FOURIER MULTIPLIERS FOR HARDY SPACES ON GRADED LIE GROUPS

QING HONG, GUORONG HU, AND MICHAEL RUZHANSKY

Abstract. In this paper, we investigate the $H^p(G) \to L^p(G)$, $0 < p \leq 1$, boundedness of multiplier operators defined via group Fourier transform on a graded Lie group G, where $H^p(G)$ is the Hardy space on G. Our main result extends those obtained in [Colloq. Math. **165** (2021), 1–30], where the $L^1(G) \to L^{1,\infty}(G)$ and $L^p(G) \to L^p(G)$, $1 < p < \infty$, boundedness of such Fourier multiplier operators were proved.

Contents

1. Introduction

Many problems in harmonic analysis and partial differential equations are related to the study of Fourier or spectral multipliers for certain function spaces. We start by recalling the classical Mihlin multiplier theorem. It says that if a function $\sigma(\xi)$ defined on $\mathbb{R}^n \backslash \{0\}$ has continuous derivatives up to $(\lfloor n/2 \rfloor + 1)$-th order, and satisfies

$$|\partial_\xi^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \tag{1.1}$$

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for all $\xi \in \mathbb{R}^n\backslash\{0\}$ and all multi-indices $\alpha \in \mathbb{N}_0^n$ with length $|\alpha| \leq \lfloor n/2 \rfloor + 1$, then the Fourier multiplier operator T_σ associated with σ, initially defined for $f \in \mathcal{S}(\mathbb{R}^n)$ via

$$T_\sigma f = \mathcal{F}^{-1}(\sigma \widehat{f}),$$

extends to a bounded operator on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. Hörmander [23] improved this result by showing that the regularity condition on $\sigma(\xi)$ could be allowed to be of fractional order. More precisely he proved that if $\sigma \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

$$\sup_{t>0} \|\eta(\cdot)\sigma(t\cdot)\|_{W_s^2(\mathbb{R}^n)} < \infty \tag{1.2}$$

for some $s > n/2$, where η is a function in $C_0^\infty(\mathbb{R}^n\backslash\{0\})$ such that $|\eta(\xi)| \geq c > 0$ on some annulus $\{r_1 < |\xi| < r_2\}$, then T_σ extends to a bounded operator on $L^p(\mathbb{R}^n)$, for all $1 < p < \infty$. Here $W_s^2(\mathbb{R}^n)$ denote the Sobolev spaces on $\mathbb{R}^n$. It is well known that condition (1.2) is weaker than condition (1.1). Calderón and Torchinsky [4] extended Mihlin and Hörmander's multiplier theorem to the case $0 < p \leq 1$, proving that if σ satisfies (1.2) for some $s > n(1/p - 1/2)$, then T_σ is bounded on the Hardy space $H^p(\mathbb{R}^n)$.

Multipliers for Lebesgue or Hardy spaces have also been studied extensively in the context of Lie groups. For spectral multipliers on Lie groups associated to one (or several) opeartors such as a sub-Laplacian, see, for example, [22, 5, 27, 17, 1, 25, 26] and the references therein. Note that the optimality of a Mihlin-Hörmander condition in terms of the topological or homogeneous dimensions for spectral multipliers on stratified groups is a very difficult problem which has so far only been solved in the case of 2-step [26, 17, 28, 25]. Concerning Fourier multipliers on Lie groups, to our best knowledge, the first work was done by Coifman and Weiss in [6], where they studied the Fourier multipliers of $SU(2)$, see also [7]. After that, investigations of Fourier multipliers on compact Lie groups has been focused on the central multipliers [32, 33, 34], untill the appearance of the recent works of the third-named author and Wirth [30, 31] and Fischer [12]. The rest of the literature concerning Fourier multipliers on Lie groups is restricted to the motion group [29] and to the Heisenberg group [8, 24, 2].

Recently, Fischer and the third-named author [13] investigated Fourier multipliers on graded Lie groups. One of their main results is the following Mihlin-type Fourier multiplier theorem for L^p spaces on graded Lie groups. (Basic concepts concerning graded Lie groups and representation theory, and the definition of difference operators will be recalled in Section 2.)

Theorem A (see [13, Theorem 1.1]). *Let G be a graded Lie group with homogeneous dimension Q. Let $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ be a measurable field of operators in $L^\infty(\widehat{G})$. Assume that there exist a positive Rockland operator $\mathcal{R}$ (of homogeneous degree ν) and an integer $N > Q/2$ divisible by the dilation weights $v_1, \cdots, v_n$ (see Section 2 for their definition) such that*

$$\sup_{\pi\in\widehat{G}} \left\|\Delta^\alpha \sigma\ \pi(\mathcal{R})^{\frac{[\alpha]}{\nu}}\right\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty \tag{1.3}$$

and

$$\sup_{\pi\in\widehat{G}} \left\|\pi(\mathcal{R})^{\frac{[\alpha]}{\nu}}\ \Delta^\alpha \sigma\right\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty, \tag{1.4}$$

hold for all $\alpha \in \mathbb{N}_0^n$ with $[\alpha] \leq N$. Then the Fourier multiplier operator T_σ defined via

$$\mathcal{F}_G(T_\sigma f)(\pi) = \sigma(\pi)\widehat{f}(\pi) \tag{1.5}$$

is of weak type $(1,1)$*, and is bounded on* $L^p(G)$ *for all* $1 < p < \infty$*.*

