\text{-converges as } s \nearrow 1 \text{ to} \quad \mathcal{F}_{1,f}$$

for $\mathcal{F}_{1,f} : L^2(\Omega) \to \bar{\mathbb{R}}$ defined by

$$\mathcal{F}_{1,f}(v) := \frac{1}{2}\Phi_H(|\nabla v|) - \int_\Omega fv \quad \forall v \in K_\psi^1 \quad \text{and} \quad \mathcal{F}_{1,f}(v) = +\infty \text{ otherwise}.$$

Proof. The lim inf inequality between the modulars $(1-s)\Phi_{s,G}(v)$ and $\Phi_H(|\nabla v|)$ follows from the previous Theorem 5.26, while the limsup inequality for the modulars follows simply by taking the constant sequence.

Next, we write the functionals as

$$\mathcal{F}_{s,f}(v) := \frac{1}{2}\Phi_{s,G}(v) - \int_\Omega fv + I_{K^s_\psi}(v)$$

and

$$\mathcal{F}_{1,f}(v) := \frac{1}{2}\Phi_H(|\nabla v|) - \int_\Omega fv + I_{K^1_\psi}(v)$$

respectively for the obstacle functionals $I_K(v) = 0$ if $v \in K$ and $+\infty$ if $v \in L^2(\Omega)\backslash K$ (see for instance, Section 3.2 of [123]). Since $K^1_\psi \subset K^{s'}_\psi \subset K^s_\psi$ for every $0 < s < s' < 1$ is decreasing as $s \nearrow 1$, we also have the lower semi-continuity of $(1-s)I_{K^s_\psi}(v)$.

Furthermore, since $H^1_0(\Omega) \subset H^s_0(\Omega)$ for all $0 < s < 1$, the convex sets $K^1_\psi \subset K^s_\psi$ for all $0 < s < 1$, so once again taking the constant sequence, we have the lim sup inequality, concluding the Γ-convergence for the functionals $(1-s)I_{K^s_\psi}(v)$.

Finally, making use of the classic results for sums of Γ-convergent sequences (see for instance, Proposition 6.17 and 6.21 of [83]), we have the Γ-convergence of $(1-s)\mathcal{F}_{s,f/(1-s)}$ to $\mathcal{F}_{1,f}$. ☐

Next, we recall the classic theorem that Γ-convergence implies the convergence of infimum of functionals (see, for instance, Proposition 7.18 and Corollary 7.20 of [83]).

Theorem 5.29. *Given a metric space (X, d), let $F, F_j : X \to \bar{\mathbb{R}}$, $j \in \mathbb{N}$ be such that F_j Γ-converges to F. Assume that for each $j \in \mathbb{N}$, there exists $u_j \in X$ such that $F_j(u_j) = \inf_X F_j$, and suppose that the sequence $\{u_j\}_{j\in\mathbb{N}} \subset X$ is precompact. Then every accumulation point of $\{u_j\}_{j\in\mathbb{N}}$ is a minimum of F and*

$$\inf_X F = \lim_{j\to\infty} \inf_X F_j.$$

Finally, we give the proof of the main Theorem 5.21.

Proof of Theorem 5.21. The proof follows similarly as in Theorem 6.9 of [113], applied to the obstacle problem. Indeed, since the existence of an infimum $u_{s_j} \in H^{s_j}_0(\Omega) \subset L^2(\Omega)$ of the functional $\mathcal{F}_{s_j,f/(1-s_j)}$ is guaranteed from Theorem 5.9, by Theorem 5.26, for $0 < s_j \nearrow 1$, the sequence of infimums $\{u_{s_j}\}_{j\in\mathbb{N}} \subset L^2(\Omega)$ of $\mathcal{F}_{s_j,f/(1-s_j)}$ is precompact. Therefore, applying the Γ-convergence of $(1-s)\mathcal{F}_{s,f/(1-s)}$ to $\mathcal{F}_{1,f}$ as shown in Theorem 5.28, by the previous theorem with $X = L^2(\Omega)$, there exists $u \in L^2(\Omega)$ such that

$$u_{s_j} \to u \quad \text{in } L^2(\Omega) \quad \text{as } s \nearrow 1$$

such that

$$\mathcal{F}_{1,f}(u) = \inf_{v\in L^2(\Omega)} \mathcal{F}_{1,f}(v).$$

But the infimum $u_{s_j} \in K^{s_j}_\psi$ is unique for each $0 < s_j \leq 1$ by Theorem 5.9, since it is the solution to the nonlocal one obstacle problem for each s_j. Therefore, in particular, $u \in L^2(\Omega)$ solves the classical variational one obstacle problem 5.14 with the potential operator corresponding to $\mathcal{F}_{1,f}$ as given by 5.15 (see, for instance, [158]).

Finally, since the convex sets K^s_ψ are sequentially closed in $H^s_0(\Omega)$, and for smooth bounded Lipschitz domains, the imbedding of $H^s_0(\Omega)$ into $L^2(\Omega)$ is compact, by the convergence of u_{s_j} to u in $L^2(\Omega)$,

$$u \in \{v \in L^2(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}.$$

Furthermore, $u \in H^1_0(\Omega)$, because $\mathcal{F}_{1,f}$ is finite only for u in that space and the classical one obstacle problem has a unique solution.

The two obstacles and N membranes problem follow similarly by defining the functional to be infinite for $v \notin K^s_{\psi,\varphi}$ and $\boldsymbol{v} \notin K^s_N$ respectively, and the Mosco convergence of the decreasing sets $K^s_{\psi,\varphi}$ to $K^1_{\psi,\varphi}$ and K^s_N to K^1_N since $H^1_0(\Omega) \subset H^{s'}_0(\Omega) \subset H^s_0(\Omega)$ for any $0 < s < s' < 1$, as well as the relation

$$u_{s_j,n-1} \geq u_{s_j,n} \geq \inf_{k\geq l} u_{s_k,n} \forall j \geq l \quad \Longrightarrow \quad \inf_{j\geq l} u_{s_j,n-1} \geq \inf_{j\geq l} u_{s_j,n} \quad \Longrightarrow \quad \liminf_j u_{s_j,n-1} \geq \liminf_j u_{s_j,n},$$

giving the Γ-convergence of the associated functionals of the obstacle-type problems. ☐

Remark 5.30. *Compare this result to the classical result of Attouch as given on pages 126–127 of [82], where the Γ-convergent functionals are given in $H_0^1(\Omega)$ with Mosco-convergent convex sets K_n.*

Remark 5.31. *In the case where $\bar{\mathcal{L}}_g^s$ corresponds to the symmetric linear nonlocal operator $\mathcal{L}_a^s$ for $u, v \in H_0^s(\Omega)$ in Lipschitz domains $\Omega \subset \mathbb{R}^d$, given by*

$$\langle \mathcal{L}_a^s u, v \rangle = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{a}(x,y) \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}}\, dx\, dy$$

for a strictly elliptic and bounded kernel $a_* \leq \hat{a}(x,y) \leq a^*$, it is possible to consider the corresponding energy functional

$$\mathcal{J}_s(u) := \frac{1}{2} \iint_{\mathbb{R}^{2d}} \hat{a}(x,y) \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\, dx\, dy$$

for $0 < s < 1$, as in [112]. Then, by the boundedness and ellipticity of $\hat{a}(x,y)$, $\mathcal{J}_s$ is equivalent, up to a constant, to the Gagliardo semi-norm $[u]_{s,\mathbb{R}^d}$. Therefore, we can show similarly that $(1-s)K_d\mathcal{J}_s$ Γ-converges to some limit $\mathcal{J}_1$ for some positive constant K_d. Extending Lemma 2.7 of [111], we have, $r > 0$,

$$\mathcal{J}_s(u + rv_s) = \mathcal{J}_s(u) + r\langle \bar{\mathcal{L}}_g^s u, v_s \rangle + o(r),$$

where $o(r)$ depends only on the Gagliardo semi-norm of v_s.

Then, following Lemma 2.8 of [111], for every $u \in H_0^1(\Omega)$ and $v_k \in H_0^{s_k}(\Omega)$ such that $v_k \to v$ strongly in $L^2(\Omega)$, we have

$$\langle \mathcal{L}_a^s u, v_k \rangle \to \langle \mathcal{L}_a^1 u, v \rangle$$

for some $\mathcal{L}_a^1$ corresponding to $\mathcal{J}_1$ up to some constants. This gives the convergence of the elliptic obstacle problems, using Minty's lemma.

However, unlike the fractional Laplacian, for general $a(x,y)$, we are not able to compute $\mathcal{J}_1$ explicitly, so the form of the corresponding limit operator $\mathcal{L}_a^1$ is not yet known. On the other hand, in the case when $a(x,y) = 1$, the limit operator can be made explicit, and we have the convergence of the fractional Laplacian to the classical Laplacian, as shown in Section 2.2 of [111]. (See also Sections 3 and 4.1 of [116].) Compare also with Theorem 2.9.

5.4 Parabolic Nonlinear Nonlocal Obstacle-Type Problems

Next, we consider the parabolic nonlinear nonlocal obstacle-type problems. We begin with the one obstacle problem, given with initial condition $u(0,x) = u_0 \geq \psi(0,x)$ in Ω for $u_0 \in H_0^s(\Omega)$ by

$$u \in \mathbb{K}_\psi^s : \quad \int_{Q_T} \left(\frac{\partial u}{\partial t} + \bar{\mathcal{L}}_g^s u \right)(v-u) \geq \int_{Q_T} f(v-u) \quad \forall v \in \mathbb{K}_\psi^s, \tag{5.16}$$

where $Q_T =]0,T[\times\Omega$, for the non-empty convex set

$$\mathbb{K}_\psi^s = \{u \in L^2(0,T; H_0^s(\Omega)) : u \geq \psi \text{ a.e. in } Q_T\} \neq \emptyset.$$

The obstacle $\psi \in L^2(0,T;H^s(\mathbb{R}^d)) \cap H^1(0,T;L^2(\Omega)) \subset C([0,T];L^2(\Omega))$ is such that $\psi(0,x) = \psi_0$ in Ω and admissible in the sense that $\psi \leq 0$ a.e. in Ω^c for a.e. t.

Parabolic fractional obstacle problems were first considered by Caffarelli and Figalli in [62], and later built upon by Barrios, Figalli and Ros-Oton in [28], both for the fractional Laplacian. However, to the best of our knowledge, nothing is yet known about parabolic obstacle problems for nonlocal operators defined with more general kernels. As observed in [28], approaches commonly used in the stationary elliptic case usually fail to apply to the evolutionary parabolic problem.

We show the main existence result. Observe that we are unable to apply the Stampacchia theorem directly in the parabolic case, unlike in the stationary case. However, we can still obtain the solution as a limiting solution to the bounded penalised problem.

Theorem 5.32. *For $0 < s < 1$, assume that*

$$f, \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}_g^s \psi - f\right)^+ \in L^2(Q_T). \tag{5.17}$$

Then, there exists a unique strong solution $u \in \mathbb{K}_\psi^s$ with the maximal regularity

$$u \in L^2(0,T;H_0^s(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$$

to the parabolic one obstacle problem (5.16).

Remark 5.33. *In the case of $s = 1$, the nonlinear operator $\bar{\mathcal{L}}_g^1$ is once again given by the classical Ladyzhenskaya-Ural'tseva operator*

$$\bar{\mathcal{L}}_g^1 = -\frac{1}{2}\nabla \cdot \big(g(x, |\nabla u|)\nabla u\big)$$

for some nonlinear coefficient $g(x, r)$. See also [159] for the $s = 1$ result.

Proof. Consider a nondecreasing Lipschitz function $\theta : \mathbb{R} \to [0,1]$ such that

$$\theta \in C^{0,1}(\mathbb{R}), \quad \theta' \geq 0, \quad \theta(+\infty) = 1 \quad \text{ and } \theta(t) = 0 \text{ for } t \leq 0;$$

$$\exists C_\theta > 0 : [1 - \theta(t)]t \leq C_\theta, \quad t > 0.$$

Then, for any $\varepsilon > 0$, consider the family of functions

$$\theta_\varepsilon(t) = \theta\left(\frac{t}{\varepsilon}\right), \quad t \in \mathbb{R}.$$

Observe that $\theta_\varepsilon(t)$ converges as $\varepsilon \to 0$ to the multi-valued Heaviside function

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ [0,1] & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}.$$

For $\zeta \in L^2(Q_T)$ such that

$$\zeta \geq \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}_g^s \psi - f\right)^+ \qquad \text{a.e. in } Q_T,$$

consider the bounded penalised problem based on θ given by

$$\frac{\partial u_\varepsilon}{\partial t} + \bar{\mathcal{L}}_g^s u_\varepsilon + \zeta \theta_\varepsilon(u_\varepsilon - \psi) = f + \zeta \text{ a.e. } Q_T, \quad u_\varepsilon = 0 \text{ in }]0,T[\times \Omega^c, \quad u_\varepsilon(0,\cdot) = u_0 \in H_0^s(\Omega). \tag{5.18}$$

(Observe that it is possible to write $\bar{\mathcal{L}}_g^s$ in this form, by Remark 5.4.) Since $\bar{\mathcal{L}}_g^s$ is strictly coercive and Lipschitz in $H_0^s(\Omega)$ and θ_ε is monotone, it is well-known (see, for instance, [160]) that this problem has a unique solution

$$u_\epsilon \in L^2(0,T;H_0^s(\Omega)) \cap H^1(0,T;L^2(\Omega)).$$

Furthermore, by the regularity of $f, \zeta \in L^2(Q_T)$, we have that $\frac{\partial u_\varepsilon}{\partial t}, \bar{\mathcal{L}}_g^s u_\varepsilon \in L^2(Q_T)$.

Furthermore, multiplying (5.18) by $\frac{\partial u_\varepsilon}{\partial t}$, which makes sense since $\frac{\partial u_\varepsilon}{\partial t} \in L^2(Q_T)$, and integrating in $Q_t =]0,t[\times\Omega$ for any $t \in]0,T[$ and taking the supremum over all t, we have

$$\int_0^T \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{g_*}{2c_{d,s}^2} \sup_{t\in]0,T[} \left\|u_\varepsilon(t)\right\|^2_{H_0^s(\Omega)} - \frac{g^*}{2c_{d,s}^2}\left\|u_\varepsilon(0)\right\|^2_{H_0^s(\Omega)}$$
$$\leq \sup_{t\in]0,T[} \int_{Q_t} \left(\frac{\partial u_\varepsilon}{\partial t} + \bar{\mathcal{L}}_g^s u_\varepsilon\right)\frac{\partial u_\varepsilon}{\partial t}$$

$$\le \sup_{t\in]0,T[} \int_{Q_T} (\zeta + f)\frac{\partial u_\varepsilon}{\partial t} + \sup_{t\in]0,T[} \int_{Q_T} \zeta\theta_\varepsilon(u_\varepsilon - \psi)\frac{\partial u_\varepsilon}{\partial t}$$

$$\le \frac{1}{4}\sup_{t\in]0,T[} \int_0^t \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2_{L^2(\Omega)} + \sup_{t\in]0,T[} \int_{Q_t} |\zeta + f|^2 + \frac{1}{4}\sup_{t\in]0,T[} \int_0^t \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2_{L^2(\Omega)} + \sup_{t\in]0,T[} \int_{Q_t} |\zeta|^2|\theta_\varepsilon(u_\varepsilon - \psi)|^2$$

$$\le \frac{1}{4}\int_0^T \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2_{L^2(\Omega)} + \int_{Q_T} |\zeta + f|^2 + \frac{1}{4}\int_0^T \left\|\frac{\partial u_\varepsilon}{\partial t}\right\|^2_{L^2(\Omega)} + \int_{Q_T} |\zeta|^2,$$

so u_ε is uniformly bounded in $H^1(0,T;L^2(\Omega))$ independent of ε.

Then, for all $v \in L^2(0,T;H^s_0(\Omega))$ such that $v \ge 0$,

$$\int_\Omega \left(\frac{\partial\psi}{\partial t} + \bar{\mathcal{L}}^s_g\psi\right)v = \int_\Omega \left(\frac{\partial\psi}{\partial t} + \bar{\mathcal{L}}^s_g\psi - f + f\right)v \le \int_\Omega \left[\left(\frac{\partial\psi}{\partial t} + \bar{\mathcal{L}}^s_g\psi - f\right)^+ + f\right]v \le \int_\Omega (\zeta + f)v \quad \text{a.e. } t.$$

Now, taking $v = (\psi - u_\varepsilon)^+$ and subtracting the penalised problem (5.18) from the above equation, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^t \frac{\partial}{\partial t}\left\|(\psi - u_\varepsilon)^+\right\|^2_{L^2(\Omega)} + \frac{g_*}{c^2_{d,s}}\int_0^t \left\|(\psi - u_\varepsilon)^+\right\|^2_{H^s_0(\Omega)} \\
\le &\int_{Q_t} \left[\frac{\partial(\psi - u_\varepsilon)^+}{\partial t} + \bar{\mathcal{L}}^s_g(\psi - u_\varepsilon)^+\right](\psi - u_\varepsilon)^+ \\
\le &\int_{Q_t} \left(\frac{\partial\psi}{\partial t} + \bar{\mathcal{L}}^s_g\psi\right)(\psi - u_\varepsilon)^+ - \int_{Q_t} \left(\frac{\partial u_\varepsilon}{\partial t} + \bar{\mathcal{L}}^s_g u_\varepsilon\right)(\psi - u_\varepsilon)^+ \\
\le &\int_{Q_t} (\zeta + f)(\psi - u_\varepsilon)^+ + \int_{Q_t} \zeta\theta_\varepsilon(u_\varepsilon - \psi)(\psi - u_\varepsilon)^+ - \int_{Q_t} (f + \zeta)(\psi - u_\varepsilon)^+ \\
= &\int_{Q_t} \zeta\theta_\varepsilon(u_\varepsilon - \psi)(\psi - u_\varepsilon)^+ \\
= &0.
\end{aligned}$$

The last equality is true because either $u_\varepsilon - \psi > 0$ which gives $(\psi - u_\varepsilon)^+ = 0$, or $u_\varepsilon - \psi \le 0$ which gives $\theta_\varepsilon(u_\varepsilon - \psi) = 0$ by the construction of θ, thus implying $\theta_\varepsilon(u_\varepsilon - \psi)(\psi - u_\varepsilon)^+ = 0$.

Taking the supremum over all $t \in]0,T[$, we have

$$\frac{1}{2}\left\|(\psi - u_\varepsilon)^+\right\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{g_*}{c^2_{d,s}}\left\|(\psi - u_\varepsilon)^+\right\|^2_{L^2(0,T;H^s_0(\Omega))} \le \frac{1}{2}\left\|(\psi(0,\cdot) - u_\varepsilon(0,\cdot))^+\right\|^2_{L^2(\Omega)} = 0$$

since $u_\varepsilon(0,\cdot) = u_0 \ge \psi(0,\cdot)$, so $u_\varepsilon \in \mathbb{K}^s_\psi$ for any $\varepsilon > 0$.

Therefore, by the uniform boundedness of $u_\varepsilon \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, passing to a subsequence if necessary, there exists a limit $\hat{u}$ such that

$$u_\varepsilon \rightharpoonup \hat{u} \text{ weakly in } H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s_0(\Omega)) \text{ and strongly in } L^2(Q_T). \tag{5.19}$$

Finally, we show that this limit is the solution to the parabolic one obstacle problem (5.16), i.e. $u_\varepsilon \to \hat{u} = u$. Taking $v = w - u_\varepsilon$ in the penalised problem (5.18) for arbitrary $w \in \mathbb{K}^s_\psi$, we have for any fixed $t \in]0,T[$,

$$\begin{aligned}
\int_{Q_t} \left(\frac{\partial u_\varepsilon}{\partial t} + \bar{\mathcal{L}}^s_g u_\varepsilon\right)(w - u_\varepsilon) &= \int_{Q_t} [f + \zeta - \zeta\theta_\varepsilon(u_\varepsilon - \psi)](w - u_\varepsilon) \\
&= \int_{Q_t} f(w - u_\varepsilon) + \int_{Q_t} \zeta[1 - \theta_\varepsilon(u_\varepsilon - \psi)](w - u_\varepsilon) \\
&\ge \int_{Q_t} f(w - u_\varepsilon) + \int_{Q_t} \zeta[1 - \theta_\varepsilon(u_\varepsilon - \psi)](\psi - u_\varepsilon) \\
&= \int_{Q_t} f(w - u_\varepsilon) - \varepsilon\int_{Q_t} \zeta[1 - \theta_\varepsilon(u_\varepsilon - \psi)]\frac{u_\varepsilon - \psi}{\varepsilon}
\end{aligned}$$

$$\geq \int_{Q_t} f(w-u_\varepsilon) - \varepsilon C_\theta \int_{Q_t} \zeta$$

since $\zeta, 1-\theta_\varepsilon, w-\psi \geq 0$ for $w \in \mathbb{K}^s_\psi$. Taking $w = \hat{u}$, we see that u_ε satisfies the assumptions of Lemma 5.34 below, so taking $\varepsilon \searrow 0$, we conclude that $\hat{u}$ is a solution of the parabolic one obstacle problem (5.16). Finally, the uniqueness follows as in the classical case, by taking the difference of the two corresponding problems and observing that the norms of the difference are equal to 0, therefore $\hat{u} = u$.

□

Lemma 5.34. *Suppose $u_n \rightharpoonup u$ weakly in $H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s_0(\Omega))$ and strongly in $L^2(Q_T)$. If $u_n(0,\cdot) \to u(0,\cdot)$ in $L^2(\Omega)$, and*

$$\limsup_{n\to\infty} \int_{Q_T} \left(\frac{\partial u_n}{\partial t} + \bar{\mathcal{L}}^s_g u_n \right)(u_n - u) \leq 0,$$

then

$u_n \to u$ *strongly in* $L^2(0,T;H^s_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, *and* $\bar{\mathcal{L}}^s_g u_n \to \bar{\mathcal{L}}^s_g u$ *strongly in* $L^2(0,T;H^{-s}(\Omega))$.

Proof. Using the coercivity of $\bar{\mathcal{L}}^s_g$ from Proposition 5.3,

$$\begin{aligned}\frac{1}{2}\|u_n - u\|^2_{L^\infty(0,T;L^2(\Omega))} &+ \frac{g_*}{c^2_{d,s}}\|u_n - u\|^2_{L^2(0,T;H^s_0(\Omega))} \\ &\leq \int_{Q_T} \frac{\partial(u_n-u)}{\partial t}(u_n - u) + \langle \bar{\mathcal{L}}^s_g u_n - \bar{\mathcal{L}}^s_g u, u_n - u\rangle + \frac{1}{2}\left\|u_n(0,\cdot) - u(0,\cdot)\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Then, taking the limsup for each term on the left, which is bounded by limsup of the sum on the right, which, using the assumptions, is given by

$$\limsup_{n\to\infty} \int_{Q_T} \frac{\partial u}{\partial t}(u - u_n) + \limsup_{n\to\infty} \langle \bar{\mathcal{L}}^s_g u, u - u_n\rangle + \frac{1}{2}\limsup_{n\to\infty}\left\|u_n(0,\cdot) - u(0,\cdot)\right\|^2_{L^2(\Omega)} = 0$$

since $u_n \rightharpoonup u$ in $H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s_0(\Omega))$. Therefore, we have the strong convergence of u_n to u in $L^2(0,T;H^s_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

Furthermore, since $\bar{\mathcal{L}}^s_g$ is a Lipschitz operator by Proposition 5.3, it follows that $\bar{\mathcal{L}}^s_g u_n \to \bar{\mathcal{L}}^s_g u$ strongly in $L^2(0,T;H^{-s}(\Omega))$. □

Remark 5.35. *In particular, the error estimate*

$$\frac{1}{2}\|u_\varepsilon - u\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{g_*}{c^2_{d,s}}\|u_\varepsilon - u\|^2_{L^2(0,T;H^s_0(\Omega))} \leq \varepsilon C_\theta \|\zeta\|_{L^1(Q_T)} \tag{5.20}$$

holds. Indeed, taking $w = u$ in the last estimate of the proof of the theorem above, we obtain

$$\int_{Q_t} \left(\frac{\partial u_\varepsilon}{\partial t} + \bar{\mathcal{L}}^s_g u_\varepsilon \right)(u - u_\varepsilon) \geq \int_{Q_t} f(u - u_\varepsilon) - \varepsilon C_\theta \int_{Q_t} \zeta,$$

but taking $v = u_\varepsilon \in \mathbb{K}^s_\psi$ in the original obstacle problem (5.16), we have

$$\int_{Q_t} \left(\frac{\partial u}{\partial t} + \bar{\mathcal{L}}^s_g u \right)(u_\varepsilon - u) \geq \int_{Q_t} f(u_\varepsilon - u).$$

Taking the difference of these two equations, we have

$$\int_{Q_t} \left(\frac{\partial(u_\varepsilon - u)}{\partial t} + \bar{\mathcal{L}}^s_g u_\varepsilon - \bar{\mathcal{L}}^s_g u \right)(u_\varepsilon - u) \leq \varepsilon C_\theta \int_{Q_t} \zeta.$$

Using the ellipticity of $\bar{\mathcal{L}}^s_g$, we take the supremum over all $t \in]0,T[$ and obtain (5.20) since $\zeta \geq 0$.

Similar to the elliptic case in Theorem 5.9, we also have a comparison property for the solution of the parabolic nonlocal one obstacle problem (5.16):

Proposition 5.36. *Suppose u is the solution of the variational inequality* (5.16) *with data f and convex set $\mathbb{K}^s_\psi$, and $\hat{u}$ be the solution with data $\hat{f}$ and convex set $\mathbb{K}^s_{\hat{\psi}}$. If $\psi \geq \hat{\psi}$ and $f \geq \hat{f}$, then $u \geq \hat{u}$ a.e. in Q_T.*

Proof. Taking $v = u \vee \hat{u} = u + (\hat{u} - u)^+ \in \mathbb{K}^s_\psi$ and $v = u \wedge \hat{u} = \hat{u} - (\hat{u} - u)^+ \in \mathbb{K}^s_{\hat{\psi}}$ for the two equations corresponding to (u, f, ψ) and $(\hat{u}, \hat{f}, \hat{\psi})$ respectively and summing, we have

$$\int_{Q_T} \left(\frac{\partial(\hat{u} - u)}{\partial t} + \bar{\mathcal{L}}^s_g \hat{u} - \bar{\mathcal{L}}^s_g u \right) (\hat{u} - u)^+ + \int_{Q_T} (f - \hat{f})(\hat{u} - u)^+ \leq 0.$$

Since $f - \hat{f} \geq 0$, by the strict T-monotonicity of the operator in Theorem 5.3, $u \geq \hat{u}$. □

Finally, we derive the Lewy-Stampacchia inequality, in a similar form to the classical case given in [128], and the case for the fractional Laplacian given in [150].

Theorem 5.37 (Lewy-Stampacchia inequality)**.** *Under the assumptions* (5.17)*, the solution $u \in \mathbb{K}^s_\psi$ of the parabolic one obstacle problem* (5.16) *satisfies*

$$f \leq \frac{\partial u}{\partial t} + \bar{\mathcal{L}}^s_g u \leq f \vee \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}^s_g \psi \right) \qquad a.e.\ in\ Q_T.$$

Proof. Choosing $\zeta = \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}^s_g \psi - f \right)^+$ in the penalised problem (5.18), and making use of the property of θ that $0 \leq 1 - \theta_\varepsilon \leq 1$, then for any $\varepsilon > 0$ and any $v \in L^2(0, T; H^s_0(\Omega))$, $v \geq 0$, we have

$$\int_{Q_T} \left(\frac{\partial u_\varepsilon}{\partial t} + \bar{\mathcal{L}}^s_g u_\varepsilon \right) v = \int_{Q_T} \left[f + \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}^s_g \psi - f \right)^+ (1 - \theta_\varepsilon) \right] v \leq \int_{Q_T} \left[f + \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}^s_g \psi - f \right)^+ \right] v$$

and also

$$\frac{\partial u_\varepsilon}{\partial t} + \bar{\mathcal{L}}^s_g u_\varepsilon = f + \zeta - \zeta \theta_\varepsilon = f + \zeta(1 - \theta_\varepsilon) \geq f$$

from the boundary value problem. Together, these give

$$\int_{Q_T} f v \leq \int_{Q_T} \left(\frac{\partial u_\varepsilon}{\partial t} + \bar{\mathcal{L}}^s_g u_\varepsilon \right) v \leq \int_{Q_T} \left[f + \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}^s_g \psi - f \right)^+ \right] v = \int_{Q_T} \left[f \vee \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}^s_g \psi - f \right) \right] v.$$

By the error estimate (5.20), $\bar{\mathcal{L}}^s_g u_\varepsilon$ converges strongly to $\bar{\mathcal{L}}^s_g u$ in $L^2(0, T; H^{-s}(\Omega))$, and so in particular, converges weakly in $L^2(Q_T)$ because $\frac{\partial u_\varepsilon}{\partial t}$ converges weakly to $\frac{\partial u}{\partial t}$ in $L^2(Q_T)$. Therefore, we can take $\varepsilon \to 0$ so that this inequality holds also for u. Since v is arbitrary, we have the result. □

Remark 5.38. *For $s = 1$, the parabolic Lewy-Stampacchia inequalities have also been obtained for pseudomonotone operators in [226] and [128].*

Remark 5.39. *For $0 < s < 1$, in the linear case, the parabolic obstacle problem defined with the stochastic fractional Laplacian has also been considered in [98], as well as in [146] for semi-Dirichlet forms which include the linear nonlocal operator $\mathcal{L}_a$.*

Remark 5.40. *In the case when $\bar{\mathcal{L}}^s_g = \mathcal{L}_a$ is linear (c.f. Remark 5.31), it is also possible to extend our results for the obstacle-type problems for weaker data $f \in L^2(0, T; H^{-s}(\Omega))$, as in Section 4.4, with more general lower semi-continuous obstacles by considering the solution in the positive cone, making use of the parabolic s-capacity as considered by Pierre in [185, 186, 187]. See also [245] and [148] for the stochastic case. However, when less regular obstacles are considered, the solution is not unique, as discussed in [172].*

Similar results applying the results of Pierre in the consideration of the multiple obstacles problem have been obtained in [146] for more general nonlocal operators corresponding to semi-Dirichlet forms, by considering various equivalent notions of nonlocal parabolic capacity in [147].

Moreover, as in the stationary case, we can extend it to multiple obstacles problems:

Theorem 5.41 (Parabolic Two Obstacles Problem)**.** *Suppose*

$$f, \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}_g^s \psi - f\right)^+, \left(\frac{\partial \varphi}{\partial t} + \bar{\mathcal{L}}_g^s \varphi - f\right)^+ \in L^2(Q_T)$$

and $\psi, \varphi \in L^2(0,T; H^s(\mathbb{R}^d)) \cap H^1(0,T; L^2(\Omega))$ *such that* $\psi \leq 0 \leq \varphi$ *a.e. in* Ω^c *for a.e.* t*. Then, there exists a unique strong solution* $u \in \mathbb{K}^s_{\psi,\varphi}$ *with initial condition* $\psi(0,x) \leq u(0,x) = u_0 \leq \varphi(0,x)$ *in* Ω*,* $u_0 \in H^s_0(\Omega)$ *for the non-empty convex set*

$$\mathbb{K}^s_{\psi,\varphi} = \{u \in L^2(0,T; H^s_0(\Omega)) : \psi \leq u \leq \varphi \text{ a.e. in } Q_T\} \neq \emptyset,$$

such that

$$u \in L^2(0,T; H^s_0(\Omega)) \cap H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$$

to the parabolic two obstacles problem

$$u \in \mathbb{K}^s_{\psi,\varphi} : \quad \int_{Q_T} \left(\frac{\partial u}{\partial t} + \bar{\mathcal{L}}_g^s u\right)(v - u) \geq \int_{Q_T} f(v-u) \quad \forall v \in \mathbb{K}^s_{\psi,\varphi}. \tag{5.21}$$

Furthermore, the solution u *satisfies*

$$f \wedge \left(\frac{\partial \varphi}{\partial t} + \bar{\mathcal{L}}_g^s \varphi\right) \leq \frac{\partial u}{\partial t} + \bar{\mathcal{L}}_g^s u \leq f \vee \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}_g^s \psi\right) \quad \text{a.e. in } Q_T.$$

Proof. The proof is similar to that of the one obstacle problem, now for two obstacles. In this case, we consider the bounded penalised problem given by

$$\frac{\partial u_\varepsilon}{\partial t} + \bar{\mathcal{L}}_g^s u_\varepsilon + \zeta_\psi \theta_\varepsilon(u_\varepsilon - \psi) - \zeta_\varphi \theta_\varepsilon(\varphi - u_\varepsilon) = f + \zeta_\psi - \zeta_\varphi \text{ a.e. } Q_T, \quad u_\varepsilon = 0 \text{ in }]0,T[\times\Omega^c$$

with initial condition $u_\varepsilon(0,\cdot) = u_0$ in Ω for $\zeta_\psi, \zeta_\varphi \in L^2(Q_T)$ such that

$$\zeta_\psi \geq \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}_g^s \psi - f\right)^+, \quad \zeta_\varphi \geq \left(\frac{\partial \varphi}{\partial t} + \bar{\mathcal{L}}_g^s \varphi - f\right)^- \quad \text{a.e. in } Q_T,$$

with $\theta_\varepsilon(t) = 1$ for $t \geq \varepsilon$. This penalised problem is known to have a unique solution

$$u_\epsilon \in L^2(0,T; H^s_0(\Omega)) \cap H^1(0,T; L^2(\Omega)),$$

uniformly bounded in this space.

Furthermore, as in the stationary case, it is easy to show that $\psi \leq u_\varepsilon \leq \varphi + \varepsilon$, and by the uniform boundedness of u_ε, it converges to a limit u, with a similar estimate to (5.20), which satisfies the parabolic two obstacles problem such that $\psi \leq u \leq \varphi$.

Finally, the Lewy-Stampacchia inequalities once again follow from choosing

$$\zeta_\psi = \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}_g^s \psi - f\right)^+ \quad \text{and} \quad \zeta_\varphi = \left(\frac{\partial \varphi}{\partial t} + \bar{\mathcal{L}}_g^s \varphi - f\right)^-$$

in the bounded penalised problem and letting $\varepsilon \to 0$. □

Then, by a simple iteration as in Subsection 4.3.2, we have the same result for the parabolic N membranes problem.

Theorem 5.42 (Parabolic N Membranes Problem)**.** *Suppose* $f^i, \dots, f^N \in L^2(Q_T)$*. Then, there exists a unique strong solution* $\boldsymbol{u} = (u_1, u_2, \dots, u_N) \in \mathbb{K}^s_N$ *with initial condition* $u_1(0,x) \geq \dots \geq u_N(0,x)$ *in* Ω *for the non-empty convex set*

$$\mathbb{K}^s_N = \{(u_1, \dots, u_N) \in [L^2(0,T; H^s_0(\Omega))]^N : u_1 \geq \dots \geq u_N \text{ a.e. in } Q_T\} \neq \emptyset$$

such that

$$u_i \in L^2(0,T;H^s_0(\Omega)) \cap H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$$

to the parabolic N membranes problem

$$\boldsymbol{u} \in \mathbb{K}^s_N : \quad \sum_{i=1}^N \int_{Q_T} \left(\frac{\partial u_i}{\partial t} + \bar{\mathcal{L}}^s_g u_i \right)(v_i - u_i) \geq \sum_{i=1}^N \int_{Q_T} f^i (v_i - u_i) \quad \forall (v_1, \dots, v_N) \in \mathbb{K}^s_N. \tag{5.22}$$

Furthermore, the solution $\boldsymbol{u} = (u_1, \dots u_N)$ satisfies a.e. in Q_T,

$$\begin{aligned}
f^1 \wedge \frac{\partial u_1}{\partial t} + \bar{\mathcal{L}}^s_g u_1 &\leq f^1 \vee \dots \vee f^N \\
f^1 \wedge f^2 \leq \frac{\partial u_2}{\partial t} + \bar{\mathcal{L}}^s_g u_2 &\leq f^2 \vee \dots \vee f^N \\
&\vdots \\
f^1 \wedge \dots \wedge f^{N-1} \leq \frac{\partial u_{N-1}}{\partial t} + \bar{\mathcal{L}}^s_g u_{N-1} &\leq f^{N-1} \vee f^N \\
f^1 \wedge \dots \wedge f^N \leq \frac{\partial u_N}{\partial t} + \bar{\mathcal{L}}^s_g u_N &\leq f^N.
\end{aligned}$$

Furthermore, when $\bar{\mathcal{L}}^s_g$ is given by the linear operator $\mathcal{L}_a$ for a symmetric $a(x,y)$, we can once again make use of the Lewy-Stampacchia inequalities to show local regularity for the three nonlocal obstacle-type problems, extending the result in Section 4.3.3 to parabolic nonlinear problems.

Suppose that

(a) $f, \frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}^s_g \psi \in L^\infty(Q_T)$ for the one obstacle problem,

(b) $f \wedge \left(\frac{\partial \varphi}{\partial t} + \bar{\mathcal{L}}^s_g \varphi \right)$ and $f \vee \left(\frac{\partial \psi}{\partial t} + \bar{\mathcal{L}}^s_g \psi \right)$ are in $L^\infty(Q_T)$ for the two obstacles problem, or

(c) $f^i \in L^\infty(Q_T)$ for $i = 1, \dots, N$ for the N membranes problem.

Then, making use of the weak Harnack inequality for $\mathcal{L}_a$ as given in Theorem 1.2 of [107] (see also [142] for more general linear non-symmetric operators), we have

Theorem 5.43. *Suppose $\bar{\mathcal{L}}^s_g = \mathcal{L}_a$ for the symmetric kernel $a(x,y)$ satisfying* (2.2) *for some $a_*, a^* > 0$. Let u denote the solutions of the parabolic one obstacle problem* (5.16)*, or the parabolic two obstacles problem* (5.21)*, or $u = u_i$ for $i = 1, \dots, N$ of the parabolic N membranes problem* (5.22)*, respectively, under the assumptions (a), (b) or (c) above. Suppose the ball of radius 2 about the origin B_2 is a subset of Ω, and $]t_0 - 1, t_0 + 1[\in]0, T[$ for some t_0. Then there is a constant $c = c(d, s, a_*, a^*)$ such that*

$$\|u\|_{L^1(]t_0-1,t_0-1+(1/2)^{2s}[\times B_{1/2})} \leq c \left(\inf_{]t_0+1-(1/2)^{2s},t_0+1[\times B_{1/2}} u + \|\mathcal{L}_a u\|_{L^\infty(]t_0-1,t_0+1[\times B_2)} \right).$$

Furthermore, as in the classical de Giorgi-Nash-Moser theory, the weak Harnack inequality implies a decay of oscillation-result and local Hölder regularity estimates for weak solutions, extending Theorem 1.2 of [107] to the nonhomogeneous case.

Theorem 5.44. *Let u denote the solutions of the parabolic one obstacle problem* (5.16)*, or the parabolic two obstacles problem* (5.21)*, or $u = u_i$ for $i = 1, \dots, N$ of the N membranes problem* (5.22)*, respectively, under the assumptions (a), (b) or (c) above, with $\bar{\mathcal{L}}^s_g = \mathcal{L}_a$ for the symmetric kernel $a(x,y)$ satisfying* (2.2)*. Then, there exists a constant $\beta = \beta(d, s, a_*, a^*)$ such that for every $Q' \Subset Q_T$, the following Hölder estimate holds:*

$$\sup_{(t,x),(\tau,y) \in Q'} \frac{|u(t,x) - u(\tau,y)|}{(|x-y| + |t-\tau|^{1/2s})^\beta} \leq \frac{1}{\eta^\beta} \left(\|u\|_{L^\infty(]0,T[\times \mathbb{R}^d)} + \|\mathcal{L}_a u\|_{L^\infty(Q_T)} \right)$$

for some $\eta = \eta(Q', Q_T) > 0$.

Consequently, u is locally C^β in space and also $C^{\beta/2s}$ in time.

Proof. This once again follows easily from the Lewy-Stampacchia inequalities for the parabolic obstacle-type problems in Theorems 5.37, 5.41 and 5.42, which hold a.e. in $Q' \Subset Q_T$, and we have the result making use of the previous weak Harnack inequality Theorem. $\square$

Remark 5.45. *The s-convergence of the parabolic obstacle-type problems remains an open problem.*

5.5 Asymptotic Behaviour as $t \to \infty$

Finally, we draw a relation between the evolution nonlinear nonlocal obstacle-type problems and the stationary ones, by analysing the behaviour of the solutions as $t \to \infty$, in the case of when the obstacles are independent of time. We first begin with the one obstacle problem.

Theorem 5.46. *Let $f_\infty \in L^2(\Omega)$ and $f \in L^\infty(0,T;L^2(\Omega))$ such that $\int_t^{t+1} \|f(t) - f_\infty\|^2_{L^2(\Omega)}\, dt \to 0$ as $t \to \infty$. Assume ψ is time-independent such that $\bar{\mathcal{L}}^s_g \psi \in L^2(\Omega)$, $\psi \in H^s(\mathbb{R}^d)$ with $\psi \leq 0$ a.e. in Ω^c. Suppose $u \in \mathbb{K}^s_\psi$ is the solution of the nonlinear nonlocal parabolic one obstacle problem (5.16), i.e.*

$$u(t) \in K^s_\psi \text{ a.e. } t: \quad \int_\Omega \left(\frac{\partial u}{\partial t}(t) + \bar{\mathcal{L}}^s_g u(t)\right)(v(t) - u(t)) \geq \int_{Q_T} f(t)(v(t) - u(t)) \quad \forall v(t) \in K^s_\psi \quad \text{a.e. } t \in]0,T[\tag{5.23}$$

with initial condition $u(0,x) = u_0 \in H^s_0(\Omega)$ satisfying $u_0 \geq \psi(x)$. Suppose also that u_∞ solves the corresponding stationary one obstacle problem (5.5), i.e.

$$u_\infty \in K^s_\psi: \quad \langle \bar{\mathcal{L}}^s_g u_\infty, v - u_\infty\rangle \geq \int_\Omega f_\infty (v - u_\infty) \quad \forall v \in K^s_\psi. \tag{5.5}$$

Then,

$$\|u(t) - u_\infty\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Proof. Take $v = u_\infty \geq \psi \in K^s_\psi$ in (5.23) and $v = u(t) \geq \psi \in K^s_\psi$ in (5.5) for a.e. t and adding, we have

$$\int_\Omega \left(\frac{\partial(u(t) - u_\infty)}{\partial t} + \bar{\mathcal{L}}^s_g u(t) - \bar{\mathcal{L}}^s_g u_\infty\right)(u(t) - u_\infty) \leq \int_\Omega (f(t) - f_\infty)(u(t) - u_\infty) \quad \text{a.e. } t.$$

Then, by the monotonicity and Lipschitzness of g and the Cauchy-Schwarz inequality, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\|u(t) - u_\infty\|^2_{L^2(\Omega)} + \frac{g_*}{c^2_{d,s}}\|u(t) - u_\infty\|^2_{H^s_0(\Omega)} \leq \frac{c^2_{d,s}C^2_P}{2g_*}\|f(t) - f_\infty\|^2_{L^2(\Omega)} + \frac{g_*}{2c^2_{d,s}C^2_P}\|u(t) - u_\infty\|^2_{L^2(\Omega)}.$$

In particular, by the inclusion $L^2(\Omega)$ into $H^s_0(\Omega)$ in Lemma 1.3,

$$\frac{\partial}{\partial t}\|u(t) - u_\infty\|^2_{L^2(\Omega)} + \frac{g_*}{c^2_{d,s}C^2_P}\|u(t) - u_\infty\|^2_{L^2(\Omega)} \leq \frac{c^2_{d,s}C^2_P}{g_*}\|f(t) - f_\infty\|^2_{L^2(\Omega)}.$$

Applying Lemma 5.47 below,

$$\|u(t) - u_\infty\|^2_{L^2(\Omega)} \leq e^{-\frac{g_*}{c^2_{d,s}C^2_P}t}\|u(0) - u_\infty\|^2_{L^2(\Omega)} + \frac{c^4_{d,s}C^4_P}{g^2_*}\left[\sup_{\tau\geq\sigma}\int_\tau^{\tau+1}\|f(\tau) - f_\infty\|^2_{L^2(\Omega)}\, d\tau'\right].$$

Taking $t \to \infty$, the result follows by the convergence of $f(t)$ to f_∞ in the hypothesis. $\square$

Lemma 5.47. *[See, for instance, Page 286 of [131]] Suppose $\varphi'(t) + \mu\varphi(t) \leq \eta(t)$ for $\mu, t > 0$, $\varphi, \eta \geq 0$. Then,*

$$\varphi(\sigma + t) \leq e^{-\mu t}\varphi(\sigma) + \frac{1}{\mu}\left[\sup_{\tau\geq\sigma}\int_\tau^{\tau+1}\eta(\tau')d\tau'\right]$$

for $t > 1$, $\sigma \geq 0$.

Proof. Given

$$\varphi'(t) + \mu\varphi(t) \leq \eta(t),$$

multiplying by the integrating factor $e^{\mu t}$, we have

$$\left(e^{\mu t}\varphi(t)\right)' \leq e^{\mu t}\eta(t).$$

Integrating this from σ to $\sigma + t$, we have

$$e^{\mu(t+\sigma)}\varphi(t+\sigma) - e^{\mu\sigma}\varphi(\sigma) \leq \int_{\sigma}^{t+\sigma} e^{\mu\tau'}\eta(\tau')\,d\tau',$$

which can be rewritten as

$$\begin{aligned}
\varphi(t+\sigma) \leq & e^{-\mu t}\varphi(\sigma) + e^{-\mu(t+\sigma)}\int_{\sigma}^{t+\sigma} e^{\mu\tau'}\eta(\tau')\,d\tau' \\
\leq & e^{-\mu t}\varphi(\sigma) + e^{-\mu(t+\sigma)}\int_{\sigma}^{\sigma+t-1}\int_{\tau}^{\tau+1} e^{\mu\tau'}\eta(\tau')\,d\tau'\,d\tau \\
\leq & e^{-\mu t}\varphi(\sigma) + e^{-\mu(t+\sigma)}\int_{\sigma}^{\sigma+t-1} e^{\mu(\tau+1)}\int_{\tau}^{\tau+1}\eta(\tau')\,d\tau'\,d\tau \\
\leq & e^{-\mu t}\varphi(\sigma) + e^{-\mu(t+\sigma)}\left[\sup_{\tau\geq\sigma}\int_{\tau}^{\tau+1}\eta(\tau')d\tau'\right]\int_{\sigma}^{\sigma+t-1} e^{\mu(\tau+1)}\,d\tau \\
= & e^{-\mu t}\varphi(\sigma) + \frac{e^{-\mu(t+\sigma)}}{\mu}\left[\sup_{\tau\geq\sigma}\int_{\tau}^{\tau+1}\eta(\tau')d\tau'\right]\left[e^{\mu(\sigma+t)} - e^{\mu(\sigma+1)}\right] \\
\leq & e^{-\mu t}\varphi(\sigma) + \frac{1}{\mu}\left[\sup_{\tau\geq\sigma}\int_{\tau}^{\tau+1}\eta(\tau')d\tau'\right].
\end{aligned}$$

□

Furthermore, with exactly the same proof modified to the convex sets $K^s_{\psi,\varphi}$, we can show the convergence of the evolution two obstacles problem to the stationary two obstacles problem, when the obstacles ψ and φ are time-independent.

Theorem 5.48. *Let $f_\infty \in L^2(\Omega)$ and $f \in L^\infty(0,T;L^2(\Omega))$ such that $\int_t^{t+1}\left\|f(t) - f_\infty\right\|^2_{L^2(\Omega)}\,dt \to 0$ as $t \to \infty$. Assume ψ, φ are time-independent such that $\bar{\mathcal{L}}^s_g\psi, \bar{\mathcal{L}}^s_g\varphi \in L^2(\Omega)$, $\psi, \varphi \in H^s(\mathbb{R}^d)$ with $\psi \leq 0 \leq \varphi$ a.e. in Ω^c. Suppose $u(t) \in K^s_{\psi,\varphi}$ is the solution of the nonlinear nonlocal parabolic two obstacles problem* (5.21) *for a.e. t satisfying $\psi(t) \leq u(t) \leq \varphi(t)$ in Ω for a.e. t with initial condition $\psi(0,x) \leq u(0,x) = u_0 \leq \varphi(0,x)$ in Ω, $u_0 \in H^s_0(\Omega)$, and $u_\infty \in K^s_{\psi,\varphi}$ solves the corresponding stationary two obstacles problem* (5.9). *Then,*

$$\left\|u(t) - u_\infty\right\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Finally, for the N membranes problem, we have a similar result:

Theorem 5.49. *For $i = 1, \ldots, N$, let $f^i_\infty \in L^2(\Omega)$ and $f^i \in L^\infty(0,T;L^2(\Omega))$ such that $\int_t^{t+1}\left\|f^i(t) - f^i_\infty\right\|^2_{L^2(\Omega)}\,dt \to 0$ as $t \to \infty$. Suppose $\boldsymbol{u}(t) \in K^s_N$ is the solution of the nonlinear nonlocal parabolic N membranes problem* (5.22) *for a.e. t satisfying $u_1(t) \geq \cdots \geq u_N(t)$ in Ω for a.e. t with initial condition $u_1(0,x) \geq \cdots \geq u_N(0,x)$ in Ω, and $\boldsymbol{u}_\infty \in K^s_N$ solves the corresponding stationary N membranes problem* (5.10). *Then,*

$$\left\|\boldsymbol{u}(t) - \boldsymbol{u}_\infty\right\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Proof. Take $\boldsymbol{v} = \boldsymbol{u}_\infty \in K^s_N$ in (5.22) where $\boldsymbol{u}_\infty = (u_{\infty,1}, \ldots, u_{\infty,N})$ satisfies

$$u_{\infty,1} \geq \cdots \geq u_{\infty,N} \text{ a.e. in } \Omega \quad \forall t \in]0,T[.$$

Take also $\boldsymbol{v} = \boldsymbol{u}(t) \in K_N^s$ in (5.10) for a.e. t, where $\boldsymbol{u}(t) = (u_1(t), \dots, u_N(t))$ satisfies for fixed $t \in]0, T[$,

$$u_1(t) \geq \cdots \geq u_N(t) \text{ a.e. in } \Omega.$$

Adding the two equations, we have the result as with the one obstacle problem, making use of the strict monotonicity and Lipschitzness of g as well as Lemma 5.47. □

Remark 5.50. *Observe that it is necessary for the obstacles ψ, φ to be time-independent in the one and two obstacles problems, so that $u \in \mathbb{K}_\psi^s$ implies $u(t) \in K_\psi^s$ for a.e. t in the one obstacle problem, and similarly for the two obstacles problem. For the N membranes problem, no such similar additional assumption is required, because the "obstacle" is given by the previous component u_{k-1}, which is time-independent for each fixed t.*

6 Anisotropic Fractional and Nonlocal Stefan-Type Problems

6.1 Introduction

The classical Stefan problem, in an open bounded Lipschitz domain $\Omega \ni x = (x_1, \ldots, x_d)$ and for time $t \in [0, T]$, can be formulated in $Q_T =]0, T[\times\Omega$ by an evolution equation involving a subdifferential operator

$$\frac{\partial}{\partial t}\beta(\vartheta) - \nabla \cdot (A\nabla\vartheta) \ni f, \tag{6.1}$$

where $\vartheta(t, x)$ is the temperature, ∇ is the gradient, $A = A(x)$ is a symmetric, strictly elliptic and bounded matrix, and β corresponds to a maximal monotone graph, such that $\beta(r) = b(r) + \lambda\chi$ for $\chi \in H(\vartheta)$ for the maximal monotone graph $H(r)$ associated with the Heaviside function, i.e. $H(r) = 0$ for $r < 0$, $H(r) = 1$ for $r > 0$, $H(0) = [0, 1]$, and b a given continuous and strictly increasing function, $\lambda > 0$ (see Figure 2) with inverse $\gamma = \beta^{-1}$ satisfying $\lim_{r\to+\infty} \gamma(r) = +\infty$ and $\lim_{r\to-\infty} \gamma(r) = -\infty$ for the two-phase problem and $\gamma(r) = 0$ for $r \leq \lambda$ for the one-phase problem. The notation $\beta(\vartheta)$ should be understood as follows: there exists a section η of the multifunction $\beta(\vartheta)$ which satisfies the required conditions. In turn, ϑ is easy to recover from η since $\beta^{-1} = \gamma$ is a single-valued mapping. For works on the variational formulation of the classical Stefan problem, see for instance [182, 136, 121], Chapter V.9 of [153], Section 3.3 of [161], [85], [227], [191], [192] and [239].

We can also consider the one-phase problem (I) as the limit of the two-phase problem (II). Indeed, physically, for large Stefan number, the liquid phase only contributes exponentially small terms to the location of the solid–melt interface. Therefore, at times close to complete solidification, the temperature in the liquid essentially vanishes and the two-phase problem reduces to the one-phase problem. For more detailed discussions, see [169]. See also [225] for the one-dimensional case in the classical setting.

Here, we consider the corresponding fractional Stefan-type problem, given in Q_T by

$$\frac{\partial}{\partial t}\beta(\vartheta) + \tilde{\mathcal{L}}^s_A\vartheta \ni f, \tag{6.2}$$

where $\tilde{\mathcal{L}}^s_A = -D^s \cdot AD^s$ is a fractional operator defined with the distributional Riesz fractional derivatives, with anisotropy given by a measurable matrix $A = A(x)$, which is symmetric, strictly uniformly elliptic and bounded independent of time satisfying

$$a_*|z|^2 \leq A(x)z \cdot z \leq a^*|z|^2 \tag{6.3}$$

for almost every $x \in \mathbb{R}^d$ and all $z \in \mathbb{R}^d$. Then, the classical problem (6.1) corresponds to the case $s = 1$, i.e. (6.2) with the operator $\tilde{\mathcal{L}}^1_A$, where $D^1 = \nabla$.

In this chapter, we are also concerned with the classical fractional Sobolev space $H^s_0(\Omega)$ in a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, for $0 < s < 1$, defined as

$$H^s_0(\Omega) := \overline{C^\infty_c(\Omega)}^{\|\cdot\|_{H^s}},$$

with

$$\|u\|^2_{H^s} = \|u\|^2_{L^2(\mathbb{R}^d)} + \|D^s u\|^2_{L^2(\mathbb{R}^d)^d}, \tag{6.4}$$

where u is extended by 0 in $\mathbb{R}^d\backslash\Omega$, so that this extension is also in $H^s(\mathbb{R}^d)$. By the classical fractional Poincaré inequality (see Lemma 1.3), we shall consider the space $H^s_0(\Omega)$ with the following equivalent norm

$$\|u\|^2_{H^s_0(\Omega)} = \|D^s u\|^2_{L^2(\mathbb{R}^d)^d}. \tag{6.5}$$

We subsequently denote the dual space of $H^s_0(\Omega)$ by $H^{-s}(\Omega)$ for $0 < s \leq 1$. Then, by the Sobolev-Poincaré inequalities, we have the compact embeddings

$$H^s_0(\Omega) \hookrightarrow L^q(\Omega), \qquad L^{q'}(\Omega) \hookrightarrow H^{-s}(\Omega) = (H^s_0(\Omega))'$$

for $1 \leq q < 2^*$, where $2^* = \frac{2d}{d-2s}$ and $q' > 2^\# = \frac{2d}{d+2s}$ when $s < \frac{d}{2}$, and if $d = 1$, $2^* = q$ for any finite q and $2^\# = \frac{q}{q-1}$ when $s = \frac{1}{2}$ and $2^* = \infty$ and $2^\# = 1$ when $s > \frac{1}{2}$. We recall that those embeddings are continuous also for $q = 2^*$ when $s < \frac{d}{2}$ (see for example, Theorem 4.54 of [93]).

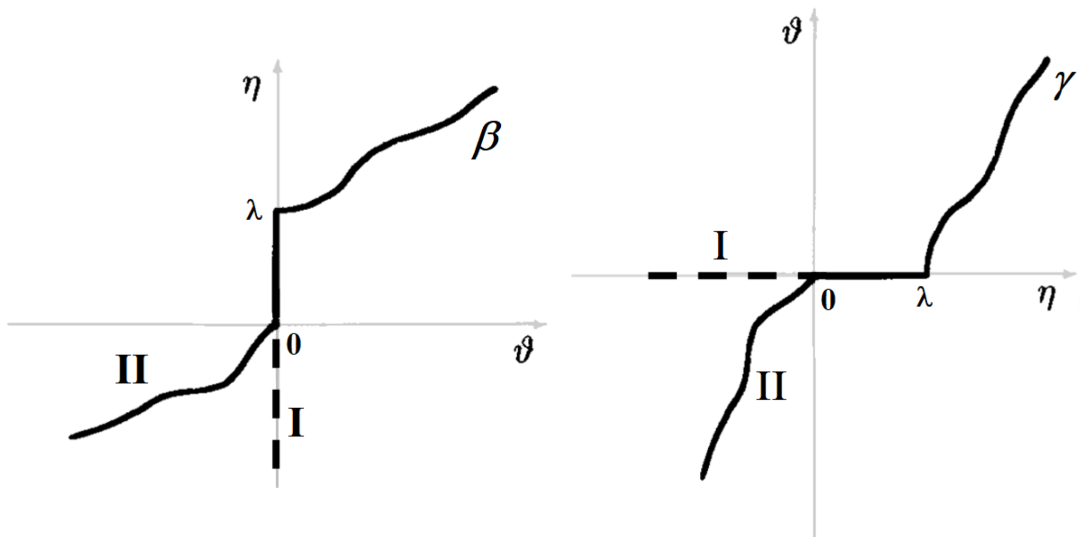

Figure 2: The maximal monotone graphs β as the inverse of the continuous monotone functions $\gamma = \beta^{-1}$ in the case of the II phases and the I phase Stefan problems

In particular, we call such problems fractional Stefan-type problems, when β is a maximal monotone graph. This includes the porous medium equation (see for instance [234]), where β^{-1} is given by a power law function

$$\beta^{-1}(r) := |r|^m sign(r)$$

for the single-valued restriction of the signum function

$$sign(x) := \begin{cases} \{-1\} & x < 0, \\ \{0\} & x = 0, \\ \{1\} & x > 0. \end{cases}$$

for $m > 1$, as well as the classical Stefan problem, where β has a jump such that β^{-1} is given by

$$\beta^{-1}(u) = c_1(u - L)^+ \quad \text{for } u \geq 0, \quad \beta^{-1}(u) = c_2 u \quad \text{for } u < 0,$$

where c_1, c_2 and L are positive constants.

Recall that the nonlocal operator $\tilde{\mathcal{L}}^s_A = -D^s \cdot AD^s$ may also be defined in the duality sense for $u \in H^s(\mathbb{R}^d)$:

$$\langle \tilde{\mathcal{L}}^s_A u, v\rangle := \int_{\mathbb{R}^d} AD^s u \cdot D^s v \quad \forall v \in H^s_0(\Omega), \tag{3.1}$$

with v extended by zero outside Ω, defining an operator from $H^s(\mathbb{R}^d)$ to $H^{-s}(\Omega)$ since $AD^s u \in L^2(\mathbb{R}^d)^d$. Also for $u \in H^s_0(\Omega)$, since we can extend it by 0 outside Ω to obtain a function in $H^s(\mathbb{R}^d)$, $\tilde{\mathcal{L}}^s_A : H^s_0(\Omega) \to H^{-s}(\Omega)$ can also be represented by

$$\tilde{\mathcal{L}}^s_A u = -D^s \cdot (AD^s u). \tag{6.6}$$

Given any $\tilde{g} \in H^s(\mathbb{R}^d)$, we introduce $g \in H^s(\mathbb{R}^d)$ defined on the whole space $\mathbb{R}^d$ which satisfies $g|_{\Omega^c} = \tilde{g}$ and is $\tilde{\mathcal{L}}^s_A$-harmonic in Ω, that is to say, we solve the Dirichlet problem with $g = \tilde{g}$ a.e. on Ω^c for the equation

$$\tilde{\mathcal{L}}^s_A g = 0 \text{ in } H^{-s}(\Omega) \tag{6.7}$$

in a weak sense, which means

$$\int_{\mathbb{R}^d} AD^s g \cdot D^s v = 0 \quad \forall v \in H^s_0(\Omega).$$

Note that this is possible and defines g a.e. in $\mathbb{R}^d$ by Lax-Milgram theorem (see Section 3.2, and also Theorem 1.13 of [213]), since A is strictly elliptic and bounded.

Next, we introduce the enthalpy function

$$\eta(t,x) \in \beta(\vartheta(t,x)) \text{ for almost every } (t,x) \in Q_T \tag{6.8}$$

with initial condition

$$\eta(0) = \eta_0 \text{ in } H^{-s}(\Omega), \tag{6.9}$$

and we prescribe a Dirichlet boundary condition

$$\vartheta(t) = \tilde{g}(t) \text{ a.e. in } \Omega^c = \mathbb{R}^d \setminus \Omega, \text{ for a.e. } t \in]0,T[, \tag{6.10}$$

for a given $\tilde{g}(t) \in H^s(\mathbb{R}^d)$. For simplicity we shall often describe this Dirichlet condition by saying that $\vartheta(t) - \tilde{g}(t) \in H^s_0(\Omega)$ for a.e. t, which is certainly clear for $s > 1/2$, by the trace theorem, and an abuse of notation for $s \leq 1/2$. Now, for almost every $t \in [0,T]$, introducing $g(t) = \tilde{g}(t)$ in Ω^c and such that $\tilde{\mathcal{L}}^s_A g(t) = 0$ in Ω in the distributional sense, assuming $f \in L^2(0,T; H^{-s}(\Omega))$, we then have the following weak formulation of the Stefan-type problem when viewed as a single-unknown problem:

$$\left\langle \frac{d\eta}{dt}, \xi \right\rangle + \langle \tilde{\mathcal{L}}^s_A(\gamma(\eta) - g), \xi \rangle = \langle f, \xi \rangle, \quad \forall \xi \in L^2(0,T; H^s_0(\Omega)) \tag{6.11}$$

with initial data (6.9), where $\langle \cdot, \cdot \rangle$ denotes the duality between $L^2(0,T; H^{-s}(\Omega))$ and $L^2(0,T; H^s_0(\Omega))$. Here the Lipschitz graph γ, which may have flat parts, is defined as the inverse of the maximal monotone graph β (see Figure 2). We call the solution η of (6.11) the *generalised solution* for the enthalpy formulation, by requiring

$$\eta \in H^1(0,T; H^{-s}(\Omega)) \cap L^2(Q_T) \text{ with } \gamma(\eta) - g \in L^2(0,T; H^s_0(\Omega)).$$

By the regularity of η, setting $\beta = b + \lambda H$, we can write $\eta = [b(\vartheta) + \lambda\chi] \in \beta(\vartheta)$ with $\chi \in H(\vartheta)$ a.e. in Q_T, i.e.

$$0 \leq \chi_{\{\vartheta > 0\}} \leq \chi \leq 1 - \chi_{\{\vartheta < 0\}} \leq 1 \quad \text{ a.e. in } Q_T.$$

Suppose we take a more regular test function ξ which additionally satisfies $\xi(T) = 0$. Then, using integration by parts in time, we also have a weak variational formulation, with $f \in L^2(0,T; L^2(\Omega))$ and $\eta_0 \in L^2(\Omega)$, for the solution $\vartheta = \gamma(\eta)$, i.e. ϑ is the *weak solution* for the temperature formulation:

$$(\vartheta, \chi) \in [L^2(Q_T)]^2, \chi \in H(\vartheta) \text{ and } \vartheta - g \in L^2(0,T; H^s_0(\Omega)) \tag{6.12}$$

satisfy

$$-\int_{Q_T} [b(\vartheta) + \lambda\chi] \frac{\partial \xi}{\partial t} + \int_{\mathbb{R}^d \times [0,T]} AD^s\vartheta \cdot D^s\xi = \int_{Q_T} f\xi + \int_\Omega \eta_0 \xi(0), \quad \forall \xi \in \Xi^s_T, \tag{6.13}$$

where

$$\Xi^s_T := \{\xi \in L^2(0,T; H^s_0(\Omega)) \cap H^1(0,T; L^2(\Omega)) : \xi(T) = 0 \text{ in } \Omega\}.$$

Compare with [182, 136] and Section V.9 of [153] for the classical case with $s = 1$.

Remark 6.1. *Note that the variational problem* (6.11) *incorporates the Dirichlet condition* (6.10) *in the original problem given in* (6.2) *because of the definition* (6.7). *Since this implies* $\int_{\mathbb{R}^d \times [0,T]} AD^s g \cdot D^s \xi = 0$ *for all* $\xi \in \Xi^s_T$, *we obtain* (6.13) *without that term.*

Although in general η, ϑ *may be nonzero outside* Ω, *except for the bilinear form* $\int_{\mathbb{R}^d \times [0,T]} AD^s\vartheta \cdot D^s\xi$, *the other integral terms in the variational formulation* (6.13) *are only integrated over* Ω *in space, since the test function* ξ *is 0 in* $]0,T[\times\Omega^c$.

Different nonlocal versions of Stefan-type problems have previously been considered, including in [50] and [71] for nonsingular integral kernels, in [240, 40, 203, 201] for the fractional Caputo derivatives, and in [231, 229, 232, 233, 134] for the fractional Laplacian and its nonlocal integral generalisation in [19]. Stefan-type problems that are fractional in the time derivative have also been considered (see, for instance, [200, 156, 67].)

Indeed, when the matrix A is a multiple of the identity matrix, the fractional Stefan-type problem (6.2) reduces to that with the fractional Laplacian as considered in [231]–[233]. Furthermore, in instances as described in Section 3.4 when the fractional operator $\tilde{\mathcal{L}}^s_A$ is replaced with a nonlocal operator $\mathcal{L}^s_a$, corresponding to a Dirichlet form with the kernel a which satisfies some coercivity and boundedness conditions, (6.2) may also be considered a nonlocal Stefan problem, as considered in [19], and will also be considered in Sections 6.8–6.8.2. However, an equivalence relation between the fractional operator with the matrix A and the nonlocal operator with the kernel a cannot be established in general except in the isotropic homogeneous case (for more details, see Section 3.4), so the two Stefan-type problems with those two operators are not equivalent.

In this Chapter, we show the existence of a unique solution for the fractional Stefan-type problem with Dirichlet boundary conditions, where the spatial operator is a general anisotropic non-local singular operator of fractional type as given by (2.1), and we keep the classical temperature-enthalpy relation illustrated in Figure 2. This relation in the classical equation (6.1) incorporates, in a generalised form, the free boundary condition relating the balance between the normal velocity of the interface and the jump of the local anisotropic heat flow. In 1-dimension, the extension of the classical free boundary Stefan condition to fractional diffusion, as in the recent paper [203] with the fractional Caputo derivative in the nonlocal diffusive term, can be easily made explicit. Similar explicit formulation can be used with the 1-dimensional fractional Riesz spatial derivative when, for each fixed time, the free boundary is a point.

However, in higher dimensions, the Riesz fractional s-gradient, as proposed in [216], is an appropriate fractional operator maintaining translational and rotational invariance, as well as homogeneity of degree s under isotropic scaling, and so the $\tilde{\mathcal{L}}^s_A$ operator gives a natural and appropriate anisotropic generalisation of the fractional Laplacian. Keeping the generalised Stefan condition in the evolution equation (6.2) involving the maximal monotone operator β is a natural generalisation for the formulation of the anisotropic Stefan problem, extending [232] and [233], which corresponds to the case where the matrix A is the identity matrix in the unbounded domain. Such an anisotropic operator is coordinate invariant, which makes it more suitable in higher dimensions. Furthermore, the use of this $\tilde{\mathcal{L}}^s_A$ operator allows us to recover the classical Stefan problem when $s = 1$, which is in accordance with a requirement of weak continuity from the nonlocal model to the local model, when $s \nearrow 1$. However, a main issue remains open in the fractional multidimensional model, namely what is the physical meaning of the Stefan condition due to the lack of a convenient interpretation and definition for the fractional heat flux across the solid-liquid interface.

In the first part of this chapter, we show the existence of a unique solution for the fractional Stefan-type problem with Dirichlet boundary conditions, where the spatial operator is a general anisotropic fractional singular operator as given by (3.1), which is a generalisation of the results obtained in [203] for the fractional Caputo derivative, and in [232, 233] for the fractional Laplacian which corresponds to the case where the matrix A is the identity matrix in the unbounded domain.

In Sections 6.2 and 6.3, we employ Hilbertian techniques to show the existence of a generalised enthalpy solution and a weak temperature solution to the initial and boundary value two-phase Stefan-type problem (6.11)–(6.13) following the approach of Damlamian [84]–[85] for the classical case $s = 1$. This is coupled with the equivalence with the variational inequality formulation for the two-phase and the one-phase problems in Section 6.4.

Making use of convergence properties of the fractional derivatives to the classical derivatives when $s \nearrow 1$, we show, in Section 6.5, that the solution of the fractional Stefan-type problem converges to the solution of the classical case corresponding to $s = 1$. Next, we consider the asymptotic behaviour of the solution as $t \to \infty$ in Section 6.6. Such convergence properties apply to both the two-phase problem, and the one-phase problem, which corresponds to the case of a nonnegative temperature. The one-phase problem (I) is recovered in Section 6.7 from the two-phase problem (II), when the maximal monotone graph for (II) (see Figure 2) degenerates to that of the one-phase problem (I).

6.2 Existence of the Generalised Enthalpy Solution η

Let $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}^s_A$ be the duality mapping defined by

$$\langle \tilde{\mathcal{L}} u, v \rangle = \int_{\mathbb{R}^d} AD^s u \cdot D^s v =: [u, v]_A = (U, V), \tag{6.14}$$

from $H_0^s(\Omega)$ to $H^{-s}(\Omega)$ with $H_0^s(\Omega)$ identified to a subspace of $L^2(\Omega)$. Here $\langle\cdot,\cdot\rangle$ is the duality between $H^{-s}(\Omega)$ with $H_0^s(\Omega)$, with u, v extended by zero outside Ω. The equality of the inner product in $H^{-s}(\Omega)$ given by $(\cdot,\cdot)$, with the topology endowed from $\tilde{\mathcal{L}}$, with the equivalent inner product $[\cdot,\cdot]_A$ in $H_0^s(\Omega)$ holds by Riesz representation theorem, with

$$U = \tilde{\mathcal{L}}u \text{ and } V = \tilde{\mathcal{L}}v \text{ respectively.} \tag{6.15}$$

This is possible by assumption (6.3) and the Poincaré inequality, as long as Ω is bounded.

In this section, we consider the two-phase problem with

$$\gamma \text{ is Lipschitz with Lipschitz constant } C_\gamma \text{ such that } \gamma(0) = 0 \quad \text{ and } \quad \liminf_{|r|\to+\infty} \frac{\gamma(r)}{r} > 0. \tag{6.16}$$

We prove an existence theorem for the enthalpy η similar to the classical case, as given in [85] and [84] (See also [239] for further developments). To do so, we need a result of Attouch-Damlamian [22]–[23] in the case where the Hilbert space H is $H^{-s}(\Omega)$.

Proposition 6.2. *[Theorem 1 of [22], and [23]] Let $(\varphi_t)_{t\in[0,T]}$ be a family of lower semi-continuous convex functions on a Hilbert space H. Assume that there exists a function $\mathfrak{a}$ belonging to $BV(0,T)$ such that the following holds:*

$$\varphi_t(V) \le \varphi_\tau(V) + |\mathfrak{a}(t) - \mathfrak{a}(\tau)|(\varphi_\tau(V) + |V| + 1), \quad \forall 0 \le \tau \le t \le T, \forall V \in H. \tag{6.17}$$

Then, for $U_0 \in D(\varphi_0) = \{U_0 \in H : \varphi_0(U_0) < +\infty\}$ and $F \in L^2(0,T;H)$, there is a unique solution $U \in H^1(0,T;H)$ satisfying

$$\frac{dU}{dt} + \partial\varphi_t(U) = F, \quad U(0) = U_0. \tag{6.18}$$

Furthermore, the following estimates hold independent of φ:

$$\|U\|_{C([0,T];H)} \le C_1\left(\|U_0\|_H, \|F\|_{L^1(0,T;H)}, \|\mathfrak{a}\|_{BV}\right), \tag{6.19}$$

$$\left\|\frac{dU}{dt}\right\|_{L^2(0,T;H)} \le C_2\left(\|U_0\|_H, \varphi_0(U_0), \|F\|_{L^2(0,T;H)}, \|\mathfrak{a}\|_{BV}\right), \tag{6.20}$$

$$\left\|\varphi_t(U)\right\|_{L^\infty(0,T)} \le C_3\left(\|U_0\|_H, \varphi_0(U_0), \|F\|_{L^2(0,T;H)}, \|\mathfrak{a}\|_{BV}\right). \tag{6.21}$$

Making use of this proposition, we can show the following existence result.

Theorem 6.3. *Let $f \in L^2(0,T;H^{-s}(\Omega))$ and $\tilde{g} \in BV(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^d))$, and define g as in (6.7), so g satisfies the same regularity as $\tilde{g}$ (see Section 3.2). Assume $\eta_0 \in L^2(\Omega)$ and γ satisfies (6.16). Then there exists a unique generalised enthalpy solution η to the problem (6.11) with initial condition (6.9), such that*

$$\eta \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^{-s}(\Omega)) \tag{6.22}$$

and it satisfies

$$\|\eta\|_{C([0,T];H^{-s}(\Omega))} \le C_1\left(\|f\|_{L^1(0,T;H^{-s}(\Omega))}, \|\eta_0\|_{H^{-s}(\Omega)}, \|g\|_{BV(0,T;L^2(\Omega))}\right), \tag{6.23}$$

$$\left\|\frac{d\eta}{dt}\right\|_{L^2(0,T;H^{-s}(\Omega))} \le C_2\left(\|f\|_{L^2(0,T;H^{-s}(\Omega))}, \|\eta_0\|_{L^2(\Omega)}, \|g\|_{BV(0,T;L^2(\Omega))}\right), \tag{6.24}$$

$$\|\eta\|_{L^\infty(0,T;L^2(\Omega))} \le C_4\left(\|f\|_{L^2(0,T;H^{-s}(\Omega))}, \|\eta_0\|_{L^2(\Omega)}, \|g\|_{BV(0,T;L^2(\Omega))}\right), \tag{6.25}$$

where C_1, C_2 are exactly the constants from (6.19)–(6.20), while C_4 depends on (6.21) and (6.17). Furthermore, the corresponding weak temperature solution $\vartheta = \gamma(\eta)$ satisfies

$$\vartheta - g \in L^2(0,T;H_0^s(\Omega)) \tag{6.26}$$

and, in addition, it solves (6.13) when $f \in L^2(0,T;L^2(\Omega))$.

Proof. We apply Proposition 6.2 with $|\mathfrak{a}(t) - \mathfrak{a}(\tau)| = \left\|g(t) - g(\tau)\right\|_{L^2(\Omega)}$ to the following functions ϕ_t on the Hilbert space $H^{-s}(\Omega)$ given for each $t \in [0,T]$ by

$$\phi_t(W) = \begin{cases} \int_\Omega (j(W) - g(t)W)\,dx & \text{for } W \in L^2(\Omega); \\ +\infty & \text{for } W \in H^{-s}(\Omega)\backslash L^2(\Omega) \end{cases} \tag{6.27}$$

where j is the primitive of γ such that $j(0) = 0$. Then, j is quadratic and the domain $D(\phi_t)$ of ϕ_t is given by

$$D(\phi_t) = \{W \in H^{-s}(\Omega) : \phi_t(W) < \infty\} = L^2(\Omega) \tag{6.28}$$

thanks to the Cauchy-Schwarz inequality and making use of the fact that W lies in $L^2(\Omega)$. It is well-known (see for instance, Theorem 17 of [52]) that ϕ_t is lower semi-continuous, convex, proper and coercive on $H^{-s}(\Omega)$. Furthermore, there exist constants δ and c such that

$$\delta\|W\|^2_{L^2(\Omega)} \le \phi_\tau(W) + c|\Omega| + \left\|g(\tau)\right\|_{L^2(\Omega)}\|W\|_{L^2(\Omega)}\,. \tag{6.29}$$

Consequently, by classical results of subdifferentials (see for instance, [52] or [144]), the subdifferential $\partial\phi_t$ is a maximal monotone operator of $H^{-s}(\Omega)$.

In fact, the subdifferential $\partial\phi_t$ is characterised as follows:

$$V \in \partial\phi_t(U) \text{ in } H^{-s}(\Omega) \quad \text{if and only if} \quad U \in L^2(\Omega) \text{ and } \tilde{\mathcal{L}}^{-1}(V) + g = \gamma(U) \text{ a.e. in } \Omega, \tag{6.30}$$

and we recall from (6.27) that

$$\gamma(U) - g = \tilde{\mathcal{L}}^{-1}(V) = v \in H^s_0(\Omega),$$

representing the Dirichlet condition in weak form in the trace sense for $s > \frac{1}{2}$ and more generally $\gamma(U) = g$ in Ω^c. Indeed, the characterisation of the subdifferential in terms of the convex conjugate functions involving (U,V) for $U, V \in H^{-s}(\Omega)$ reads as:

$$V \in \partial\phi_t(U) \iff \phi_t(U) + \phi_t^*(V) = (U,V) \tag{6.31}$$

where $\phi_t^*(V) = \sup_W\{(W,V) - \phi_t(W)\}$. Then for a given $V \in H^{-s}(\Omega)$,

$$\begin{aligned} \phi_t^*(V) &= \sup_{W\in L^2(\Omega)} \{\langle W, \tilde{\mathcal{L}}^{-1}V\rangle - \phi_t(W)\} \\ &= \sup_{W\in L^2(\Omega)} \left\{\langle W, \tilde{\mathcal{L}}^{-1}V\rangle - \int_\Omega j(W) - gW\,dx\right\} \\ &= \sup_{W\in L^2(\Omega)} \left\{\int_\Omega W(\tilde{\mathcal{L}}^{-1}V + g) - \int_\Omega j(W)\,dx\right\}. \end{aligned}$$

Set $J(W) = \int_\Omega j(W)$. Recognising the evaluation at the point $\tilde{\mathcal{L}}^{-1}V + g$ with the convex conjugate on $L^2(\Omega)$ of $j(W)$, by well-known results (see for example Lemma 1 of [53], or [189]), we can associate the convex conjugate $J^*(U)$ with $\int_\Omega j^*(U)$, so we have

$$\phi_t^*(V) = \int_\Omega j^*(\tilde{\mathcal{L}}^{-1}V + g)\,dx,$$

where j^* is the convex conjugate of j on $\mathbb{R}$. From (6.31), this means that

$$\int_\Omega j(U) - gU + j^*(\tilde{\mathcal{L}}^{-1}V + g) = \langle\tilde{\mathcal{L}}^{-1}V, U\rangle,$$

or

$$\int_\Omega j(U) + j^*(\tilde{\mathcal{L}}^{-1}V + g) - U(\tilde{\mathcal{L}}^{-1}V + g) = 0. \tag{6.32}$$

Recall (see for example, [21]) that for dual convex functions j and j^*,

$$j(a) + j^*(b) \ge ab$$

for all numbers a, b. Therefore, the integrand in (6.32) must be non-negative, and so it is almost everywhere zero, i.e.

$$j(U) + j^*(\tilde{\mathcal{L}}^{-1}V + g) - U(\tilde{\mathcal{L}}^{-1}V + g) = 0.$$

This means that the points U and $\tilde{\mathcal{L}}^{-1}V + g$ are conjugated, i.e. $\tilde{\mathcal{L}}^{-1}V + g \in \partial j(U)$. By definition of j as the primitive of γ, we have $\tilde{\mathcal{L}}^{-1}V + g = \partial j(U) = \gamma(U)$.

Now, we are ready to apply Proposition 6.2 in the space $H^{-s}(\Omega)$ with the convex functions ϕ_t. For $W \in D(\phi_\tau) \cap D(\phi_t) = D(\phi_0)$ since the domain $D(\phi_t)$ as given in (6.28) is independent of t, we have, by (6.27),

$$\phi_t(W) - \phi_\tau(W) = -\int_\Omega W(g(t) - g(\tau)),$$

so, by the Cauchy-Schwarz inequality,

$$|\phi_t(W) - \phi_\tau(W)| \leq \left\|g(t) - g(\tau)\right\|_{L^2(\Omega)} \|W\|_{L^2(\Omega)}. \tag{6.33}$$

Also, from (6.29), we have that

$$\|W\|_{L^2(\Omega)} \leq C_5(1 + \phi_\tau(W)), \tag{6.34}$$

where C_5 depends only on δ, $|\Omega|$ and $\|g\|_{BV(0,T;L^2(\Omega))}$. Therefore, with the given regularity of g inherited from $\tilde{g}$, (6.17) is satisfied, hence we can apply Proposition 6.2 to solve the Cauchy problem

$$\frac{d\eta}{dt} + \partial\phi_t(\eta(t)) \ni f(t) \quad \text{for almost all } t \in [0,T], \quad \eta(0) = \eta_0 \text{ in } H^{-s}(\Omega) \tag{6.35}$$

with $\eta_0 \in D(\phi_0)$, i.e. $\eta_0 \in L^2(\Omega)$ and $j(\eta_0) \in L^1(\Omega)$, obtaining a unique

$$\eta \in H^1(0,T; H^{-s}(\Omega)).$$

Moreover, the estimates in Proposition 6.2 and (6.34) give

$$\eta \in L^\infty(0,T; L^2(\Omega)).$$

Also, setting $\vartheta = \gamma(\eta)$ gives

$$\tilde{\mathcal{L}}(\vartheta - g) \in L^2(0,T; H^{-s}(\Omega)),$$

so that

$$\vartheta - g \in L^2(0,T; H^s_0(\Omega)),$$

and by (6.30),

$$\partial\phi_t(\eta(t)) = \tilde{\mathcal{L}}(\gamma(\eta) - g).$$

Therefore, multiplying (6.35) by a test function $\xi \in L^2(0,T; H^s_0(\Omega))$, since $\eta \in H^1(0,T; H^{-s}(\Omega))$, we have

$$\left\langle \frac{d\eta}{dt}, \xi \right\rangle + \langle \tilde{\mathcal{L}}(\gamma(\eta) - g), \xi \rangle = \langle f, \xi \rangle,$$

which is (6.11).

Finally, for $\xi \in \Xi^s_T$, we can integrate in time by parts and obtain (6.13). □

Remark 6.4. *To apply Proposition 6.2, we see from* (6.33) *that it is sufficient to require* $g \in BV(0,T; L^2(\Omega))$, *as in [85]. However, we require additionally that* $g \in L^2(0,T; H^s(\mathbb{R}^d))$ *so that* (6.11)–(6.13) *is well-defined.*

Remark 6.5. *We observe that the general result of the above proposition and theorem applies to general maximal monotone operators of subdifferential type with different functions* γ, *and so, besides two-phase Stefan-type problems, it applies also to other models including the porous medium equation. In fact, different assumptions on* γ *can be used (see page 12 of [84] for more details), generalising the case of the assumption* (6.16).

Remark 6.6. *Considering the above proposition in the case where the Hilbert space H is $H^{-s}(\Omega)$, the solution to the Cauchy problem* (6.18) *in $H^{-s}(\Omega)$ with the convex function ϕ_t, with domain $L^2(\Omega)$, is obtained by considering the approximated problem with the convex function given by its Yosida approximation $\phi_{t,\lambda}(V) = \frac{1}{\lambda}(Id + (Id + \lambda\partial\phi_t)^{-1})V$. Since the estimate* (6.17) *carries over to $\phi_{t,\lambda}$, we can apply the Gronwall's inequality to obtain the estimates* (6.19) *and* (6.21) *for the solutions to the approximated problem as in Part 3 of the proof of Theorem 1 in [22]. Next, we make use of the absolute continuity of the map $t \mapsto \phi_{t,\lambda}(V)$ to apply to the* (6.18) *to obtain the estimate* (6.20) *from the time derivative. Passing to the limit for the approximated problems give the corresponding constants C_1, C_2 and C_3 for the problem* (6.18) *in $H^{-s}(\Omega)$.*

Therefore, for $\sigma \leq s \leq 1$, recalling that we have the continuity of the inclusions $H^{-\sigma}(\Omega) \subset H^{-s}(\Omega) \subset H^{-1}(\Omega)$ as a consequence of Lemma 6.19 below, we can bound the $H^{-s}(\Omega)$ norms with the $H^{-\sigma}(\Omega)$ norms, thereby obtaining the solution to (6.18) *for all s, $\sigma \leq s \leq 1$ with the corresponding estimates* (6.19)–(6.21) *for the constants C_1, C_2 and C_3 depending only on σ and independent of s.*

Since the constant C_4 is obtained from (6.21) *and* (6.34)*, similarly, we can once again consider the problem* (6.11) *in $H^{-s}(\Omega)$ for each s, $\sigma \leq s \leq 1$, and such that the constant C_4 in* (6.25) *may be chosen depending only on σ and not on s.*

Note that we are unable to obtain strict T-monotonicity, since we do not have such a result for the fractional operator $\tilde{\mathcal{L}}^s_A$ (Compare with Section 2.3 for the nonlocal operator $\mathcal{L}_a$). However, we have the following continuous dependence result (see also Lemma 3.2 of [86]).

Proposition 6.7. *Let η and $\hat{\eta}$ denote two generalised enthalpy solutions of the fractional Stefan-type problem* (6.11) *corresponding to (f, g, η_0) and $(\hat{f}, g, \hat{\eta}_0)$ respectively, where $f, \hat{f}, g$, and $\eta_0, \hat{\eta}_0$ are as in the assumptions of Theorem 6.3. Then, for any $0 \leq t \leq T$:*

$$\left\|\eta(t) - \hat{\eta}(t)\right\|_{H^{-s}(\Omega)} \leq \|\eta_0 - \hat{\eta}_0\|_{H^{-s}(\Omega)} + \int_0^t \left\|f(\tau) - \hat{f}(\tau)\right\|_{H^{-s}(\Omega)} d\tau \tag{6.36}$$

and furthermore

$$\left\|\vartheta - \hat{\vartheta}\right\|_{L^2(0,T;L^2(\Omega))} \leq \sqrt{C_\gamma}\|\eta_0 - \hat{\eta}_0\|_{H^{-s}(\Omega)} + \sqrt{\frac{3C_\gamma}{2}}\left\|f - \hat{f}\right\|_{L^1(0,T;H^{-s}(\Omega))}. \tag{6.37}$$

Proof. Writing $\vartheta = \gamma(\eta)$ and $\hat{\vartheta} = \gamma(\hat{\eta})$, we have in $H^{-s}(\Omega)$,

$$\frac{d\eta}{dt}(\tau) = -\tilde{\mathcal{L}}^s_A(\vartheta(\tau) - g(\tau)) + f(\tau) \tag{6.38}$$

and

$$\frac{d\hat{\eta}}{dt}(\tau) = -\tilde{\mathcal{L}}^s_A(\hat{\vartheta}(\tau) - g(\tau)) + \hat{f}(\tau) \tag{6.39}$$

for a.e. $\tau \in [0, T]$. Taking the difference of these two equations and multiplying by $\eta - \hat{\eta}$, we have

$$\begin{aligned}
&\frac{d}{d\tau}\left\|\eta(\tau) - \hat{\eta}(\tau)\right\|^2_{H^{-s}(\Omega)} = 2\left(\eta'(\tau) - \hat{\eta}'(\tau), \eta(\tau) - \hat{\eta}(\tau)\right)\\
&= -2\left(\tilde{\mathcal{L}}^s_A(\vartheta(\tau) - \hat{\vartheta}(\tau) - g(\tau) + g(\tau)), \eta(\tau) - \hat{\eta}(\tau)\right) + 2\left(f(\tau) - \hat{f}(\tau), \eta(\tau) - \hat{\eta}(\tau)\right)\\
&= -2\left(\tilde{\mathcal{L}}^s_A(\vartheta(\tau) - \hat{\vartheta}(\tau)), \eta(\tau) - \hat{\eta}(\tau)\right) + 2\left(f(\tau) - \hat{f}(\tau), \eta(\tau) - \hat{\eta}(\tau)\right)
\end{aligned}$$

for a.e. $\tau \in [0, T]$. Recalling by Theorem 6.3 that $\vartheta(\tau) - \hat{\vartheta}(\tau) \in H^s_0(\Omega) \subset L^2(\Omega)$ and $\eta(\tau) - \hat{\eta}(\tau) \in L^2(\Omega)$ for a.e. τ, observe that the Lipschitz property of γ give

$$\left(\tilde{\mathcal{L}}^s_A(\vartheta(\tau) - \hat{\vartheta}(\tau)), \eta(\tau) - \hat{\eta}(\tau)\right) = \int_\Omega \left(\vartheta(\tau) - \hat{\vartheta}(\tau)\right)\left(\eta(\tau) - \hat{\eta}(\tau)\right) \geq \frac{1}{C_\gamma}\left\|\vartheta(\tau) - \hat{\vartheta}(\tau)\right\|^2_{L^2(\Omega)}$$

by (6.14) and by identifying the duality $\langle\cdot,\cdot\rangle$ with the $L^2(\Omega)$-inner product in the framework of the Gelfand triple $H^s_0(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$. Therefore, we deduce that

$$\frac{d}{d\tau}\left\|\eta(\tau) - \hat{\eta}(\tau)\right\|^2_{H^{-s}(\Omega)} + \frac{2}{C_\gamma}\left\|\vartheta(\tau) - \hat{\vartheta}(\tau)\right\|^2_{L^2(\Omega)} \leq 2\left(f(\tau) - \hat{f}(\tau), \eta(\tau) - \hat{\eta}(\tau)\right) \tag{6.40}$$

for a.e. $\tau \in [0,T]$. Integrating both sides of (6.40) over $[0,t] \subset [0,T]$ for any $T > 0$ gives

$$
\begin{aligned}
&\left\|\eta(t) - \hat{\eta}(t)\right\|_{H^{-s}(\Omega)}^2 + \frac{2}{C_\gamma}\int_0^t \left\|\vartheta(\tau) - \hat{\vartheta}(\tau)\right\|_{L^2(\Omega)}^2 \, d\tau \\
&\leq \|\eta_0 - \hat{\eta}_0\|_{H^{-s}(\Omega)}^2 + 2\int_0^t \left(f(\tau) - \hat{f}(\tau), \eta(\tau) - \hat{\eta}(\tau)\right) \, d\tau \\
&\leq \|\eta_0 - \hat{\eta}_0\|_{H^{-s}(\Omega)}^2 + 2\int_0^t \left\|f(\tau) - \hat{f}(\tau)\right\|_{H^{-s}(\Omega)} \left\|\eta(\tau) - \hat{\eta}(\tau)\right\|_{H^{-s}(\Omega)} \, d\tau
\end{aligned}
\tag{6.41}
$$

by the Cauchy-Schwarz inequality. Finally, recalling (6.23), we apply these estimates and a Gronwall-type inequality (see Lemma 6.8 below) to obtain the result (6.36).