Examining the proof of Theorem A (given in [13]), we find that the condition (1.3) is sufficient to give the weak $(1,1)$ estimate of T_σ. The latter along with the $L^2(G)$-boundedness of T_σ (which follows from the Plancherel theorem) and an interpolation argument yields the $L^p(G)$-boundedness of T_σ for $1 < p \leq 2$. Note that $T_\sigma^* = T_{\sigma^*}$, where $\sigma^* = \{\sigma(\pi)^*, \pi \in \widehat{G}\}$, and that if σ satisfies (1.4) then σ^* satisfies (1.3). Hence, if σ satisfies (1.3) and (1.4), then both T_σ and T_σ^* are bounded on $L^p(G)$ for $1 < p \leq 2$, which implies that T_σ is bounded on $L^p(G)$ for all $1 < p < \infty$.

The purpose of the present paper is to extend Theorem A to the case $0 < p \leq 1$ by investigating the $H^p(G) \to L^p(G)$ boundedness of T_σ, where $H^p(G)$ is the Hardy space on G. Our main results is the following

Theorem 1.1. *Let* G *be a graded Lie group with homogeneous dimension* Q*. Let* $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ *be a measurable field of operators in* $L^\infty(\widehat{G})$*. Let* $0 < p \leq 1$*. Assume that there exist a positive Rockland operator* $\mathcal{R}$ *(of homogeneous degree* ν*) and an integer* $N > Q(1/p - 1/2)$ *divisible by the dilation weights* $v_1, \cdots, v_n$ *such that*

$$\sup_{\pi \in \widehat{G}} \left\| \Delta^\alpha \sigma \ \pi(\mathcal{R})^{\frac{[\alpha]}{\nu}} \right\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty$$

holds for all $\alpha \in \mathbb{N}_0^n$ *with* $[\alpha] \leq N$*, Then the Fourier multiplier operator* T_σ *defined by (1.5) is bounded from* $H^p(G)$ *to* $L^p(G)$*.*

Some remarks concerning Theorem 1.1 are in order.

(1) Taking $p = 1$ in Theorem 1.1, we have the $H^1(G) \to L^1(G)$ boundedness of T_σ, under the assumption that σ satisfies (1.3) for some integer $N > Q/2$ which is divisible by the dilation weights $v_1, \cdots, v_n$. Thus (by interpolation) our result also implies the $L^p(G)$-boundedness of T_σ stated in Theorem A under the same assumptions.

(2) In the abelian Euclidean setting, that is, $(\mathbb{R}^n, +)$ with the usual isotropic dilation with $\mathcal{R}$ being the Laplace operator, (1.3) is equivalent to (1.4), and each of them reduces to (1.1). Indeed, the Euclidean abelian setting, all the dilations weights $v_1, \cdots, v_n$ are equal to 1, and $\pi(\mathcal{R})$ reduces to $|\xi|^2$, where ξ is the (Fourier) dual variable.

(3) As we mentioned before, the optimality of the Mihlin-Hörmander condition for multipliers on Lie groups is a very deep problem. It is known that on any 2-step stratified group the sufficient and necessary condition for L^p-boundedness of a spectral multiplier $F(\mathcal{L})$ (where $\mathcal{L}$ is a sub-Laplaican) is that F satisfies a scale-invariant smoothness condition of order $s > n/2$, where n is the topological dimension of the group (see [26]). It is natural to ask whether the condition $N > Q(1/p - 1/2)$ in Theorem 1.1 can be replaced by $N > n(1/p - 1/2)$. However, we do not indent to study this problem in the present paper.

To prove Theorem 1.1 we shall mainly use an atomic decomposition of $H^p(G)$, the Littlewood–Paley decomposition, and a Taylor formula with integral remainder on homogeneous groups which is due to Bonfiglioli [3]. Hulanicki's theorem will also play an important role in our proof.

This paper is organized as follows. In Section 2, we recall basic notions concerning graded Lie groups, basic representation theory, the group Fourier transform, Rockland

operators and difference operators. In Section 3, we recall some basic facts about Hardy spaces on graded Lie groups, including their atomic decomposition. The proof of our main theorem will be given in Section 4.

Notation. We use $\mathbb{N}_0$ to denote the set of all nonnegative integers. For a nonnegative number s, we denote by $\lfloor s \rfloor$ the largest integer less than or equal to s. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces, we denote by $\mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ the Banach space of the bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. When $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ then we write $\mathscr{L}(\mathcal{H}_1, \mathcal{H}_2) = \mathscr{L}(\mathcal{H})$. The letter C will denote positive constants, which are independent of the main variables involved and whose value may vary at every occurrence. By writing $f \lesssim g$ we mean that $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$, we also write $f \sim g$.