Furthermore, applying the Cauchy-Schwarz inequality again, we obtain, applying (6.36) to (6.41),

$$
\begin{aligned}
\frac{2}{C_\gamma}\int_0^T \left\|\vartheta(t) - \hat{\vartheta}(t)\right\|_{L^2(\Omega)}^2 \, dt &\leq \|\eta_0 - \hat{\eta}_0\|_{H^{-s}(\Omega)}^2 + 2\|\eta_0 - \hat{\eta}_0\|_{H^{-s}(\Omega)} \int_0^T \left\|f(t) - \hat{f}(t)\right\|_{H^{-s}(\Omega)} \, dt \\
&\quad + 2\int_0^T \left\|f(t) - \hat{f}(t)\right\|_{H^{-s}(\Omega)} \left(\int_0^t \left\|f(\tau) - \hat{f}(\tau)\right\|_{H^{-s}(\Omega)} \, d\tau\right) dt \\
&\leq \|\eta_0 - \hat{\eta}_0\|_{H^{-s}(\Omega)}^2 + 2\|\eta_0 - \hat{\eta}_0\|_{H^{-s}(\Omega)} \left\|f - \hat{f}\right\|_{L^1(0,T;H^{-s}(\Omega))} \\
&\quad + 2\int_0^T \left\|f(t) - \hat{f}(t)\right\|_{H^{-s}(\Omega)} \left(\left\|f - \hat{f}\right\|_{L^1(0,T;H^{-s}(\Omega))}\right) dt \\
&\leq 2\|\eta_0 - \hat{\eta}_0\|_{H^{-s}(\Omega)}^2 + 3\left\|f - \hat{f}\right\|_{L^1(0,T;H^{-s}(\Omega))}^2
\end{aligned}
$$

which gives (6.37). ☐

Lemma 6.8. *Let $F \in L^1(0,T)$ and $y \in L^\infty(0,T)$ be non-negative functions and $C > 0$ a constant such that*

$$
y^2(t) \leq \int_0^t F(\tau) y(\tau) \, d\tau + C \qquad \text{for } t \in]0,T[.
$$

Then we have

$$
y(t) \leq \frac{1}{2}\int_0^t F(\tau) \, d\tau + \sqrt{C} \qquad \text{for } t \in [0,T].
$$

Proof. Let $x(t) = \int_0^t F(\tau) y(\tau) \, d\tau + C$. Then $x' = Fy \leq F\sqrt{x}$. Integrating in time of the relation $\frac{d}{dt}(\sqrt{x}) = \frac{x'}{2\sqrt{x}} \leq \frac{F}{2}$, we have the result. ☐

Remark 6.9. *In general, for $\gamma \not\equiv \hat{\gamma}$, $g \neq \hat{g}$ and an arbitrary time interval $0 \leq t_1 < t_2 \leq T$, with a similar argument we have the fractional version of the continuous dependence property corresponding to Lemma 3.2 of [86] for the classical case $s = 1$:*

$$
\begin{aligned}
&\left\|\eta(t_2) - \hat{\eta}(t_2)\right\|_{H^{-s}(\Omega)}^2 + \frac{2}{C_\gamma}\int_{t_1}^{t_2} \left\|\vartheta(\tau) - \gamma(\hat{\eta})(\tau)\right\|_{L^2(\Omega)}^2 \, d\tau + 2\int_{t_1}^{t_2} \langle \gamma(\hat{\eta})(\tau) - \hat{\vartheta}(\tau), \eta(\tau) - \hat{\eta}(\tau)\rangle \, d\tau \\
&\leq \left\|\eta(t_1) - \hat{\eta}(t_1)\right\|_{H^{-s}(\Omega)}^2 + 2\int_{t_1}^{t_2} \left(f(\tau) - \hat{f}(\tau), \eta(\tau) - \hat{\eta}(\tau)\right) d\tau + 2\int_{t_1}^{t_2}\int_\Omega (g(\tau) - \hat{g}(\tau))(\eta(\tau) - \hat{\eta}(\tau)) \, dx \, d\tau.
\end{aligned}
\tag{6.42}
$$

As a consequence, we immediately see that if $f = \hat{f}$, $g = \hat{g}$ and $\gamma = \hat{\gamma}$, then

$$
\left\|\eta(t_2) - \hat{\eta}(t_2)\right\|_{H^{-s}(\Omega)}^2 + \frac{2}{C_\gamma}\int_{t_1}^{t_2} \left\|\vartheta(\tau) - \hat{\vartheta}(\tau)\right\|_{L^2(\Omega)}^2 \, d\tau \leq \left\|\eta(t_1) - \hat{\eta}(t_1)\right\|_{H^{-s}(\Omega)}^2
$$

for any $0 \leq t_1 \leq t_2 \leq T$. Furthermore, in this case, the map $t \mapsto \left\|\eta(t) - \hat{\eta}(t)\right\|_{H^{-s}(\Omega)}$ is non-increasing in $t \in [0,T]$ for the same given data.

Also as a consequence of (6.42) with $\gamma = \hat{\gamma}$ and the estimates leading to (6.22) of Theorem 6.3, we have the following corollary:

Corollary 6.10. *The solution of the variational Stefan-type problem* (6.11) *on the interval* $[0,T]$ *depends continuously on* f, g *and* η_0 *in the following sense: if a sequence* $f_m \in L^2(0,T;H^{-s}(\Omega))$, $g_m \in BV(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^d))$ *and* $\eta_{0,m} \in L^2(\Omega)$, *is such that the* g_m*'s and the* $\eta_{0,m}$*'s are uniformly bounded in those spaces and* $f_m \to f$ *in* $L^2(0,T;H^{-s}(\Omega))$ *and* $g_m \to g$ *in* $L^2(0,T;L^2(\Omega))$ *and* $\eta_{0,m} \to \eta_0$ *in* $H^{-s}(\Omega)$, *then the solution* η_m *converges to* η *in* $L^2(0,T;H^{-s}(\Omega))$ *and* $\vartheta_m = \gamma(\eta_m)$ *converges to* $\vartheta = \gamma(\eta)$ *in* $L^2(0,T;L^2(\Omega))$.

6.3 Regularity of the Weak Temperature Solution ϑ

If we further assume that g has two time derivatives, by the Lipschitz continuity of γ, we can achieve higher regularity of the weak temperature solution $\vartheta = \gamma(\eta)$ in (6.13). The proof makes use of the Faedo-Galerkin method, and follows closely Chapter 6 of [84], and we include it here for completeness.

Let $(F_n)_{n\in\mathbb{N}}$ be an increasing set of finite dimensional subspaces of $H^s_0(\Omega)$, such that their union is dense in $H^s_0(\Omega)$, generated by the eigenvectors of the operator $\tilde{\mathcal{L}}^{-1}|_{L^2(\Omega)}$. This is possible since the inverse of $\tilde{\mathcal{L}}$ is compact in $L^2(\Omega)$, by the compactness of the injection $H^s_0(\Omega) \hookrightarrow L^2(\Omega)$. We denote $F_n^* = \tilde{\mathcal{L}}(F_n) \subset H^{-s}(\Omega)$ and set

$$\phi_{t,n} = \phi_t + I_{F_n^*} \text{ in } H^{-s}(\Omega),$$

where $I_{F_n^*}$ is the indicator function of F_n^*, i.e. $I_{F_n^*} = 0$ in F_n^*, $I_{F_n^*} = +\infty$ elsewhere.

We first recall a result of Attouch (Theorem 1.10 of [20]), which relates the Mosco convergence of the convex functionals and the convergence of the solutions of the Cauchy problem in the space $H = H^{-s}(\Omega)$.

Proposition 6.11. *Let* H *be a real Hilbert space with a scalar product and associated norm. Let* $\varphi_n \xrightarrow{M} \varphi$ *be a set of lower semi-continuous convex functions in* $L^2(0,T;H)$ *that converges in the Mosco sense in* H. *Denote by* η_n *the solutions of the evolution equations*

$$\frac{d\eta_n}{dt} + \partial\varphi_n(\eta_n) \ni f_n, \quad \eta_n(0) = \eta_{0,n} \tag{6.43}$$

where $f_n \in L^2(0,T;H)$, $\eta_{0,n} \in \overline{D(\varphi_n)}$. *Suppose that* $\eta_{0,n} \to \eta_0$ *in* H, $f_n \to f$ *in* $L^2(0,T;H)$. *Assume also that* $\frac{d\eta_n}{dt}$ *is bounded in* $L^2(0,T;H)$. *Then there exists a limit* $\eta \in H^1(0,T;H)$, *such that* $\eta_n \rightharpoonup \eta$ *weakly in* $H^1(0,T;H)$, *where* η *is the solution of*

$$\frac{d\eta}{dt} + \partial\varphi(\eta) \ni f, \quad \eta(0) = \eta_0. \tag{6.44}$$

With this proposition, our approach would be to determine the subdifferental of $\phi_{t,n}$ and show that they converge to ϕ_t in the sense of Mosco. We recall that $\varphi_n \xrightarrow{M} \varphi$ if for every $x \in D(\varphi)$, there exists an approximating sequence of elements $x_n \in D(\varphi_n)$, converging strongly to x, such that $\limsup_{n\to\infty} \varphi_n(x_n) \leq \varphi(x)$, and for any subsequence φ_{n_k} of φ_n such that $x_k \rightharpoonup x$ in H, we have $\liminf_{k\to\infty} \varphi_{n_k}(x_k) \geq \varphi(x)$. Then applying Proposition 6.11 to our Faedo-Galerkin approximation, and with the additional estimates we obtain from Proposition 6.2, we can pass to the limit to get the additional regularity to the solution for the limit problem.

For simplicity, we drop the parameter t and consider t to be fixed in $]0,T[$, and we denote $\phi_{t,n}$ as ϕ_n and $\phi_t = \phi$. Denote i to be the compact injection of $H^s_0(\Omega)$ into $L^2(\Omega)$ and take $E_n = i(F_n)$ by considering F_n as a subspace of $H^s_0(\Omega)$. It is clear that i^{-1} is an isomorphism between E_n and F_n, with norm depending on n.

Proposition 6.12. $\phi_n \xrightarrow{M} \phi$ *in* $H^{-s}(\Omega)$.

Proof. Denote i^* to be the injection map from $L^2(\Omega)$ to $H^{-s}(\Omega)$. Then $i^*(E_n) = i^* \circ i(F_n) = F_n^*$. Indeed, for an eigenvector u of $\tilde{\mathcal{L}}|_{L^2(\Omega)}$ corresponding to an eigenvalue μ, we have, by definition, $\tilde{\mathcal{L}}u = \mu i^* \circ i(u)$ in $H^{-s}(\Omega)$, hence the result.

For $i^*(U) \in D(\phi)$, we define $i^*(U_n)$, where $U_n = \mathbb{P}_{E_n} U$ is the projection of U into E_n in $L^2(\Omega)$. Since $\overline{\cup E_n}^{L^2} = L^2(\Omega)$ by construction, so $U_n \to U$ in $L^2(\Omega)$, and therefore $i^*(U_n) = \mathbb{P}_{F_n^*} i^*(U) \to i^*(U)$ in $H^{-s}(\Omega)$.

In addition, since γ satisfies the growth condition (6.16) at $\pm\infty$, its primitive j is quadratic at $\pm\infty$ (so that $j(r)/|r|^2$ and its inverse remain bounded as $r \to \pm\infty$). Therefore, by the dominated convergence theorem, the map $U \mapsto \int_\Omega j(U)$ is continuous in $L^2(\Omega)$, and since $i^*(U_n) \in F_n^*$, so $\phi_n(i^*(U_n)) = \phi(i^*(U_n)) \to \phi(i^*(U))$.

On the other hand, the sequence ϕ_n is decreasing (since F_n is increasing), so we conclude the Mosco convergence of ϕ_n to ϕ given that ϕ is known to be lower semi-continuous. □

Next, we want to obtain a solution of the approximate Cauchy problem for η_n, making use of Proposition 6.2 as in the proof of Theorem 6.3.

Proposition 6.13. *Setting* $V = \tilde{\mathcal{L}} v$,

$V \in \partial\phi_n(U)$ *in* $H^{-s}(\Omega)$ *if and only if* $U \in D(\phi) \cap F_n^*, \gamma(U) - g \in L^2(\Omega)$ *and* $i(v) + g - \gamma(U) \perp E_n$ *in* $L^2(\Omega)$.

Proof. Denote the inf-convolution of two convex functions by the composition operator ∇. Then by definition, we know that the convex conjugate $\phi_n^* = (\phi^* \nabla I_{F_n^*}^*)^{**}$, where the double asterisk ** stands for the regularised l.s.c. function of $\psi_n = \phi^* \nabla I_{F_n^*}^*$.

Since F_n^* is a subspace of $H^{-s}(\Omega)$, we have $I_{F_n^*}^* = I_{(F_n^*)^\perp}$, where the orthogonality is inherited from the duality between $H_0^s(\Omega)$ and $H^{-s}(\Omega)$. Since $\tilde{\mathcal{L}}(F_n) = F_n^*$, $(F_n^*)^\perp$ is also the orthogonal of F_n in $H_0^s(\Omega)$. We therefore have

$$\psi_n(w) = \phi^* \nabla I_{F_n^*}^*(w) = \phi^* \nabla I_{(F_n^*)^\perp}(w) = \inf_{\mathbb{P}_{F_n}(z-w)=0} \int_\Omega j^*(g+z).$$

Since γ is globally Lipschitz, β satisfies the growth assumption (6.16) at infinity, so the function j^* is quadratic at infinity and therefore $z \mapsto \int_\Omega j^*(z)$ is continuous in $L^2(\Omega)$. Furthermore, the function $z \mapsto \int_\Omega j^*(z)$ is coercive in $L^2(\Omega)$.

Henceforth, we deduce that there exists $z = z(v)$ in $L^2(\Omega)$, not necessarily unique, such that $\psi_n(v) = \int_\Omega j^*(g + z(v))$ with $z(v) - i(v)$ in $L^2(\Omega)$, such that $z(v) - i(v) \perp E_n$ in $L^2(\Omega)$. Indeed, $z - v \perp F_n$ in $H_0^s(\Omega)$ so $\langle \tilde{\mathcal{L}}\xi, z - v\rangle = 0$ for all ξ in the basis of F_n. Hence, taking a vector ξ in that basis, we have $\tilde{\mathcal{L}}\xi = i^*(\tilde{\mathcal{L}}\xi) = \mu i^* \circ i(\xi)$, so $0 = \int_\Omega i(\xi) i(z - v)$ which means that $i(z) - i(v)$ is orthogonal to E_n in $L^2(\Omega)$. Since $z(v)$ is the weak limit in $L^2(\Omega)$, considering a minimising sequence of such $i(z)$, we have the result.

Futhermore, using the coercivity of the integral of j^* in $L^2(\Omega)$ again, we see that ψ_n is lower semi-continuous in $H_0^s(\Omega)$, so $\psi_n = \phi_n^*$.

Therefore, $V \in \partial\phi_n(U)$ if and only if $i^*(U) \in F_n^*$, and there exists $z(v) \in L^2(\Omega)$ with $z(v) - i(v) \in E_n^\perp$ and, as in (6.32),

$$\int_\Omega j(U) + j^*(g+z) = \langle U, g+z\rangle.$$

But since $U \in D(\phi) \cap F_n^* \subset L^2(\Omega)$, we can rewrite this as

$$\int_\Omega j(U) + j^*(g+z) - U(g+z) = 0,$$

so, as in the proof of Theorem 6.3, we have that the points U and $g+z$ are conjugated by j, thus $z(v) + g = \partial j(U) = \gamma(U)$ a.e. in Ω. The reverse is also clearly true. □

Now, setting $f_n = \mathbb{P}_{E_n} f$ for $f \in L^2(0,T; L^2(\Omega))$ and for $\eta_0 \in L^2(\Omega)$, we apply the Proposition 6.2 for $(\phi_{t,n})_{t \in [0,T]}$ to solve

$$\frac{d\eta_n}{dt} + \partial\phi_{t,n}(\eta_n) \ni i^*(f_n), \quad \eta_n(0) = \eta_{0,n}, \tag{6.45}$$

where $\eta_{0,n}$ is constructed as in the proof of Proposition 6.12 such that $\eta_{0,n} \in D(\phi_{0,n})$ with $\eta_{0,n} \to \eta_0 \in D(\phi_0)$ strongly in $H^{-s}(\Omega)$ and $\phi_{0,n}(\eta_{0,n}) \to \phi_0(\eta_0)$. Then by (6.20), $\frac{d\eta_n}{dt}$ is bounded in $L^2(0,T; H^{-s}(\Omega))$. Moreover, as in Proposition 6.12, for all $U \in L^2(0,T; H^{-s}(\Omega))$, we have

$$\varphi_n(U) := \int_0^T \phi_{t,n}(U(t))\,dt \xrightarrow{M} \varphi(U) := \int_0^T \phi_t(U(t))\,dt$$

in the sense of Mosco.

Therefore, applying Proposition 6.11, we conclude that η_n converges weakly in $H^1(0,T;H^{-s}(\Omega))$ to the solution η of

$$\frac{d\eta}{dt} + \partial\phi_t(\eta) \ni i^*(f), \quad \eta(0) = \eta_0.$$

Having obtained the approximation $\eta_n \rightharpoonup \eta$ for the enthalpy η, we want to pass to the limit in the temperatures $\vartheta_n = \gamma(\eta_n) \to \vartheta = \gamma(\eta)$. To do so, we require some estimates on the derivative of the temperatures.

Proposition 6.14. *Suppose $f \in L^2(0,T;L^2(\Omega))$ and $\tilde{g} \in W^{2,1}(0,T;L^2(\mathbb{R}^d)) \cap L^\infty(0,T;H^s(\mathbb{R}^d))$. Assume $\eta_0 \in L^2(\Omega)$ and, setting $\vartheta(0) = \gamma(\eta_0)$, assume $\vartheta(0) - g(0) \in H^s_0(\Omega)$. Denote by $\eta_n \in H^1(0,T;F_n^*)$, and $\tilde{\eta}_n \in H^1(0,T;E_n)$ such that $\eta_n = i^*(\tilde{\eta}_n)$, the generalised solution associated to the approximate Cauchy problem* (6.45)*, corresponding to the Faedo-Galerkin method as described above. Then, the integral*

$$\int_0^T \int_\Omega \left| \frac{\partial \gamma(\tilde{\eta}_n)}{\partial t} \right|^2 \leq C_6, \quad \left\| \mathbb{P}_{F_n}(\gamma(\tilde{\eta}_n) - g) \right\|_{L^\infty(0,T;H^s_0(\Omega))} \leq C_7 \tag{6.46}$$

is uniformly bounded in n, with the bounds C_6, C_7 dependent on the Lipschitz constant C_γ and the given data f, g, η_0.

Proof. Since $\eta_n \in H^1(0,T;F_n^*)$, there exists $\tilde{\eta}_n \in H^1(0,T;E_n)$ such that $\eta_n = i^*(\tilde{\eta}_n)$, $v_n = \gamma(\tilde{\eta}_n) - g$, and, by Proposition 6.13 applied to $\tilde{\eta}_n$, satisfies

$$\frac{\partial \eta_n}{\partial t} + \tilde{\mathcal{L}} v_n = i^*(f_n), v_n \in L^2(0,T;H^s_0(\Omega)) \text{ with } i(v_n) + g - \gamma(\tilde{\eta}_n) \perp E_n \text{ in } L^2(\Omega). \tag{6.47}$$

Since γ is Lipschitz, we have $\gamma(\tilde{\eta}_n) \in H^1(0,T;L^2(\Omega))$ and $\gamma(\tilde{\eta}_n) - g \in H^1(0,T;L^2(\Omega))$. Let $h_n = \mathbb{P}_{F_n} v_n$ and $\tilde{h}_n = \mathbb{P}_{E_n}(\gamma(\tilde{\eta}_n) - g)$. Then

$$h_n \in H^1(0,T;F_n) \text{ and } \tilde{h}_n \in H^1(0,T;E_n). \tag{6.48}$$

Indeed, we have $\gamma(\tilde{\eta}_n) - g \in H^1(0,T;L^2(\Omega))$, so $\tilde{h}_n = \mathbb{P}_{E_n}(\gamma(\tilde{\eta}_n) - g) \in H^1(0,T;E_n)$. Since $\tilde{h}_n = \mathbb{P}_{E_n} i(v_n)$, so by the choice of F_n, we have $\mathbb{P}_{E_n} \circ i = i \circ \mathbb{P}_{F_n}$, and we deduce that $i(h_n) = \tilde{h}_n$. Therefore, since i gives an isomorphism between F_n and E_n, we obtain the properties in (6.48).

Making use of these properties, we can therefore multiply (6.47) by $\frac{\partial h_n}{\partial t} \in L^2(0,T;F_n)$ to obtain

$$\int_\Omega \frac{\partial \tilde{\eta}_n}{\partial t} \left[i\left(\frac{\partial h_n}{\partial t} \right) \right] + \left[v_n, \frac{\partial h_n}{\partial t} \right]_A = \int_\Omega f_n \left[i\left(\frac{\partial h_n}{\partial t} \right) \right] \tag{6.49}$$

by (6.15). Now,

$$\left[\frac{\partial h_n}{\partial t}, v_n \right]_A = \left[\frac{\partial h_n}{\partial t}, \mathbb{P}_{F_n} v_n \right]_A = \left[\frac{\partial h_n}{\partial t}, h_n \right]_A = \int_{\mathbb{R}^d} AD^s \frac{\partial h_n}{\partial t} \cdot D^s h_n = \frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}^d} AD^s h_n \cdot D^s h_n,$$

and from $f_n = \mathbb{P}_{E_n} f$, we obtain

$$\int_\Omega \frac{\partial \tilde{\eta}_n}{\partial t} \frac{\partial \tilde{h}_n}{\partial t} + \frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}^d} AD^s h_n \cdot D^s h_n = \int_\Omega f_n \frac{\partial \tilde{h}_n}{\partial t}. \tag{6.50}$$

Now, recalling the definition of $\tilde{h}_n$, we observe that

$$\int_\Omega \left(\frac{\partial \tilde{\eta}_n}{\partial t} - f_n \right) \frac{\partial \tilde{h}_n}{\partial t} = \int_\Omega \left(\frac{\partial \tilde{\eta}_n}{\partial t} - f_n \right) \frac{\partial}{\partial t} \mathbb{P}_{E_n}(\gamma(\tilde{\eta}_n) - g) = \int_\Omega \left(\frac{\partial \tilde{\eta}_n}{\partial t} - f_n \right) \mathbb{P}_{E_n} \frac{\partial}{\partial t}(\gamma(\tilde{\eta}_n) - g),$$

so since $\frac{\partial \tilde{\eta}_n}{\partial t} - f_n \perp L^2(\Omega) \backslash E_n$, we have

$$\int_\Omega \left(\frac{\partial \tilde{\eta}_n}{\partial t} - f_n \right) \frac{\partial}{\partial t}(\gamma(\tilde{\eta}_n) - g) + \frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}^d} AD^s h_n \cdot D^s h_n = 0. \tag{6.51}$$

Integrating this over $[0,t]$ for $t \le T$, we obtain, by the coercivity of A in (6.3) and integrating by parts in time,

$$
\begin{aligned}
&\int_0^t \int_\Omega \frac{\partial \tilde{\eta}_n}{\partial t} \frac{\partial \gamma(\tilde{\eta}_n)}{\partial t} + \frac{1}{2} a_* \big\| h_n(t) \big\|^2_{H^s_0(\Omega)} \\
\le\, & \frac{1}{2} a^* \big\| h_n(0) \big\|^2_{H^s_0(\Omega)} + \int_0^t \int_\Omega f_n \frac{\partial \gamma(\tilde{\eta}_n)}{\partial t} + \int_0^t \int_\Omega \frac{\partial \tilde{\eta}_n}{\partial t} \frac{\partial g}{\partial t} - \int_0^t \int_\Omega f_n \frac{\partial g}{\partial t} \\
=\, & \frac{1}{2} a^* \big\| h_n(0) \big\|^2_{H^s_0(\Omega)} + \int_0^t \int_\Omega f_n \frac{\partial \gamma(\tilde{\eta}_n)}{\partial t} - \int_0^t \int_\Omega \tilde{\eta}_n \frac{\partial^2 g}{\partial t^2} \\
&+ \int_\Omega \tilde{\eta}_n(t) \frac{\partial g}{\partial t}(t) - \int_\Omega \tilde{\eta}_n(0) \frac{\partial g}{\partial t}(0) - \int_0^t \int_\Omega f_n \frac{\partial g}{\partial t}.
\end{aligned}
\tag{6.52}
$$

Now, we know by (6.21) that $\phi_{t,n}(\eta_n(t))$ is bounded independent of n and t, so $\|\tilde{\eta}_n\|_{L^\infty(0,T;L^2(\Omega))}$ is bounded independent of n (see also (6.23)). Then, by the Cea-type lemma (see, for instance, Proposition 2.5 of [15]) given by

$$
\|\mathbb{P}_{F_n} w\|^2_{H^s_0(\Omega)} \le \frac{a^*}{a_*} \|w\|^2_{H^s_0(\Omega)} \quad \forall w \in H^s_0(\Omega),
$$

we have, by the compatibility of the initial condition giving $h_n(0) = \mathbb{P}_{F_n}(\gamma(\tilde{\eta}_n(0)) - g(0)) = \mathbb{P}_{F_n}(\vartheta(0) - g(0))$,

$$
\begin{aligned}
&\int_0^t \int_\Omega \frac{\partial \tilde{\eta}_n}{\partial t} \frac{\partial \gamma(\tilde{\eta}_n)}{\partial t} + \frac{1}{2} a_* \big\| h_n(t) \big\|^2_{H^s_0(\Omega)} \\
&\qquad \le \frac{1}{2} \frac{a^{*2}}{a_*} \big\| \vartheta(0) - g(0) \big\|^2_{H^s_0(\Omega)} + \int_0^t \int_\Omega f_n \frac{\partial \gamma(\tilde{\eta}_n)}{\partial t} - \int_0^t \int_\Omega \tilde{\eta}_n \frac{\partial^2 g}{\partial t^2} \\
&\qquad\qquad + \int_\Omega \tilde{\eta}_n(t) \frac{\partial g}{\partial t}(t) - \int_\Omega \tilde{\eta}_n(0) \frac{\partial g}{\partial t}(0) - \int_0^t \int_\Omega f_n \frac{\partial g}{\partial t}.
\end{aligned}
\tag{6.53}
$$

Now, letting C_γ be the Lipschitz constant of γ, we have

$$
\frac{\partial \tilde{\eta}_n}{\partial t} \frac{\partial \gamma(\tilde{\eta}_n)}{\partial t} \ge \frac{1}{C_\gamma} \left| \frac{\partial \gamma(\tilde{\eta}_n)}{\partial t} \right|^2 \quad \text{a.e. } Q_T.
\tag{6.54}
$$

Also, observe the boundedness of $\tilde{\eta}_n$ in $L^\infty(0,T;L^2(\Omega))$, since η_n is obtained as a solution to the Faedo-Galerkin finite dimensional approximated problem (6.45) and therefore also satisfies (6.25). Therefore, applying the Cauchy-Schwarz inequality to the term $\int_0^t \int_\Omega f_n \frac{\partial \gamma(\tilde{\eta}_n)}{\partial t}$ and making use of the assumption $\vartheta(0) - g(0) \in H^s_0(\Omega)$ gives the first uniform bound $\int_0^T \int_\Omega \left| \frac{\partial \gamma(\tilde{\eta}_n)}{\partial t} \right|^2 \le C_6$.

Using again (6.53), we can easily take the supremum over all time to obtain the second uniform bound $\big\| \mathbb{P}_{F_n}(\gamma(\tilde{\eta}_n) - g) \big\|_{L^\infty(0,T;H^s_0(\Omega))} = \|h_n\|_{L^\infty(0,T;H^s_0(\Omega))} \le C_7$. ☐

Remark 6.15. *For fixed $\sigma > 0$ and s such that $\sigma \le s \le 1$, similarly to Remark 6.6, we observe that $\tilde{\eta}_n \in L^\infty(0,T;L^2(\Omega))$, and $\tilde{\eta}_n$ can be bounded for each s by a constant depending on σ but independent of s, by the continuity of the eigenfunctions (in Section 3.3), and depending explicity on T and γ. Similarly, by Section 3.3, the η_n's are bounded independent of $s \ge \sigma$ in $H^1(0,T;F_n^*)$. This allows us to consider the convergence of the variational problem as s varies.*

In addition, when we have a sequence of Lipschitz functions γ_n, we can also obtain (6.54) *by considering a Lipschitz constant C_γ given by the supremum of all the Lipschitz constants C_{γ_n}.*

Now, we can finally proceed to show the existence of more regular solutions to the variational problem (6.13). Indeed, we have the following result:

Theorem 6.16. *Let $f \in L^2(0,T;L^2(\Omega))$ and $\tilde{g} \in W^{2,1}(0,T;L^2(\mathbb{R}^d)) \cap L^\infty(0,T;H^s(\mathbb{R}^d))$, and define g as in* (6.7) *with the same regularity (see Section 3.2). Assume $\eta_0 \in L^2(\Omega)$, and setting $\vartheta(0) = \gamma(\eta_0)$ assume $\vartheta(0) - g(0) \in H^s_0(\Omega)$. Then there exists a unique weak temperature solution ϑ to the variational problem* (6.11)–(6.13)*, such that*

$$
\vartheta \in L^\infty(0,T;H^s(\mathbb{R}^d)) \cap H^1(0,T;L^2(\Omega)).
\tag{6.55}
$$

Proof. From Proposition 6.14, h_n is bounded in $L^\infty(0,T;H_0^s(\Omega))$. Furthermore, if we recall the definition of $\tilde{h}_n$ as the projection onto E_n, we have

$$\left\|\frac{\partial \tilde{h}_n}{\partial t}\right\|_{L^2(\Omega)} \leq \left\|\frac{\partial}{\partial t}(\gamma(\tilde{\eta}_n)-g)\right\|_{L^2(\Omega)},$$

so $\tilde{h}_n$ is bounded in $H^1(0,T;L^2(\Omega))$.

By Proposition 6.11, we know that $\eta_n \rightharpoonup \eta$ in $H^1(0,T;H^{-s}(\Omega))$ and $\tilde{\eta}_n \rightharpoonup \eta$ weakly* in $L^\infty(0,T;L^2(\Omega))$, and

$$\tilde{\mathcal{L}}(v_n) = \partial\phi_{t,n}(\eta_n) = i^*(f_n) - \frac{\partial \eta_n}{\partial t} \rightharpoonup i^*(f) - \frac{\partial \eta}{\partial t} = \partial\phi_t(\eta) = \tilde{\mathcal{L}}v \text{ in } L^2(0,T;H^{-s}(\Omega)).$$

Therefore, on applying $\tilde{\mathcal{L}}^{-1}$, v_n tends to $v=\gamma(\eta)-g$ weakly in $L^2(0,T;H_0^s(\Omega))$.

Since $v_n = h_n + k_n$ for some $k_n \in F_n^\perp$, we deduce that $k_n \rightharpoonup 0$ in $L^2(0,T;H_0^s(\Omega))$ and $h_n \rightharpoonup \gamma(\eta)-g$ also in this space, so $\tilde{h}_n \rightharpoonup i(\gamma(\eta)-g)$ in $L^2(0,T;L^2(\Omega))$. Therefore, by (6.46), $\gamma(\eta)-g$ lies in $L^\infty(0,T;H_0^s(\Omega))$ and $i(\gamma(\eta)-g)\in H^1(0,T;L^2(\Omega))$. Finally as $\vartheta=\gamma(\eta)$, we have the desired regularity (6.55). □

Remark 6.17. *It can be seen that the bounds in* (6.46) *can be made to depend only on $\sigma>0$ and independent of s for $\sigma\leq s\leq 1$, by the continuity of the eigenfunctions as shown in Section 3.3. Then, as in Remark 6.15, the bounds $\left\|D^s(\vartheta-g)\right\|_{L^\infty(0,T;L^2(\mathbb{R}^d)^d)}$ and $\left\|\frac{\partial\vartheta}{\partial t}\right\|_{L^2(0,T;L^2(\Omega))}$ in* (6.55) *only depend only on σ and independent of s, allowing us to consider the convergence of the variational problem as s varies.*

6.4 The Variational Inequality Formulations

We observe that the formulation given in (6.13) can be formally transformed into a variational inequality formulation with fractional derivatives (see for example [192] or Chapter VII of [84]). Indeed, consider an element $w\in H_0^s(\Omega)$ independent of t and taking in (6.11) the test function $\xi(\tau,x)=w(x)$ for $\tau\in]t-\epsilon,t+\epsilon[$ and $\xi(\tau,x)=0$, dividing by 2ϵ and letting $\epsilon\to 0$, denoting now by $\langle\cdot,\cdot\rangle$ the duality between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$, we obtain

$$\left\langle \frac{d\eta}{dt}(t),w\right\rangle + \langle \tilde{\mathcal{L}}_A^s(\gamma(\eta(t))-g(t)),w\rangle = \langle f(t),w\rangle \quad \text{for a.e. } t \text{ for all } w\in H_0^s(\Omega).$$

Then, integrating with respect to time and using the regularity of η and its initial condition, we have,

$$\int_\Omega \eta(t)w + \int_0^t\int_{\mathbb{R}^d} AD^s(\vartheta)\cdot D^s w = \int_0^t\int_\Omega fw + \int_\Omega \eta_0 w \tag{6.56}$$

for almost all $t\in[0,T]$ and $w\in H_0^s(\Omega)$ by recalling that $\int_0^t\int_{\mathbb{R}^d} AD^s g\cdot D^s w = 0$ for all w. We write $\eta(t)=b(\vartheta(t))+\lambda\chi(t)$ for a.e. t for $\lambda>0$ and b a given continuous and increasing function (see Figure 2). Then, denoting

$$\Theta(t)=\int_0^t \vartheta(\tau)\,d\tau \quad \text{and} \quad \mathfrak{F}(t)=\int_0^t f(\tau)\,d\tau,$$

we observe that $b(\vartheta(t)) = b\left(\frac{\partial\Theta}{\partial t}(t)\right)\in L^2(\Omega)$ a.e. t. On the other hand, since $H(r)$ is the subdifferential of the convex function r^+, we have the inequality

$$s\chi \leq (r+s)^+ - r^+. \tag{6.57}$$

So, we obtain from (6.56) the nonlocal variational inequality

$$\int_\Omega b\left(\frac{\partial\Theta}{\partial t}(t)\right)w + \int_{\mathbb{R}^d} AD^s\Theta(t)\cdot D^s w + \int_\Omega \lambda\left(\frac{\partial\Theta}{\partial t}(t)+w\right)^+ \geq \int_\Omega \lambda\left(\frac{\partial\Theta}{\partial t}(t)\right)^+ + \int_\Omega (\mathfrak{F}(t)+\eta_0)w \tag{6.58}$$

for all $w\in H_0^s(\Omega)$ for a.e. t.

By Theorem 6.3, $\vartheta - g \in L^2(0,T;H^s_0(\Omega))$, so Θ satisfies

$$\Theta \in H^1(0,T;H^s(\mathbb{R}^d)), \quad \Theta(0)=0, \quad \text{and } \Theta(t) - \int_0^t g(\tau)\,d\tau = 0 \text{ in } \Omega^c \text{ for a.e. } t, \tag{6.59}$$

and defining

$$\mathbb{K}(t) := H^s_0(\Omega) + g(t) \text{ for a.e. } t \in]0,T[,$$

from (6.58) with $w = \tilde{w}(t) - \frac{\partial\Theta}{\partial t}(t)$, where $\tilde{w}(t) \in \mathbb{K}(t)$, we obtain, for almost every t,

$$\int_\Omega b\left(\frac{\partial\Theta}{\partial t}\right)\left(\tilde{w} - \frac{\partial\Theta}{\partial t}\right) + \int_{\mathbb{R}^d} AD^s\Theta \cdot D^s\left(\tilde{w} - \frac{\partial\Theta}{\partial t}\right) + \int_\Omega \lambda\tilde{w}^+ - \int_\Omega \lambda\left(\frac{\partial\Theta}{\partial t}\right)^+ \geq \int_\Omega (\mathfrak{F}(t)+\eta_0)\left(\tilde{w} - \frac{\partial\Theta}{\partial t}\right), \quad \forall \tilde{w}(t) \in \mathbb{K}(t), \tag{6.60}$$

which corresponds to the variational inequality formulations of Duvaut and Frémond (see [84, 227, 228, 192]). With the same assumptions on f, $\tilde{g}$ and η_0, we can obtain a solution Θ to (6.60), (6.59) using the Faedo-Galerkin method (refer to [228] or Chapter 3 of [192] for a proof starting from the variational inequality formulation (6.60), using the special basis of Appendix 3.3. A similar result can also be obtained using the Rothe method (refer to Section 3.1 of [239]).

Making use of the notation given in (6.60), we can derive a continuous dependence relation of Θ on f, g and η_0.

Proposition 6.18. *Let Θ and $\hat{\Theta}$ denote the weak temperature solutions of two fractional Stefan-type problems (6.60) corresponding to (γ, f, g, η_0) and $(\gamma, \hat{f}, \hat{g}, \hat{\eta}_0)$ respectively, where $f, \hat{f}, g, \hat{g}$, and $\eta_0, \hat{\eta}_0$ satisfy the assumptions of Theorem 6.16. Then,*

$$\frac{b_*}{4}\int_0^t\int_\Omega \left|\frac{\partial\Theta}{\partial t} - \frac{\partial\hat{\Theta}}{\partial t}\right|^2 + \frac{a_*}{2}\int_{\mathbb{R}^d} |D^s(\Theta(t)-\hat{\Theta}(t))|^2$$
$$\leq \left(\frac{b^*}{2b_*} + \frac{b_*}{4} + 2b\right)\int_0^t\int_\Omega \left|\frac{\partial\mathfrak{G}}{\partial t} - \frac{\partial\hat{\mathfrak{G}}}{\partial t}\right|^2 + \frac{2}{b_*}\int_0^t\int_\Omega |\mathfrak{F}-\hat{\mathfrak{F}}|^2 + \frac{a^*}{2}\int_{\mathbb{R}^d} |D^s(\mathfrak{G}(t)-\hat{\mathfrak{G}}(t))|^2 + \frac{2t}{b_*}\int_\Omega |\eta_0 - \hat{\eta}_0|^2. \tag{6.61}$$

Proof. Suppose Θ and $\hat{\Theta}$ are two solutions of (6.60) corresponding to $(\mathfrak{F}, \mathfrak{G}, \eta_0)$ and $(\hat{\mathfrak{F}}, \hat{\mathfrak{G}}, \hat{\eta}_0)$ respectively, i.e.

$$\int_\Omega b\left(\frac{\partial\Theta}{\partial t}\right)\left(\tilde{w} - \frac{\partial\Theta}{\partial t}\right) + \int_{\mathbb{R}^d} AD^s(\Theta - \mathfrak{G}) \cdot D^s\left(\tilde{w} - \frac{\partial\Theta}{\partial t}\right) + \int_\Omega b\tilde{w}^+ - \int_\Omega b\left(\frac{\partial\Theta}{\partial t}\right)^+ \geq \int_\Omega (\mathfrak{F}+\eta_0)\left(\tilde{w} - \frac{\partial\Theta}{\partial t}\right),$$

and

$$\int_\Omega b\left(\frac{\partial\hat{\Theta}}{\partial t}\right)\left(\tilde{w} - \frac{\partial\hat{\Theta}}{\partial t}\right) + \int_{\mathbb{R}^d} AD^s(\hat{\Theta} - \hat{\mathfrak{G}}) \cdot D^s\left(\tilde{w} - \frac{\partial\hat{\Theta}}{\partial t}\right) + \int_\Omega b\tilde{w}^+ - \int_\Omega b\left(\frac{\partial\hat{\Theta}}{\partial t}\right)^+ \geq \int_\Omega (\hat{\mathfrak{F}}+\hat{\eta}_0)\left(\tilde{w} - \frac{\partial\hat{\Theta}}{\partial t}\right),$$

Taking the test function $\tilde{w}$ to be $\frac{\partial\hat{\Theta}}{\partial t} + \bar{g} = \frac{\partial\hat{\Theta}}{\partial t} + g(t) - \hat{g}(t) = \frac{\partial\hat{\Theta}}{\partial t} + \frac{\partial\mathfrak{G}}{\partial t} - \frac{\partial\hat{\mathfrak{G}}}{\partial t}$ in the first equation, and $\frac{\partial\Theta}{\partial t} - \bar{g}$ in the second, we compare the two equations to obtain

$$\int_\Omega b\left(\frac{\partial\Theta}{\partial t}\right)\left(\frac{\partial\hat{\Theta}}{\partial t} + \bar{g} - \frac{\partial\Theta}{\partial t}\right) + \int_{\mathbb{R}^d} AD^s(\Theta - \mathfrak{G}) \cdot D^s\left(\frac{\partial\hat{\Theta}}{\partial t} + \bar{g} - \frac{\partial\Theta}{\partial t}\right)$$

$$
\begin{aligned}
&\quad + \int_\Omega b\left(\frac{\partial\hat\Theta}{\partial t}+\bar g\right)^+ - \int_\Omega b\left(\frac{\partial\Theta}{\partial t}\right)^+ - \int_\Omega (\mathfrak{F}+\eta_0)\left(\frac{\partial\hat\Theta}{\partial t}+\bar g-\frac{\partial\Theta}{\partial t}\right)\\
&\geq -\int_\Omega b\left(\frac{\partial\hat\Theta}{\partial t}\right)\left(\frac{\partial\Theta}{\partial t}-\bar g-\frac{\partial\hat\Theta}{\partial t}\right) - \int_{\mathbb{R}^d} AD^s(\hat\Theta-\hat{\mathfrak{G}})\cdot D^s\left(\frac{\partial\Theta}{\partial t}-\bar g-\frac{\partial\hat\Theta}{\partial t}\right)\\
&\quad - \int_\Omega b\left(\frac{\partial\Theta}{\partial t}-\bar g\right)^+ + \int_\Omega b\left(\frac{\partial\hat\Theta}{\partial t}\right)^+ + \int_\Omega (\hat{\mathfrak{F}}+\hat\eta_0)\left(\frac{\partial\Theta}{\partial t}-\bar g-\frac{\partial\hat\Theta}{\partial t}\right)\\
&= \int_\Omega b\left(\frac{\partial\hat\Theta}{\partial t}\right)\left(\frac{\partial\hat\Theta}{\partial t}+\bar g-\frac{\partial\Theta}{\partial t}\right) + \int_{\mathbb{R}^d} AD^s(\hat\Theta-\hat{\mathfrak{G}})\cdot D^s\left(\frac{\partial\hat\Theta}{\partial t}+\bar g-\frac{\partial\Theta}{\partial t}\right)\\
&\quad - \int_\Omega b\left(\frac{\partial\Theta}{\partial t}-\bar g\right)^+ + \int_\Omega b\left(\frac{\partial\hat\Theta}{\partial t}\right)^+ + \int_\Omega (\hat{\mathfrak{F}}+\hat\eta_0)\left(\frac{\partial\Theta}{\partial t}-\bar g-\frac{\partial\hat\Theta}{\partial t}\right).
\end{aligned}
$$

Observe that for any r, s,

$$
r^+ - s^+ = \frac{|r|+r}{2} - \frac{|s|-s}{2} = \frac{1}{2}[(|r|-|s|)+(r-s)] \leq \frac{1}{2}[|r-s|+|r-s|] = |r-s|
$$

by the triangle inequality, therefore

$$
\begin{aligned}
&\int_\Omega \left(\frac{\partial\hat\Theta}{\partial t}+\bar g\right)^+ - \int_\Omega \left(\frac{\partial\Theta}{\partial t}\right)^+ + \int_\Omega \left(\frac{\partial\Theta}{\partial t}-\bar g\right)^+ - \int_\Omega \left(\frac{\partial\hat\Theta}{\partial t}\right)^+\\
&= \left[\int_\Omega \left(\frac{\partial\hat\Theta}{\partial t}+\bar g\right)^+ - \int_\Omega \left(\frac{\partial\hat\Theta}{\partial t}\right)^+\right] + \left[\int_\Omega \left(\frac{\partial\Theta}{\partial t}-\bar g\right)^+ - \int_\Omega \left(\frac{\partial\Theta}{\partial t}\right)^+\right]\\
&\leq \int_\Omega |\bar g| + \int_\Omega |-\bar g| = 2\int_\Omega |\bar g|.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\int_{\mathbb{R}^d} AD^s(\Theta-\mathfrak{G})\cdot D^s\left(\frac{\partial\hat\Theta}{\partial t}+\bar g-\frac{\partial\Theta}{\partial t}\right) - \int_{\mathbb{R}^d} AD^s(\hat\Theta-\hat{\mathfrak{G}})\cdot D^s\left(\frac{\partial\hat\Theta}{\partial t}+\bar g-\frac{\partial\Theta}{\partial t}\right)\\
&= \int_{\mathbb{R}^d} AD^s(\Theta-\mathfrak{G}-\hat\Theta+\hat{\mathfrak{G}})\cdot D^s\left[\frac{\partial}{\partial t}\left(\hat\Theta+\mathfrak{G}-\hat{\mathfrak{G}}-\Theta\right)\right]\\
&\leq -a_*\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^d} |D^s(\Theta-\mathfrak{G}-\hat\Theta+\hat{\mathfrak{G}})|^2.
\end{aligned}
$$

Finally, suppose that b is bilipschitz, i.e. there exists $b^* \geq b_* > 0$ such that

$$
b_*|p-q|^2 \leq [b(p)-b(q)](p-q) \leq b^*|p-q|^2.
$$

Then

$$
\begin{aligned}
&\int_\Omega b\left(\frac{\partial\Theta}{\partial t}\right)\left(\frac{\partial\hat\Theta}{\partial t}+\bar g-\frac{\partial\Theta}{\partial t}\right) - \int_\Omega b\left(\frac{\partial\hat\Theta}{\partial t}\right)\left(\frac{\partial\hat\Theta}{\partial t}+\bar g-\frac{\partial\Theta}{\partial t}\right)\\
&= \int_\Omega \left[b\left(\frac{\partial\Theta}{\partial t}\right)-b\left(\frac{\partial\hat\Theta}{\partial t}\right)\right]\left(\frac{\partial\hat\Theta}{\partial t}-\frac{\partial\Theta}{\partial t}\right) + \int_\Omega \left[b\left(\frac{\partial\Theta}{\partial t}\right)-b\left(\frac{\partial\hat\Theta}{\partial t}\right)\right]\bar g\\
&\leq -\int_\Omega b_*\left|\frac{\partial\hat\Theta}{\partial t}-\frac{\partial\Theta}{\partial t}\right|^2 + \int_\Omega \left[b\left(\frac{\partial\Theta}{\partial t}\right)-b\left(\frac{\partial\hat\Theta}{\partial t}\right)\right]\bar g
\end{aligned}
$$

$$\leq -\int_\Omega b_* \left|\frac{\partial\hat\Theta}{\partial t} - \frac{\partial\Theta}{\partial t}\right|^2 + \frac{b_*}{2b^*}\int_\Omega \left|b\left(\frac{\partial\Theta}{\partial t}\right) - b\left(\frac{\partial\hat\Theta}{\partial t}\right)\right|^2 + \frac{b^*}{2b_*}\int_\Omega |\bar g|^2$$

$$\leq -\int_\Omega b_* \left|\frac{\partial\hat\Theta}{\partial t} - \frac{\partial\Theta}{\partial t}\right|^2 + \frac{b_*}{2b^*}\int_\Omega b^* \left|\frac{\partial\Theta}{\partial t} - \frac{\partial\hat\Theta}{\partial t}\right|^2 + \frac{b^*}{2b_*}\int_\Omega |\bar g|^2$$

$$= -\frac{b_*}{2}\int_\Omega \left|\frac{\partial\Theta}{\partial t} - \frac{\partial\hat\Theta}{\partial t}\right|^2 + \frac{b^*}{2b_*}\int_\Omega |\bar g|^2.$$

Combining these results, we have

$$\frac{b_*}{2}\int_\Omega \left|\frac{\partial\Theta}{\partial t} - \frac{\partial\hat\Theta}{\partial t}\right|^2 + \frac{a_*}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^d} |D^s(\Theta - \mathfrak{G} - \hat\Theta + \hat{\mathfrak{G}})|^2$$
$$\leq \frac{b^*}{2b_*}\int_\Omega |\bar g|^2 + 2\int_\Omega b|\bar g| + \int_\Omega (\hat{\mathfrak{F}} - \mathfrak{F})\left(\frac{\partial\hat\Theta}{\partial t} + \bar g - \frac{\partial\Theta}{\partial t}\right) + \int_\Omega (\hat\eta_0 - \eta_0)\left(\frac{\partial\hat\Theta}{\partial t} + \bar g - \frac{\partial\Theta}{\partial t}\right)$$
$$\leq \left(\frac{b^*}{2b_*} + 2b\right)\int_\Omega |\bar g| + \frac{2}{b_*}\int_\Omega |\mathfrak{F} - \hat{\mathfrak{F}}|^2 + \frac{b_*}{4}\int_\Omega \left|\frac{\partial\Theta}{\partial t} - \frac{\partial\mathfrak{G}}{\partial t} - \frac{\partial\hat\Theta}{\partial t} + \frac{\partial\hat{\mathfrak{G}}}{\partial t}\right|^2 + \frac{2}{b_*}\int_\Omega |\eta_0 - \hat\eta_0|^2.$$

Integrating with respect to time from 0 to t, we have

$$\frac{b_*}{2}\int_0^t\int_\Omega \left|\frac{\partial\Theta}{\partial t} - \frac{\partial\hat\Theta}{\partial t}\right|^2 + \frac{a_*}{2}\int_{\mathbb{R}^d} |D^s(\Theta - \hat\Theta)|^2 - \frac{a^*}{2}\int_{\mathbb{R}^d} |D^s(\mathfrak{G} - \hat{\mathfrak{G}})|^2$$
$$\leq \frac{b_*}{2}\int_0^t\int_\Omega \left|\frac{\partial\Theta}{\partial t} - \frac{\partial\hat\Theta}{\partial t}\right|^2 + \frac{a_*}{2}\int_{\mathbb{R}^d} |D^s(\Theta - \mathfrak{G} - \hat\Theta + \hat{\mathfrak{G}})|^2$$
$$\leq \left(\frac{b^*}{2b_*} + 2b\right)\int_0^t\int_\Omega |\bar g|^2 + \frac{2}{b_*}\int_0^t\int_\Omega |\mathfrak{F} - \hat{\mathfrak{F}}|^2 + \frac{b_*}{4}\int_0^t\int_\Omega \left|\frac{\partial\Theta}{\partial t} - \frac{\partial\mathfrak{G}}{\partial t} - \frac{\partial\hat\Theta}{\partial t} + \frac{\partial\hat{\mathfrak{G}}}{\partial t}\right|^2 + \frac{2}{b_*}\int_0^t\int_\Omega |\eta_0 - \hat\eta_0|^2$$
$$\leq \left(\frac{b^*}{2b_*} + 2b\right)\int_0^t\int_\Omega |\bar g|^2 + \frac{2}{b_*}\int_0^t\int_\Omega |\mathfrak{F} - \hat{\mathfrak{F}}|^2 + \frac{b_*}{4}\int_0^t\int_\Omega \left|\frac{\partial\Theta}{\partial t} - \frac{\partial\hat\Theta}{\partial t}\right|^2 + \frac{b_*}{4}\int_0^t\int_\Omega \left|\frac{\partial\mathfrak{G}}{\partial t} - \frac{\partial\hat{\mathfrak{G}}}{\partial t}\right|^2$$
$$+ \frac{2t}{b_*}\int_\Omega |\eta_0 - \hat\eta_0|^2$$
$$= \left(\frac{b^*}{2b_*} + 2b\right)\int_0^t\int_\Omega |\bar g|^2 + \frac{2}{b_*}\int_0^t\int_\Omega |\mathfrak{F} - \hat{\mathfrak{F}}|^2 + \frac{b_*}{4}\int_0^t\int_\Omega \left|\frac{\partial\Theta}{\partial t} - \frac{\partial\hat\Theta}{\partial t}\right|^2 + \frac{b_*}{4}\int_0^t\int_\Omega |\bar g|^2 + \frac{2t}{b_*}\int_\Omega |\eta_0 - \hat\eta_0|^2$$

by the triangle inequality, which can be rewritten as

$$\frac{b_*}{4}\int_0^t\int_\Omega \left|\frac{\partial\Theta}{\partial t} - \frac{\partial\hat\Theta}{\partial t}\right|^2 + \frac{a_*}{2}\int_{\mathbb{R}^d} |D^s(\Theta - \hat\Theta)|^2$$
$$\leq \left(\frac{b^*}{2b_*} + \frac{b_*}{4} + 2b\right)\int_0^t\int_\Omega \left|\frac{\partial\mathfrak{G}}{\partial t} - \frac{\partial\hat{\mathfrak{G}}}{\partial t}\right|^2 + \frac{2}{b_*}\int_0^t\int_\Omega |\mathfrak{F} - \hat{\mathfrak{F}}|^2 + \frac{a^*}{2}\int_{\mathbb{R}^d} |D^s(\mathfrak{G} - \hat{\mathfrak{G}})|^2 + \frac{2t}{b_*}\int_\Omega |\eta_0 - \hat\eta_0|^2.$$

Recalling that $f, \hat f \in L^2([0,T]; H^{-s}(\Omega))$, $g, \hat g \in W^{2,1}([0,T]; L^2(\Omega)) \cap L^\infty([0,T]; H^s(\mathbb{R}^d))$, and $\eta_0, \hat\eta_0 \in L^2(\Omega)$, we obtain continuous dependence of Θ on $\mathfrak{F}, \mathfrak{G}$ and η_0. □

Similarly, for the one phase problem (in Section 6.7 below) we can also obtain an equivalent variational inequality formulation, now of obstacle type. Indeed, governed by γ^o, the weak temperature solution ϑ^o

obtained in (6.13_{1ph}) is non-negative at all times $t \in [0,T]$. Therefore, its primitive

$$\Theta^o(t) = \int_0^t \vartheta^o(\tau)\, d\tau$$

is also always non-negative, and satisfies

$$\Theta^o \in H^1(0,T;H^s(\mathbb{R}^d)), \quad \Theta^o(0) = 0 \text{ and } \Theta^o(t) \geq 0, \quad \Theta^o(t) - \int_0^t g(\tau)\, d\tau = 0 \text{ in } \Omega^c \text{ for a.e. } t \in]0,T[, \tag{6.59^o}$$

and from (6.56), denoting $\chi_o \in H(\vartheta^o)$,

$$\int_\Omega b\left(\frac{\partial \Theta^o}{\partial t}(t)\right) w + \int_{\mathbb{R}^d} AD^s\Theta^o(t)\cdot D^s w + \int_\Omega \lambda \chi_o(t) w = \int_\Omega \mathfrak{F}(t) w + \int_\Omega \eta_0 w, \qquad \text{for a.e. } t, \forall w \in H^s_0(\Omega). \tag{6.56^o}$$

Now introduce

$$\mathbb{K}^+(t) := \left\{ v \in H^s(\mathbb{R}^d) : v \geq 0 \text{ a.e. in } \Omega, v = \int_0^t g(\tau)\, d\tau \text{ in } \Omega^c \right\}, \text{ for a.e. } t \in]0,T[.$$

Assuming that $\chi_{\{\vartheta^o(t)>0\}} = \chi_{\{\Theta^o(t)>0\}}$ and $\chi_{\{\vartheta^o(t)<0\}} = \chi_{\{\Theta^o(t)<0\}}$ for a.e. $t \in]0,T[$, we can once again make use of the inequality (6.57) to obtain

$$\lambda\chi_o(v - \Theta^o) \leq \lambda(v^+ - \Theta^{o+}) = \lambda(v - \Theta^o)$$

when $v(t), \Theta^o(t) \geq 0$. Therefore, we can rewrite the equation (6.56^o) with $w = v - \Theta^o(t)$ for $v \in \mathbb{K}^+(t)$ as a variational inequality to obtain the following evolutionary obstacle-type problem for $\Theta^o(t) \in \mathbb{K}^+(t)$:

$$\int_\Omega b\left(\frac{\partial \Theta^o}{\partial t}(t)\right)(v - \Theta^o(t)) + \int_{\mathbb{R}^d} AD^s\Theta^o(t)\cdot D^s(v - \Theta^o(t)) \geq \int_\Omega (\mathfrak{F}(t) + \eta_0 - \lambda)(v - \Theta^o(t)) \quad \forall v \in \mathbb{K}^+(t).$$

This corresponds to the nonlocal version of the parabolic variational inequality obtained by Duvaut [102] for the one-phase Stefan problem for the classical case $s = 1$. See also [191, 192] or [239].

6.5 Convergence to the Classical Problem as $s \nearrow 1$

Next, as $s \nearrow 1$ the s-fractional derivatives converge to the classical derivatives, we show that the corresponding solutions to the fractional Stefan-type problem converge in appropriate spaces to the classical one.

To consider the convergence of the problem as $s \nearrow 1$, we start with a continuous dependence property of the Riesz derivatives as s varies, which can be easily shown using Fourier transform first for $u(t) \in C_c^\infty(\Omega)$, and extended by density as in Lemma 1.5 adopted to the time-dependent case.

Lemma 6.19. *For $u \in L^\infty(0,T;H_0^{s'}(\Omega))$, $D^s u$ is continuous in $L^\infty(0,T;L^2(\mathbb{R}^d)^d)$ as s varies in $[\sigma, s']$ for $0 < \sigma < s' \leq 1$. As a consequence, we have the following estimate: for $\sigma \leq s \leq 1$,*

$$\left\| D^\sigma u(t) \right\|_{L^2(\mathbb{R}^d)^d} \leq c_\sigma \left\| D^s u(t) \right\|_{L^2(\mathbb{R}^d)^d}, \tag{6.62}$$

for any $u(t) \in H^s_0(\Omega)$ for a.e. $t \in [0,T]$, where the constant c_σ is independent of s and t.

Consequently, we have a continuous transition from the fractional Stefan-type problem to the classical Stefan-type problem as $s \nearrow 1$ in the following sense.

Theorem 6.20. *Let (η_s, ϑ_s) be the solution to the fractional Stefan-type problem for $0 < \sigma \leq s < 1$ for $f_s \in L^2(0,T;L^2(\Omega))$, $\tilde{g}_s \in W^{2,1}(0,T;L^2(\mathbb{R}^d)) \cap L^\infty(0,T;H^s(\mathbb{R}^d))$, i.e. $\vartheta_s = \gamma(\eta_s)$ for a.e. $x,t \in Q_T$ and*

$$-\int_{Q_T} \eta_s \frac{\partial \xi}{\partial t} + \int_{\mathbb{R}^d\times[0,T]} AD^s\vartheta_s \cdot D^s\xi = \int_{Q_T} f_s\xi + \int_\Omega \eta_{0,s}\xi(0), \quad \forall \xi \in \Xi^s_T \tag{6.13}$$

with Dirichlet boundary condition $\vartheta_s = g_s$ *on* $]0,T[\times\Omega^c$*, initial condition* $\eta_s(0) = \eta_{0,s} \in L^2(\Omega)$*, and setting* $\vartheta_s(0) = \gamma(\eta_{0,s})$ *assume* $\vartheta_s(0) - g_s(0) \in H^s_0(\Omega)$ *is bounded uniformly in* s *for* $0 < \sigma \le s < 1$*. Suppose that there exists* $\eta_0 \in L^2(\Omega)$*,* $f \in L^2(0,T;L^2(\Omega))$ *and* $\tilde{g} \in W^{2,1}(0,T;L^2(\mathbb{R}^d)) \cap L^\infty(0,T;H^1(\mathbb{R}^d))$ *such that*

$$\begin{aligned} &\eta_{0,s} \rightharpoonup \eta_0 \text{ in } L^2(\Omega), \\ &f_s \rightharpoonup f \text{ in } L^2(0,T;L^2(\Omega)), \quad \text{and} \\ &\tilde{g}_s \rightharpoonup \tilde{g} \text{ in } W^{2,1}(0,T;L^2(\mathbb{R}^d))\text{-weak and in } L^\infty(0,T;H^\sigma(\mathbb{R}^d))\text{-weak}^*. \end{aligned} \tag{6.63}$$

Then, the sequence $(\eta_s,\vartheta_s)_s$ *converges weakly to* (η,ϑ) *in the sense that*

$$\eta_s \rightharpoonup \eta \text{ in } L^\infty(0,T;L^2(\Omega))\text{-weakly}^* \text{ and in } H^1(0,T;H^{-1}(\Omega))\text{-weak}, \tag{6.64}$$

and

$$\vartheta_s \rightharpoonup \vartheta \text{ in } L^\infty(0,T;H^\sigma(\Omega))\text{-weak}^*, \text{ in } H^1(0,T;L^2(\Omega))\text{-weak and in } C([0,T];L^2(\Omega)) \tag{6.65}$$

as $s \nearrow 1$*, where* (η,ϑ) *solves uniquely the Stefan problem for* $s = 1$ *with* $\vartheta = \gamma(\eta)$ *and initial condition* $\eta(0) = \eta_0$ *in* Ω*, and Dirichlet boundary condition* $\vartheta = g$ *on* $]0,T[\times\partial\Omega$*, and*

$$-\int_{Q_T} \eta\frac{\partial\xi}{\partial t} + \int_{Q_T} AD\vartheta\cdot D\xi = \int_{Q_T} f\xi + \int_\Omega \eta_0\xi(0), \quad \forall\xi\in\Xi^1_T. \tag{6.66}$$

Proof. Recall that $\eta_s \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ independent of ϕ_t, by Remark 6.6. Moreover, invoking the continuity of the inclusions $H^{-\sigma}(\Omega) \subset H^{-s}(\Omega) \subset H^{-1}(\Omega)$, we have, by (6.25),

$$\|\eta_s\|_{L^\infty(0,T;L^2(\Omega))} \le C_4\left(\|\eta_{0,s}\|_{L^2(\Omega)}, \|f_s\|_{L^2(0,T;H^{-s}(\Omega))}, \|g_s\|_{BV(0,T;L^2(\Omega))}\right) \le C_4' \tag{6.67}$$

for a constant C_4' depending on σ but independent of s by assumption (6.63). Then, by Lemma 1.3

$$\|f_s\|_{L^2(0,T;H^{-s}(\Omega))} \le \frac{C_P}{\sigma}\|f_s\|_{L^2(0,T;L^2(\Omega))}.$$

Similarly, by (6.24) and (6.63),

$$\frac{1}{c_1}\left\|\frac{\partial\eta_s}{\partial t}\right\|_{L^2(0,T;H^{-1}(\Omega))} \le \left\|\frac{\partial\eta_s}{\partial t}\right\|_{L^2(0,T;H^{-s}(\Omega))} \le C_2\left(\|\eta_{0,s}\|_{L^2(\Omega)}, \|f_s\|_{L^2(0,T;L^2(\Omega))}, \|g_s\|_{BV(0,T;L^2(\Omega))}\right). \tag{6.68}$$

Therefore, η_s is bounded in $L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ uniformly with respect to s, and, up to a subsequence, $(\eta_s)_s$ is converging in $H^1(0,T;H^{-1}(\Omega))$-weak and in $L^\infty(0,T;L^2(\Omega))$-weak* to some η as in (6.64).

Furthermore, for $\vartheta_s - g_s \in L^\infty(0,T;H^s_0(\Omega))$, we have

$$\left\|D^s(\vartheta_s - g_s)\right\|_{L^\infty(0,T;L^2(\mathbb{R}^d)^d)} \le C \tag{6.69}$$

by (6.63) and Remarks 6.15 and 6.17 for some constant C independent of s depending on $\sigma \le s$ and on the data. By the Poincaré inequality, $\vartheta_s - g_s$ is also bounded, so

$$\vartheta_s - g_s \xrightharpoonup[s\nearrow 1]{} \vartheta - g \text{ in } L^\infty(0,T;L^2(\mathbb{R}^d))\text{-weak}^* \quad \text{and} \quad D^s(\vartheta_s - g_s) \xrightharpoonup[s\nearrow 1]{} \zeta \text{ in } L^\infty(0,T;L^2(\mathbb{R}^d)^d)\text{-weak}^*$$

for some ϑ, ζ.

Now, by the convergence Lemma 6.19, for all $\Phi \in L^2(0,T;C^\infty_c(\Omega)^d)$, denoting by $\tilde{\Phi}$ the zero extension of Φ outside Ω,

$$D^s\cdot\Phi \xrightarrow[s\nearrow 1]{} \widetilde{D\cdot\Phi} = D\cdot\tilde{\Phi} \quad \text{in } L^2(0,T;L^2(\mathbb{R}^d)^d),$$

therefore,

$$\int_0^T\int_{\mathbb{R}^d} D^s(\vartheta_s - g_s)\cdot\tilde{\Phi} = -\int_0^T\int_{\mathbb{R}^d}(\vartheta_s - g_s)(D^s\cdot\Phi) \xrightarrow[s\nearrow 1]{} -\int_0^T\int_{\mathbb{R}^d}(\vartheta - g)(\widetilde{D\cdot\Phi}).$$

But by the a priori estimate on $D^s(\vartheta_s - g_s)$,

$$\left|\int_0^T \int_{\mathbb{R}^d} D^s(\vartheta_s - g_s)\cdot\Phi\right| \leq C\|\Phi\|_{L^2(0,T;L^2(\mathbb{R}^d)^d)},$$

which implies, in the limit, that

$$\left|\int_0^T \int_\Omega (\vartheta - g)(D\cdot\Phi)\right| = \left|\int_0^T \int_{\mathbb{R}^d} (\vartheta - g)(\widetilde{D\cdot\Phi})\right| \leq C\|\Phi\|_{L^2(0,T;L^2(\mathbb{R}^d)^d)}.$$

Therefore we have $D(\vartheta - g) \in L^2(0,T;[L^2(\Omega)]^d)$ and hence

$$-\int_0^T \int_{\mathbb{R}^d} (\vartheta - g)(\widetilde{D\cdot\Phi}) = \int_0^T \int_{\mathbb{R}^d} D(\vartheta - g)\cdot\tilde{\Phi}$$

so $\zeta = D(\vartheta - g)$. Moreover, since $\vartheta - g = w - \lim_{s\nearrow 1}(\vartheta_s - g_s) = 0$ outside Ω, and the boundary of Ω being Lipschitz, we may conclude $\vartheta - g \in L^\infty(0,T;H^1_0(\Omega))$.

We claim that (η, ϑ) satisfies the Stefan-type problem for $s = 1$. Indeed, for any $\xi \in \Xi^1_T \subset \Xi^s_T$,

$$-\int_{Q_T} \eta\frac{\partial\xi}{\partial t} + \int_0^T \int_\Omega AD(\vartheta - g)\cdot D\xi = -\int_{Q_T} \eta\frac{\partial\xi}{\partial t} + \int_0^T \int_{\mathbb{R}^d} AD(\vartheta - g)\cdot\widetilde{D\xi}$$
$$= \lim_{s\nearrow 1}\left\{-\int_{Q_T} \eta_s\frac{\partial\xi}{\partial t} + \int_0^T \int_{\mathbb{R}^d} AD^s(\vartheta_s - g_s)\cdot D^s\xi\right\} = \lim_{s\nearrow 1}\left\{\int_{Q_T} f\xi + \int_\Omega \eta_{0,s}\xi(0)\right\} = \int_{Q_T} f\xi + \int_\Omega \eta_0\xi(0)$$

since $D^s(\vartheta_s - g_s) \rightharpoonup D(\vartheta - g)$ in $L^\infty(0,T;L^2(\mathbb{R}^d)^d)$-weak*, $\eta_s \rightharpoonup \eta$ in $L^\infty(0,T;L^2(\Omega))$-weak*, and $D^s\xi \to \widetilde{D\xi}$ strongly in $L^2(0,T;L^2(\mathbb{R}^d)^d)$ by Lemma 6.19. Therefore, (η, ϑ) satisfies (6.66).

Moreover, by Remark 6.17, $\frac{\partial\vartheta_s}{\partial t}$ is bounded in $L^2(0,T;L^2(\Omega))$, so we can take the limit as $s \nearrow 1$ to obtain that

$$\frac{\partial\vartheta_s}{\partial t} \rightharpoonup \frac{\partial\vartheta}{\partial t} \text{ in } L^2(0,T;L^2(\mathbb{R}^d))\text{-weak.}$$

Since $\frac{\partial g_s}{\partial t} \rightharpoonup \frac{\partial g}{\partial t}$ in $L^2(0,T;L^2(\mathbb{R}^d))$-weak,

$$\vartheta_s - g_s \rightharpoonup \vartheta - g \text{ in } L^\infty(0,T;H^\sigma_0(\Omega))\text{-weak}^* \text{ and in } H^1(0,T;L^2(\Omega))\text{-weak}$$

as $s \nearrow 1$, and so by compactness (see, for instance, Corollary 4 of [220]),

$$\vartheta_s - g_s \to \vartheta - g \text{ in } C([0,T];L^2(\Omega)),$$

giving the convergence (6.65) as desired using the convergence of g_s to g in (6.63).

Finally, it remains to show that $\vartheta = \gamma(\eta)$ a.e. in $]0,T[\times\Omega$, or equivalently $\eta \in \beta(\vartheta)$. Indeed, since $\vartheta_s = \gamma(\eta_s)$ a.e. in $]0,T[\times\Omega$ with $\eta_s \rightharpoonup \eta$ weakly in $L^2(0,T;L^2(\Omega))$ and $\vartheta_s \to \vartheta$ in $C([0,T];L^2(\Omega))$, by the maximal monotonicity of β (see, for instance, Proposition 2.5 of [54]), we have $\eta \in \beta(\vartheta)$ and $\eta_0 \in \beta(\vartheta(0))$ satisfying (6.66). Subsequently, we obtain the solution $\vartheta = \gamma(\eta)$ a.e. in $]0,T[\times\Omega$, with initial condition $\vartheta(0) = \lim_{s\nearrow 1}\gamma(\eta_{0,s}) = \gamma(\eta_0)$ by the convergence of $\eta_{0,s}$ to η_0 in $L^2(\Omega)$.

□

6.6 Asymptotic Behaviour as $t \to \infty$

In this section, we derive the asymptotic behaviour of the weak solutions as $t \to \infty$, following the approach of the classical case in [86]. We first begin with a well-known asymptotic convergence result for the solutions of differential equations with maximal monotone operators.

Proposition 6.21 (See, for instance, Theorem 3.11 of [54]). *Let φ be a lower semi-continuous convex functional on a Hilbert space H. Suppose that for all $C \in \mathbb{R}$, the set $\{x \in H : \varphi(x) + |x|^2 \leq C\}$ is compact. Let $f_\infty \in H$ and let $f(t)$ be a function such that $f - f_\infty \in L^1(t_0,\infty;H)$. Suppose $U \in C(t_0,\infty;H)$ is a weak solution to the equation $\frac{dU}{dt} + \partial\varphi(U) \ni f$. Then $\lim_{t\to+\infty} U(t) = U_\infty$ in H exists and $f_\infty \in \partial\varphi(U_\infty)$.*

With this proposition, we can directly obtain the convergence of the generalised enthalpy solutions $\eta(t) \to \eta_\infty$ in the case where $\tilde{g}(t) = \tilde{g}_\infty$ for all $t \geq t_0$, i.e. the Dirichlet data is independent of time, with $f(t) - f_\infty \in L^1(t_0, \infty; H^{-s}(\Omega))$. For more general $\tilde{g}(t)$ converging to some $\tilde{g}_\infty$, we may also have a characterisation of the asymptotic behaviour of the generalised enthalpy solution towards the stationary solution, which can be written in terms of the stationary Dirichlet problem $\vartheta_\infty = g_\infty$ in Ω^c for the temperature ϑ_∞:

$$\int_{\mathbb{R}^d} AD^s\vartheta_\infty \cdot D^s\xi = \langle f_\infty, \xi \rangle, \quad \forall \xi \in H^s_0(\Omega). \tag{6.70}$$

Theorem 6.22. *Let f, $\tilde{g}$ and η_0 satisfy the assumptions in Theorem 6.3 such that $f - f_\infty \in L^1(0,\infty; H^{-s}(\Omega)) \cap L^2(0,\infty; H^{-s}(\Omega))$ and $\tilde{g} - \tilde{g}_\infty \in W^{1,1}(0,\infty; L^2(\Omega))$ for given $f_\infty \in H^{-s}(\Omega)$ and $\tilde{g}_\infty \in H^s(\mathbb{R}^d)$. (We can subsequently define g_∞ and $g(t)$ in the same spaces using (6.7) as explained in the Section 3.2.) Let $\eta \in \beta(\vartheta)$ be the generalised enthalpy solution to the fractional Stefan-type problem (6.11) for all $T > 0$. Then, there exists an $\eta_\infty \in L^2(\Omega)$ such that*

$$\eta(t) \to \eta_\infty \text{ strongly in } H^{-s}(\Omega) \text{ and weakly in } L^2(\Omega) \text{ as } t \to \infty,$$

where η_∞ is such that $\vartheta_\infty = \gamma(\eta_\infty)$ satisfies (6.70) with $\vartheta_\infty = g_\infty$ in Ω^c.

Proof. We first note that, while η_∞ is not unique in general, there exists a unique weak temperature solution $\vartheta_\infty = \gamma(\eta_\infty)$ to (6.70) with $\vartheta_\infty = g_\infty$ in Ω^c by the Riesz representation theorem for A coercive and bounded, since we have the equivalent norms (1.17) in $H^s_0(\Omega)$.

Furthermore, under our assumptions, by a similar approach to the Proposition 3.2 and its Corollary in [86], there is a positive constant M such that

$$\sup_{t\geq 0} \|\eta(t)\|_{L^2(\Omega)} \leq M. \tag{6.71}$$

Let ϵ be any positive number. Since g is bounded, we can take a number t_ϵ such that

$$\int_{t_\epsilon}^\infty \|g(\tau) - g_\infty\|_{L^2(\Omega)} + \|f(\tau) - f_\infty\|_{H^{-s}(\Omega)} \, d\tau \leq \epsilon.$$

Also, let w_ϵ be the solution of the fractional Stefan-type problem (6.11) corresponding to (f_∞, g_∞) on $[t_\epsilon, \infty[$ with initial value $w_\epsilon(t_\epsilon) = \eta(t_\epsilon)$, i.e.

$$\frac{dw_\epsilon}{dt}(t) + \partial\phi_\infty(w_\epsilon) = \frac{dw_\epsilon}{dt}(t) + \tilde{\mathcal{L}}^s_A(\gamma(w_\epsilon(t)) - g_\infty) = f_\infty. \tag{6.72}$$

By Proposition 6.21 in the interval $[t_\epsilon, \infty[$ with $H = H^{-s}(\Omega)$ and φ given by the convex functional ϕ_∞ as defined in (6.27) for the Dirichlet boundary condition g_∞, since the set $\{W \in H^{-s}(\Omega) : \phi_\infty(W) + |W|^2 \leq C\}$ is a bounded set in L^2 and therefore compact in $H^{-s}(\Omega)$, we have that

$$w_\epsilon(t) \text{ converges in } H^{-s}(\Omega) \text{ as } t \to \infty \text{ to a point } w_\epsilon^\infty \in L^2(\Omega)$$

satisfying

$$f_\infty \in \partial\phi_\infty(w_\epsilon^\infty), \quad \text{or equivalently} \quad \tilde{\mathcal{L}}^s_A(\gamma(w_\epsilon^\infty) - g_\infty) = f_\infty. \tag{6.73}$$

Therefore, there is a number $t'_\epsilon \geq t_\epsilon$ such that

$$\|w_\epsilon(t) - w_\epsilon(\tau)\|_{H^{-s}(\Omega)} \leq \epsilon \quad \forall t, \tau \geq t'_\epsilon.$$

Also, as in Remark 6.9, we have that

$$\frac{1}{2}\frac{d}{d\tau}\|\eta(\tau) - w_\epsilon(\tau)\|^2_{H^{-s}(\Omega)} + \frac{2}{C_\gamma}\|\gamma(\eta)(\tau) - \gamma(w_\epsilon)(\tau)\|^2_{L^2(\Omega)}$$
$$\leq 2\left(f(\tau) - f_\infty, \eta(\tau) - w_\epsilon(\tau)\right) + 2\int_\Omega (g(\tau) - g_\infty)(\eta(\tau) - w_\epsilon(\tau)),$$

so in particular,

$$\frac{d}{d\tau}\left\|\eta(\tau)-w_\epsilon(\tau)\right\|^2_{H^{-s}(\Omega)} \le K\left(\left\|f(\tau)-f_\infty\right\|_{H^{-s}(\Omega)}+\left\|g(\tau)-g_\infty\right\|_{L^2(\Omega)}\right)$$

for some constant K for a.e. $\tau \ge t_\epsilon$. Integrating both sides over $[t_\epsilon, t]$, we have

$$\left\|\eta(t)-w_\epsilon(t)\right\|^2_{H^{-s}(\Omega)} \le K\epsilon \tag{6.74}$$

for any $t \ge t_\epsilon$. Therefore, if $t, s \ge t'_\epsilon$,

$$\left\|\eta(t)-\eta(s)\right\|_{H^{-s}(\Omega)} \le \left\|\eta(t)-w_\epsilon(t)\right\|_{H^{-s}(\Omega)}+\left\|w_\epsilon(t)-w_\epsilon(s)\right\|_{H^{-s}(\Omega)}+\left\|w_\epsilon(s)-\eta(s)\right\|_{H^{-s}(\Omega)} \le 2\sqrt{K\epsilon}+\epsilon.$$

This implies that $\eta(t)$ converges in $H^{-s}(\Omega)$ as $t \to \infty$ to some $\eta_\infty \in H^{-s}(\Omega)$. Also, since (6.74) holds for all $t \ge t_\epsilon$ and $\lim_{t\to\infty} w_\epsilon(t) = w_\epsilon^\infty$, we have that $w_\epsilon^\infty \to \eta_\infty$ in $H^{-s}(\Omega)$ as $\epsilon \searrow 0$. Since w_ϵ^∞ satisfies (6.73), so does η_∞.

Finally, defining $\vartheta_\infty = \gamma(\eta_\infty)$, taking the limit in ϵ in (6.73), we have $\vartheta_\infty = (\tilde{\mathcal{L}}^s_A)^{-1} f_\infty + g_\infty$. ☐

Remark 6.23. *In addition, if we assume $f - f_\infty \in W^{1,1}(0,\infty; H^{-s}(\Omega))$, we have that the solution η to the fractional Stefan-type problem (6.11) satisfies $\eta - \eta_\infty \in H^1(0,\infty, H^{-s}(\Omega))$. This follows as in the proof of Theorem 2.1 of [86], and it can be shown that the energy functional $J(t)$ given by*

$$J(t) := \phi_t(\eta(t)) + \int_0^t \left\|\frac{d\eta}{dt}(\tau)\right\|^2_{H^{-s}(\Omega)} d\tau - C\int_0^t \left(\left\|\frac{\partial g}{\partial t}(\tau)\right\|_{L^2(\Omega)} + \left\|\frac{df}{dt}(\tau)\right\|_{H^{-s}(\Omega)}\right) d\tau \quad \text{for } t \ge 0$$

is bounded and non-increasing on $]0,\infty[$. So $\lim_{t\to\infty} J(t)$ exists and

$$\frac{d\eta}{dt} \in L^2(0,\infty, H^{-s}(\Omega)). \tag{6.75}$$

We can also increase the regularity of $\tilde{g}$ as in Theorem 6.16 to obtain the convergence of ϑ.

Theorem 6.24. *Let $f - f_\infty \in L^1(0,\infty; H^{-s}(\Omega)) \cap L^2(0,\infty; L^2(\Omega))$ and $\tilde{g} - \tilde{g}_\infty \in W^{2,1}(0,\infty; L^2(\mathbb{R}^d)) \cap H^1(0,\infty; L^2(\mathbb{R}^d)) \cap L^2(0,\infty; H^s(\mathbb{R}^d))$ (and so similarly with $g - g_\infty$), and $\eta_0 \in L^2(\Omega)$, where $f_\infty \in L^2(\Omega)$ and $\tilde{g}_\infty \in H^s(\mathbb{R}^d)$. Suppose that ϑ is the weak temperature solution to the fractional Stefan-type problem (6.13), and ϑ_∞ is the stationary weak temperature solution to (6.70) with $\vartheta_\infty = g_\infty$ in Ω^c. Then*

$$\vartheta(t) \to \vartheta_\infty \text{ in } L^2(\Omega) \text{ and in } H^s(\mathbb{R}^d)\text{-weak as } t \to \infty.$$

In addition, if $f - f_\infty \in W^{1,1}(0,\infty; H^{-s}(\Omega))$, we have

$$\vartheta(t) - g(t) \to \vartheta_\infty - g_\infty \text{ strongly in } H^s_0(\Omega) \text{ as } t \to \infty.$$

In particular, if $g(t) \to g_\infty$ in $H^s(\mathbb{R}^d)$ as $t \to \infty$, then $\vartheta(t) \to \vartheta_\infty$ strongly in $H^s(\mathbb{R}^d)$ as $t \to \infty$.

Proof. Let (η, ϑ) be the solution to the fractional Stefan-type problem (6.13), so that their finite-dimensional approximations $(\eta_n, \gamma(\eta_n))$ satisfy the inequality (6.53). Since the η_n's are uniformly bounded in $L^\infty(0,\infty; L^2(\Omega))$ by Theorem 6.22 applied to the approximated problem, we have

$$\left|\int_0^\infty \int_\Omega \tilde{\eta}_n \frac{\partial^2 g}{\partial t^2}\right| \le \|\tilde{\eta}_n\|_{L^\infty(0,\infty;L^2(\Omega))} \left\|\frac{\partial^2 g}{\partial t^2}\right\|_{L^1(0,\infty;L^2(\Omega))},$$

$$\lim_{t\to\infty}\int_\Omega \tilde{\eta}_n(t)\frac{\partial g}{\partial t}(t) = 0 \quad \text{since } \frac{\partial g}{\partial t} \to 0 \text{ in } L^2(\Omega),$$

$$\left|\int_\Omega \tilde{\eta}_n(0)\frac{\partial g}{\partial t}(0)\right| \le \left\|\tilde{\eta}_n(0)\right\|_{L^2(\Omega)} \left\|\frac{\partial g}{\partial t}(0)\right\|_{L^2(\Omega)},$$

and

$$\left|\int_0^\infty \int_\Omega f_n \frac{\partial g}{\partial t}\right| \leq \|f_n\|_{L^2(0,\infty;L^2(\Omega))} \left\|\frac{\partial g}{\partial t}\right\|_{L^2(0,\infty;L^2(\Omega))}$$

$$= \|\mathbb{P}_{E_n} f\|_{L^2(0,\infty;L^2(\Omega))} \left\|\frac{\partial g}{\partial t}\right\|_{L^2(0,\infty;L^2(\Omega))} \leq \frac{a^*}{a_*} \|f\|_{L^2(0,\infty;L^2(\Omega))} \left\|\frac{\partial g}{\partial t}\right\|_{L^2(0,\infty;L^2(\Omega))}$$

and, passing to the limit in n in (6.53), we conclude

$$\vartheta - g \in L^\infty(0,\infty; H^s_0(\Omega)) \cap H^1(0,\infty; L^2(\Omega)). \tag{6.76}$$

Let w^* be any accumulation point of $\{\vartheta(t) - g(t)\}$ in $H^s_0(\Omega)$ for the weak topology as $t \to \infty$, and let $\{t_n\}_n$ be a sequence in $[0,\infty[$ such that $t_n \nearrow \infty$ and $\vartheta(t_n) - g(t_n) \rightharpoonup w^*$ weakly in $H^s_0(\Omega)$ as $n \to \infty$. Then, by the convergence of g and the compactness of $H^s_0(\Omega)$ in $L^2(\Omega)$,

$$\vartheta(t_n) \to w^* + g_\infty \text{ in } L^2(\Omega).$$

Also, from Theorem 6.22, there exists an η_∞ such that

$$\eta(t_n) \rightharpoonup \eta_\infty \text{ in } L^2(\Omega)\text{-weak}.$$

As $\vartheta(t_n) = \gamma(\eta(t_n))$, by the property of maximal monotone operators in $L^2(\Omega)$, the limit of any subsequence as $t_n \to \infty$ satisfies

$$w^* + g_\infty = \gamma(\eta_\infty) = \vartheta_\infty.$$

Therefore, $w^* = \vartheta_\infty - g_\infty$, and we have the convergence

$$\vartheta(t) \to \vartheta_\infty \text{ in } L^2(\Omega) \text{ as } t \to \infty \tag{6.77}$$

and

$$\vartheta(t) - g(t) \rightharpoonup \vartheta_\infty - g_\infty \text{ in } H^s_0(\Omega)\text{-weak as } t \to \infty. \tag{6.78}$$

In order to obtain the strong convergence in (6.78), we define the function $E(t)$ by

$$E(t) := \frac{1}{C_\gamma}\int_0^t \left\|\frac{\partial \vartheta(\tau)}{\partial t}\right\|^2_{L^2(\Omega)} d\tau + \frac{1}{2}\langle \tilde{\mathcal{L}}^s_A(\vartheta(t) - g(t)), \vartheta(t) - g(t)\rangle - \int_\Omega \eta(t)\frac{\partial g(t)}{\partial t} - \big\langle f(t), \vartheta(t) - g(t)\big\rangle \tag{6.79}$$

for $t \geq 0$. Then, using again the inequality (6.53) in the limit $n \to \infty$ with the integral taken over the interval $[t_1, t_2]$ and incorporating the Lipschitz property in (6.54), we obtain

$$\frac{1}{C_\gamma}\int_{t_1}^{t_2} \left\|\frac{\partial \vartheta(\tau)}{\partial t}\right\|^2_{L^2(\Omega)} d\tau + \frac{1}{2}\langle \tilde{\mathcal{L}}^s_A(\vartheta(t_2) - g(t_2)), \vartheta(t_2) - g(t_2)\rangle$$
$$- \int_\Omega \eta(t_2)\frac{\partial g}{\partial t}(t_2) - \big\langle f(t_2), \vartheta(t_2) - g(t_2)\big\rangle$$
$$\leq \frac{1}{2}\langle \tilde{\mathcal{L}}^s_A(\vartheta(t_1) - g(t_1)), \vartheta(t_1) - g(t_1)\rangle - \int_\Omega \eta(t_1)\frac{\partial g}{\partial t}(t_1) - \big\langle f(t_1), \vartheta(t_1) - g(t_1)\big\rangle$$
$$- \int_{t_1}^{t_2} \left\langle \frac{\partial f(\tau)}{\partial t}, \vartheta(\tau) - g(\tau)\right\rangle d\tau - \int_{t_1}^{t_2}\int_\Omega \eta(\tau)\frac{\partial^2 g(\tau)}{\partial t^2}\, d\tau,$$

or

$$E(t_2) \leq E(t_1) - \int_{t_1}^{t_2} \left\{ \left\langle \frac{\partial f(\tau)}{\partial t}, \vartheta(\tau) - g(\tau)\right\rangle + \int_\Omega \eta(\tau)\frac{\partial^2 g(\tau)}{\partial t^2}\right\} d\tau. \tag{6.80}$$

Recalling (6.71) and (6.76), we have $\eta \in L^\infty(0,T; L^2(\Omega))$ and $\vartheta - g \in L^\infty(0,T; H^s_0(\Omega))$, and so

$$\int_{t_1}^{t_2} \left(\left\langle \frac{\partial f(\tau)}{\partial t}, \vartheta(\tau) - g(\tau)\right\rangle + \int_\Omega \eta(\tau)\frac{\partial^2 g(\tau)}{\partial t^2}\right) d\tau \leq K_1 \int_{t_1}^{t_2} \left\|\frac{\partial f(\tau)}{\partial t}\right\|_{H^{-s}(\Omega)} d\tau + K_2 \int_{t_1}^{t_2} \left\|\frac{\partial^2 g(\tau)}{\partial t^2}\right\|_{L^2(\Omega)} d\tau$$

for some constants $K_1, K_2 \geq 0$ for any $t_2 \geq t_1 \geq 0$. Setting H to be the function

$$H(\cdot) := K_1 \left\| \frac{\partial f(\cdot)}{\partial t} \right\|_{H^{-s}(\Omega)} + K_2 \left\| \frac{\partial^2 g(\cdot)}{\partial t^2} \right\|_{L^2(\Omega)} \in L^1(0,\infty),$$

it follows that

$$E(t_2) - \int_0^{t_2} H(\tau)\, d\tau \leq E(t_1) - \int_0^{t_1} H(\tau)\, d\tau \quad \text{for all } t_2 \geq t_1 \geq 0.$$

This implies that $\lim_{t\to\infty} E(t)$ exists, which we write as E_∞ and, by definition (6.79),

$$\lim_{t\to\infty} \langle \tilde{\mathcal{L}}^s_A(\vartheta(t) - g(t)), \vartheta(t) - g(t)\rangle = 2E_\infty - \frac{2}{C_\gamma} \int_0^\infty \left\| \frac{\partial \vartheta(\tau)}{\partial t} \right\|^2_{L^2(\Omega)} d\tau + 2\langle f_\infty, \vartheta_\infty - g_\infty\rangle =: l_\infty \tag{6.81}$$

since η is bounded in $L^2(\Omega)$ and $\frac{\partial g(t)}{\partial t} \to 0$ in $L^2(\mathbb{R}^d)$ as $t \to \infty$.

Next, taking a sequence $\{t_n\}_n$ with $t_n \to \infty$ so that

$$\frac{d\eta}{dt}(t_n) \to 0 \text{ in } H^{-s}(\Omega),$$

which is always possible by (6.75), we have, recalling that ϑ_∞ is the weak temperature solution to (6.70),

$$\tilde{\mathcal{L}}^s_A(\vartheta(t_n) - g(t_n)) = f(t_n) - \frac{d\eta}{dt}(t_n) \to f_\infty = \tilde{\mathcal{L}}^s_A(\vartheta_\infty - g_\infty) \text{ in } H^{-s}(\Omega). \tag{6.82}$$

Therefore, by (6.78), (6.82) and (6.81),

$$\langle \tilde{\mathcal{L}}^s_A(\vartheta(t_n) - g(t_n)), \vartheta(t_n) - g(t_n)\rangle \to \langle \tilde{\mathcal{L}}^s_A(\vartheta_\infty - g_\infty), \vartheta_\infty - g_\infty\rangle = l_\infty. \tag{6.83}$$

Finally, since the duality in the left-hand-side of (6.81) is equivalent to the square of the $H^s_0(\Omega)$ norm of $\vartheta(t) - g(t)$ by (6.14), we may conclude the strong convergence result

$$\vartheta(t) - g(t) \to \vartheta_\infty - g_\infty \text{ in } H^s_0(\Omega) \text{ as } t \to \infty.$$

□

Remark 6.25. *Since $\eta(t) = b(\vartheta(t)) + \chi(t)$, $\chi(t) \in H(\vartheta(t))$, and $\eta(t) \xrightarrow[t\to\infty]{} \eta_\infty$ in $L^2(\Omega)$-weak and $\vartheta(t) \to \vartheta_\infty$ in $L^2(\Omega)$, we have the existence of a $\chi_\infty \in H(\vartheta_\infty)$, such that $\chi(t) \xrightarrow[t\to\infty]{} \chi_\infty$ in $L^\infty(\Omega)$-weak*.*

Remark 6.26. *Similar asymptotic results as $t \to \infty$ for the case $s = 1$ have been obtained in [86] considering other variants on the asymptotic behaviour of f and $\tilde{g}$.*

Earlier asymptotic behaviour results for $s = 1$ were obtained in Remarks 9 and 11 of [228] in the variational inequality form in a special case.

6.7 From Two Phases to One Phase

Let ν be a parameter such that (6.13) written with the Lipschitz graph γ^ν corresponds to the two-phase problem when $\nu > 0$, and to the one-phase problem when $\nu = 0$. In this section, we obtain the solution to the one-phase problem, making use of the solution to the two-phase problem.

Consider the one-phase problem given with data f^o, $\tilde{g}^o \geq 0$ by

$$-\int_{Q_T} \eta^o \frac{\partial \xi}{\partial t} + \int_{\mathbb{R}^d} AD^s\vartheta^o \cdot D^s\xi = \int_{Q_T} f^o\xi + \int_\Omega \eta^o_0 \xi(0), \quad \forall \xi \in \Xi^s_T \tag{6.13$_{1ph}$}$$

with initial condition $\eta^o(0,x) = \eta^o_0(x)$ with regularity as in Theorem 6.16 and $\vartheta^o = \gamma^o(\eta^o)$ such that $\vartheta^o(0) - g^o(0) \in H^s_0(\Omega)$. In this section, we use the lower subscript o to indicate the one-phase problem, and the upper superscript 0 to indicate the initial condition. We first show that there exists a solution to this problem, by obtaining the solution as the limit of a sequence of solutions to two-phase problems. The main idea is that we flatten the left leg of the monotone Lipschitz graph γ to obtain γ^o which has range $[0,\infty[$. Then γ^o will still satisfy the same conditions (6.16) at $r=+\infty$. Furthermore, we define the convex functional ϕ_t^o by

$$\phi_t^o(W)=\begin{cases}\int_\Omega(j^o(W)-g^o(t)W)\,dx & \text{for } W\in L^2(\Omega);\\ +\infty & \text{for } W\in H^{-s}(\Omega)\backslash L^2(\Omega)\end{cases}$$

for the primitive j^o of γ^o chosen such that j^o vanishes at 0.

Remark 6.27. *Observe that the image of γ^o is $[0,\infty[$. Therefore, given any $\eta_0^o\in L^2(\Omega)$, $\vartheta^o(0)=\gamma^o(\eta_0^o)\geq 0$. This also applies to $\eta^o(t)\in L^2(\Omega)$ at general time $t\in[0,T]$, so we have $\vartheta^o(t)=\gamma^o(\eta^o(t))\geq 0$ for all t. As such, it is necessary that the Dirichlet boundary condition g^o is non-negative in $]0,T[\times\Omega^c$.*

Theorem 6.28. *Let $f^o\in L^2(0,T;L^2(\Omega))$ and $\tilde{g}^o\in W^{2,1}(0,T;L^2(\mathbb{R}^d))\cap L^\infty(0,T;H^s(\mathbb{R}^d))$, and define g^o as in* (6.7) *(and subsequently with the same regularity). Assume $\eta_0^o\in L^2(\Omega)$ and, setting $0\leq\vartheta^o(0)=\gamma^o(\eta_0^o)$, assume $\tilde{g}^o\geq 0$ in $]0,T[\times\Omega^c$ and $\vartheta^o(0)-g^o(0)\in H_0^s(\Omega)$. Then, there exist a unique generalised enthalpy solution η^o and a weak temperature solution ϑ^o to the variational problem (6.13$_{1ph}$) with*

$$\eta^o\in\beta^o(\vartheta^o)\quad \text{and}\quad \vartheta^o=\gamma^o(\eta^o)\geq 0,$$

such that

$$\eta^o\in L^\infty(0,T;L^2(\Omega))\cap H^1(0,T;H^{-s}(\Omega)) \tag{6.84}$$

and

$$\vartheta^o\in L^\infty(0,T;H^s(\mathbb{R}^d))\cap H^1(0,T;L^2(\Omega)) \tag{6.85}$$

with $\vartheta^o=g^o$ in Ω^c.

Proof. We construct η^o and ϑ^o as the limit of an approximating sequence of η^ν and ϑ^ν. (See also the proof of Theorem A.1 in [233].)

Indeed, since γ^o is non-negative,

$$\lim_{|r|\to+\infty}\frac{\gamma^o(r)}{r}\geq 0.$$

Then, consider the strictly increasing approximation given by

$$\gamma^\nu(r)=\gamma^o(r)+\nu r \tag{6.86}$$

for $\nu>0$. Assuming γ^o is Lipschitz continuous, so is γ^ν. Also, γ^ν clearly converges to γ^o uniformly on compact sets as ν tends to zero. Furthermore,

$$\liminf_{|r|\to+\infty}\frac{\gamma^\nu(r)}{r}\geq\nu+\liminf_{|r|\to+\infty}\frac{\gamma^o(r)}{r}>0,$$

so (6.16) is satisfied. The corresponding maximal monotone graph β^ν is then given by

$$\beta^\nu(r)=\frac{1}{\nu}(r-(Id+\nu\beta^o)^{-1}(r)), \tag{6.87}$$

which is Lipschitz continuous with constant $\frac{1}{\nu}$. Therefore, from Theorem 6.3 and Theorem 6.16, we obtain the unique generalised enthalpy and weak temperature solutions η^ν and ϑ^ν of the approximate regularised problem with approximating compatible functions f^ν, g^ν and η_0^ν in the same spaces as the ones of the data

$$-\int_{Q_T}\eta^\nu\frac{\partial\xi}{\partial t}+\int_{\mathbb{R}^d\times[0,T]}AD^s\vartheta^\nu\cdot D^s\xi=\int_{Q_T}f^\nu\xi+\int_\Omega\eta_0^\nu\xi(0),\quad\forall\xi\in\Xi_T^s, \tag{6.13$^\nu$}$$

such that $\eta^\nu=\beta^\nu(\vartheta^\nu)$ are uniformly bounded in $H^1(0,T;H^{-s}(\Omega)\cap L^\infty(0,T;L^2(\Omega))$ for $\nu<1$, since the estimates (6.67)–(6.68) are independent of ν with

$$\phi_{t,n}^\nu(\eta_0^\nu)=\int_\Omega(j^\nu(\eta_0^\nu)+g^\nu(0)\eta_0^\nu)+I_{F_n^*}$$

$$= \int_\Omega (j^o(\eta_0^\nu) + \nu|\eta_0^\nu|^2 + g^\nu(0)\eta_0^\nu) + I_{F_n^*}$$
$$\leq \int_\Omega (j^o(\eta_0^\nu) + |\eta_0^\nu|^2 + g^\nu(0)\eta_0^\nu) + I_{F_n^*}$$

for uniformly bounded $\eta_0^\nu, g^\nu(0) \in L^2(\Omega)$. We recall that $(F_n)_{n\in\mathbb{N}}$ is an increasing set of finite dimensional subspaces of $H_0^s(\Omega)$, $F_n^* = \mathcal{L}(F_n) \subset H^{-s}(\Omega)$, and $I_{F_n^*}$ is the indicator function of F_n^*, i.e. $I_{F_n^*} = 0$ in F_n^*, $I_{F_n^*} = +\infty$ elsewhere.

Henceforth, taking $C_\gamma = C_{\gamma^o} + 1$ in (6.54) and making use of (6.53) at the limit $n \to \infty$, we obtain that $\frac{\partial \vartheta^\nu}{\partial t}$ is bounded in $L^2(0,T;L^2(\Omega))$ and $\vartheta^\nu - g^\nu$ is bounded in $L^\infty(0,T;H_0^s(\Omega))$ independently of ν. Passing to the limit as ν tends to zero, since η^ν is bounded in $H^1(0,T;H^{-s}(\Omega))$ as a solution to (6.13$_\nu$), we have $(\eta^{\nu_n})_n$ converging in $H^1(0,T;H^{-s}(\Omega))$-weak and in $L^\infty(0,T;L^2(\Omega))$ in the weak* topology, to some η^o. Similarly, $(\vartheta^{\nu_n})_n = \big(\gamma^{\nu_n}(\eta^{\nu_n})\big)_n$ converges weakly in $H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^s(\mathbb{R}^d))$, and by compactness also in $C([0,T];L^2(\Omega))$, to some ϑ^o such that $\vartheta^o(t) - g^o(t) \in H_0^s(\Omega)$ a.e. t. Passing to the limit, ϑ^o satisfies (6.13$_{1ph}$) with the required regularity (6.85). Also, by the maximal monotonicity of β^o and the Mosco convergence of β^ν to β^o, we have $\eta^o \in \beta^o(\vartheta^o)$ and $\eta_0^o \in \beta^o(\vartheta^o(0))$ satisfying (6.13$_{1ph}$) and (6.84). Subsequently, $\vartheta^o = \gamma^o(\eta^o)$ a.e. in $]0,T[\times\Omega$ and $\vartheta^o(0) = \lim_{\nu\to 0}\gamma^\nu(\eta_0^\nu) = \gamma^o(\eta_0^o)$ by the convergence of η_0^ν to η_0^o in $L^2(\Omega)$. Since the range of γ^o is $[0,\infty[$, $\vartheta \geq 0$ and we obtain the solution of the one-phase problem. ☐

Having obtained a unique solution to the limiting one-phase problem, we now show that the solutions of the two-phase problem given by

$$-\eta^\nu \int_{Q_T} \frac{\partial \xi}{\partial t} + \int_{\mathbb{R}^d\times[0,T]} AD^s(\vartheta^\nu - g^\nu)\cdot D^s\xi = \int_{Q_T} f^\nu \xi + \int_\Omega \eta_0^\nu \xi(0) \quad \forall \xi \in \Xi_T^s \tag{6.13$_{2ph}$}$$

with $\vartheta^\nu = \gamma^\nu(\eta^\nu)$ in fact converges to the one-phase problem (6.13$_{1ph}$). For the classical case of $s = 1$, see also [225], as well as the proof of Theorem 6.1 on pages 44-45 of [84]).

Theorem 6.29. *Assume that for each $\nu \geq 0$, $f^\nu \in L^2(0,T;L^2(\Omega))$, $\tilde{g}^\nu \in W^{2,1}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^s(\mathbb{R}^d))$ bounded independently of ν, and $\eta_0^\nu \in \overline{D(\phi_t^\nu)}$. Writing $\vartheta^\nu = \gamma^\nu(\eta^\nu)$ for the Lipschitz graph γ^ν with a uniform Lipschitz constant C_γ for all $\nu \geq 0$, assume that $\eta_0^\nu \in L^2(\Omega)$ and, setting $0 \leq \vartheta^\nu(0) = \gamma^\nu(\eta_0^\nu)$, assume $\tilde{g}^\nu \geq 0$ in $]0,T[\times\Omega^c$ and $\vartheta^\nu(0) - g^\nu(0) \in H_0^s(\Omega)$ is bounded uniformly in ν for $\nu \geq 0$. Let $(\eta^\nu, \vartheta^\nu)$ be the unique solution of the fractional two-phase Stefan-type problem (6.13$_{2ph}$), while (η^o, ϑ^o) is the unique solution of the fractional one-phase Stefan-type problem (6.13$_{1ph}$) with $0 \leq \vartheta^o = \gamma^o(\eta^o)$. Suppose that $\eta_0^\nu \rightharpoonup \eta_0^o$ in $L^2(\Omega)$, $f^\nu \rightharpoonup f^o$ in $L^2(0,T;L^2(\Omega))$, $g^\nu \rightharpoonup g^o$ in $W^{2,1}(0,T;L^2(\mathbb{R}^d))$-weak and in $L^\infty(0,T;H^s(\mathbb{R}^d))$-weak*, and γ^ν converges to γ^o uniformly on compact sets as ν tends to zero. Then,*

$$\eta^\nu \rightharpoonup \eta^o \textit{ in } H^1(0,T;H^{-s}(\Omega))\textit{-weak and in } L^\infty(0,T;L^2(\Omega))\textit{-weak}^* \textit{ as } \nu \searrow 0$$

and

$$\vartheta^\nu \rightharpoonup \vartheta^o \textit{ in } H^1(0,T;L^2(\Omega))\textit{-weak, in } L^\infty(0,T;H^s(\mathbb{R}^d))\textit{-weak}^* \textit{ and in } C([0,T];L^2(\Omega)) \textit{ as } \nu \searrow 0.$$

Proof. Indeed, as in the previous theorem, since $\eta^\nu \in \beta^\nu(\vartheta^\nu)$ is a solution to (6.13$_{2ph}$), it is bounded in $H^1(0,T;H^{-s}(\Omega))$. Passing to a subsequence, we have $(\eta^{\nu_n})_n$ converging in $H^1(0,T;H^{-s}(\Omega))$-weak and in $L^\infty(0,T;L^2(\Omega))$ in the weak* topology, to some η^o.

Furthermore,

$$\tilde{\mathcal{L}}_A^s(\gamma^\nu(\eta^\nu) - g^\nu) = \partial\phi_t^\nu(\eta^\nu) = f^\nu - \frac{\partial \eta^\nu}{\partial t} \rightharpoonup f^o - \frac{\partial \eta^o}{\partial t} = \partial\phi_t^o(\eta^o) = \tilde{\mathcal{L}}_A^s(\gamma^o(\eta^o) - g^o) \text{ weakly in } L^2(0,T;H^{-s}(\Omega).$$

Therefore, by applying $(\tilde{\mathcal{L}}_A^s)^{-1}$, $w^\nu = \gamma^\nu(\eta^\nu) - g^\nu$ converges weakly to $w^o = \gamma^o(\eta^o) - g^o$ in $L^2(0,T;H_0^s(\Omega))$. But η^ν is in $L^\infty(0,T;L^2(\Omega))$ for each ν by Theorem 6.3 since η^ν is the generalised enthalpy solution to the Stefan-type problem (6.13$_{2ph}$), bounded independent of $\nu > 0$ for ν small enough. Therefore, by the assumptions, we can again obtain a priori estimates on $\vartheta_\nu = \gamma^\nu(\eta^\nu)$ in $L^\infty(0,T;H^s(\mathbb{R}^d)) \cap H^1(0,T;L^2(\Omega))$, and the conclusion follows as in the proof of the previous theorem. ☐

Remark 6.30. *Similarly to the convergence of the two-phase problem, it is possible to extend the results of Sections 4 and 5 to the one-phase problem.*

6.8 Nonlocal Stefan-Type Problems

In this next part of the chapter, we consider the Stefan-type problems defined instead with the nonlocal operator $\mathcal{L}_a$ from Chapter 2. Much work has been done to investigate such problems, in particular in the case where γ is given by the power law as for porous medium-type equations. In particular, the author in [237] made use of the fractional Laplacian to replace the non-linear graph in the porous medium equation (see also [11],[65], [64], [210], [235], [38], [63], [222], [7] and [149] for other works considering nonlocal interaction terms, as well as [66] for a derivation, from microscopic dynamics, of the equation defined with the fractional Laplacian), which was shown to be in fact equivalent to the nonlocal bilinear form of the fractional Laplacian. Assuming an m-accretive operator on $L^1(\Omega)$, he generalised the result to various types of non-degenerate and degenerate kernels, as well as to the fractional p-Laplacian, which is a generalisation of results from [236]. Similarly, the homogeneous porous medium equation with no source term and no boundary data with the spectral fractional Laplacian was considered in [43] and [36] with no further assumptions on the operator, and a generalisation of the spectral fractional Laplacian to the killed semigroup of Brownian motion was considered for the Cauchy problem in [170]. A porous medium-type equation with nonlocal operators has also been considered in [105] for operators that are obtained as generators of Lévy processes with even kernels of the form $a(x,y) = a(x-y) = a(y-x)$, and in [81] for nonlocal operators with a finite horizon, while more general nonlocal operators similar to ours have been considered in [209], [230] and [137] for the homogeneous Cauchy problem on unbounded domains.

In this section, we first show the existence and uniqueness of the Stefan-type problem given similarly by (6.11), but with the nonlocal operator $\mathcal{L}_a$. This follows exactly the same proof as for the fractional $\tilde{\mathcal{L}}_A$, with simple minor modifications, and is a generalisation of the result given in [42] which considered only the restricted fractional Laplacian and the spectral fractional Laplacian.

Next, extending the T-monotonicity of $\mathcal{L}_a$ given in Section 2.3, we have, in addition, that $\mathcal{L}_a \circ \gamma$ is T-accretive, thereby allowing us to extend to the case with L^1 data in the next section 6.8.2 to obtain mild solutions using the results of [44]. Finally, we conclude this section by observing some properties of the mild solutions.

6.8.1 Existence and uniqueness of solution to nonlocal Stefan-type problems

Indeed, repeating the proof of Theorem 6.3 with the same convex function ϕ_t, we have the equivalent of (6.30) given by

$$V \in \partial\phi_t(U) \text{ in } H^{-s}(\Omega) \quad \text{if and only if} \quad U \in L^2(\Omega) \text{ and } \mathcal{L}_a^{-1}(V) + g = \gamma(U) \text{ a.e. in } \Omega, \tag{6.88}$$

which gives

$$\left\langle \frac{d\eta}{dt}, \xi \right\rangle + \langle \mathcal{L}_a(\gamma(\eta) - g), \xi\rangle = \langle f, \xi\rangle, \quad \forall \xi \in L^2(0,T; H^s_0(\Omega))$$

and the regularity

$$\eta \in L^\infty(0,T; L^2(\Omega)) \cap H^1(0,T; H^{-s}(\Omega))$$

with

$$\gamma(\eta) - g \in L^2(0,T; H^s_0(\Omega)).$$

Therefore, we have the following theorem

Theorem 6.31. *Suppose $a(x,y)$ is symmetric. Let $f \in L^2(0,T;H^{-s}(\Omega))$ and $g \in BV(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\mathbb{R}^d))$. Assume $\eta_0 \in L^2(\Omega)$ and γ satisfies* (6.16)*. Then there exists a unique generalised solution u to the problem*

$$\left\langle \frac{d\eta}{dt}, \xi \right\rangle + \langle \mathcal{L}_a(\gamma(\eta) - g), \xi\rangle = \langle f, \xi\rangle, \quad \forall \xi \in L^2(0,T; H^s_0(\Omega)) \tag{6.89}$$

with initial data

$$\eta(0) = \eta_0 \text{ in } H^{-s}(\Omega),$$

such that

$$\eta \in L^\infty(0,T; L^2(\Omega)) \cap H^1(0,T; H^{-s}(\Omega)), \quad \gamma(\eta) - g \in L^2(0,T; H^s_0(\Omega)).$$

Furthermore, the generalised solution satisfies (6.23)–(6.25).

Remark 6.32. *Observe that the symmetry of the kernel $a(x,y)$ is necessary so that we can use the operator $\mathcal{L}_a^{-1}$ in order to work with the inner product in dual space.*

This result generalises that of [19] and [42] which only considered homogeneous data and zero boundary conditions. See also Theorem 1.1 of [190] or Corollary 1.2 of [39] for similar results for Dirichlet forms using other approaches.

Next, we have the important result that the operator $\mathcal{L}_a \circ \gamma$ is T-accretive in L^2 for $a(x,y)$ symmetric, i.e.

Proposition 6.33. *Suppose $a(x,y)$ is symmetric. Let η and $\hat{\eta}$ denote two solutions of the nonlocal Stefan-type problems corresponding to (f,η_0) and $(\hat{f},\hat{\eta}_0)$ respectively with the same Dirichlet boundary condition g, where $f,\hat{f} \in L^2(0,T;L^2(\Omega))$ such that $f-\hat{f} \in L^1(Q_T)$ and $\eta_0,\hat{\eta}_0$ satisfying the same assumptions as Theorem 6.31. Then, the following energy estimates hold for every $t>0$:*

$$\int_\Omega (\eta(t)-\hat{\eta}(t))^+ \leq \int_\Omega (\eta_0-\hat{\eta}_0)^+ + \int_0^t \int_\Omega (f-\hat{f})^+$$

and

$$\int_\Omega |\eta(t)-\hat{\eta}(t)| \leq \int_\Omega |\eta_0-\hat{\eta}_0| + \int_0^t \int_\Omega |f-\hat{f}|.$$

Proof. Multiplying (6.11) by a test function $\xi \in L^2(0,T;H_0^s(\Omega))$, we obtain, for the difference of the two equations corresponding to η and $\hat{\eta}$,

$$\int_0^T \left\langle \frac{\partial}{\partial t}(\eta-\hat{\eta}),\xi \right\rangle + \langle \mathcal{L}_a(v-\hat{v}),\xi\rangle = \int_0^T \left\langle f-\hat{f},\xi \right\rangle, \quad \forall \xi \in L^2(0,T;H_0^s(\Omega)),$$

where we have written $v := \gamma(\eta)$. Now we take the test function ξ to be $\varphi_\delta(w(\tau,x))\chi_{[0,t]}(\tau)$, where $\varphi_\delta(w)$ is the approximation of the $sign^+$ (Heaviside) function, given by

$$\varphi_\delta(w) = \begin{cases} \frac{1}{\delta}(w-(Id+\delta sign^+)^{-1}w) = sign^+(w) & \text{for } w \geq \delta \\ w/\delta & \text{for } w \leq \delta \\ 0 & \text{for } w \leq 0, \end{cases}$$

where $w = v-\hat{v}$.

Recall that for symmetric $a(x,y) \geq 0$,

$$\langle \mathcal{L}_a u, v\rangle = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (v(x)-v(y))(\eta(x)-\eta(y))a(x,y)\,dx\,dy \quad a.e. x \neq y.$$

Since φ_δ is monotone in w and $\varphi_\delta(0)=0$, we have

$$(\varphi_\delta(w(x))-\varphi_\delta(w(y)))(w(x)-w(y)) = (\varphi_\delta(w(x))-\varphi_\delta(w(y)))(w(x)-w(y)) \geq 0 \quad \forall w \in H_0^s(\Omega).$$

Making use of the positivity of $a(x,y)$, we thus have

$$\langle \mathcal{L}_a w, \varphi_\delta(w)\rangle = \frac{1}{2}(\tau)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\varphi_\delta(w(x))-\varphi_\delta(w(y)))(w(x)-w(y))a(x,y)\,dx\,dy \geq 0 \quad \forall w \in H_0^s(\Omega),$$

so the bilinear term with test function $\varphi_\delta(w(\tau,x))\chi_{[0,t]}(\tau)$ is non-negative. Therefore, rewriting $w = (v-g)-(\hat{v}-g) = v-\hat{v}$, we have

$$\int_0^t \left\langle \frac{\partial}{\partial t}(\eta-\hat{\eta}),\varphi_\delta(v-\hat{v}) \right\rangle \leq \int_0^t \left\langle f-\hat{f},\varphi_\delta(v-\hat{v}) \right\rangle.$$

Observe that since γ is strictly increasing, $sign^+(v-\hat{v}) = sign^+(\eta-\hat{\eta})$. Thus, observing that $0 \leq \varphi_\delta \leq 1$, letting $\delta \to 0$ gives

$$\int_0^t \left\langle \frac{\partial}{\partial t}(\eta-\hat{\eta}), sign^+(\eta-\hat{\eta}) \right\rangle$$

$$
\begin{aligned}
&= \int_0^t \left\langle \frac{\partial}{\partial t}(\eta - \hat{\eta}), \lim_{\delta \to 0} \varphi_\delta(v - \hat{v}) \right\rangle \\
&= \lim_{\delta \to 0} \int_0^t \left\langle \frac{\partial}{\partial t}(\eta - \hat{\eta}), \varphi_\delta(v - \hat{v}) \right\rangle \\
&\leq \lim_{\delta \to 0} \int_0^t \left\langle f - \hat{f}, \varphi_\delta(v - \hat{v}) \right\rangle \\
&= \int_0^t \left\langle (f - \hat{f})^+, sign^+(\eta - \hat{\eta}) \right\rangle - \int_0^t \left\langle (f - \hat{f})^-, sign^+(\eta - \hat{\eta}) \right\rangle \\
&\leq \int_0^t \left\langle (f - \hat{f})^+, sign^+(\eta - \hat{\eta}) \right\rangle \\
&\leq \int_0^t \int_\Omega (f - \hat{f})^+
\end{aligned}
$$

by Lebesgue's dominated convergence theorem, since we can pass to the limit in δ using the weak lower semi-continuity of the L^1 norm on $L^2(\Omega)$. Computing the integral over time on the left-hand-side gives

$$
\left\|(\eta(t) - \hat{\eta}(t))^+\right\|_{L^1(\Omega)} \leq \left\|(\eta_0 - \hat{\eta}_0)^+\right\|_{L^1(\Omega)} + \int_0^t \left\|(f - \hat{f})^+\right\|_{L^1(\Omega)},
$$

which can be rewritten as the first estimate.

To obtain the second estimate, we exchange η and $\hat{\eta}$, and then add the two inequalities. □

Remark 6.34. *Observe that g is fixed, since we only have a relationship between η and v, and not between η and $v - g$. Suppose otherwise that we have g and $\hat{g}$. Then the first term (time-derivative term) involves a duality between $\frac{\partial}{\partial t}(\eta - \hat{\eta})$ and $(v - \hat{v}) - (g - \hat{g})$, which needs to be considered separately, but when considered separately, is no longer in $H_0^s(\Omega)$ and the duality no longer makes sense.*

However, having obtained $\mathcal{L}_a \circ \gamma$ is T-accretive, we also have the operator $\mathcal{A}$ defined by $\mathcal{A}\eta := \mathcal{L}_a \circ \gamma(\eta) - \mathcal{L}_a g$ being T-accretive, since $\mathcal{L}_a g = 0$ a.e. in Ω. As a result, we are still able to consider inhomogeneous Dirichlet boundary conditions.

Remark 6.35 (Comparison Principle)**.** *Consequently, if $f \geq 0$ and $g \geq -\lambda$ for any fixed $\lambda > 0$, then the weak solution $v = \gamma(\eta)$ satisfies $v \geq -\lambda$ a.e. in $]0, T[\times\Omega$, provided $v(0) \geq -\lambda$.*

Remark 6.36. *It is not known whether the bilinear term $\langle \mathcal{L}_a w, \varphi_\delta(w) \rangle$ is non-negative when $a(x, y)$ is not symmetric. However, the solution to the generalised Stefan-type problem only exists when $a(x, y)$ is symmetric, so we only consider the symmetric case.*

By the T-accretivity of the operator $\mathcal{L}_a \circ \gamma$ in $L^2(\Omega)$, we can obtain that the solution exponentially decays to the stationary solution, making use of the classical results of Pazy [184] (see also Proposition 19.1 of [34]).

Corollary 6.37 (Asymptotic Behaviour of Solution in L^2)**.** *There exists a unique $\eta_\infty \in L^2(\Omega)$ such that $\mathcal{L}_a(\gamma(\eta_\infty)) \ni f$. Moreover, for every solution η,*

$$
\lim_{t \to \infty} e^{-t(\mathcal{L}_a \circ \gamma)}\eta = \eta_\infty.
$$

As a result, we have that the functions $t \mapsto |\eta(t)|$ and $t \mapsto \left|\frac{du}{dt}(t)\right|$ are nonincreasing on $[0, +\infty[$.

6.8.2 The L^1 framework for the nonlocal Stefan-type problem

In this next subsection, we will extend the results from the previous section, to the L^1 framework. Much of the analysis has been done in [44], which assumed that the operator has L^1-contraction properties. They gave the nonlocal operator $\mathcal{L}_a$ as an example of such an operator, in Section 3.3 of [44], but did not show that $\mathcal{L}_a$ does possess the L^1-contraction property. Here, we complete the proof and show that, in fact, the nonlocal operator $\mathcal{L}_a$ with a not necessarily symmetric kernel $a(x,y)$ can be used to extend the L^1-contraction property from the Hilbertian framework in Proposition 6.33.

As such, the Stefan-type problem defined with $\mathcal{L}_a$ for a symmetric $a(x,y)$ possesses the properties derived in [44], which is a generalisation of the classical result given in [234], a generalisation of the results obtained in [183] for the fractional Laplacian which corresponds to a specific kernel as given in (2.5), and is also more general than the kernels considered in [105]. It is not known how a similar theory for Neumann boundary conditions can be developed, but works based on non-singular kernels such as Chapter 5 of [13] or for the spectral Laplacian such as in [1] may provide some insights.

Consider the operator $\mathcal{L} = \mathcal{L}_a : H_0^s(\Omega) \to H^{-s}(\Omega)$, and the stationary homogeneous Dirichlet problem

$$\mathcal{L}_p u = f \quad \text{ for } f \in L^p(\Omega), \quad 1 \le p < \infty$$

such that $\mathcal{L}_p$ is the restriction of the image of $\mathcal{L}_a$ to $L^p(\Omega)$. For $p = 2$, the natural definition of the domain $D(\mathcal{L}_2)$ is given by

$$D(\mathcal{L}_2) = \{u \in H_0^s(\Omega) : \mathcal{L}_2 u \in L^2(\Omega)\}.$$

In this case, $\mathcal{L}_2$ corresponds to $\mathcal{L}_a$, so the Lax-Milgram theorem gives a solution to the Dirichlet problem by Theorem 2.5.

Furthermore, we note that the existence result in Theorem 6.31 holds for all $f \in L^2(0,T; L^p(\Omega))$ for $2 \ge p > 2^\# = \frac{2d}{d+2s}$ by the Sobolev embedding theorem 1.1, while the L^1-contraction Proposition 6.33 given above holds for all $f \in L^2(0,T; L^2(\Omega))$. We want to further extend these results to $f \in L^2(0,T; L^p(\Omega))$ for all $1 \le p \le 2$. To do so, we recall some definitions:

Recall that for bounded domains $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, we can define

$$H_0^s(\Omega) := \{u \in H^s(\mathbb{R}^d) : u = 0 \text{ a.e. in } \mathbb{R}^d \backslash \Omega\} \tag{6.90}$$

with norm

$$\|u\|^2_{H_0^s(\Omega)} := \iint_{D_\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}}\, dx\, dy \tag{1.18}$$

for $D_\Omega = \mathbb{R}^d \times \mathbb{R}^d \backslash (\Omega^c \times \Omega^c)$. Also, for $p \neq 2$, we denote $H_0^{s,p}(\Omega)$ to be the closure of $C_c^\infty(\Omega)$ with respect to the norm $\left\|D^s(\cdot)\right\|_{L^p(\mathbb{R}^d)}$, as given in Section 1.2.

For $f \in L^2(\Omega)$, we first recall that $\mathcal{L}_a$ with a (not necessarily symmetric) kernel $a(x,y)$ is T-monotone, i.e. it satisfies

$$\langle \mathcal{L}_a v, v^+ \rangle > 0 \quad \forall v \in H_0^s(\Omega) \text{ such that } v^+ \neq 0,$$

as given in Theorem 2.8 of Section 2.3.

For $f \in L^1(\Omega)$, the natural domain is

$$D(\mathcal{L}_1) = \{u \in H_0^{s,1}(\Omega) : u = 0 \text{ in } \Omega^c \text{ and } u, \mathcal{L}_1 u \in L^1(\Omega)\}.$$

Our main result is to show the L^1-contraction of the nonlocal linear operator $\mathcal{L}_1$, which will directly lead to the existence of unique mild solutions by the Crandall-Liggett theory (Theorem A and Theorem I of [78]), as shown in [44]. This is given as follows:

Theorem 6.38. *(a) $D(\mathcal{L}_1)$ is dense, and $(Id + \lambda\mathcal{L}_1)^{-1}$ is a contraction on $L^1(\Omega)$ for each $\lambda > 0$.*

(b) $D(\mathcal{L}_1) \subset W_0^{1,q}(\Omega)$ for $1 \le q < d/(d-s)$ and there is a constant $c = c(q) > 0$ such that $c\|u\|_{W^{1,q}(\Omega)} \le \|\mathcal{L}_1 u\|_{L^1(\Omega)}$ for $u \in D(\mathcal{L}_1)$.

(c) $\mathcal{L}_1$ is the L^1-closure $\bar{\mathcal{L}}_2$ of $\mathcal{L}_2$.

(d) $\sup_\Omega (Id + \lambda\mathcal{L}_1)^{-1} f \le \sup_\Omega f^+$ for each $\lambda > 0$ and $f \in L^1(\Omega)$, that is $\left\|(Id + \lambda\mathcal{L}_1)^{-1} f\right\|_{L^\infty(\Omega)} \le \left\|f^+\right\|_{L^\infty(\Omega)}$.

We will show this via a series of lemmas and theorems.

Lemma 6.39. *$D(\mathcal{L}_2) \subset D(\mathcal{L}_1)$.*

Proof. It is clear that if $\mathcal{L}u \in L^2(\Omega)$, then $\mathcal{L}u \in L^1(\Omega)$ for a bounded set Ω, by the Cauchy-Schwarz inequality. We also claim that $H_0^{s,2}(\Omega) \subset H_0^{s,1}(\Omega)$, so we have $D(\mathcal{L}_2) \subset D(\mathcal{L}_1)$ since $u = 0$ in Ω^c for both $D(\mathcal{L}_2)$ and $D(\mathcal{L}_1)$. Indeed, by Hölder's inequality and an elementary estimate involving the measure of finite measurable sets Ω (see, for example, Lemma 6.1 of [95]), and recalling $u \equiv 0$ outside Ω, we have that

$$\begin{aligned}
\frac{1}{c_{d,s}} \int_{\mathbb{R}^d \setminus B_R(\Omega)} |D^s u(x)|\, dx &= \int_{\mathbb{R}^d \setminus B_R(\Omega)} \left| \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+s}}\, dy \right| dx \\
&= \int_{\mathbb{R}^d \setminus B_R(\Omega)} \left| \int_{\mathbb{R}^d} \frac{-u(y)}{|x-y|^{d+s}}\, dy \right| dx \\
&= \int_{\mathbb{R}^d \setminus B_R(\Omega)} \left| \int_{\Omega} \frac{-u(y)}{|x-y|^{d+s}}\, dy \right| dx \\
&\le \int_{\mathbb{R}^d \setminus B_R(\Omega)} \int_{\Omega} \frac{|u(y)|}{|x-y|^{d+s}}\, dy\, dx \\
&\le \int_{\mathbb{R}^d \setminus B_R(\Omega)} \left(\int_{\Omega} |u(y)|^{\frac{d}{d-s}}\, dy \right)^{\frac{d-s}{d}} \left(\int_{\Omega} |x-y|^{-\frac{d}{s}(d+s)}\, dy \right)^{\frac{s}{d}} dx \\
&\le \|u\|_{L^{\frac{d}{d-s}}(\Omega)} \int_{\mathbb{R}^d \setminus B_R(\Omega)} \left(\int_{\Omega} |x-y|^{-\frac{d}{s}(d+s)} \right)^{\frac{s}{d}} dy\, dx \\
&\le \|u\|_{L^{\frac{d}{d-s}}(\Omega)} \int_{\mathbb{R}^d \setminus B_R(y)} \left(\int_{\Omega} |x-y|^{-\frac{d}{s}(d+s)}\, dy \right)^{\frac{s}{d}} dx \\
&\le C(\omega) \|u\|_{L^{\frac{d}{d-s}}(\Omega)} R^{-s}
\end{aligned}$$

since $B_R(y) \subseteq B_R(\Omega)$ for $y \in \Omega$, where $C(\omega)$ is a constant depending on the spherical measure ω. Then, by the Cauchy-Schwarz inequality and the Sobolev inequality (Lemma 1.1) since $\frac{d}{d-s} < \frac{2d}{d-2s}$ (see also Lemma 8 of [211]), for a constant $C(|\Omega|)$ depending on the measure of Ω,

$$\begin{aligned}
\int_{\mathbb{R}^d} |D^s u(x)|\, dx &= \int_{\mathbb{R}^d \setminus B_R(\Omega)} |D^s u(x)|\, dx + \int_{B_R(\Omega)} |D^s u(x)|\, dx \\
&\le C(\omega) c_{d,s} \|u\|_{L^{\frac{d}{d-s}}(\Omega)} R^{-s} + C(|\Omega|) \|D^s u\|_{L^2(B_R(\Omega))} \\
&\le C'(\omega, s, d) \|D^s u\|_{L^1(\mathbb{R}^d)} R^{-s} + C(|\Omega|) \|D^s u\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Taking R large enough such that $R^s = 2C'(\omega, s, d)$. Then we have $\|D^s u\|_{L^1(\mathbb{R}^d)} \le 2C(|\Omega|)\|D^s u\|_{L^2(\mathbb{R}^d)}$ and $H_0^{s,2}(\Omega) \subset H_0^{s,1}(\Omega)$. Since $\|u\|_{L^1(\mathbb{R}^d)} \le \|u\|_{L^2(\mathbb{R}^d)}$, we have $D(\mathcal{L}_2) \subset D(\mathcal{L}_1)$. ☐

Having obtained the inclusions of the domains, we now want to show the inclusions of the operators, i.e. $\mathcal{L}_1 \supset \bar{\mathcal{L}}_2$, where $\bar{\mathcal{L}}_2$ is the closure of $\mathcal{L}_2$ with respect to the L^1-topology. Observe that, see also Proposition 4 of [154] for the symmetric case, for a Lipschitz convex function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(0) = 0$,

$$\mathcal{E}_a(\Phi(u), v) \le \mathcal{E}_a(u, v\Phi'(u)) \quad \text{for } v \ge 0, v \in H_0^s(\Omega), \text{ weakly in } \Omega \tag{6.91}$$

for $u \in H_0^s(\Omega)$, where

$$\mathcal{E}_a(u, v) = \langle \mathcal{L}_a u, v \rangle.$$

Indeed, since $\Phi(a) - \Phi(b) \le \Phi'(a)(a-b)$ for all a, b if Φ is convex, we have, by the positivity of $a(x, y)$,

$$P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v(x)(\Phi(u(x)) - \Phi(u(y))) a(x,y)\, dy\, dx \le P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v(x) \Phi'(u(x))(u(x) - u(y)) a(x,y)\, dy\, dx \quad \forall v \ge 0.$$

The reverse inequality holds true for Φ concave.

We can then adapt the proof of Theorem 13 of [154] using the Moser technique to obtain the following theorem.

Theorem 6.40. *Let $f \in L^m(\Omega)$ with $m > d/2s$. Suppose $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary. Then, there exists a constant C, depending only on a_*, a^*, d, Ω, $\|u\|_{H_0^s(\Omega)}$, $\|g\|_{L^p(\Omega)}$ and s, such that the weak solution of*

$$\mathcal{L}u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \Omega^c$$

satisfies

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Proof. Assume, without loss of generality, that $|\Omega| = 1$. For $\beta > 1$ and $T > 0$ large, set the following convex function

$$\Phi(\sigma) = \Phi_T(\sigma) = \begin{cases} \sigma^\beta & \text{if } \quad 0 \leq \sigma < T, \\ \beta T^{\beta-1}\sigma - (\beta-1)T^\beta, & \text{if } \quad \sigma \geq T. \end{cases}$$

Since Φ is Lipschitz (with Lipschitz constant $\beta T^{\beta-1}$) and $\Phi(0) = 0$, we have $\Phi(u) \in H_0^s(\Omega)$ and as previously discussed, we have

$$\mathcal{E}_a(\Phi(u), v) \leq \mathcal{E}_a(u, v\Phi'(u)) \quad \forall v \geq 0, v \in H_0^s(\Omega). \tag{6.91}$$

Since Φ is positive, taking $v = \Phi(u)$ in the bilinear form, we use the Sobolev-Poincaré Lemma 1.1 and the symmetric form in Proposition 2.2 to obtain

$$\begin{aligned} \frac{a_*}{c_{d,s}^2 C_S^2}\left\|\Phi(u)\right\|_{L^{2^*}(\Omega)}^2 &\leq \frac{a_*}{2}\iint_{D_\Omega} \frac{|\Phi(u(x)) - \Phi(u(y))|^2}{|x-y|^{d+2s}}\,dx\,dy \\ &\leq \frac{1}{2}\iint_{D_\Omega} [\Phi(u(x)) - \Phi(u(y))]^2 a^{sym}(x,y)\,dx\,dy \\ &= \iint_{D_\Omega} \Phi(u(x))[\Phi(u(x)) - \Phi(u(y))]a(x,y)\,dx\,dy \\ &= \langle \Phi(u), \mathcal{L}\Phi(u)\rangle. \end{aligned}$$

Applying (6.91) then gives

$$\begin{aligned} \frac{a_*}{c_{d,s}^2 C_S^2}\left\|\Phi(u)\right\|_{L^{2^*}(\Omega)}^2 &\leq \langle \Phi(u), \mathcal{L}\Phi(u)\rangle \\ &\leq \langle \Phi(u)\Phi'(u), \mathcal{L}u\rangle \\ &= \langle \Phi(u)\Phi'(u), f\rangle \\ &\leq \left\|\Phi(u)\Phi'(u)\right\|_{L^{\frac{m}{m-1}}(\Omega)}\|f\|_{L^m(\Omega)} \\ &\leq \left\|\beta u^{2\beta-1}\right\|_{L^{\frac{m}{m-1}}(\Omega)}\|f\|_{L^m(\Omega)} \end{aligned}$$

since $\Phi(r) \leq r^\beta$ and $\Phi'(r) \leq \beta r^{\beta-1}$.

Setting $q = 2m'$ where m' satisfies $\frac{1}{m} + \frac{1}{m'} = 1$, and letting $T \to \infty$, we have, by the lower-semicontinuity of liminfs as $\Phi(r) \leq r^\beta$ and the Cauchy-Schwarz inequality,

$$\begin{aligned} \left(\int_\Omega |u|^{2^*\beta}\right)^{\frac{1}{2^*\beta}} &= \|u\|_{L^{2^*\beta}(\Omega)} \\ &= \liminf_{T\to\infty}\left\|\Phi(u)\right\|_{L^{2^*}(\Omega)}^{\frac{1}{\beta}} \leq \left(\frac{\beta c_{d,s}^2 C_S^2}{a_*}\|f\|_{L^m(\Omega)}\right)^{\frac{1}{2\beta}}\left(\int_\Omega |u|^{\frac{2\beta-1}{2}q}\right)^{\frac{1}{\beta q}} \\ &\leq \left(\frac{\beta c_{d,s}^2 C_S^2}{a_*}\|f\|_{L^m(\Omega)}\right)^{\frac{1}{2\beta}}\left(\int_{\{|u|\leq 1\}} |u|^{\frac{2\beta-1}{2}q} + \int_{\{|u|\geq 1\}} |u|^{\frac{2\beta-1}{2}q}\right)^{\frac{1}{\beta q}} \\ &\leq \left(\frac{\beta c_{d,s}^2 C_S^2}{a_*}\|f\|_{L^m(\Omega)}\right)^{\frac{1}{2\beta}}\left(\int_{\{|u|\leq 1\}} 1 + \int_{\{|u|\geq 1\}} |u|^{\beta q}\right)^{\frac{1}{\beta q}} \\ &\qquad \text{since } \beta, q \geq 1 \end{aligned}$$

$$\leq \left(\frac{\beta c_{d,s}^2 C_S^2}{a_*}\|f\|_{L^m(\Omega)}\right)^{\frac{1}{2\beta}} \left(|\Omega| + \int_\Omega |u|^{\beta q}\right)^{\frac{1}{\beta q}}$$
$$= \left(\frac{\beta c_{d,s}^2 C_S^2}{a_*}\|f\|_{L^m(\Omega)}\right)^{\frac{1}{2\beta}} \left(1 + \int_\Omega u^{\beta q}\right)^{\frac{1}{\beta q}}$$

The rest follows by a standard iteration process. Indeed, take $\beta_0 = 1$ and define $\beta_j = \left(\frac{2^*}{q}\right)^j$ (which is greater than 1 since $m > \frac{d}{2s}$) so that

$$\beta_{j+1} q = 2^* \beta_j.$$

Denote

$$A_j = \left(\int_\Omega u^{2^*\beta_j}\right)^{\frac{1}{2^*\beta_j}}, \quad c_j = \left(\frac{\beta_j c_{d,s}^2 C_S^2}{a_*}\|f\|_{L^m(\Omega)}\right)^{\frac{1}{2\beta_j}}.$$

Then, we have the recurrence formula

$$A_{j+1} \leq c_{j+1}(1 + A_j^{2^*\beta_j})^{\frac{1}{2^*\beta_j}}.$$

Renormalising if necessary, we may assume, without loss of generality, that $A_0 = 1$. Then, since we are in a bounded domain Ω, by L^p-embeddings (Hölder's inequality), $A_j \geq 1$ for all j. Taking logarithms,

$$\log A_{j+1} \leq \log c_{j+1} + \frac{1}{2^*\beta_j} \log\left(1 + A_j^{2^*\beta_j}\right) \leq \log c_{j+1} + \frac{1}{2^*\beta_j} + \log A_j,$$

by making use of the inequality $\log(1+x) \leq 1 + \log x$ if $x \geq 1$. By iteration,

$$\log A_{j+1} \leq \sum_{k=1}^{j+1} \log c_{j+1} + \sum_{k=1}^{j} \frac{1}{2^*\beta_j} + \log A_0. \tag{6.92}$$

Recall that by Sobolev embedding theorem (Lemma 1.1), $A_0 \leq \|u\|_{H_0^s(\Omega)}$. Also, observe that the two series

$$\sum_{k=1}^{\infty} \log c_{j+1} = \sum_{k=1}^{\infty} \frac{1}{2\beta_k} \log\left(\frac{\beta_k c_{d,s}^2 C_S^2}{a_*}\|f\|_{L^m(\Omega)}\right), \quad \sum_{k=1}^{\infty} \frac{1}{2^*\beta_j}$$

are convergent, since $\beta_k > 1$ for all k, so the ratio of the $k+1$-th term to the k-th term is always less than 1. Taking (6.92) to infinity gives the result, since $\lim_{j\to\infty} A_j = \|u\|_{L^\infty(\Omega)}$. □

Now, we take $f \in L^m(\Omega)$ for $1 \leq m \leq \frac{2d}{d+2s}$ and consider the following problem

$$\mathcal{L}u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \Omega^c$$

in distributional sense, i.e. $u \in L^1(\Omega)$ is a *weak solution* to the problem if for $f \in L^1(\Omega)$,

$$\int_\Omega u\psi \, dx = \int_\Omega f\phi \, dx, \tag{6.93}$$

for any

$$\phi \in \mathcal{T}(\Omega) := \{\phi : \mathcal{L}\phi = \psi \text{ in } \Omega, \phi = 0 \text{ in } \Omega^c, \psi \in C_c^\infty(\Omega)\}$$

with $\psi \in C_c^\infty(\Omega)$.

Remark 6.41. *Observe that if we consider $\psi \in L^m(\Omega)$, $m > d/2$ for the test function space, we recover the classical definition of duality solution as in [221] (see also, for instance, [6]). Therefore, we can obtain a solution of the adjoint problem if we have a weak solution in $L^1(\Omega)$.*

Indeed, using Theorem 6.40, ϕ is bounded and consequently all the terms in the identity (6.93) make sense.

Next, we develop the existence theorem for L^1-data, following the ideas of [154] but with a non-symmetric kernel. We need to introduce the truncation function $T_k := -k \vee (k \wedge u)$ for every $k \geq 0$, which will be useful for the proof.

Remark 6.42. *Observe that for $v \geq 0$, the given definition of T_k gives*

$$[v(x) - T_k(v(x))][T_k(v(x)) - T_k(v(y))] \geq 0 \qquad a.e.\ in\ \Omega \times \Omega.$$

Indeed,

$$\begin{cases} v(x) - T_k(v(x)) = v(x) - v(x) = 0 & \text{if } v(x) \leq k \text{ since } T_k(v(x)) = v(x), \\ T_k(v(x)) - T_k(v(y)) = k - k = 0 & \text{if } v(x) \geq k \text{ and } v(y) \geq k, \\ [v(x) - T_k(v(x))][T_k(v(x) - T_k(v(y))] = [v(x) - k][k - v(y)] \geq 0 & \text{if } v(x) \geq k \text{ and } v(y) \leq k. \end{cases}$$

In particular, this means that

$$v(x)[T_k(v(x) - T_k(v(y))] \geq T_k(v(x))[T_k(v(x) - T_k(v(y))] \qquad a.e.\ in\ \Omega \times \Omega,$$

or, by the positivity of $a(x,y)$,

$$\mathcal{E}_a(T_k(v), v) \geq \mathcal{E}_a(T_k(v), T_k(v)).$$

By symmetry and a similar analysis, we also have

$$|v(x) - y(y)|^2 \geq [v(x) - v(y)][T_k(v(x) - T_k(v(y))] \geq [T_k(v(x) - T_k(v(y))]^2 \qquad a.e.\ in\ \Omega \times \Omega.$$

Also, observe that $T_k(v)$ is a Lipschitz function of v with Lipschitz constant 1, so for $v \geq 0$, since $T_k(v) \geq 0$, we have

$$T_k(v(x))[T_k(v(x)) - T_k(v(y))] \leq T_k(v(x))[v(x) - v(y)] \qquad a.e.\ in\ \Omega \times \Omega.$$

Theorem 6.43. *There is a unique weak solution to*

$$\mathcal{L}u = f \tag{6.94}$$

for any $f \in L^1(\Omega)$, which is positive if f is positive. Furthermore,

$$\forall k \geq 0, \quad T_k(u) := -k \vee (k \wedge u) \in H_0^s(\Omega), \tag{6.95}$$

$$u \in L^q(\Omega) \quad \forall q \in \left]1, \frac{d}{d-2s}\right[\tag{6.96}$$

and

$$|D^s u| \in L^r(\Omega) \quad \forall r \in \left]1, \frac{d}{d-s}\right[. \tag{6.97}$$

Proof. The uniqueness follows since for $f_1 = f_2$, $\int_\Omega (u_1 - u_2)\psi\, dx = 0$ for any $\psi \in C_c^\infty(\Omega)$, so $u_1 \equiv u_2$.

For positivity, we first observe that for any v,

$$\mathcal{E}_a(v^+, v^-) = P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v^-(x)(v^+(x) - v^+(y)) a(x,y)\, dx\, dy = -P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} v^-(x) v^+(y) a(x,y)\, dx\, dy \leq 0,$$

since $v^+(x)v^-(x) = 0$ as v^+ and v^- cannot both be nonzero at the same point x, and $v^+, v^-, a(x,y) \geq 0$. Therefore, considering the test function $v = u^-$ for positive f, we have

$$0 \leq \int_\Omega f u^- = \langle \mathcal{L}u, u^- \rangle = \mathcal{E}_a(u, u^-) = \mathcal{E}_a\left(u^+, u^-\right) - \mathcal{E}_a\left(u^-, u^-\right) \leq -\mathcal{E}_a\left(u^-, u^-\right) \leq 0$$

by the positivity of a, so $u^- \equiv 0$.

Therefore, we can restrict the problem (6.94) to positive data which give positive solutions. The general case can then be obtained by decomposing the data into its positive and negative parts and then dealing with the two data separately, making use of the linearity of the operator.

To prove the existence of a solution, we obtain the solution as a limit of solutions to approximated problems.

Consider $f_n \in L^\infty(\Omega)$ $(f_n \geq 0)$ such that $f_n \to f$ in $L^1(\Omega)$, and let $u_n \in H_0^s(\Omega)$ be the solution to the problem

$$\mathcal{L}u_n = f_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{in } \Omega^c.$$

Multiplying this equation by $T_k(u_n)$ for $k \geq 0$ and integrating over Ω, we have

$$\begin{aligned}
&\frac{a_*}{c_{d,s}^2}\left\|T_k(u_n)\right\|_{L^{2^*}(\Omega)}^2 \\
&\leq \frac{a_*}{c_{d,s}^2} C_S \left\|D^s T_k(u_n)\right\|_{L^2(\mathbb{R}^d)}^2 && \text{by Lemma 1.1} \\
&\leq \frac{1}{2} C_S \, P.V. \iint_{D_\Omega} (T_k(u_n)(x) - T_k(u_n)(y))^2 a(x,y)\, dy\, dx && \text{by the coercivity of } a \text{ in (2.2)} \\
&= C_S \, P.V. \iint_{D_\Omega} T_k(u_n)(x)(T_k(u_n)(x) - T_k(u_n)(y)) a(x,y)\, dy\, dx \\
&\leq C_S \, P.V. \iint_{D_\Omega} T_k(u_n)(x)(u_n(x) - u_n(y)) a(x,y)\, dy\, dx && \text{by Remark 6.42} \\
&= C_S \langle \mathcal{L}u_n, T_k(u_n)\rangle \\
&= C_S \langle f_n, T_k(u_n)\rangle \\
&\leq C_S k \|f_n\|_{L^1(\Omega)} && \text{since } T_k \leq k
\end{aligned} \tag{6.98}$$

since u_n can be assumed to be positive, by taking positive f by the first part of this proof. Also, by the definition of T_k and 2^*, we have

$$k\, m\{x \in \Omega : u_n(x) \geq k\}^{\frac{d-2s}{2d}} = \left\|T_k(u_n)\right\|_{L^{2^*}(\{u_n \geq k\})} \leq \left\|T_k(u_n)\right\|_{L^{2^*}(\Omega)}.$$

Combining both inequalities, we have that

$$m\{x \in \Omega : u_n(x) \geq k\} \leq C \left(\frac{\|f_n\|_{L^1(\Omega)}}{k}\right)^{\frac{d}{d-2s}} \tag{6.99}$$

for some constant C, so u_n is bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{d}{d-2s}}(\Omega)$ (see, for example, Chapter V.3 of [224] for the definition of Marcinkiewicz spaces), and consequently (6.96) holds true.

It remains to show (6.97). Inspired by the results of [154] for the symmetric Dirichlet form, we prove a similar result here for the more general non-symmetric case. Fix $\lambda > 0$, we want to estimate the measure of the set $\{x \in \Omega : |D^s u_n(x)| \geq \lambda\}$. Observe that this set can be rewritten, for any positive k, as

$$\{x \in \Omega : |D^s u_n(x)| \geq \lambda\} = \{x \in \Omega : |D^s u_n(x)| \geq \lambda, u_n < k\} \cup \{x \in \Omega : |D^s u_n(x)| \geq \lambda, u_n \geq k\},$$

and therefore

$$\{x \in \Omega : |D^s u_n(x)| \geq \lambda\} \subset \{x \in \Omega : |D^s u_n(x)| \geq \lambda, u_n < k\} \cup \{x \in \Omega : u_n \geq k\}. \tag{6.100}$$

For the first set, applying the reverse inequality of (6.91) to the concave function $\Phi(u_n) = T_k(u_n)$ when restricted to positive values for the kernel

$$k_{\mathbb{I}} := \frac{1}{|x-y|^{d+2s}}$$

which corresponds to the fractional Laplacian (up to a constant), we have, for non-negative $u_n \in H_0^s(\Omega)$,

$$\mathcal{E}_{\mathbb{I}}(T_k(u_n), v) \geq \mathcal{E}_{\mathbb{I}}\left(u_n, v\chi_{\{x \in \Omega, u_n < k\}}\right) \quad \forall v \in H_0^s(\Omega), v \geq 0.$$

Taking $v = T_k(u_n)$ (since $u_n \geq 0$ implies $T_k(u_n) \geq 0$) with the kernel function $k_{\mathbb{I}}$ gives

$$\begin{aligned}
&\left\|D^s T_k(u_n)\right\|_{L^2(\mathbb{R}^d)} \\
&= \frac{1}{2} c_{d,s}^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|T_k(u_n(x)) - T_k(u_n(y))|^2}{|x-y|^{d+2s}} \, dx \, dy \\
&= c_{d,s}^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} T_k(u_n(x))(T_k(u_n(x)) - T_k(u_n(y))) \frac{1}{|x-y|^{d+2s}} \, dx \, dy \\
&\geq c_{d,s}^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} T_k(u_n(x)) \chi_{\{x \in \Omega, u_n < k\}} (u_n(x) - u_n(y)) \frac{1}{|x-y|^{d+2s}} \, dx \, dy \\
&= c_{d,s}^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_n(x) \chi_{\{x \in \Omega, u_n < k\}} (u_n(x) - u_n(y)) \frac{1}{|x-y|^{d+2s}} \, dx \, dy
\end{aligned}$$

since $T_k(u_n) = u_n$ for $0 < u_n < k$. Therefore, we have

$$m\{x \in \Omega : |D^s u_n(x)| \geq \lambda, u_n < k\} \leq \frac{1}{\lambda^2} \int_{\{x \in \Omega, u_n < k\}} |D^s u_n|^2 = \frac{1}{\lambda^2} \left\|D^s T_k(u_n)\right\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{k c_{d,s}^2}{\lambda^2 a_*} \|f_n\|_{L^1(\Omega)}$$

by (6.98).

For the second set, we just make use of the inequality (6.99). Therefore, combining the inequalities for both sets of (6.100), we have that, for every $k > 0$,

$$m\{x \in \Omega : |D^s u_n(x)| \geq \lambda\} \leq \frac{k c_{d,s}^2}{\lambda^2 a_*} \|f_n\|_{L^1(\Omega)} + C \left(\frac{\|f_n\|_{L^1(\Omega)}}{k} \right)^{\frac{d}{d-2s}}.$$

Minimising in k, we obtain

$$m\{x \in \Omega : |D^s u_n(x)| \geq \lambda\} \leq C' \left(\frac{\|f_n\|_{L^1(\Omega)}}{\lambda} \right)^{\frac{d}{d-s}} \tag{6.101}$$

for a constant C' depending on C and a_*. Therefore, $|D^s u_n|$ is bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{d}{d-s}}(\Omega)$, and so (6.97) holds true.

Finally, we want to pass to the limit, and show that the limit is the solution of the original problem (6.94). By linearity, for any $m, n \in \mathbb{N}$, $u_n - u_m$ solves

$$\mathcal{L}(u_n - u_m) = f_n - f_m \quad \text{in } \Omega, \quad u_n = u_m = 0 \quad \text{in } \Omega^c.$$

Choosing, for any $k > 0$, $T_k(u_n - u_m)$ as the test function, we can deduce, in a similar manner as done for (6.99), that

$$m\{x \in \Omega : |u_n - u_m| \geq k\} \leq C \left(\frac{\|f_n - f_m\|_{L^1(\Omega)}}{k} \right)^{\frac{d}{d-2s}}.$$

As $f_n, f_m \to f$ in $L^1(\Omega)$, for n, m large enough, we obtain that $\{u_n\}$ is a Cauchy sequence in measure. Consequently, up to subsequences, it converges in Ω almost everywhere, towards some function u.

Also, using the embedding of $\mathcal{M}^p(\Omega)$ spaces into $L^p(\Omega)$ for p finite, we have that u_n also converges to u in $L^q(\Omega)$ for any $1 \leq q \leq \frac{d}{d-2s}$. This allows us to pass to the limit in the equation and obtain a weak solution of (6.94), which, by uniqueness, is u. Since the whole sequence converges to u in $L^q(\Omega)$, (6.96) holds for u.

Similarly, since (6.101), we have that

$$m\{x \in \Omega : |D^s(u_n - u_m)| \geq \lambda\} \leq C' \left(\frac{\|f_n - f_m\|_{L^1(\Omega)}}{\lambda} \right)^{\frac{d}{d-s}},$$

thus $D^s u_n$ is a Cauchy sequence in measure in Ω, and therefore, up to a subsequence, $D^s u_n$ converges a.e. in Ω. By Fatou's lemma, (6.95) follows from (6.98). Again, by Fatou's lemma and (6.101), (6.97) follows. □

Remark 6.44. *This Theorem 6.43 corresponds, for $s = 1$, to the classical result where the weak solutions u are such that $|\nabla u| \in L^r(\Omega)$ for $r \in \left]1, \frac{d}{d-1}\right[$ and $u \in L^q(\Omega)$ for $q \in \left]1, \frac{d}{d-2}\right[$. Furthermore, similar embedding results for fractional Sobolev spaces can be obtained for fractional-type operators using semigroup theory (See [166]).*

As a result of this theorem, we have the following result.

Lemma 6.45. *$\mathcal{L}_1 \supset \bar{\mathcal{L}}_2$ and $\bar{\mathcal{L}}_2$ satisfies Proposition 6.33(b). Here $\bar{\mathcal{L}}_2$ is the closure of $\mathcal{L}_2$ with respect to the L^1-topology.*

Proof. Indeed, since the graph of $\mathcal{L}_1$ is closed in $H_0^{s,1}(\Omega) \times L^1(\Omega)$, it follows that $\bar{\mathcal{L}}_2 \subset \mathcal{L}_1$. □

Now, by Theorem 2.8, we have the L^1-contraction for $\bar{\mathcal{L}}_2$, and we want to extend this to $\mathcal{L}_1$ using the density of the operators, which will require the following two results.

Lemma 6.46. *$\mathcal{L}_1$ is one-to-one.*

Proof. Since Theorem 6.43 holds, by Remark 6.41, for each $f \in L^p(\Omega)$, $p > d/2$, there is a solution v of the adjoint problem associated to the formal adjoint $\mathcal{L}'$ of $\mathcal{L}$:

$$v \in D(\mathcal{L}_1) : \langle \mathcal{L}' v, w \rangle = \int_\Omega f w, \quad \forall w \in C_c^\infty(\Omega).$$

By density and the definition of $H_0^{s,p}(\Omega)$, we can let $w \in H_0^{s,1}(\Omega)$. Taking $w = u$ and supposing that $\mathcal{L}_1 u = 0$, we obtain

$$\int_\Omega f u = \langle v, \mathcal{L}_1 u \rangle = 0$$

for any $f \in L^p(\Omega)$, so $u = 0$. □

As a corollary, we have the following result.

Corollary 6.47. *$Id + \lambda \mathcal{L}_1$ is one-to-one for each $\lambda > 0$.*

We now complete the proof of the following theorem for L^1 contraction for $\mathcal{L}_1$.

Proof of Theorem 6.38. We have shown that $\bar{\mathcal{L}}_2 \subset \mathcal{L}_1$, $Id + \lambda \bar{\mathcal{L}}_2$ is onto and $Id + \lambda \mathcal{L}_1$ is one-to-one. Thus $\bar{\mathcal{L}}_2 = \mathcal{L}_1$, and (a) and (c) are proved since $\bar{\mathcal{L}}_2$ satisfies (a) by Proposition 2.8. Also, Theorem 6.43 gives (b), so all of Proposition 6.38 is proved except (d). But we know that $\mathcal{L}_2$ satisfies (d) by Proposition 2.8. To show that $\mathcal{L}_1$ satisfies (d), let $f \in L^1(\Omega)$ and choose $f_\varepsilon \in L^2(\Omega)$ such that $f_\varepsilon \to f$ in $L^1(\Omega)$, $f_\varepsilon(x) \to f(x)$, $f_\varepsilon(x) \leq f(x)^+$, and $u_\varepsilon(x) = (Id + \lambda \mathcal{L}_2)^{-1} f_\varepsilon(x) \to u(x)$ a.e. $x \in \Omega$. Then let $n \to \infty$ to obtain $u_\varepsilon(x) \leq \left\| {f_\varepsilon}^+ \right\|_{L^\infty(\Omega)} \leq \|f^+\|_{L^\infty(\Omega)}$, and we obtain the result. □

Remark 6.48. *Note that Proposition 6.33(b) gives a regularity result.*

Having obtained the L^1 contraction for $f \in L^1(\Omega)$, we want to conduct interpolation, which will require an analogue of the Gagliardo-Nirenberg-Sobolev inequality (stated as Theorem 6 in [154]) and the corresponding interpolation between $W^{s,p}$ spaces.

Theorem 6.49. *Let u be a function in $L^p(\mathbb{R}^d)$ such that $D^s u \in L^q(\mathbb{R}^d)$ with $p, q \geq 1$. Then there exists a positive constant $c = c(d, s, p, q)$ such that*

$$\|u\|_{L^r(\mathbb{R}^d)} \leq c \|D^s u\|_{L^q(\mathbb{R}^d)}^\theta \|u\|_{L^p(\mathbb{R}^d)}^{1-\theta}$$

for every r and θ satisfying $1 \leq r < +\infty$ and $0 \leq \theta \leq 1$ such that

$$\frac{1}{r} = \theta \left(\frac{1}{q} - \frac{s}{d} \right) + (1 - \theta) \frac{1}{p}.$$

Lemma 6.50 (Corollary III.2 of [55]). *Assume that* $u \in W^{s_1,p_1}(\mathbb{R}^d) \cap W^{s_2,p_2}(\mathbb{R}^d)$ *for some* $s_i, p_i > 0$, $i = 1, 2$. *Then, for* $0 \le s_1 < s_2 < \infty$, $1 < p_1, p_2 < \infty$, *we have*

$$\|u\|_{W^{s,p}(\mathbb{R}^d)} \le c\|u\|_{W^{s_1,p_1}(\mathbb{R}^d)}\|u\|_{W^{s_2,p_2}(\mathbb{R}^d)}$$

for some constant c, where

$$s = \theta s_1 + (1-\theta)s_2, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

for any $0 \le \theta \le 1$.

As a result, by interpolation between $W^{s,1}(\mathbb{R}^d)$ and $W^{s,2}(\mathbb{R}^d)$, we have the following corollary.

Corollary 6.51. *The associated operator* $\mathcal{L}_p$ *with domain*

$$D(\mathcal{L}_p) = \{u \in H_0^{s,p}(\Omega) : u, \mathcal{L}_p u \in L^p(\Omega), u = 0 \text{ in } \Omega^c\} \quad \text{for } 1 \le p \le 2$$

is closed and densely-defined in $L^p(\Omega)$, *i.e. there is a solution u for the Dirichlet problem* $\mathcal{L}_p u = f \in L^p(\Omega)$ *for each* $1 \le p \le 2$. *Furthermore,* $(Id + \lambda\mathcal{L}_p)^{-1}$ *is a contraction for each* $\lambda > 0$.

Observe that all the results in this section does not require the kernel $a(x, y)$ to be symmetric, since the non-symmetric operator $\mathcal{L}_a$ is T-accretive. Symmetry is only required for the nonlinear operator $\mathcal{L}_a \circ \gamma$, as in Proposition 6.33.

6.8.3 Existence of unique mild solutions and other properties

We first recall (from Definition 1.3(ii) of [34] or Definition 10.6(ii) of [234]) that for a Banach space B, $u :]0, T[\to B$ is called a *mild* solution of the Cauchy problem

$$\frac{du}{dt} + \mathcal{A}u \ni f \quad \text{in }]0, T[$$

for an operator $\mathcal{A}$, if there exists a sequence $\{(u_n, f_n)\}$ such that u_n is a strong solution of the problem with f_n for any n, and for $f_n \to f$ strongly in $L^1(0, T; B)$, $u_n \to u$ in B locally uniformly in $]0, T[$. Furthermore, we have the following classical theorem characterising mild solutions (see for example, Theorem 3.3 of [34], Theorem 1.17 of [234], or Theorem 5.6.1 of [239]).

Theorem 6.52. *Let B be a Banach space and* $\mathcal{A} : B \to 2^B$ *be an m-accretive operator. Then if* $f \in L^1(0, T; B)$ *and* $u_0 \in \overline{D(\mathcal{L})}$, *then the corresponding Cauchy problem with initial data* u_0 *has a unique mild solution that satisfies* L^1*-contraction property.*

Remark 6.53. *The idea of mild solutions follow from the classical method of* change of pivot spaces $H_0^s(\Omega) \subset L^2(\Omega) \subset H^{-s}(\Omega)$ *(see, for instance, Chapter 2.3 of [161] or Section II.6 of [238]). Indeed, by making use of the Riesz representation given in* (6.14) *applied to the nonlocal operator* $\mathcal{L}_a$, *we have*

$$\langle \mathcal{L}_a u, v\rangle = [u, v]_a = (U, V). \tag{6.14}$$

As a result, the equations

$$\left\langle \frac{d\eta}{dt}, \xi \right\rangle + \langle \mathcal{L}_a(\gamma(\eta)), \xi\rangle = \langle f, \xi\rangle, \quad \forall \xi \in L^2(0, T; H_0^s(\Omega))$$

and

$$\left(\frac{d\eta}{dt}, \xi\right) + \langle \gamma(\eta), \xi\rangle = (f, \xi), \quad \forall \xi \in L^2(0, T; L^2(\Omega))$$

are equivalent.

As a result, we have the existence, uniqueness of mild solutions to the nonlocal Stefan-type problem (6.89) using the classic Brezis-Strauss theory (see, for instance, Corollary 12 of [56] and Theorem II.9.2 of [215] for the classical case), as in [44], which also showed other quantitative properties such as absolute upper bounds, smoothing effects, and weighted L^1 estimates.

Furthermore, considering instead the operator $\mathcal{L}_a u - \mathcal{L}_a g - f$, which is equally T-accretive, we can include Dirichlet boundary conditions and a source function to obtain

Theorem 6.54. *Let $f \in L^1(Q_T)$ and $g \in BV(0,T;L^2(\Omega)) \cap L^2(0,T;W^{s,1}(\mathbb{R}^d))$ such that $\mathcal{L}_a g \in L^1(\Omega)$. Assume $u_0 \in L^1(\Omega)$ such that γ satisfies* (6.16)*. Then there exists a unique mild solution η to the problem* (6.89) *such that $\gamma(\eta) = g$ on Ω^c.*

Remark 6.55. *Since γ is a general maximal monotone operator satisfying some Lipschitz condition, this includes the Stefan problem as well as the porous medium equation.*

Remark 6.56. *Recall also that this mild solution can be explicitly written out as $\eta(t) = \lim_{n\to\infty}(Id + \frac{t}{n}(\mathcal{L}_a \circ \gamma))^{-n}x$ as in [78], if we enlarge the state space. Otherwise, this solution need not be in the domain of $\mathcal{L}$, or even be differentiable for any $t \geq 0$, as noted on page 243 of [184]*

Remark 6.57. *For the case $\Omega = \mathbb{R}^d$, it is well-known, from [33], that there exists a unique mild solution to the classical Stefan-type problem with the classical derivative with data in $L^1(\mathbb{R}^d)$. A similar existence theorem for linear parabolic equations with data in L^1 has been obtained in [110] for unbounded domains Ω. However, the proof of the boundedness of the solution in Theorem 6.40 using the Moser technique requires that Ω is a bounded Lipschitz domain. To extend the existence result for nonlinear parabolic equations to unbounded domains, we may require similar concentration-compactness principles as obtained in [110], making use of asymptotics derived in [12]. Otherwise, solutions may have infinite speed of propagation, as shown in [41].*

On the other hand, the spectral fractional Laplacian has already previously been considered on geometrically non-trivial spaces, (see for instance, [1], [35], [180]), and for more general models for bounded domains (see, for example, [133]).

Remark 6.58. *The result can also similarly be extended to kernels that correspond to the p-Laplacian, making use of the L^2 Hilbertian theory given in [236], probably in a similar way as was done for the half-Laplacian in [183].*

Finally, once again, by the results of Pazy [184], we have a similar asymptotic behaviour for these mild solutions

Corollary 6.59 (Asymptotic Behaviour of Solution in L^1)**.** *There exists a unique $\eta_\infty \in L^1(\Omega)$ such that $\mathcal{L}_a(\gamma(\eta_\infty)) \ni f$. Moreover, for every mild solution η,*

$$\lim_{t\to\infty} e^{-t(\mathcal{L}_a \circ \gamma)}\eta = \eta_\infty.$$

At the same time, the solutions have an infinite speed of propagation, as shown in [41].

Remark 6.60. *Since we require L^2-norms for the s-convergence theory, we are unable to apply the results of Section 5.3 to L^1-solutions.*

7 Global Nonlocal Non-Isotropic Quasilinear Diffusion Systems

7.1 Introduction

Consider the quasilinear diffusion problem for $\boldsymbol{u} = (u^1, \dots, u^m) = \boldsymbol{u}(t,x)$

$$\begin{cases} \boldsymbol{u}' + \Pi(t,x,\boldsymbol{u},\Sigma\boldsymbol{u})\mathbb{A}\boldsymbol{u} = \boldsymbol{f}(t,x,\boldsymbol{u},\Sigma\boldsymbol{u}) & \text{in }]0,T[\times\Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{in }]0,T[\times\Omega^c, \\ \boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0(\cdot) & \text{in } \Omega \end{cases} \tag{7.1}$$

for an open (bounded or unbounded) set $\Omega \subset \mathbb{R}^d$, $\boldsymbol{u}_0 \in \mathbf{H}^s_0(\Omega) := [H^s_0(\Omega)]^m$ and any $T \in]0,\infty[$, where $\Sigma\boldsymbol{u} \in \mathbb{R}^q$ for $0 < q \le m \times d$ represents fractional or nonlocal derivatives in the form $D^\sigma\boldsymbol{u}$ or $\mathcal{D}^\sigma\boldsymbol{u}$ for $\sigma < 2s$, $0 < s \le 1$, σ possibly equal or greater than 1 including the classical gradient, $\mathbb{A}$ is symmetric time-independent local or nonlocal operator, which is bounded and $\mathbf{L}^2(\Omega) = [L^2(\Omega)]^m$-coercive, i.e.

$$\begin{aligned} &\langle \mathbb{A}\boldsymbol{u}, \boldsymbol{v}\rangle \le a^*\|\boldsymbol{u}\|_{\mathbf{H}^s_0(\Omega)}\|\boldsymbol{v}\|_{\mathbf{H}^s_0(\Omega)} \text{ for some } a^* > 0,\ \forall \boldsymbol{u},\boldsymbol{v} \in \mathbf{H}^s_0(\Omega), \text{ and} \\ &\langle \mathbb{A}\boldsymbol{u}, \boldsymbol{u}\rangle + \mu\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \ge a_*\|\boldsymbol{u}\|^2_{\mathbf{H}^s_0(\Omega)} \text{ for some } \mu \ge 0,\ a_* > 0,\ \forall \boldsymbol{u} \in \mathbf{H}^s_0(\Omega), \end{aligned} \tag{7.2}$$

for the classical Sobolev space $\mathbf{H}^s_0(\Omega), 0 < s \le 1$, so that $\mathbb{A} : \mathbf{H}^s_0(\Omega) \to \mathbf{H}^{-s}(\Omega)$ is linear and continuous. Suppose also that $\boldsymbol{f} :]0,T[\times\Omega\times\mathbb{R}^m\times\mathbb{R}^q \to \mathbb{R}^m$ is measurable such that it is continuous with respect to $\boldsymbol{u}$ and $\Sigma\boldsymbol{u}$ for almost every (t,x) and satisfies a linear growth condition with respect to the last variable, and $\Pi :]0,T[\times\Omega\times\mathbb{R}^m\times\mathbb{R}^q \to \mathbb{R}^{m\times m}$ is a measurable, coercive, invertible matrix such that it is continuous with respect to $\boldsymbol{u}$ and $\Sigma\boldsymbol{u}$ for almost every (t,x) and

$$\underline{\gamma}|\xi|^2 \le \Pi\xi\cdot\xi \quad \text{and} \quad \Pi\xi\cdot\xi^* \le \bar{\gamma}|\xi||\xi^*| \quad \text{for all } \xi,\xi^* \in \mathbb{R}^m \tag{7.3}$$

for all $\boldsymbol{u}$ and $\Sigma\boldsymbol{u}$ and almost all (t,x) with $0 < \underline{\gamma} \le \bar{\gamma}$.

The main purpose of this chapter is to prove the existence of a solution $\boldsymbol{u}$ to Problem (7.1) in the space

$$H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2_{\mathbb{A}}) \cap C([0,T];\mathbf{H}^s_0(\Omega)).$$

Here $\mathbf{L}^2_{\mathbb{A}} = D(\mathbb{A})$ is the domain of the operator $\mathbb{A}$, associated with homogeneous Dirichlet boundary condition when $\mathbb{A}u \in \mathbf{L}^2(\Omega)$, given by

$$\mathbf{L}^2_{\mathbb{A}} = D(\mathbb{A}) := \{\boldsymbol{u} \in \mathbf{H}^s_0(\Omega) : \mathbb{A}\boldsymbol{u} \in \mathbf{L}^2(\Omega)\},$$

as $\mathbb{A}$ may be regarded as an operator in the classical framework $\mathbf{H}^\sigma_0(\Omega) \subset \mathbf{L}^2(\Omega) \subset \mathbf{H}^{-s}(\Omega)$. Then, because the operator $\mathbb{A}$ is closed, the space $\mathbf{L}^2_{\mathbb{A}}$ is a Hilbert space when equipped with the graph norm for any $\Omega \subseteq \mathbb{R}^d$. Subsequently, the Bochner space $L^2(0,T;\mathbf{L}^2_{\mathbb{A}})$ is also a Hilbert space.

Note that here $\boldsymbol{f}_{\boldsymbol{u}}(t,x) = \boldsymbol{f}(t,x,\boldsymbol{u}(t,x),\Sigma\boldsymbol{u}(t,x))$ and $\Pi_{\boldsymbol{u}}(t,x) = \Pi(t,x,\boldsymbol{u}(t,x),\Sigma\boldsymbol{u}(t,x))$ are functions in $L^2(0,T;\mathbf{L}^2(\Omega))$ and $L^\infty(]0,T[\times\Omega)$ respectively. Problem 7.1 generalises the quasilinear equation defined with the classical gradient in [16] to systems of equations with more general derivatives.

Recall from Section 1.1.1, and extending to the vectorial case as in [30], for $0 < s < 1$, the Riesz *fractional gradient* $D^s\boldsymbol{u}$ may be defined component-wise in integral form for vectors $\boldsymbol{u} = (u^1, u^2, \dots, u^m) \in \mathbf{H}^s_0(\Omega)$, respectively, by

$$D^s_i u^j(x) := c_{d,s}\int_{\mathbb{R}^d} \frac{u^j(x) - u^j(y)}{|x-y|^{d+s}}\frac{x_i - y_i}{|x-y|}\,dy, \qquad i = 1,\dots,d, \quad j = 1,\dots,m \tag{7.4}$$

where $c_{d,s} = 2^s\pi^{-\frac{d}{2}}\frac{\Pi\left(\frac{d+s+1}{2}\right)}{\Pi\left(\frac{1-s}{2}\right)}$ is given in terms of the Gamma-function, and $\boldsymbol{u}$ is extended by $\boldsymbol{0}$ outside Ω, supposed to satisfy the extension property if $s > 1/2$, so that we can assume that the extension of $\boldsymbol{u}$ is in $\mathbf{H}^s(\mathbb{R}^d)$ whenever $\boldsymbol{u} \in \mathbf{H}^s_0(\Omega)$. Similarly, the *nonlocal gradient* $\mathcal{D}^s\boldsymbol{u}$ is defined using Section 1.1.2 component-wise, as in [101], by

$$\mathcal{D}^s u^j(x,y) := \frac{u^j(x) - u^j(y)}{|x-y|^{\frac{d}{2}+s}}, \qquad j = 1,\dots,m. \tag{7.5}$$

For $1 < s < 2$, we consider only the fractional gradients, defined by

$$D_i^s u^j = D_i^{s-1}(\partial_i u^j), \tag{7.6}$$

for the classical partial derivative $\partial_i = \frac{\partial}{\partial x_i}$. This is possible by making use of the semigroup property of the Riesz potentials and the property of the distributional Riesz fractional gradients which can be given through the convolution with it (Theorem 1.2 of [213]). Note that it is not possible to define a higher order nonlocal gradient, since $\mathcal{D}^s u^j \notin L^2(\mathbb{R}^d \times \mathbb{R}^d)$ for $s > 1$.

$\mathbb{A}$ may be given by linear combinations of the classical gradient ∂, D^s or $\mathcal{D}^s$, as long as it is bounded and $\mathbf{L}^2(\Omega)$-coercive, as in the following examples. When $s = 1$, this includes the local operator given by

$$\langle \mathbb{A}\boldsymbol{u}, \boldsymbol{v} \rangle = \langle \mathbb{L}\boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{\alpha,\beta,i,j} \int_\Omega A_{ij}^{\alpha\beta} \partial_\beta u^j \cdot \partial_\alpha v^i \tag{7.7}$$

with a bounded, coercive tensor $A = (A_{ij}^{\alpha\beta}(x))$, symmetric in α and β, where $\langle \mathbb{A}\boldsymbol{u}, \boldsymbol{v} \rangle$ is understood as the duality between $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}_0^1(\Omega)$. The sum here is taken between 1 to d for α and β, and between 1 to m for i and j. $\mathbb{A}$ may also be the anisotropic fractional operator, extended from Chapter 3 to the vectorial case,

$$\langle \mathbb{A}\boldsymbol{u}, \boldsymbol{v} \rangle = \langle \tilde{\mathcal{L}}_A^s \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{\alpha,\beta,i,j} \int_{\mathbb{R}^d} A_{ij}^{\alpha\beta} D_\beta^s u^j \cdot D_\alpha^s v^i \tag{7.8}$$

for $s \leq 1$, where D_α^s coincides with ∂_α in the classical case of $s = 1$, where $\langle \mathbb{A}\boldsymbol{u}, \boldsymbol{v} \rangle$ is understood as the duality between $\mathbf{H}^{-s}(\Omega)$ and $\mathbf{H}_0^s(\Omega)$.

We can also consider the anisotropic nonlocal operator $\mathbb{A} : \mathbf{H}_0^s(\Omega) \to \mathbf{H}^{-s}(\Omega)$, similarly extending that from Chapter 2 to the vectorial case,

$$\mathbb{A}\boldsymbol{u} = \mathbb{L}_A^s \boldsymbol{u} = P.V. \int_{\mathbb{R}^d} A(x,y) \frac{\boldsymbol{u}(x) - \boldsymbol{u}(y)}{|x-y|^{d+2s}}\, dy \tag{7.9}$$

defined for a symmetric, bounded, coercive matrix kernel $A = A_{ij}(x,y)$, i.e. for almost all (x,y) in $\mathbb{R}^d \times \mathbb{R}^d$,

$$a_*|\xi|^2 \leq A\xi \cdot \xi \leq a^*|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^m,$$

for $s < 1$, so that, for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}_0^s(\Omega)$,

$$\langle \mathbb{L}_A^s \boldsymbol{u}, \boldsymbol{v} \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} A(x,y) \frac{\boldsymbol{u}(x) - \boldsymbol{u}(y)}{|x-y|^{\frac{d}{2}+s}} \cdot \frac{\boldsymbol{v}(x) - \boldsymbol{v}(y)}{|x-y|^{\frac{d}{2}+s}}\, dy\, dx \leq a^* \|\boldsymbol{u}\|_{\mathbf{H}_0^s(\Omega)} \|\boldsymbol{v}\|_{\mathbf{H}_0^s(\Omega)},$$

and

$$\langle \mathbb{L}_A^s \boldsymbol{u}, \boldsymbol{u} \rangle \geq a_* \|\boldsymbol{u}\|_{\mathbf{H}_0^s(\Omega)}^2.$$

Recall from Section 1.2 that the fractional Sobolev spaces $H^s(\mathbb{R}^d)$ for all real s are defined by

$$H^s(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : \{\xi \mapsto (1+|\xi|^2)^{s/2}\hat{u}(\xi)\} \in L^2(\mathbb{R}^d)\},$$

with norm

$$\|u\|_{H^s(\mathbb{R}^d)} = \left\|(1+|\xi|^2)^{s/2}\hat{u}\right\|_{L^2(\mathbb{R}^d)},$$

where $\hat{u}$ is the Fourier transform of u. For $0 < s < 1$, this norm is well known to be equivalent to

$$\|u\|_{H^s(\mathbb{R}^d)}^2 = \|u\|_{L^2(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}}\, dx\, dy =: \|u\|_{L^2(\mathbb{R}^d)}^2 + [u]_{H^s(\mathbb{R}^d)}^2. \tag{1.13}$$

On the other hand, as it was shown in [213] and (1.8), the $H^s(\mathbb{R}^d)$-norm given by (1.13) is in fact equal to

$$\|u\|_{H^s(\mathbb{R}^d)}^2 = \|u\|_{L^2(\mathbb{R}^d)}^2 + \frac{2}{c_{d,s}^2} \int_{\mathbb{R}^d} |D^s u|^2 = \|u\|_{L^2(\mathbb{R}^d)}^2 + \frac{2}{c_{d,s}^2} \|D^s u\|_{L^2(\mathbb{R}^d)}^2. \tag{1.14}$$

Then, if Ω has Lipschitz boundary, hence satisfying the extension property, $H^s(\Omega)$ coincides with the space of restrictions to Ω of the elements of $H^s(\mathbb{R}^d)$ as in [162] and [93], with norm

$$\|u\|_{H^s(\Omega)} = \inf_{U=u \text{ a.e. } \Omega} \|U\|_{H^s(\mathbb{R}^d)} . \tag{1.15}$$

Here, the subspace $H_0^s(\Omega)$ is the usual Sobolev space, for $0 < s \leq 1$, given by the closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$ for general open sets $\Omega \subset \mathbb{R}^d$, as in [162], and $H^{-s}(\Omega)$ its dual. Since $C_c^\infty(\Omega)$ is dense in $H^s(\Omega)$ if and only if $s \leq \frac{1}{2}$, in this case, $H_0^s(\Omega) = H^s(\Omega)$. Otherwise, if $s > \frac{1}{2}$, $H_0^s(\Omega)$ is strictly contained in $H^s(\Omega)$. On the other hand, as in [93], for bounded sets with Lipschitz boundary, $\mathcal{O} \subset \mathbb{R}^d$, $C_c^\infty(\bar{\mathcal{O}})$ is dense in $H^s(\mathcal{O})$ for all $s \geq 0$.

This can be further extended for $s > 1$, by an abuse of notation, by defining $H_0^s(\Omega)$ to be the space

$$H_0^s(\Omega) := \{u \in H^s(\mathbb{R}^d) : \text{ supp } u \subset \bar{\Omega}\}.$$

Consider the *maximal regularity space*

$$MR := H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2_{\mathbb{A}}),$$

equipped with the norm for $0 < s \leq 1$

$$\|\boldsymbol{u}\|_{MR}^2 := \int_0^T \left\|\boldsymbol{u}'(t)\right\|_{\mathbf{L}^2(\Omega)}^2 + \int_0^T \left\|\boldsymbol{u}(t)\right\|_{\mathbf{H}_0^s(\Omega)}^2 + \int_0^T \left\|\mathbb{A}\boldsymbol{u}(t)\right\|_{\mathbf{L}^2(\mathbb{R}^d)}^2 , \tag{7.10}$$

so that the linear inhomogeneous problem

$$\boldsymbol{u}'(t) + \mathbb{A}\boldsymbol{u}(t) = \boldsymbol{f}(t) \quad \text{ for a.e. } t \in]0,T[, \quad \boldsymbol{u}(0) = \mathbf{0}$$

is well-defined with a source term $\boldsymbol{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$.

Classically, parabolic quasilinear systems in non-divergence form have frequently been considered (see [120], [104] [153], [135], [10], [188], [130], [18] and their references), with multiple physical, chemical and biological applications such as in reaction-diffusion systems (see, for example, [174]), phase-field models (see, for example, [179]) and population models (see, for instance, [155] and [32]). Parabolic equations have also been extended to the case of nonlocal reaction-diffusion (see, for instance, [29] and [58]).

In Section 7.2, we will first consider the linear problem, extending the approach of [16] to systems by introducing a suitable time-dependent matrix Υ, thereby obtaining the solution to the non-autonomous linear problem given with the well-known maximal regularity and exemplifying with three linear systems of the above type, which may have additional regularity in bounded Lipschitz domains. Next, we will extend the result by a fixed point argument to obtain the result for the quasilinear problem. Limited by the currently known regularity of the Dirichlet problems associated to the operator $\mathbb{A}$, in Section 7.3, we obtain the existence of a solution for the global quasilinear nondivergent systems for the general operators $\mathbb{A}$ satisfying (7.2), first for $\sigma < s$ and also for particular operators satisfying additional regularity properties, up to and including $\sigma = s \leq 1$. This extends the nonlocal vectorial problem with no source function considered in [152], as well as the vectorial semilinear case in [9], [174] and [17]. This also generalises [16] to systems of the form (7.1) defined in a bounded or unbounded open set $\Omega \subset \mathbb{R}^d$, for more general derivatives that can take any positive order less than s, which is an improvement even in the classical case of $s = 1$.

This result is then further generalised to larger $s < \sigma < 2s$ in the case of Ω bounded with Lipschitz boundary, making use of known regularity results for vectorial local and nonlocal operators in Section 7.4, in particular generalising to quasilinear diffusion systems the classical scalar Dirichlet case of [16]. As a result, we can also consider quasilinear diffusion equations and systems with derivatives of order $\sigma > s$ such that σ may be greater than 1, generalising the results of [16], [17] and [152]. These results may provide useful applications, particularly in population models and advection-diffusion systems, as we try to exemplify in model problems.

7.2 A Non-autonomous Linear Problem

We first consider the linear problem for a system, for $0 < s \leq 1$ up to and including the classical case of $s = 1$, as in [17] but for a different maximal regularity space, by extending the approach of [16] to systems by introducing a suitable matrix Υ.

We first observe that, by the definition of $\mathbb{A}$ as a symmetric time-independent operator, for $\boldsymbol{u} \in MR$, $\mathbb{A}\boldsymbol{u}(t) \in \mathbf{L}^2(\Omega)$ for a.e. $t \in]0,T[$, so we have the following well-known result (see, for instance, [89], page 480), which we include here for completeness.

Lemma 7.1. *Let $\boldsymbol{u} \in MR$. Then $\int_\Omega \mathbb{A}\boldsymbol{u}(\cdot)\cdot\boldsymbol{u}(\cdot) \in W^{1,1}(0,T)$ and*

$$\frac{d}{dt}\int_\Omega \mathbb{A}\boldsymbol{u}(t)\cdot\boldsymbol{u}(t) = 2\int_\Omega \mathbb{A}\boldsymbol{u}(t)\cdot\boldsymbol{u}'(t) \quad \text{for a.e. } t\in]0,T[.$$

Furthermore, the continuous embedding holds

$$MR \hookrightarrow C([0,T];\mathbf{H}^s_0(\Omega)).$$

Proof. We first take $\boldsymbol{u} \in C^1([0,T];\mathbf{L}^2_{\mathbb{A}})$. Then, since $\mathbb{A}$ is symmetric and time-independent, we have

$$\int_\Omega \mathbb{A}\boldsymbol{u}(t)\cdot\boldsymbol{u}'(t) = \frac{1}{2}\left(\int_\Omega \mathbb{A}\boldsymbol{u}(t)\cdot\boldsymbol{u}'(t) + \int_\Omega \boldsymbol{u}(t)\cdot\mathbb{A}\boldsymbol{u}'(t)\right) = \frac{1}{2}\frac{d}{dt}\int_\Omega \mathbb{A}\boldsymbol{u}(t)\cdot\boldsymbol{u}(t).$$

Then, the result holds for arbitrary $\boldsymbol{u}\in MR$ by an approximation by density.

The second part of the lemma may be proved as in Proposition 3.6 of [96]. Indeed, since $\int_\Omega \mathbb{A}\boldsymbol{u}(\cdot)\cdot\boldsymbol{u}(\cdot) \in W^{1,1}(0,T) \subset C([0,T])$, together with the continuous embedding $MR \hookrightarrow C([0,T];\mathbf{L}^2(\Omega))$ and the coercivity (7.2) yields $MR \subset L^\infty(0,T;\mathbf{H}^s_0(\Omega))$. Now, it is well-known that $C([0,T];\mathbf{L}^2(\Omega)) \cap L^\infty(0,T;\mathbf{H}^s_0(\Omega)) \subset C([0,T];\mathbf{H}^s_0(\Omega)\text{-weak})$ (see, for instance, Lemma 3.3 of [96]). Then, as $\tau \to t$ for fixed t,

$$\begin{aligned}
a_*\big\|\boldsymbol{u}(t)-\boldsymbol{u}(\tau)\big\|^2_{\mathbf{H}^s_0(\Omega)} &\le \int_\Omega \mathbb{A}(\boldsymbol{u}(t)-\boldsymbol{u}(\tau))\cdot(\boldsymbol{u}(t)-\boldsymbol{u}(\tau)) + \mu\big\|\boldsymbol{u}(t)-\boldsymbol{u}(\tau)\big\|^2_{\mathbf{L}^2(\Omega)}\\
&= 2\int_\Omega \mathbb{A}(\boldsymbol{u}(t)-\boldsymbol{u}(\tau))\cdot\boldsymbol{u}(t) + \int_\Omega \big[\mathbb{A}\boldsymbol{u}(\tau)\cdot\boldsymbol{u}(\tau) - \mathbb{A}\boldsymbol{u}(t)\cdot\boldsymbol{u}(t)\big]\\
&\quad + \mu\big\|\boldsymbol{u}(t)-\boldsymbol{u}(\tau)\big\|^2_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

The three terms tend to 0: the first one by the weak continuity of $\boldsymbol{u}(\cdot)$ in $\mathbf{H}^s_0(\Omega)$, the second one by the continuity of the map $\tau \mapsto \int_\Omega \mathbb{A}\boldsymbol{u}(\tau)\cdot\boldsymbol{u}(\tau)$ and the third one again by the embedding $MR \hookrightarrow C([0,T];\mathbf{L}^2(\Omega))$. ☐

Recall, for instance from [89], the following well-known maximal regularity result: for all $\boldsymbol{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\boldsymbol{u}_0 \in \mathbf{H}^s_0(\Omega)$, there exists a unique solution to the autonomous problem

$$\begin{aligned}
&\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2_{\mathbb{A}}),\\
&\boldsymbol{u}'(t) + \mathbb{A}\boldsymbol{u}(t) = \boldsymbol{f}(t) \quad \text{for a.e. } t\in]0,T[,\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0.
\end{aligned} \tag{7.11}$$

We consider now a linear non-autonomous problem, obtained by a multiplicative perturbation.

Theorem 7.2. *Let $\Upsilon = \Upsilon(t,x) :]0,T[\times\Omega \to \mathbb{R}^{m\times m}$ is a measurable, coercive, invertible matrix satisfying (7.3). Then, for every $\boldsymbol{f}\in L^2(0,T;\mathbf{L}^2(\Omega))$, $\boldsymbol{u}_0 \in \mathbf{H}^s_0(\Omega)$, there exists a unique solution of the problem*

$$\begin{aligned}
&\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2_{\mathbb{A}}) \cap C([0,T];\mathbf{H}^s_0(\Omega)),\\
&\boldsymbol{u}'(t) + \Upsilon(t,\cdot)\mathbb{A}\boldsymbol{u}(t) = \boldsymbol{f}(t) \quad \text{for a.e. } t\in]0,T[,\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0.
\end{aligned} \tag{7.12}$$

Moreover, there exists a constant $c = c(\underline{\gamma},\bar{\gamma},a_*,a^*,\mu,T) > 0$ independent of $\boldsymbol{f}$ and $\boldsymbol{u}_0$ such that

$$\|\boldsymbol{u}\|_{MR} \le c\left(\|\boldsymbol{f}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{u}_0\|_{\mathbf{H}^s_0(\Omega)}\right) \tag{7.13}$$

for each solution $\boldsymbol{u}$ of (7.12)

Proof. We use the method of continuity (c.f. Section 5.2 of [126]) as in Theorem 3.2 of [16]. For every $\lambda \in [0,1]$, consider the matrix $\Upsilon_\lambda := (1-\lambda)\mathbb{I} + \lambda\Upsilon$ for the identity matrix $\mathbb{I}$ of dimension $m \times m$ and the bounded operator

$$B_\lambda : MR \to L^2(0,T;\mathbf{L}^2(\Omega)) \times \mathbf{H}^s_0(\Omega)$$

given by

$$B_\lambda \boldsymbol{u} = \int_\Omega (\boldsymbol{u}' + \Upsilon_\lambda \mathbb{A}\boldsymbol{u}) \cdot \boldsymbol{u}_0.$$

Then $B : [0,1] \to \mathcal{L}(MR, L^2(0,T;\mathbf{L}^2(\Omega)) \times \mathbf{H}^s_0(\Omega))$ (where $\mathcal{L}$ denotes the space of linear bounded operators) is continuous and B_0 is invertible by the maximal regularity result for the linear autonomous problem (7.11). Therefore, by Theorem 5.2 of [126], it suffices to prove the a priori estimate

$$\|\boldsymbol{u}\|_{MR} \leq \|B_\lambda \boldsymbol{u}\| = c_1 \left(\left\|\boldsymbol{u}' + \Upsilon_\lambda \mathbb{A}\boldsymbol{u}\right\|_{L^2(0,T;\mathbf{L}^2(\mathbb{R}^d))} + \|\boldsymbol{u}_0\|_{\mathbf{H}^s_0(\Omega)} \right) \forall \lambda \in [0,1], \forall \boldsymbol{u} \in MR, \tag{7.14}$$

for some constant $c_1 = c_1(\underline{\gamma}, \bar{\gamma}, a_*, a^*, T) > 0$, which gives (7.13) for $\lambda = 1$.

Let $\lambda \in [0,1]$. Let $\boldsymbol{u} \in MR$ be such that

$$\boldsymbol{u}' + \Upsilon_\lambda \mathbb{A}\boldsymbol{u} = \boldsymbol{f} \quad \text{and} \quad \boldsymbol{u}(0) = \boldsymbol{u}_0.$$

Then, multiplying the equation by $[\Upsilon^*_\lambda]^{-1}\boldsymbol{u}'(t)$, where $[\Upsilon^*_\lambda]^{-1}$ is the inverse of the adjoint of Υ_λ, we have, for almost every $t \in [0,T]$,

$$\int_\Omega [\Upsilon^*_\lambda]^{-1}\boldsymbol{u}'(t) \cdot \boldsymbol{u}'(t) + \int_\Omega \Upsilon_\lambda \mathbb{A}\boldsymbol{u} \cdot [\Upsilon^*_\lambda]^{-1}\boldsymbol{u}'(t) = \int_\Omega \boldsymbol{f} \cdot [\Upsilon^*_\lambda]^{-1}\boldsymbol{u}'(t),$$

which by Lemma 7.1 and the Cauchy-Schwarz inequality, gives

$$\int_\Omega [\Upsilon^*_\lambda]^{-1}\boldsymbol{u}'(t) \cdot \boldsymbol{u}'(t) + \frac{1}{2}\frac{d}{dt}\int_\Omega \mathbb{A}\boldsymbol{u} \cdot \boldsymbol{u}'(t) \leq \frac{\bar{\gamma}}{2}\int_\Omega \left[[\Upsilon^*_\lambda]^{-1}\boldsymbol{f}(t)\right]^2 + \frac{1}{2\bar{\gamma}}\int_\Omega \left[\boldsymbol{u}'(t)\right]^2.$$

Integrating over time on $]0,t[$ for every finite $t \in]0,T[$ and using the estimate (7.3), it follows by (7.2) that

$$\frac{1}{2\bar{\gamma}}\int_0^t \left\|\boldsymbol{u}'(\tau)\right\|^2_{\mathbf{L}^2(\Omega)} d\tau + \frac{a_*}{2}\left\|\boldsymbol{u}(t)\right\|^2_{\mathbf{H}^s_0(\Omega)} \leq \frac{a^*}{2}\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \frac{\mu}{2}\left\|\boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} + \frac{\bar{\gamma}}{2\underline{\gamma}^2}\int_0^t \left\|\boldsymbol{f}(\tau)\right\|^2_{\mathbf{L}^2(\Omega)} d\tau,$$

Observe that by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\left\|\boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} &= \left\|\boldsymbol{u}(0)\right\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \frac{d}{d\tau}\left\|\boldsymbol{u}(\tau)\right\|^2_{\mathbf{L}^2(\Omega)} d\tau \\
&= \|\boldsymbol{u}_0\|^2_{\mathbf{L}^2(\Omega)} + 2\int_0^t \int_\Omega \boldsymbol{u}(\tau) \cdot \boldsymbol{u}'(\tau)\, d\tau \\
&\leq \|\boldsymbol{u}_0\|^2_{\mathbf{L}^2(\Omega)} + 2\mu\bar{\gamma}\int_0^t \left\|\boldsymbol{u}(\tau)\right\|^2_{\mathbf{L}^2(\Omega)} d\tau + \frac{1}{2\mu\bar{\gamma}}\int_0^t \left\|\boldsymbol{u}'(\tau)\right\|^2_{\mathbf{L}^2(\Omega)} d\tau,
\end{aligned} \tag{7.15}$$

and so we have

$$\begin{aligned}
&\frac{1}{2\bar{\gamma}}\int_0^t \left\|\boldsymbol{u}'(\tau)\right\|^2_{\mathbf{L}^2(\Omega)} d\tau + a_*\left\|\boldsymbol{u}(t)\right\|^2_{\mathbf{H}^s_0(\Omega)} \\
&\quad \leq a^*\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \mu\|\boldsymbol{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \frac{\bar{\gamma}}{\underline{\gamma}^2}\|\boldsymbol{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + 2\mu^2\bar{\gamma}\int_0^t \left\|\boldsymbol{u}(\tau)\right\|^2_{\mathbf{L}^2(\Omega)} d\tau \\
&\quad \leq a^*\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \mu\|\boldsymbol{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \frac{\bar{\gamma}}{\underline{\gamma}^2}\|\boldsymbol{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + 2\mu^2\bar{\gamma}c_S\int_0^t \left\|\boldsymbol{u}(\tau)\right\|^2_{\mathbf{H}^s_0(\Omega)} d\tau
\end{aligned} \tag{7.16}$$

where c_S is the constant for the Sobolev embedding $\mathbf{H}^s_0(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$. Applying the integral form of Gronwall's lemma to the second term on the left-hand-side, there exists a constant $c_2 = c_2(\underline{\gamma}, \bar{\gamma}, a_*, a^*, \mu, T) > 0$ such that

$$\sup_{t \in [0,T[}\left\|\boldsymbol{u}(t)\right\|^2_{\mathbf{H}^s_0(\Omega)} \leq c_2\left(\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \|\boldsymbol{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}\right).$$

Inserting this into (7.16), we obtain that

$$\int_0^T \left\|\boldsymbol{u}'(\tau)\right\|^2_{\mathbf{L}^2(\Omega)}\, d\tau \le c_3 \left(\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \|\boldsymbol{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}\right)$$

for some constant $c_3 = c_3(\underline{\gamma}, \bar{\gamma}, a_*, a^*, \mu, T) > 0$.

Finally, since

$$\int_0^T \left\|\mathbb{A}\boldsymbol{u}(\tau)\right\|^2_{\mathbf{L}^2(\mathbb{R}^d)}\, d\tau \le \frac{1}{\underline{\gamma}^2}\int_0^T \left\|\Upsilon_\lambda \mathbb{A}\boldsymbol{u}(\tau)\right\|^2_{\mathbf{L}^2(\mathbb{R}^d)}\, d\tau = \frac{1}{\underline{\gamma}^2}\int_0^T \left\|\boldsymbol{u}'(\tau) - \boldsymbol{f}(\tau)\right\|^2_{\mathbf{L}^2(\Omega)}\, d\tau,$$

the MR norm of $\boldsymbol{u}$ can be estimated giving (7.13) and by Lemma 7.1, the proof is complete.

□

Remark 7.3. *If $\mu = 0$ in (7.2), this theorem is a special case of Theorem 1.1 of [27].*

Next, we identify local and nonlocal vectorial operators to which we can apply Theorem 7.2.

Example 1: Local operators

As a first example of $\mathbb{A}$, we consider the local operator $\mathbb{L}$, such that the i-th component of $\mathbb{L}\boldsymbol{u}$ is given by

$$(\mathbb{L}\boldsymbol{u})^i = -\sum_{\alpha,\beta,j} \partial_\alpha (A^{\alpha\beta}_{ij}\partial_\beta u^j) + b_{ij}u^j \tag{7.7}$$

for $b_{ij} \in L^\infty(\Omega)$ and $A^{\alpha\beta}_{ij} \in L^\infty(\Omega)$. Then, the linear non-autonomous problem for $\boldsymbol{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $\boldsymbol{u}_0 \in \mathbf{H}^1_0(\Omega)$ given by

$$\boldsymbol{u}'(t) + \Upsilon(t,\cdot)\mathbb{L}\boldsymbol{u}(t) = \boldsymbol{f}(t), \quad \text{for a.e. } t \in]0,T[$$

has a solution $\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2_{\mathbb{L}})$. This extends the results of [16].

Furthermore, we can explicitly write out the space $\mathbf{L}^2_{\mathbb{L}}$ for the following special case of $\mathbb{A} = \mathbb{L}$ using the following proposition:

Proposition 7.4 (Theorem 4.9 of [124])**.** *Let Ω be an open domain in $\mathbb{R}^d$. Suppose in addition that $A^{\alpha\beta}_{ij} \in C^{0,1}_{loc}(\Omega)$ is continuous up to the boundary of $\bar{\Omega}$ and satisfies the Legendre-Hadamard condition*

$$\sum_{\alpha,\beta,i,j} A^{\alpha\beta}_{ij}\nu_\alpha\nu_\beta\xi^i\xi^j \ge a_*|\xi|^2|\nu|^2 \quad \forall \xi \in \mathbb{R}^m, \nu \in \mathbb{R}^d. \tag{7.17}$$

Then, for all weak solutions $\boldsymbol{u}$ of the equation with $\boldsymbol{f} \in \mathbf{L}^2_{loc}(\Omega)$,

$$\mathbb{L}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega,$$

$\boldsymbol{u} \in \mathbf{H}^2_{loc}(\Omega)$.

If, in addition, by Theorem 6 of [97], Ω is bounded with $C^{1,1}$ boundary and $A^{\alpha\beta}_{ij} \in C^{0,1}(\bar{\Omega})$, we can extend the result globally up to the boundary of Ω for the unique solution of the homogeneous Dirichlet problem for $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, so that the unique solution $\boldsymbol{u} \in \mathbf{H}^1_0(\Omega)$ lies in $\mathbf{H}^2(\Omega)$.

It is well-known (see for instance, Section 5 of [117]) to show using Fourier transform that the Legendre-Hadamard condition (7.17) for tensors A continuous up to the boundary of $\bar{\Omega}$ implies coercivity (7.2), which we recall, is given by Gårding's inequality

$$\langle \mathbb{L}\boldsymbol{u}, \boldsymbol{u}\rangle + \mu\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \ge a_*\|\boldsymbol{u}\|^2_{\mathbf{H}^1_0(\Omega)} \quad \forall \boldsymbol{u} \in \mathbf{H}^1_0(\Omega). \tag{7.2}$$

(Recall also that this is not true if $\boldsymbol{u}$ does not have support in $\bar{\Omega}$.) Therefore, as a corollary, we have

Corollary 7.5. *Suppose $\mathbb{L}$ is of the form (7.7) such that $A_{ij}^{\alpha\beta}$ is locally Lipschitz and continuous up to the boundary of $\bar{\Omega}$ satisfying the Legendre-Hadamard condition (7.17), then the linear non-autonomous Cauchy problem for $\boldsymbol{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $\boldsymbol{u}_0 \in \mathbf{H}_0^1(\Omega)$ given by*

$$\boldsymbol{u}'(t) + \Upsilon(t,\cdot)\mathbb{L}\boldsymbol{u}(t) = \boldsymbol{f}(t), \quad \text{for a.e. } t \in]0,T[$$

has a solution $\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap C([0,T];\mathbf{H}_0^1(\Omega)) \cap L^2(0,T;\mathbf{H}_{loc}^2(\Omega))$.

If, in addition, Ω is bounded with $C^{1,1}$ boundary and $A_{ij}^{\alpha\beta} \in C^{0,1}(\bar{\Omega})$, then $\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap C([0,T];\mathbf{H}_0^1(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega))$.

Example 2: Anisotropic fractional operators

Vectorial fractional operators $\tilde{\mathcal{L}}_A^s : \mathbf{H}_0^s(\Omega) \to \mathbf{H}^{-s}(\Omega)$ can also be considered, given for $0 < s \leq 1$ by

$$\langle \tilde{\mathcal{L}}_A^s \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{\alpha,\beta,i,j} \int_{\mathbb{R}^d} A_{ij}^{\alpha\beta} D_\beta^s u^j \cdot D_\alpha^s v^i \tag{7.8}$$

for a bounded, coercive tensor $A_{ij}^{\alpha\beta}$ symmetric in α, β, i.e.

$$\sum_{\alpha,i} a_* |\xi_i^\alpha|^2 \leq \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \xi_i^\alpha \cdot \xi_j^\beta \leq a^* \sum_{\alpha,i} |\xi_i^\alpha|^2 \quad \text{for all } \xi \in \mathbb{R}^{m\times d}.$$

Here, D_α^s coincides with the classical derivative ∂_α in the classical case of $s = 1$, and $\tilde{\mathcal{L}}_A^s$ in (7.8) reduces to $\mathbb{L}$ in (7.7) when $s = 1$.

Then, the linear non-autonomous problem for $\boldsymbol{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $\boldsymbol{u}_0 \in \mathbf{H}_0^s(\Omega)$ given by

$$\boldsymbol{u}'(t) + \Upsilon(t,\cdot)\tilde{\mathcal{L}}_A^s \boldsymbol{u}(t) = \boldsymbol{f}(t), \quad \text{for a.e. } t \in]0,T[$$

has a solution $\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2_{\tilde{\mathcal{L}}_A^s})$.

In the particular case when $A^{\alpha\beta}$ is given by a diagonal constant matrix, $\tilde{\mathcal{L}}_A^s$ corresponds to a system of equations defined with the fractional Laplacian.

Recall that the fractional Laplacian is defined, for $u \in H_0^s(\Omega)$, by

$$(-\Delta)^s u(x) := c_{d,s}^2 \, P.V. \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}} \, dy. \tag{2.4}$$

Then, by Theorem 7.1 of [127], or Theorem 4.1 and Remark 7 of [46], we have

Proposition 7.6. *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Let $f \in L^2(\Omega)$ and $s \in]\frac{1}{2}, 1[$. Then the solution to the homogeneous Dirichlet problem*

$$(-\Delta)^s u = f \text{ in } \Omega, \quad u = 0 \text{ in } \Omega^c$$

lies in the Besov space

$$u \in \dot{B}_{2,\infty}^{s+1/2}(\Omega) \subset H^{\min\{2s, s+1/2\}-\epsilon}(\Omega)$$

for any positive $\epsilon < \min\{2s, s+1/2\}$. (For the definition and more discussion on Besov spaces, see [46] and the references therein.)

Considering the vectorial fractional Laplacian $(-\Delta)_m^s$ defined by

$$(-\Delta)_m^s = \begin{bmatrix} c_1(-\Delta)^s & & 0 \\ & \ddots & \\ 0 & & c_m(-\Delta)^s \end{bmatrix}$$

for constants $c_1, \cdots, c_m > 0$. Then, applying Proposition 7.6 component-wise, we have

Corollary 7.7. *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. The linear non-autonomous Cauchy problem for $\boldsymbol{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $\boldsymbol{u}_0 \in \mathbf{H}_0^s(\Omega)$ given by*

$$\boldsymbol{u}'(t) + \Upsilon(t,\cdot)(-\Delta)_m^s \boldsymbol{u}(t) = \boldsymbol{f}(t), \quad \textit{for a.e. } t \in]0,T[$$

has a solution $\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap C([0,T];\mathbf{H}_0^s(\Omega)) \cap L^2(0,T;\mathbf{H}^{\min\{2s,s+1/2\}-\epsilon}(\Omega))$ for any positive $\epsilon <$ $\min\{2s, s+1/2\}$.

Example 3: Anisotropic nonlocal operators

Next, we consider the anisotropic nonlocal operator $\mathbb{L}_A^s : \mathbf{H}_0^s(\Omega) \to \mathbf{H}^{-s}(\Omega)$

$$\mathbb{L}_A^s \boldsymbol{u} = P.V. \int_{\mathbb{R}^d} A(x,y) \frac{\boldsymbol{u}(x) - \boldsymbol{u}(y)}{|x-y|^{d+2s}}\, dy \tag{7.9}$$

defined for a symmetric, bounded, coercive matrix kernel $A(x,y)$ for $0 < s < 1$. Then, once again, the linear non-autonomous problem for $\boldsymbol{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $\boldsymbol{u}_0 \in \mathbf{H}_0^s(\Omega)$ given by

$$\boldsymbol{u}'(t) + \Upsilon(t,\cdot)\mathbb{L}_A^s \boldsymbol{u}(t) = \boldsymbol{f}(t), \quad \text{for a.e. } t \in]0,T[$$

has a solution $\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2_{\mathbb{L}_A^s})$.

Suppose in addition, $m = d$ and the kernel $A(x,y)$ is a measurable, translation-invariant matrix of the form

$$A(x,y) = \frac{\hat{a}(x-y)}{|x-y|^{d+2s}} \chi_{\mathcal{C}\cap B_r(0)}(x-y) \left(\frac{x-y}{|x-y|} \otimes \frac{x-y}{|x-y|} \right), \tag{7.18}$$

where $\hat{a}$ is an even, coercive and bounded function such that $0 < a_* \leq \hat{a} \leq a^* < \infty$ for some constants $a_*, a^* > 0$, $0 < r \leq \infty$, and $\mathcal{C}$ is a double cone with apex, i.e.

$$\mathcal{C} = \Big\{ h \in \mathbb{R}^d \backslash \{0\} : \frac{h}{|h|} \in \mathcal{O} \cup (-\mathcal{O}) \text{ for any open subset } \mathcal{O} \text{ of the unit sphere } S^{d-1} \text{ with positive Hausdorff measure} \Big\}.$$

Defining the space $\mathbf{H}_{loc}^s(\Omega)$ by $\{\boldsymbol{u} \in \mathbf{L}^2(\Omega) : \eta \boldsymbol{u} \in \mathbf{H}^s(\Omega)\, \forall \eta \in C_c^\infty(\Omega)\}$, by Theorem 3.1 of [141], we have the following local regularity result for $\mathbb{L}_A^s$ for this special case:

Proposition 7.8. *Let $\Omega \subset \mathbb{R}^d$ be an open set, and $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$ extended by 0 outside. Then the weak solution to*

$$\mathbb{L}_A^s \boldsymbol{u} = \boldsymbol{f} \ \textit{in } \Omega, \quad \boldsymbol{u} = \boldsymbol{0} \ \textit{in } \Omega^c$$

lies in $\mathbf{H}_{loc}^{2s}(\Omega)$. Moreover, for any $\eta \in C_c^\infty(\Omega)$, there exists a constant C such that

$$\|\eta \boldsymbol{u}\|_{\mathbf{H}^{2s}(\mathbb{R}^d)} \leq C \|\boldsymbol{f}\|_{\mathbf{L}^2(\Omega)}.$$

Remark 7.9. *Observe that $\mathbb{L}_A^s$ can be viewed as the nonlocal version of Example 1, as explained in pages 1304–1305 of [141]. See also Lemma 3.1 of [108].*

As a corollary, we once again have

Corollary 7.10. *The linear non-autonomous Cauchy problem for $\boldsymbol{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $\boldsymbol{u}_0 \in \mathbf{H}_0^s(\Omega)$ given by*

$$\boldsymbol{u}'(t) + \Upsilon(t,\cdot)\mathbb{L}_A^s \boldsymbol{u}(t) = \boldsymbol{f}(t), \quad \textit{for a.e. } t \in]0,T[$$

has a solution $\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap C([0,T];\mathbf{H}_0^s(\Omega)) \cap L^2(0,T;\mathbf{H}_{loc}^{2s}(\Omega))$.

7.3 The Nonlinear Problem $\sigma \leq s \leq 1$

We next consider the quasilinear vectorial problem, when $0 < \sigma < s \leq 1$, extending the nonlocal vectorial problem with no source function considered in [152], as well as the vectorial semilinear case in [9], [174] and [17]. This also generalises [16] to systems of the form (7.1) defined in a bounded or unbounded open set $\Omega \subset \mathbb{R}^d$. We will apply the Schaefer fixed point theorem, which is a generalisation of the Leray-Schauder fixed point theorem to locally convex spaces, to the approximating bounded subsets $\Omega_k \subset \mathring{\Omega}$, as in [16], so that the regularity of the boundary of Ω can be ignored.

Assume

$$D(\mathbb{A}) = \mathbf{L}^2_{\mathbb{A}} \subset \mathbf{H}^{s+\theta}_{loc}(\Omega) \quad \text{for some } \theta \geq 0. \tag{7.19}$$

Note that this assumption is weaker than the one given in Equation (4.2) of [16], and allows us to cover the fractional derivatives as well.

Then, we have the following main result:

Theorem 7.11. *Suppose $\Omega \subset \mathbb{R}^d$ is an open set. Let $\mathbb{A}$ satisfy (7.19) for $\theta \geq 0$, and*

$$\Pi :]0,T[\times \Omega \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^{m\times m}$$

be a measurable, invertible matrix $\Pi = \Pi(t, x, \boldsymbol{u}, \boldsymbol{p})$ satisfying

$$\underline{\gamma}|\xi|^2 \leq \Pi\xi \cdot \xi \quad \text{and} \quad \Pi\xi \cdot \xi^* \leq \bar{\gamma}|\xi||\xi^*|, \quad 0 < \underline{\gamma} \leq \bar{\gamma}, \quad \text{for all } \xi, \xi^* \in \mathbb{R}^m, \tag{7.3}$$

such that Π is continuous in $\boldsymbol{u}$ and $\boldsymbol{p}$ for almost every (t, x). Let

$$\boldsymbol{f} :]0,T[\times \Omega \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^m$$

be a measurable vector function that is continuous in $\boldsymbol{u}$ and $\boldsymbol{p}$ for almost every (t, x), satisfying

$$|\boldsymbol{f}(t, x, \boldsymbol{u}, \boldsymbol{p})| \leq F(t, x) + \Lambda_1|\boldsymbol{u}| + \Lambda_2|\boldsymbol{p}| \quad \text{for some } F \in L^2(0,T; L^2(\Omega)), \Lambda_1, \Lambda_2 \geq 0. \tag{7.20}$$

Then for every $\boldsymbol{u}_0$ such that $\boldsymbol{u}_0 \in \mathbf{H}^s_0(\Omega)$, there exists

$$\boldsymbol{u} \in H^1(0,T; \mathbf{L}^2(\Omega)) \cap L^2(0,T; \mathbf{L}^2_{\mathbb{A}}) \cap C([0,T]; \mathbf{H}^s_0(\Omega)), \tag{7.21}$$

solving the problem

$$\begin{aligned} &\boldsymbol{u}'(t) + \Pi(t, x, \boldsymbol{u}, \Sigma\boldsymbol{u})\mathbb{A}\boldsymbol{u}(t) = \boldsymbol{f}(t, x, \boldsymbol{u}, \Sigma\boldsymbol{u}) \quad \text{for a.e. } t \in]0,T[, \\ &\boldsymbol{u}(0) = \boldsymbol{u}_0 \end{aligned} \tag{7.22}$$

where Σ represents fractional derivatives of order $\sigma \leq s$ with $0 < \sigma < s + \theta$ for $0 < s \leq 1$. Moreover, there exists a constant $c' = c'(\underline{\gamma}, \bar{\gamma}, a_*, a^*, \Lambda_1, \Lambda_2, T) > 0$ such that for every solution $\boldsymbol{u}$ of (7.22),

$$\|\boldsymbol{u}\|_{MR} \leq c' \left(\|F\|_{L^2(0,T;L^2(\Omega))} + \|\boldsymbol{u}_0\|_{\mathbf{H}^s_0(\Omega)} \right). \tag{7.23}$$

Remark 7.12. *In general, this solution is not unique.*

Remark 7.13. *This extends the results for the classical derivatives with $s = \theta = 1$ so that $s + \theta = 2$ with $\sigma = 1$ as considered in [16] for the scalar problem, as well as [17] and [152] for the semilinear vectorial problem and quasilinear vectorial problem respectively. In particular, we can consider fractional or nonlocal derivatives of any order $\sigma \leq s \leq 1$. This generalises the classical gradient, and is conceptually similar to the ideas of Boussandel (see [48] and [49]), where he considers the classical gradient weighted by a measure.*

For general operators $\mathbb{A}$ satisfying (7.2), the theorem applies with $\theta = 0$ and $\sigma < s \leq 1$. For special operators satisfying (7.19) with $\theta > 0$, we can consider derivatives of order σ up to and including $\sigma = s$ for $s \leq 1$, as in Section 7.3.2. This includes the classical vectorial operator $\mathbb{L}$ with $s = \theta = 1$ as given in Proposition 7.4, as well as the nonlocal vectorial operator $\mathbb{L}^s_A$ with $\theta = s < 1$ in Proposition 7.8. Observe that the latter includes the case of the fractional Laplacian $(-\Delta)^s_m$ of Proposition 7.6, with $0 < \theta < \frac{1}{2}$ for $s > \frac{1}{2}$ and $\theta < s$ for $s < \frac{1}{2}$. Furthermore, when $\mathbb{A} = (-\Delta)^s_m$, $\Sigma\boldsymbol{u}$ can involve both the fractional and the nonlocal derivatives. This is because the fractional Laplacian $(-\Delta)^s$ can be represented by both the fractional derivatives D^s and the nonlocal derivatives $\mathcal{D}^s$ i.e. $(-\Delta)^s u = -D^s \cdot D^s u = c^2_{d,s}\mathcal{D}_s \cdot \mathcal{D}^s u$ when considered in $\mathbb{R}^d$.

We shall also use the Schaefer fixed point theorem, as it is reproduced in Theorem 2.2 of [15], which we state here for reference. Note that if E is a Banach space, this theorem reduces to the Leray-Schauder fixed point theorem (see, for instance, Theorem 11.3 of [126]).

Theorem 7.14 (Schaefer Fixed Point Theorem)**.** *Let E be a complete locally convex vector space and let $S : E \to E$ be a continuous mapping. Assume that there exists a continuous seminorm $p : E \to \mathbb{R}^+$, a constant $R > 0$, and a compact set $\mathcal{K} \subset E$ such that the Schaefer set*

$$\mathscr{S} = \{\boldsymbol{u} \in E : \boldsymbol{u} = \lambda S\boldsymbol{u} \text{ for some } \lambda \in [0,1]\}$$

is included in

$$\mathcal{C} := \{\boldsymbol{u} \in E : p(\boldsymbol{u}) < R\}$$

such that

$$S\mathcal{C} \subset \mathcal{K}.$$

Then S has a fixed point.

Recalling that $\Sigma\boldsymbol{u} \in \mathbb{R}^q$ for $0 < q \leq m \times d$ represents fractional or nonlocal derivatives in the form $D^\sigma\boldsymbol{u}$ or $\mathcal{D}^\sigma\boldsymbol{u}$, we observe that for any Lipschitz bounded open set $\mathcal{O}$ such that $\bar{\mathcal{O}} \subset \Omega \subset \mathbb{R}^d$, for every $\boldsymbol{v} \in L^2(0,T;\mathbf{H}^\sigma(\mathcal{O}))$, the extension $\tilde{\boldsymbol{v}} \in L^2(0,T;\mathbf{H}^\sigma(\mathbb{R}^d))$ and

$$\|\Sigma\tilde{\boldsymbol{v}}\|_{L^2(0,T;\mathbf{L}^2(\mathcal{O}))} \leq C\|\boldsymbol{v}\|_{L^2(0,T;\mathbf{H}^\sigma(\mathcal{O}))} \tag{7.24}$$

for some constant C depending on $\mathcal{O}$.

7.3.1 Proof of Theorem 7.11

Let $(\Omega_k)_k$ be an increasing sequence of open bounded subsets of $\mathbb{R}^d$ with Lipschitz boundaries such that $\overline{\Omega_k} \subset \Omega$ and $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$. Consider the locally convex space

$$\begin{aligned} E &:= L^2(0,T;\mathbf{H}^\sigma_{loc}(\Omega)) \\ &:= \{\boldsymbol{u} \in \mathbf{L}^2_{loc}(]0,T[\times\Omega) : \boldsymbol{u}|_{]0,T[\times\Omega_k} \in L^2(0,T;\mathbf{H}^\sigma(\Omega_k)) \text{ for every } k \in \mathbb{N}\}, \end{aligned}$$

which is a Fréchet space for the sequence of seminorms given by $\|\cdot\|_{L^2(0,T;\mathbf{H}^\sigma(\Omega_k))}$, $k \in \mathbb{N}$, as defined in (1.15) for each Ω_k.

Recall that for a Lipschitz open bounded set $\mathcal{O} \subset \mathbb{R}^d$ (c.f. Theorem 7.26 of [126]), the Sobolev embedding $\mathbf{H}^{\sigma'}(\mathcal{O}) \hookrightarrow \mathbf{H}^\sigma(\mathcal{O})$ is compact for $\sigma < \sigma'$ by the Rellich-Kondrachov theorem. Then, by Aubin-Lions lemma (Lemma II.7.7 of [202]), we have the compact embedding

$$H^1(0,T;\mathbf{L}^2(\mathcal{O})) \cap L^2(0,T;\mathbf{H}^{\sigma'}(\mathcal{O})) \hookrightarrow L^2(0,T;\mathbf{H}^\sigma(\mathcal{O})), \tag{7.25}$$

and this embedding is continuous by the closed graph theorem. Since $\mathbf{L}^2_{\mathbb{A}} \subset \mathbf{H}^{s+\theta}_0(\Omega)$ by Assumption (7.19), applying (7.25) for $\sigma' = s + \theta$ for $\theta \geq 0$ for the open bounded sets Ω_k, it follows that

$$MR = H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2_{\mathbb{A}}) \hookrightarrow L^2(0,T;\mathbf{H}^\sigma_{loc}(\Omega)) = E \tag{7.26}$$

for $\sigma < s + \theta$ is also compact, for any set $\Omega \subseteq \mathbb{R}^d$.

Fix T and $\boldsymbol{u}_0 \in \mathbf{H}^s_0(\Omega)$. We first show the result for every k, i.e. for every k, the problem for $\boldsymbol{u} = \boldsymbol{u}_k$

$$\begin{aligned} &\boldsymbol{u} \in MR = H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2_{\mathbb{A}}), \\ &\boldsymbol{u}'(t) + \Pi(t,x,\boldsymbol{u},\Sigma\boldsymbol{u})\mathbb{A}\boldsymbol{u}(t) = \chi_{\Omega_k}(x)\boldsymbol{f}(t,x,\boldsymbol{u},\Sigma\boldsymbol{u}) \quad \text{for a.e. } t \in]0,T[, \\ &\boldsymbol{u}(0) = \boldsymbol{u}_0 \end{aligned} \tag{7.27}$$

admits a solution such that for every solution $\boldsymbol{u}$, we have

$$\|\boldsymbol{u}\|_{MR} \leq c'\left(\|F\|_{L^2(0,T;L^2(\Omega))} + \|\boldsymbol{u}_0\|_{\mathbf{H}^s_0(\Omega)}\right) \tag{7.23}$$

for some constant $c' = c'(\underline{\gamma}, \bar{\gamma}, a_*, a^*, \mu, \Lambda_1, \Lambda_2, T) > 0$ independent of k. Here $\chi_{\Omega_k}(x)$ is the scalar characteristic function which is 1 if $x \in \Omega_k$ and 0 otherwise.

For each fixed $k \in \mathbb{N}$ and for every $\boldsymbol{v} \in E$, we set

$$\Pi_{\boldsymbol{v}}(t,x) := \Pi(t,x,\boldsymbol{v}(t,x),\Sigma\tilde{\boldsymbol{v}}(t,x)), \quad \text{and}$$
$$\boldsymbol{f}_{\boldsymbol{v},k}(t,x) := \chi_{\Omega_k}(x)\boldsymbol{f}(t,x,\boldsymbol{v}(t,x),\Sigma\tilde{\boldsymbol{v}}(t,x)).$$

Then, $\Pi_{\boldsymbol{v}}$ inherits the same properties as Π, while $\boldsymbol{f}_{\boldsymbol{v},k}$ is measurable and satisfies

$$\begin{aligned}\|\boldsymbol{f}_{\boldsymbol{v},k}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} &\leq \int_0^T \int_{\Omega_k} F(t,x) + \Lambda_1^2|\boldsymbol{v}|^2 + \Lambda_2^2|\Sigma\boldsymbol{v}|^2 \\ &\leq \|F\|^2_{L^2(0,T;L^2(\Omega))} + \Lambda_1^2\|\boldsymbol{v}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega_k))} + C_k^2\Lambda_2^2\|\boldsymbol{v}\|^2_{L^2(0,T;\mathbf{H}^\sigma(\Omega_k))} < \infty\end{aligned}$$

for the same constant $C_k = C(\Omega_k)$ as in (7.24). Then, by Theorem 7.2, there exists a unique solution $\boldsymbol{u} =: \mathcal{T}_k\boldsymbol{v} \in MR$ of the problem

$$\begin{aligned}&\boldsymbol{u}'(t) + \Pi_{\boldsymbol{v}}(t,\cdot)\mathbb{A}\boldsymbol{u}(t) = \boldsymbol{f}_{\boldsymbol{v},k}(t,\cdot) \quad \text{for a.e. } t \in]0,T[, \text{ and} \\ &\boldsymbol{u}(0) = \boldsymbol{u}_0\end{aligned} \tag{7.28}$$

such that $\boldsymbol{u} \in MR$, satisfying the inequality

$$\begin{aligned}\|\boldsymbol{u}\|_{MR} &\leq c\left(\|\boldsymbol{f}_{\boldsymbol{v},k}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{u}_0\|_{\mathbf{H}^s_0(\Omega)}\right) \\ &\leq c_4\left(\|F\|^2_{L^2(0,T;L^2(\Omega))} + \|\boldsymbol{v}\|^2_{L^2(0,T;\mathbf{H}^\sigma(\Omega_k))} + \|\boldsymbol{u}_0\|_{\mathbf{H}^s_0(\Omega)}\right)\end{aligned} \tag{7.29}$$

for some constant $c_4 = c_4(c, k, \Lambda_1, \Lambda_2)$, where c is the same constant from Theorem 7.2. In this way, we have defined an operator $\mathcal{T}_k : E \to MR \subset E$.

Next, let $\boldsymbol{v}_i \to \boldsymbol{v}$ in E, and denote $\boldsymbol{u}_i = \mathcal{T}_k\boldsymbol{v}_i$ and $\boldsymbol{u} = \mathcal{T}_k\boldsymbol{v}$. We want to show that $\mathcal{T}_k$ is continuous, i.e. $\boldsymbol{u}_i \to \boldsymbol{u}$ in E. Since $(\boldsymbol{u}_i)_i$ is bounded in MR by the estimate (7.29) which is uniform in i for fixed k, and since MR is a Hilbert space, we may assume, after passing to a subsequence, that there exists a $\boldsymbol{w} \in E$ such that

$$\boldsymbol{u}_i \rightharpoonup \boldsymbol{w} \quad \text{in } MR. \tag{7.30}$$

Passing to a further subsequence, we may in addition assume that

$$\boldsymbol{u}_i' \rightharpoonup \boldsymbol{w}' \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega)), \text{ and} \tag{7.31}$$

$$\mathbb{A}\boldsymbol{u}_i \rightharpoonup \mathbb{A}\boldsymbol{w} \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega)). \tag{7.32}$$

We show that $\boldsymbol{w} = \boldsymbol{u}$. Since $\boldsymbol{v}_i \to \boldsymbol{v}$ in E, passing to a further subsequence and using a diagonalisation argument, there exists a function $V_k \in L^2(]0,T[\times\Omega_k)$ such that

$$\begin{aligned}(\boldsymbol{v}_i, \Sigma\boldsymbol{v}_i) \to (\boldsymbol{v}, \Sigma\boldsymbol{v}) &\quad \text{a.e. on }]0,T[\times\Omega, \quad \forall i \in \mathbb{N}, \text{ and} \\ |\boldsymbol{v}_i| + |\Sigma\boldsymbol{v}_i| \leq V_k &\quad \text{a.e. on }]0,T[\times\Omega_k, \quad \forall i \in \mathbb{N}\end{aligned} \tag{7.33}$$

by the continuity of Σ which involves the ∂, D^s and $\mathcal{D}^s$ operators.

By the continuity of Π and $\boldsymbol{f}$, we have

$$\Pi_{\boldsymbol{v}_i}(t,x) := \Pi(t,x,\boldsymbol{v}_i,\Sigma\boldsymbol{v}_i) \to \Pi(t,x,\boldsymbol{v},\Sigma\boldsymbol{v}) =: \Pi_{\boldsymbol{v}}(t,x), \quad \text{and}$$
$$\boldsymbol{f}_{\boldsymbol{v}_i,k}(t,x) := \chi_{\Omega_k}(x)\boldsymbol{f}(t,x,\boldsymbol{v}_i,\Sigma\boldsymbol{v}_i) \to \chi_{\Omega_k}(x)\boldsymbol{f}(t,x,\boldsymbol{v},\Sigma\boldsymbol{v}) =: \boldsymbol{f}_{\boldsymbol{v},k}(t,x) \quad \text{a.e. on }]0,T[\times\Omega.$$

Moreover, by the growth assumption on $\boldsymbol{f}$ in (7.20) and uniform domination of $\boldsymbol{v}_i$ by V_k in (7.33), we have

$$|\boldsymbol{f}_{\boldsymbol{v}_i,k}| \leq F + (\Lambda_1 + \Lambda_2)V_k \quad \text{a.e. in }]0,T[\times\Omega_k, \quad \forall i \in \mathbb{N}. \tag{7.34}$$

Recall that, for every $i \in \mathbb{N}$, $\boldsymbol{u}_i$ satisfies the problem

$$\boldsymbol{u}_i' + \Pi_{\boldsymbol{v}_i}\mathbb{A}\boldsymbol{u}_i = \boldsymbol{f}_{\boldsymbol{v}_i}. \tag{7.35}$$

By the Dominated Convergence Theorem and (7.34),

$$\boldsymbol{f}_{\boldsymbol{v}_i,k} \to \boldsymbol{f}_{\boldsymbol{v},k} \quad \text{strongly in } L^2(0,T;\mathbf{L}^2(\Omega)).$$

Also, by the Dominated Convergence Theorem, since $\Pi_{\boldsymbol{v}_i}$ is uniformly bounded as in (7.3), we have, for every $\varphi \in L^2(0,T;\mathbf{L}^2(\Omega))$,

$$\Pi^*_{\boldsymbol{v}_i}\varphi \to \Pi^*_{\boldsymbol{v}}\varphi \quad \text{in } L^2(]0,T[\times\Omega).$$

By (7.32), it follows that for every $\varphi \in L^2(0,T;\mathbf{L}^2(\Omega))$,

$$\int_0^T\int_\Omega \Pi_{\boldsymbol{v}_i}\mathbb{A}\boldsymbol{u}_i\cdot\varphi = \int_0^T\int_\Omega \mathbb{A}\boldsymbol{u}_i\cdot\Pi^*_{\boldsymbol{v}_i}\varphi \to \int_0^T\int_\Omega \mathbb{A}\boldsymbol{w}\cdot\Pi^*_{\boldsymbol{v}}\varphi = \int_0^T\int_\Omega \Pi_{\boldsymbol{v}}\mathbb{A}\boldsymbol{w}\cdot\varphi,$$

or equivalently

$$\Pi_{\boldsymbol{v}_i}\mathbb{A}\boldsymbol{u}_i \rightharpoonup \Pi_{\boldsymbol{v}}\mathbb{A}\boldsymbol{w} \quad \text{weakly in } L^2(0,T;\mathbf{L}^2(\Omega)). \tag{7.36}$$

Therefore, taking $i\to\infty$ in (7.35) gives

$$\boldsymbol{w}'(t) + \Pi_{\boldsymbol{v}}\mathbb{A}\boldsymbol{w}(t) = \boldsymbol{f}_{\boldsymbol{v},k}(t) \quad \text{in } \Omega \text{ for a.e. } t\in]0,T[.$$

Since $MR \hookrightarrow C([0,T];\mathbf{H}^s_0(\Omega))$ by Lemma 7.1, the weak convergence of $\boldsymbol{w} \rightharpoonup \boldsymbol{u}$ in MR gives

$$\boldsymbol{w}(0) = \lim_{i\to\infty}\boldsymbol{u}_i(0) = \boldsymbol{u}_0.$$

But $\boldsymbol{u}$ is also the solution of the problem (7.28) which is unique by Theorem 7.2, so $\boldsymbol{w} = \boldsymbol{u}$.

Since $\boldsymbol{u}_i \rightharpoonup \boldsymbol{u}$ in MR, by the compact embedding $MR \hookrightarrow E$, we obtain, passing to a subsequence if necessary,

$$\boldsymbol{u}_i \to \boldsymbol{u} \quad \text{strongly in } E,$$

so $\mathcal{T}_k$ is continuous.

In the next step, we show that there exists a non-negative constant depending on $\underline{\gamma}, \bar{\gamma}, a_*, a^*, \mu, \Lambda_1, \Lambda_2$ and T independent of k such that for every element $\boldsymbol{u}$ in the Schaefer set

$$\mathscr{S}_k = \{\boldsymbol{v}\in E : \boldsymbol{v} = \alpha\mathcal{T}_k\boldsymbol{v} \text{ for some } \alpha\in[0,1]\},$$

the estimate (7.23) holds.

Assume that $\boldsymbol{u} = \alpha\mathcal{T}_k(\boldsymbol{u})$ for some $\alpha\in[0,1]$, i.e. $\boldsymbol{u}$ satisfies

$$\begin{aligned}&\boldsymbol{u}'(t) + \Pi(t,\cdot,\boldsymbol{u},\Sigma\boldsymbol{u})\mathbb{A}\boldsymbol{u}(t) = \alpha\chi_{\Omega_k}(x)\boldsymbol{f}(t,\cdot,\boldsymbol{u},\Sigma\boldsymbol{u}) \quad \text{for a.e. } t\in]0,T[, \text{ and}\\ &\boldsymbol{u}(0) = \boldsymbol{u}_0.\end{aligned} \tag{7.37}$$

Multiplying the equation by $[\Pi^*_{\boldsymbol{u}}]^{-1}\boldsymbol{u}'(t)$ and integrating over Ω, we obtain, by Lemma 7.1 and the Cauchy-Schwarz inequality,

$$\begin{aligned}\int_\Omega [\Pi^*_{\boldsymbol{u}}]^{-1}\boldsymbol{u}'(t)\cdot\boldsymbol{u}'(t) + \frac{1}{2}\frac{d}{dt}\int_\Omega \mathbb{A}\boldsymbol{u}\cdot\boldsymbol{u} &= \int_\Omega [\Pi^*_{\boldsymbol{u}}]^{-1}\boldsymbol{u}'(t)\cdot\boldsymbol{u}'(t) + \int_\Omega \mathbb{A}\boldsymbol{u}\cdot\boldsymbol{u}'(t)\\ &= \alpha\int_{\Omega_k}\boldsymbol{f}\cdot[\Pi^*_{\boldsymbol{u}}]^{-1}\boldsymbol{u}'(t) \le \frac{\bar{\gamma}}{2}\int_\Omega |[\Pi^*_{\boldsymbol{u}}]^{-1}\boldsymbol{f}|^2 + \frac{1}{2\bar{\gamma}}\int_\Omega|\boldsymbol{u}'(t)|^2 \le \frac{\bar{\gamma}}{2\underline{\gamma}^2}\int_\Omega|\boldsymbol{f}|^2 + \frac{1}{2\bar{\gamma}}\int_\Omega|\boldsymbol{u}'(t)|^2,\end{aligned}$$

by the positivity of Π and since $\alpha\in[0,1]$. Making use of the coercivity and boundedness of $\Pi_{\boldsymbol{u}}$ in (7.3), we integrate over time on $]0,t[$ for every finite $t\in]0,T[$ to obtain, by (7.2) and (7.15), that

$$\begin{aligned}
&\frac{1}{\bar\gamma}\int_0^t \left\|\boldsymbol{u}'(\tau)\right\|^2_{\mathbf{L}^2(\Omega)}\,d\tau + a_*\left\|\boldsymbol{u}(t)\right\|^2_{\mathbf{H}^s_0(\Omega)}\\
&\le a^*\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \mu\left\|\boldsymbol{u}(t)\right\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2\underline{\gamma}}\int_0^t \left\|\boldsymbol{f}(\tau)\right\|^2_{\mathbf{L}^2(\Omega)}\,d\tau\\
&\le a^*\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + 2\mu^2\bar\gamma\int_0^t \left\|\boldsymbol{u}(\tau)\right\|^2_{\mathbf{L}^2(\Omega)}\,d\tau + \frac{1}{2\bar\gamma}\int_0^t \left\|\boldsymbol{u}'(\tau)\right\|^2_{\mathbf{L}^2(\Omega)}\,d\tau\\
&\quad + \frac{\bar\gamma}{2\underline{\gamma}^2}\int_0^t \left\|F(\tau)\right\|^2_{L^2(\Omega)}\,d\tau + \frac{\Lambda_1^2\bar\gamma}{2\underline{\gamma}^2}\int_0^t \left\|\boldsymbol{u}(\tau)\right\|^2_{\mathbf{L}^2(\Omega)}\,d\tau + \frac{\Lambda_2^2\bar\gamma}{2\underline{\gamma}^2}\int_0^t [\boldsymbol{u}(\tau)]^2_{\mathbf{H}^\sigma(\mathbb{R}^d)}\,d\tau\\
&\le a^*\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \frac{\bar\gamma}{2\underline{\gamma}^2}\|F\|^2_{L^2(0,T;L^2(\Omega))}\\
&\quad + \left(2\mu^2\bar\gamma + \bar\gamma\frac{\Lambda_1^2 + c_S\Lambda_2^2}{2\underline{\gamma}^2}\right)\int_0^t \left\|\boldsymbol{u}(\tau)\right\|^2_{\mathbf{H}^s_0(\Omega)}\,d\tau + \frac{1}{2\bar\gamma}\int_0^t \left\|\boldsymbol{u}'(\tau)\right\|^2_{\mathbf{L}^2(\Omega)}\,d\tau
\end{aligned}\tag{7.38}$$

by the Sobolev embedding $\mathbf{H}^\sigma_0(\Omega)\hookrightarrow\mathbf{H}^s_0(\Omega)$ with Sobolev constant c_S for $\sigma\le s$. Then, applying Gronwall's lemma, we can argue as in the proof of Theorem 7.2 to get the estimate (7.23) for every $\boldsymbol{u}\in\mathscr{S}_k$.

This means that $\mathscr{S}_k$ is bounded in MR. By the definition of the MR norm, this implies that there exists an $R>0$ such that

$$\mathscr{S}_k\subset\mathcal{C}_k := \{\boldsymbol{v}\in E : \|\boldsymbol{v}\|_{L^2(0,T;\mathbf{H}^\sigma(\Omega_k))} < R\},$$

because clearly $\|\cdot\|_{L^2(0,T;\mathbf{H}^\sigma(\Omega_k))}\le\|\cdot\|_{L^2(0,T;\mathbf{H}^\sigma(\Omega))}$. It follows from the definition of $\mathcal{T}_k$ and (7.29) that $\mathcal{T}_k\mathcal{C}_k$ is contained in a bounded subset of MR. By compactness of the embedding (7.26), $\mathcal{T}_k\mathcal{C}_k$ is contained in a compact subset of E. Therefore, by Schaefer's fixed point theorem (Theorem 7.14), the mapping $\mathcal{T}_k$ admits a fixed point $\boldsymbol{u}$ such that $\boldsymbol{u}\in MR$. By the definition of $\mathcal{T}_k$, this element $\boldsymbol{u}$ is a solution of the problem (7.27), and since $\boldsymbol{u}\in\mathscr{S}_k$, $\boldsymbol{u}$ satisfies (7.23).

Finally, we extend the result to show that (7.22) admits a solution. For every $k\in\mathbb{N}$, we choose a solution $\boldsymbol{u}_k$ of the problem (7.27). Since every such solution is an element of $\mathscr{S}_k$ and satisfies the estimate (7.23) which is independent of k, the sequence $(\boldsymbol{u}_k)_k$ is bounded in MR. Since MR is a Hilbert space, we may assume (after passing to a subsequence) that there exists a limit $\boldsymbol{u}\in E$ such that $\boldsymbol{u}_k\rightharpoonup\boldsymbol{u}$ in MR. By the compactness of the embedding (7.26), passing to a subsequence again if necessary, we obtain, through a diagonalisation argument, that

$$\begin{aligned}
&\boldsymbol{u}_k'\rightharpoonup\boldsymbol{u}' \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega)),\\
&\mathbb{A}\boldsymbol{u}_k\rightharpoonup\mathbb{A}\boldsymbol{u} \quad \text{in } L^2(0,T;\mathbf{L}^2(\Omega)),\\
&(\boldsymbol{u}_k,\Sigma\boldsymbol{u}_k)\to(\boldsymbol{u},\Sigma\boldsymbol{u}) \text{ a.e. on }]0,T[\times\Omega, \text{ and}\\
&|\boldsymbol{u}_k| + |\Sigma\boldsymbol{u}_k|\le U \quad \text{a.e. on }]0,T[\times\Omega, \quad \forall k\in\mathbb{N},
\end{aligned}\tag{7.39}$$

for some $U\in L^2_{loc}(]0,T[\times\Omega)$.

By continuity of Π and $\boldsymbol{f}$, since Ω_k is increasing to Ω,

$$\Pi(t,x,\boldsymbol{u}_k,\Sigma\boldsymbol{u}_k)\to\Pi(t,x,\boldsymbol{u},\Sigma\boldsymbol{u}), \quad \text{and}$$

$$\chi_{\Omega_k}(x)\boldsymbol{f}(t,x,\boldsymbol{u}_k,\Sigma\boldsymbol{u}_k)\to\boldsymbol{f}(t,x,\boldsymbol{u},\Sigma\boldsymbol{u}) \text{ a.e. on }]0,T[\times\Omega.$$

By the uniform boundedness of $\boldsymbol{f}_{\boldsymbol{u}_k,k}$ in (7.34) and the domination of $\boldsymbol{u}_k$ by U in (7.39), we have

$$|\chi_{\Omega_k}(x)\boldsymbol{f}(t,x,\boldsymbol{u}_k,\Sigma\boldsymbol{u}_k)|\le F + (\Lambda_1+\Lambda_2)U \quad \text{a.e. on }]0,T[\times\Omega, \quad \forall k\in\mathbb{N}.$$

Also, as in (7.36), the convergences in (7.39) imply that

$$\Pi(t,x,\boldsymbol{u}_k,\Sigma\boldsymbol{u}_k)\mathbb{A}\boldsymbol{u}_k\rightharpoonup\Pi(t,x,\boldsymbol{u},\Sigma\boldsymbol{u})\mathbb{A}\boldsymbol{u} \quad \text{weakly in } L^2(0,T;\mathbf{L}^2(\Omega)).\tag{7.40}$$

Therefore,

$$\chi_{\Omega_k}(x)\boldsymbol{f}(t,x,\boldsymbol{u}_k,\Sigma\boldsymbol{u}_k) = \boldsymbol{u}_k' + \Pi(t,x,\boldsymbol{u}_k,\Sigma\boldsymbol{u}_k)\mathbb{A}\boldsymbol{u}_k \text{ converges weakly in } L^2(0,T;\mathbf{L}^2(\Omega)).$$

On the other hand, for every $\varphi \in L^2(0,T;\mathbf{C}_c(\Omega))$ compactly supported in $]0,T[\times\Omega$, we have

$$\int_0^T \int_\Omega \chi_{\Omega_k}\boldsymbol{f}(t,x,\boldsymbol{u}_k,\Sigma\boldsymbol{u}_k)\cdot\varphi \to \int_0^T\int_\Omega \boldsymbol{f}(t,x,\boldsymbol{u},\Sigma\boldsymbol{u})\cdot\varphi$$

by the dominated convergence theorem. Since compactly supported functions are dense in $L^2(0,T;\mathbf{L}^2(\Omega))$, we have the weak convergence

$$\chi_{\Omega_k}\boldsymbol{f}(t,x,\boldsymbol{u}_k,\Sigma\boldsymbol{u}_k) \rightharpoonup \chi_\Omega\boldsymbol{f}(t,x,\boldsymbol{u},\Sigma\boldsymbol{u}) = \boldsymbol{f}(t,x,\boldsymbol{u},\Sigma\boldsymbol{u}) \quad \text{weakly in } L^2(0,T;\mathbf{L}^2(\Omega)).$$

Letting $k\to\infty$ in the problem (7.27), we therefore obtain that $\boldsymbol{u}$ satisfies the original problem

$$\boldsymbol{u}' + \Pi(t,x,\boldsymbol{u},\Sigma\boldsymbol{u})\mathbb{A}\boldsymbol{u} = \boldsymbol{f}(t,x,\boldsymbol{u},\Sigma\boldsymbol{u}) \quad \text{in } \Omega \text{ for a.e. } t\in]0,T[.$$

Furthermore, invoking the continuity $MR \hookrightarrow C([0,T];\mathbf{H}^s_0(\Omega))$ by Lemma 7.1 as before, $\boldsymbol{u}_k(0)\rightharpoonup\boldsymbol{u}(0)$ in $\mathbf{H}^s_0(\Omega)$, so $\boldsymbol{u}(0)=\boldsymbol{u}_0$. Thus, $\boldsymbol{u}$ is a solution to the problem (7.23). Furthermore, since the estimate (7.38) is independent of k, we can pass to the limit to obtain the estimate (7.23).

Remark 7.15. *It is also possible to consider a different nonlocal vectorial operator* $\mathbb{A}\boldsymbol{u} = (\mathbb{A}_1u^1,\ldots,\mathbb{A}_mu^m)$ *for each equation in the system*

$$\boldsymbol{u}' + \Pi(t,x,\boldsymbol{u},\Sigma\boldsymbol{u})\mathbb{A}\boldsymbol{u} = \boldsymbol{f}(t,x,\boldsymbol{u},\Sigma\boldsymbol{u}) \quad \text{in }]0,T[\times\Omega,$$

for $\mathbb{A}_i$ *given by (possibly different) scalar operators satisfying* (7.2)*, which may be of the form* (7.7)*,* (7.8) *or* (7.9)*, and* Π *satisfy the same assumptions.*

Remark 7.16. *The results in Theorem 7.11 can in fact be extended to the inhomogeneous Dirichlet boundary problem* $\boldsymbol{u}=\boldsymbol{g}$ *in* $]0,T[\times\Omega^c$.

Indeed, writing $MR(\mathbb{R}^d)$ *for*

$$MR(\mathbb{R}^d) := H^1(0,T;\mathbf{L}^2(\mathbb{R}^d)) \cap \{\boldsymbol{u}\in\mathbf{H}^s(\mathbb{R}^d) : \mathbb{A}\boldsymbol{u}\in\mathbf{L}^2(\mathbb{R}^d)\},$$

let $\boldsymbol{g}\in MR(\mathbb{R}^d)\cap L^2(0,T;\mathbf{H}^{s+\theta}(\mathbb{R}^d))\cap C([0,T];\mathbf{H}^s(\mathbb{R}^d))$*, such that* $\boldsymbol{g}(0)\in\mathbf{H}^s(\mathbb{R}^d)$*. Considering* $\bar{\boldsymbol{u}} = \boldsymbol{u}-\boldsymbol{g}$*, we can solve the problem for* $\bar{\boldsymbol{u}}\in MR(\Omega)$*, for the corresponding translated problem.*

7.3.2 Examples

Quasilinear System with the Classical Laplacian. As a first example, we consider the classical Laplacian Δ in the case of $s=1$ as in Example 5.1 of [16], extended to the case of a system of equations.

Considering the vectorial Laplacian $(-\Delta)_m$ defined by

$$(-\Delta)_m = \begin{bmatrix} -c_1\Delta & & 0 \\ & \ddots & \\ 0 & & -c_m\Delta \end{bmatrix}$$

for constants $c_1,\cdots,c_m>0$. Then, applying Theorem 7.11 for $\mathbb{A}=(-\Delta)_m$, we have

Corollary 7.17. *Suppose* Π *and* $\boldsymbol{f}$ *satisfy the assumptions of Theorem 7.11 with* Ω *being an open bounded Lipschitz domain. Then, writing for the gradient* $\partial=(\partial_1,\ldots,\partial_n)$*, for every* $\boldsymbol{u}_0\in\mathbf{H}^1_0(\Omega)$*, the nonlinear problem given by*

$$\begin{cases} \boldsymbol{u}'(t) + \Pi(t,x,\boldsymbol{u},\partial\boldsymbol{u})(-\Delta)_m\boldsymbol{u}(t) = \boldsymbol{f}(t,x,\boldsymbol{u},\partial\boldsymbol{u}) & \text{for a.e. } t\in]0,T[, \\ \boldsymbol{u}=\boldsymbol{0} \text{ a.e. on }]0,T[\times\partial\Omega, \qquad \boldsymbol{u}(0,\cdot)=\boldsymbol{u}_0(\cdot) \text{ a.e. in } \Omega \end{cases}$$

has a solution $\boldsymbol{u}\in H^1(0,T;\mathbf{L}^2(\Omega))\cap L^2(0,T;\mathbf{H}^2(\Omega))\cap C([0,T];\mathbf{H}^1_0(\Omega))$ *for any* $T\in]0,\infty[$*. Moreover, there exists a constant* $c'=c'(\underline{\gamma},\bar{\gamma},\Lambda_1,\Lambda_2,T)>0$ *such that every solution* $\boldsymbol{u}$ *satisfies*

$$\|\boldsymbol{u}\|_{MR} \le c'\left(\|F\|_{L^2(0,T;L^2(\Omega))} + \|\boldsymbol{u}_0\|_{\mathbf{H}^1_0(\Omega)}\right).$$

In particular, this extends the results of [16] to system of equations.

Approximation Models for Interacting Nonlocal Diffusive Species Populations. We also consider a nonlocal version of cross-diffusive systems modelling two interacting species, given for $0 < s \leq 1$ by

$$\begin{aligned}
&u' = -D_1(u, v, \Sigma u, \Sigma v)(-\Delta)^s u + R_1(u, v, \Sigma u, \Sigma v), \\
&v' = -D_2(u, v, \Sigma u, \Sigma v)(-\Delta)^s v + R_2(u, v, \Sigma u, \Sigma v), && x \in \Omega, t > 0 \\
&u(0, x) = u_0(x), \quad v(0, x) = v_0(x), && x \in \Omega, \\
&u(t, x) = v(t, x) = 0 && x \in \Omega^c, t > 0,
\end{aligned} \tag{7.41}$$

where Σ has order σ with $0 < \sigma \leq s < 1$. The diffusion coefficients D_1 and D_2 are bounded and strictly positive, describing a controlled nonlocal nonlinear spreading of the biological population which is dependent both on the size and density of the species itself and the other species. The interaction between the two species is both in terms of space competition with regard to diffusion, as well as in the linear bounded reaction terms $R_1(u, v)$ and $R_2(u, v)$. Here, the spreading can be represented by any of the operators in Section 7.2, and can be both nonlocal, as described by the nonlocal operator $\mathbb{L}_A^s$, or local, given by the classical operator $\mathbb{L}$. Such systems with constant diffusion coefficients may appear in activator-inhibitor systems with linear or sublinear kinetic functions (see, for instance, Chapter 9 of [243]).

This model can also be obtained as an approximation of Lotka–Volterra-type models, where the reaction terms are obtained from linearising quadratic terms describing predator-prey, competition or cooperation interactions. For instance, the Shigesada-Kawasaki-Teramoto system (see, for instance, [157] and [72]) given by

$$\boldsymbol{u}' + D(-\Delta)_2^s \boldsymbol{u} = R \quad x \in \Omega$$

for $\boldsymbol{u} = (u_1, u_2)$ with diffusion matrix

$$D = \begin{bmatrix} \begin{matrix} d_1 + q\rho_{11}u_1^q + \rho_{12}u_2^q \\ +q\rho_{13}(\mathcal{D}^s u_1)^q + \rho_{14}(\mathcal{D}^s u_2)^q \end{matrix} & q\rho_{12}u_1u_2^{q-1} \\ \\ q\rho_{21}u_1^{q-1}u_2 & \begin{matrix} d_2 + q\rho_{22}u_2^q + \rho_{21}u_1^q \\ +q\rho_{23}(\mathcal{D}^s u_2)^q + \rho_{24}(\mathcal{D}^s u_1)^q \end{matrix} \end{bmatrix}$$

and reaction term

$$R = (R_1, R_2), \quad R_i = (a_{1,i} - b_{1,i}u_1 - c_{1,i}u_2)u_i + (a_{2,i} - b_{2,i}\mathcal{D}^s u_1 - c_{2,i}\mathcal{D}^s u_2)\mathcal{D}^s u_i$$

can be approximated via the logistic function, which is a bounded nonlinearity,

$$u_i \sim \frac{u_i}{1 + \epsilon|u_i|} =: \tilde{u}_i \quad \epsilon > 0$$

$$\mathcal{D}^s u_i \sim \frac{\mathcal{D}^s u_i}{1 + \epsilon|\mathcal{D}^s u_i|} =: \widetilde{\mathcal{D}^s u_i} \quad \epsilon > 0$$

to obtain the system

$$\boldsymbol{u}' + D_{approx}(-\Delta)^s \boldsymbol{u} = R_{approx} \quad x \in \Omega$$

where D_{approx} is of the form

$$D_{approx} = \begin{bmatrix} \begin{matrix} d_1 + q\rho_{11}\tilde{u}_1^q + \rho_{12}\tilde{u}_2^q \\ +q\rho_{13}(\widetilde{\mathcal{D}^s u_1})^q + \rho_{14}(\widetilde{\mathcal{D}^s u_2})^q \end{matrix} & q\rho_{12}\tilde{u}_1\tilde{u}_2^{q-1} \\ \\ q\rho_{21}\tilde{u}_1^{q-1}\tilde{u}_2 & \begin{matrix} d_2 + q\rho_{22}\tilde{u}_2^q + \rho_{21}\tilde{u}_1^q \\ +q\rho_{23}(\widetilde{\mathcal{D}^s u_2})^q + \rho_{24}(\widetilde{\mathcal{D}^s u_1})^q \end{matrix} \end{bmatrix}$$

and

$$R_{approx} = \begin{bmatrix} (a_{1,1} - b_{1,1}\tilde{u}_1 - c_{1,1}\tilde{u}_2)u_1 + (a_{2,1} - b_{2,1}\widetilde{\mathcal{D}^s u_1} - c_{2,1}\widetilde{\mathcal{D}^s u_2})\mathcal{D}^s u_1 \\ (a_{1,2} - b_{1,2}\tilde{u}_1 - c_{1,2}\tilde{u}_2)u_2 + (a_{2,2} - b_{2,2}\widetilde{\mathcal{D}^s u_1} - c_{2,2}\widetilde{\mathcal{D}^s u_2})\mathcal{D}^s u_2 \end{bmatrix}$$

so that D_{approx} is bounded, under appropriate assumptions on ρ_{ij} and $\boldsymbol{u}$, and R_{approx} has a linear growth on u_1 and u_2 for any fixed ϵ, fulfilling our assumptions.

Supposing D_{approx} satisfies (7.3), which is obtained by taking the sufficient assumption that u_1, u_2 are positive and that the diffusion coefficients $\rho_{ij} > 0$ are bounded, and the derivatives are of order s, by Theorem 7.11, this problem admits a global solution $\boldsymbol{u}$ in

$$\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^{s'}(\Omega)) \cap C([0,T];\mathbf{H}^s_0(\Omega)),$$

where $s' = \min\{2s, s+\frac{1}{2}\} - \epsilon$ for any $\epsilon > 0$. This means that we can consider a more general case of cross-diffusion involving nonlocal operators, and obtaining a regularity result that is comparable to the classical cases in Theorem B of [157] and Theorem 1.3 of [32].

7.4 The Nonlinear Problem $s < \sigma < 2s \leq 2$ with Ω Bounded

In this section, we want to further extend the result to higher order derivatives $\sigma > s > 0$. In particular, σ may be greater than 1, generalising the scalar quasilinear diffusion equations in the classical case in [16]. Here, we focus on the classical elliptic operator $\mathbb{L}$ as given in Example 1 of Section 7.2 defined by (7.7), as well as the nonlocal fractional Laplacian defined in Example 2 of Section 7.2 defined by (2.4), since we have additional regularity results for those cases. Then, by the results of [97], and [127] and [46], we know that there exists a unique solution to the Dirichlet problem associated with $\mathbb{L}$ and with $(-\Delta)^s$, given by Propositions 7.4 and 7.6 respectively. Therefore, the spaces $\mathbf{L}^2_{\mathbb{L}}$ and $\mathbf{L}^2_{(-\Delta)^s_m}$ make sense. Furthermore, it is clear that $\mathbb{L}$ and $(-\Delta)^s_m$ are bounded and $\mathbf{L}^2(\Omega)$-coercive.

We first recall the following Poincaré inequality concerning the embedding of $\mathbf{L}^2(\Omega)$ in $\mathbf{H}^s_0(\Omega)$. See, for instance, Theorem 2.2 of [30], which gives the vectorial case of Lemma 1.3.

Lemma 7.18 (Poincaré inequality)**.** *Let $s \in]0,1]$. Then for any open bounded set $\Omega \subset \mathbb{R}^d$, there exists a constant $C_P > 0$ depending only on Ω, d and s such that*

$$\|\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \leq C_P \|D^s\boldsymbol{u}\|^2_{L^2(\mathbb{R}^d)}\,.$$

for all $\boldsymbol{u} \in \mathbf{H}^s_0(\Omega)$. In particular, we have the equivalence of the norms $\|\cdot\|_{\mathbf{L}^2(\Omega)}$ and $\|\cdot\|_{\mathbf{H}^s_0(\Omega)}$.

Assume $\mathbb{A}$ satisfies $\mathbf{L}^2_{\mathbb{A}} := \{\boldsymbol{u} \in \mathbf{H}^s_0(\Omega) : \mathbb{A}\boldsymbol{u} \in \mathbf{L}^2(\Omega)\} \subset \mathbf{H}^{\sigma'}(\Omega)$ for some $s < \sigma' < 2s$ for Ω bounded and Lipschitz domain, i.e. there exists a constant $C_{\mathbb{A}} > 0$ and $\mu' \geq 0$ such that

$$\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}^{\sigma'}(\Omega))} \leq C_{\mathbb{A}} \|\mathbb{A}\boldsymbol{u}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \mu'\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\,. \tag{7.42}$$

In particular, $\sigma' = 2$ for $\mathbb{A} = \mathbb{L}$, and $\sigma' = \min\{2s, s+\frac{1}{2}\}$ for $\mathbb{A} = (-\Delta)^s_m$ where $0 < s < 1$. Therefore, applying the compact embedding (7.25), we obtain that

$$MR = H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2_{\mathbb{A}}) \hookrightarrow L^2(0,T;\mathbf{H}^\sigma(\Omega)) = \tilde{E} \tag{7.43}$$

is compact for any open Lipschitz bounded set $\Omega \subseteq \mathbb{R}^d$, for any $\sigma < \sigma'$.

Also, by the Sobolev embeddings, there exists a Sobolev constant $0 < c_S < 1$ depending on $s < \sigma' < 2s$, σ and Ω, such that

$$c_S\|\boldsymbol{v}\|^2_{L^2(0,T;\mathbf{H}^\sigma(\Omega))} \leq \|\boldsymbol{v}\|^2_{L^2(0,T;\mathbf{H}^{\sigma'}(\Omega))} \quad \forall \boldsymbol{v} \in L^2(0,T;\mathbf{H}^{\sigma'}(\Omega)). \tag{7.44}$$

Then, assuming $\boldsymbol{f} :]0,T[\times \Omega \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^m$ satisfy the assumptions of Theorem 7.11 such that

$$|\boldsymbol{f}(t,x,\boldsymbol{u},\boldsymbol{p})| \leq F(t,x) + \Lambda_1|\boldsymbol{u}| + \Lambda_2|\boldsymbol{p}|^\alpha \tag{7.45}$$

for some $F \in L^2(0,T;L^2(\Omega))$, $\Lambda_1, \Lambda_2 \geq 0$, such that either

(i) $0 < \alpha < 1$, or

(ii) $\alpha = 1$ with

$$0 < \Lambda_2 \leq \underline{\gamma}\sqrt{\frac{c_S}{C_{\mathbb{A}}}}, \tag{7.46}$$

we have the following result:

Theorem 7.19. *Suppose $\Omega \subset \mathbb{R}^d$ is a Lipschitz bounded open set. Let $\Pi :]0,T[\times\Omega\times\mathbb{R}^m\times\mathbb{R}^q \to \mathbb{R}^{m\times m}$ satisfy the assumptions of Theorem 7.11, and $\boldsymbol{f} :]0,T[\times\Omega\times\mathbb{R}^m\times\mathbb{R}^q \to \mathbb{R}^m$ satisfy the assumptions (7.45) above for either condition (i) or (ii). Suppose $\mathbb{A}$ satisfies (7.42) for some $s < \sigma' < 2s \leq 2$. Then, for any $\sigma < \sigma'$ and every $\boldsymbol{u}_0$ such that $\boldsymbol{u}_0 \in \mathbf{H}^s_0(\Omega) \cap \mathbf{H}^\sigma(\Omega)$, there exists*

$$\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{L}^2_{\mathbb{A}}) \cap L^2(0,T;\mathbf{H}^\sigma(\Omega)) \cap C([0,T];\mathbf{H}^s_0(\Omega)) \tag{7.47}$$

solving the problem

$$\begin{aligned} &\boldsymbol{u}'(t) + \Pi(t,x,\boldsymbol{u},\Sigma\boldsymbol{u})\mathbb{A}\boldsymbol{u}(t) = \boldsymbol{f}(t,x,\boldsymbol{u},\Sigma\boldsymbol{u}) \quad \text{for a.e. } t \in]0,T[, \\ &\boldsymbol{u}(0) = \boldsymbol{u}_0, \end{aligned} \tag{7.48}$$

where Σ represents fractional derivatives of order σ which may be greater than 1. Moreover, there exists a constant $c'' = c''(\Omega,\underline{\gamma},\bar{\gamma},a_*,a^*,C_{\mathbb{A}},\Lambda_1,\Lambda_2,T,\alpha) > 0$ such that for every solution $\boldsymbol{u}$ of (7.48),

$$\|\boldsymbol{u}\|_{MR} \leq c''\left(\|F\|_{L^2(0,T;L^2(\Omega))} + \|\boldsymbol{u}_0\|_{\mathbf{H}^\sigma_0(\Omega)}\right). \tag{7.49}$$

Proof. Most of the proof follows the argument of Theorem 7.11, this time applying the Leray-Schauder fixed point theorem for the fixed point constructed in the Banach space $\tilde{E}$ in (7.43), for $\sigma < \sigma'$, where MR is compactly embedded. In particular, this means that we do not have to consider the sequence of sets Ω_k, and we can directly consider the compact map $\mathcal{T}$ defined by $\boldsymbol{u} =: \mathcal{T}\boldsymbol{v} \in MR$ of the problem

$$\begin{aligned} &\boldsymbol{u}'(t) + \Pi_{\boldsymbol{v}}(t,\cdot)\mathbb{A}\boldsymbol{u}(t) = \boldsymbol{f}_{\boldsymbol{v}}(t,\cdot) \quad \text{for a.e. } t \in]0,T[, \text{ and} \\ &\boldsymbol{u}(0) = \boldsymbol{u}_0 \end{aligned}$$

A major modification lies in the proof that the Leray-Schauder set

$$\mathscr{S} = \{\boldsymbol{u} \in \tilde{E} : \boldsymbol{u} = \lambda\mathcal{T}\boldsymbol{u} \text{ for some } \lambda \in [0,1]\} \tag{7.50}$$

is bounded. In particular, the proof of the a priori estimate in (7.38) needs to be modified for the case of $\sigma \geq s$.

Indeed, we obtain the bound on $\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}^\sigma_0(\Omega))}$ for $\sigma \geq s$ as follows: Multiplying the equation (7.37) by $\mathbb{A}\boldsymbol{u}$ and integrating over Ω, we obtain, by the bounds (7.3) and making use of Lemma 7.1 and the Cauchy-Schwarz inequality, for a.e. $t > 0$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega \boldsymbol{u}\cdot\mathbb{A}\boldsymbol{u} + \underline{\gamma}\|\mathbb{A}\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \\ \leq \int_\Omega \boldsymbol{u}'\cdot\mathbb{A}\boldsymbol{u} + \int_\Omega \Pi_u\mathbb{A}\boldsymbol{u}\cdot\mathbb{A}\boldsymbol{u} = \int_\Omega \boldsymbol{f}\cdot\mathbb{A}\boldsymbol{u} \\ \leq \frac{1}{2\underline{\gamma}}\|\boldsymbol{f}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\underline{\gamma}}{2}\|\mathbb{A}\boldsymbol{u}\|^2_{\mathbf{L}^2(\mathbb{R}^d)}. \end{aligned}$$

Integrating over time on $]0,t[$ for any $t \leq T$, it follows by (7.2) that

$$a_*\big\|\boldsymbol{u}(t)\big\|^2_{\mathbf{H}^s_0(\Omega)} + \underline{\gamma}\int_0^t \|\mathbb{A}\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} \leq a^*\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \mu\big\|\boldsymbol{u}(t)\big\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{\underline{\gamma}}\int_0^t \|\boldsymbol{f}\|^2_{\mathbf{L}^2(\Omega)},$$

and so, taking the supremum over $t \in]0,T[$ and making use of (7.15), we have

$$\begin{aligned} a_*\|\boldsymbol{u}\|^2_{L^\infty(0,T;\mathbf{H}^s_0(\Omega))} + \underline{\gamma}\|\mathbb{A}\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} \leq a^*\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \mu\|\boldsymbol{u}_0\|^2_{\mathbf{L}^2(\Omega)} + 2\mu^2\bar{\gamma}\|\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} \\ + \frac{1}{2\bar{\gamma}}\big\|\boldsymbol{u}'\big\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{\underline{\gamma}}\|\boldsymbol{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}. \end{aligned}$$

Considering only the term $\|\mathbb{A}\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}$ on the left-hand-side of the inequality and applying Assumptions (7.42) and (7.20) then gives

$$\begin{aligned}\frac{\underline{\gamma}}{C_{\mathbb{A}}}\|\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{H}^{\sigma'}(\Omega))} &\le \frac{\mu'}{C_{\mathbb{A}}}\|\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + a^*\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \mu\|\boldsymbol{u}_0\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad + 2\mu^2\bar{\gamma}\|\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{2\bar{\gamma}}\|\boldsymbol{u}'\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{\underline{\gamma}}\|\boldsymbol{f}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} \\ &\le a^*\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \mu\|\boldsymbol{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \left(\frac{\mu'}{C_{\mathbb{A}}} + 2\mu^2\bar{\gamma} + \frac{\Lambda_1^2}{\underline{\gamma}}\right)\|\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} \\ &\quad + \frac{1}{2\bar{\gamma}}\|\boldsymbol{u}'\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{\underline{\gamma}}\|F\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\Lambda_2^2}{\underline{\gamma}}\int_0^T \|D^\sigma\boldsymbol{u}\|^{2\alpha}_{\mathbf{L}^2(\mathbb{R}^d)}\,.\end{aligned}$$

Next, we argue as in estimate (7.38) to control the $H^1(0,T;\mathbf{L}^2(\Omega))$-norm of $\boldsymbol{u}$, and there exists a constant $c_5 = c_5(\underline{\gamma},\bar{\gamma},a_*,a^*,\mu,\Lambda_1,\Lambda_2,T) > 0$ such that

$$\frac{\underline{\gamma}}{C_{\mathbb{A}}}\|\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{H}^{\sigma'}(\Omega))} \le c_5\left(\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \|F\|^2_{L^2(0,T;L^2(\Omega))}\right) + \frac{\Lambda_2^2}{\underline{\gamma}}\int_0^T \|D^\sigma\boldsymbol{u}\|^{2\alpha}_{\mathbf{L}^2(\mathbb{R}^d)}\,.$$

Therefore, by (7.44),

$$\frac{c_S\underline{\gamma}}{C_{\mathbb{A}}}\|\boldsymbol{u}\|^2_{L^2(0,T;\mathbf{H}^{\sigma}(\Omega))} \le c_5\left(\|\boldsymbol{u}_0\|^2_{\mathbf{H}^s_0(\Omega)} + \|F\|^2_{L^2(0,T;L^2(\Omega))}\right) + \frac{\Lambda_2^2}{\underline{\gamma}}\|\boldsymbol{u}\|^{2\alpha}_{L^2(0,T;\mathbf{H}^{\sigma}(\Omega))}\,.$$

By the Assumption (7.45) with either Condition (i) with $\alpha < 1$ or Condition (ii) with $\alpha = 1$ and $\Lambda_2 < \underline{\gamma}\sqrt{\frac{c_S}{C_{\mathbb{A}}}}$, we obtain an a priori bound on the term $\|\boldsymbol{u}\|_{L^2(0,T;\mathbf{H}^\sigma(\Omega))}$.

Also, as in the estimate (7.38), the Leray-Schauder set $\mathscr{S}$ given by (7.50) is bounded in MR. Making use of the Aubin-Lions compactness lemma $MR \hookrightarrow \tilde{E}$, by the Leray-Schauder principle, $\mathcal{T}$ has a fixed point $\boldsymbol{u}$ satisfying (7.47) and (7.49) solving the problem (7.48). ☐

Remark 7.20. *The theorem holds with* $\sigma' = 2$ *for* $\mathbb{A} = \mathbb{L}$ *as well as with* $\sigma' = \min\{2s, s+\frac{1}{2}\}$ *for* $\mathbb{A} = (-\Delta)^s_m$ *for* $0 < s < 1$, *with derivatives of order* $s < \sigma < \sigma'$, *which may possibly be of order greater than 1. However, while derivatives of order less than 1 may take the form of* D^s *or* $\mathcal{D}^s$ *as defined by* (7.4) *and* (7.5) *respectively, the derivatives of order greater than 1 is only defined by* (7.6), *and* $\mathcal{D}^s\boldsymbol{u}$ *is not defined for* $s > 1$. *The derivatives of order equal to 1 is just the classical gradient.*

Remark 7.21. *As in Remark 7.16, the results in Theorem 7.19 can also be extended to the inhomogeneous Dirichlet boundary problem* $\boldsymbol{u} = \boldsymbol{g}$ *in* $]0,T[\times\Omega^c$, *for* $\boldsymbol{g} \in MR(\mathbb{R}^d) \cap L^2(0,T;\mathbf{H}^{s+\theta}(\mathbb{R}^d)) \cap C([0,T];\mathbf{H}^s(\mathbb{R}^d))$ *such that* $\boldsymbol{g}(0) \in \mathbf{H}^s(\mathbb{R}^d)$.

As a result, we can consider quasilinear diffusion equations and systems with derivatives of order $\sigma > s$ such that σ may be greater than 1, generalising the results of [16], [17] and [152]. This provides many useful applications, particularly in advection-diffusion systems, as seen in Section 7.4.1.

7.4.1 Examples

A System with the Classical Laplacian with D^σ-Quasilinearity $1 < \sigma < 2$. As a first application, we take a relook at the vectorial classical Laplacian $(-\Delta)_m$ in Section 7.3.2, this time, with the D^σ fractional derivatives for any $1 < \sigma < 2$ as defined in (7.6).

Corollary 7.22. *Suppose* Π *and* $\boldsymbol{f}$ *satisfy the assumptions of Theorem 7.19 with* $\boldsymbol{f}$ *fulfilling either Conditions (i) or (ii) of* (7.45). *Then, for* $1 < \sigma < 2$ *and every* $\boldsymbol{u}_0 \in \mathbf{H}^1_0(\Omega) \cap \mathbf{H}^\sigma(\Omega)$, *the nonlinear problem given by*

$$\boldsymbol{u}'(t) + \Pi(t,x,\boldsymbol{u},D^\sigma\boldsymbol{u})(-\Delta)_m\boldsymbol{u}(t) = \boldsymbol{f}(t,x,\boldsymbol{u},D^\sigma\boldsymbol{u}) \quad \text{for a.e. } t \in]0,T[,$$

has a solution $\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega)) \cap C([0,T];\mathbf{H}^1_0(\Omega))$ *satisfying* (7.49) *for any* $T \in]0,\infty[$.

In particular, this further extends the results of [16] to include the fractional derivatives D^σ of order $1 < \sigma < 2$.

Anisotropic Advection-Diffusion Fractional Equations for $s > \frac{1}{2}$. Our last application is a semilinear anisotropic advection-diffusion system of equations. Such a system may be useful to transport models with fractional diffusion and is inspired, in the scalar case, by the 2-dimensional forced subcritical surface quasi-geostrophic flows with nonlocal dissipation (see, for instance, [75]) and the 2-dimensional Navier-Stokes equation.

Suppose $s > \frac{1}{2}$. Let $\boldsymbol{v}(t,x)$ be a bounded velocity field in $]0,T[\times\Omega$ in a bounded $\Omega \subset \mathbb{R}^d$ such that

$$\|\boldsymbol{v}\|_{L^\infty(]0,T[\times\Omega)} \leq C_\# < \infty, \quad C_\# \text{ depending on } \Omega, \underline{\gamma}, s \text{ and } \mathbb{A} \text{ as in (7.46).} \tag{7.51}$$

For $\boldsymbol{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $\boldsymbol{u}_0 \in \mathbf{H}^s_0(\Omega) \cap \mathbf{H}^1(\Omega)$, the equation is given by

$$\boldsymbol{u}'(t,x) + \Pi\mathbb{A}\boldsymbol{u}(t,x) = -\sum_{\alpha=1}^{d} v^\alpha(t,x)\partial_\alpha \boldsymbol{u}(t,x) + \boldsymbol{f}(t,x,\boldsymbol{u}), \quad (t,x) \in]0,T[\times\Omega$$

$$\boldsymbol{u}(t,x) = \mathbf{0}, \quad (t,x) \in]0,T[\times\Omega^c,$$

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), \quad x \in \Omega,$$

where $\mathbb{A} = (-\Delta)^s_m$ or $\mathbb{L}$. Observe that this means that since $\frac{1}{2} < s < 1$, we have a convective term given by the classical gradient of $\boldsymbol{u}$.

Since $\boldsymbol{v}$ is bounded as in (7.51), we can apply Theorem 7.19 with $\sigma = 1$ and with the source function given by the term $-\sum_\alpha v^\alpha(t,x)\partial_\alpha \boldsymbol{u}(t,x) + \boldsymbol{f}(t,x,\boldsymbol{u})$, such that (7.45) is satisfied with $\alpha = 1$. As a result, the problem admits a global solution

$$\boldsymbol{u} \in H^1(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^1(\Omega)) \cap C([0,T];\mathbf{H}^s_0(\Omega))$$

for $0 < s \leq 1$ with $1 = \sigma < \sigma' < 2s = 2$ for $\mathbb{A} = \mathbb{L}$ and $1 = \sigma < \sigma' < \min\{2s, s + \frac{1}{2}\}$ for $\mathbb{A} = (-\Delta)^s_m$.

Furthermore, we do not require that $\boldsymbol{v}$ is divergence-free, which means that our result applies to compressible fluids as well. Such a result is new, as far as we know, since $\boldsymbol{v}$ is different from those considered in other works such as [241] and [91]. However, by (7.51), $\boldsymbol{v}$ must be bounded, which is a severe restriction, and therefore, in general, it may not cover the subcritical quasi-geostrophic model where $\boldsymbol{v}$ is given by the vorticity function of the Riesz transform of $\boldsymbol{u}$.

Moreover, limited by the elliptic regularity of $(-\Delta)^s$ in Proposition 7.6, we are only able to consider the subcritical $s > \frac{1}{2}$ case, and unable to obtain the critical $s = \frac{1}{2}$ nor the supercritical $s < \frac{1}{2}$ cases.

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