2. Preliminaries

2.1. **Graded Lie groups and their homogeneous structure.** A Lie group G is said to be graded if it is connected and simply connected, and its Lie algebra $\mathfrak{g}$ is endowed with a vector space decomposition $\mathfrak{g} = \oplus_{k=1}^{\infty} \mathfrak{g}_k$ (where all but finitely many of the $\mathfrak{g}_k$'s are $\{0\}$) such that $[\mathfrak{g}_k, \mathfrak{g}_{k'}] \subset \mathfrak{g}_{k+k'}$ for all $k, k' \in \mathbb{N}$. Such a group is necessarily nilpotent, and the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism. Examples of graded Lie groups include the Euclidean space $\mathbb{R}^n$, the Heisenberg group $\mathbb{H}^n$ and, more generally, all stratified Lie groups.

We choose and fix a basis $\{X_1, \cdots, X_n\}$ of $\mathfrak{g}$, so that it is adapted to the gradation, i.e., $\{X_1, \cdots, X_{n_1}\}$ (possibly $\emptyset$) is a basis of $\mathfrak{g}_1$, $\{X_{n_1+1}, \cdots, X_{n_1+n_2}\}$ (possibly $\emptyset$) is a basis of $\mathfrak{g}_2$, and so on. Via the map

$$\mathbb{R}^n \ni (x_1, \cdots, x_n) \mapsto \exp(x_1 X_1 + \cdots + x_n X_n) \equiv x \in G, \tag{2.1}$$

each point $(x_1, \cdots, x_n) \in \mathbb{R}^n$ is identified with the point $x \in G$. This map takes the Lebesgue measure on $\mathbb{R}^n$ to a bi-invariant Haar measure μ on G. We denote the group identity of G by e.

The Lie algebra $\mathfrak{g}$ is equipped with a natural family of dilations $\{\delta_r\}_{r>0}$ which are linear mappings from $\mathfrak{g}$ to $\mathfrak{g}$ determined by

$$\delta_r X = r^k X \quad \text{for } X \in \mathfrak{g}_k$$

For each $j \in \{1, \cdots, n\}$, let v_j be the unique positive integer such that $X_j \in \mathfrak{g}_{v_j}$. Then we have $\delta_r X_j = r^{v_j} X_j$, $j = 1, \cdots, n$. The associated group dilation is given by

$$\delta_r x = (r^{v_1} x_1, \cdots, r^{v_n} x_n),$$

for $x = (x_1, \cdots, x_n) \in G$ and $r > 0$. The integers $v_1, \cdots, v_n$ are referred to as weights of the dilations $\{\delta_t\}_{t>0}$, and the positive integer

$$Q := \sum_{k=1}^{\infty} k(\dim \mathfrak{g}_k) = \sum_{j=1}^{n} v_j$$

is called the homogeneous dimension of G.

A homogeneous quasi-norm on G is a continuous function $x \to |x|$ from G to $[0, \infty)$ which vanishes only at e and satisfies that $|x^{-1}| = |x|$ and $|\delta_r x| = r|x|$ for all $x \in G$

and $r > 0$. An example of homogeneous quasi-norm on G is given by

$$|x|_\kappa = \left(\sum_{j=1}^{n} x_j^{2\kappa/v_j} \right)^{1/(2\kappa)}, \tag{2.2}$$

where κ is the smallest common multiple to the weights $v_1, \cdots, v_n$. Any two homogeneous quasi-norms on G are equivalent (see [16]). Henceforth we fix a homogenous quasi-norm $|\cdot|$ on G. It satisfies a quasi-triangle inequality: there exists a constant $\gamma \geq 1$ such that

$$|xy| \leq \gamma(|x| + |y|) \tag{2.3}$$

for all $x, y \in G$.

There is an analogue of polar coordinates on homogeneous groups with the homogeneous dimension Q replacing the topological dimension n, see [16]:

$$\forall f \in L^1(G) \quad \int_G f(x) d\mu(x) = \int_0^\infty \int_{\mathfrak{S}} f(\delta_r y) r^{Q-1} d\sigma(y) dr,$$

where $d\sigma$ is a (unique) positive Borel measure on the unit sphere $\mathfrak{S} := \{x \in G : |x| = 1\}$. This implies that for $0 < r < R < \infty$ and $\theta \in \mathbb{R}$,

$$\int_{r \leq |x| \leq R} |x|^{\theta - Q} d\mu(x) = \begin{cases} C\theta^{-1}(R^\theta - r^\theta) & \text{if } \theta \neq 0, \\ C \log(R/r) & \text{if } \theta = 0. \end{cases} \tag{2.4}$$

Consequently, if $\theta > 0$ then $|\cdot|^{\theta - Q}$ is integrable near the group identity e, and if $\theta < 0$ then $|\cdot|^{\theta - Q}$ is integrable near ∞.

Since G has been identified with $\mathbb{R}^n$ via the map given in (2.1), functions on G can be viewed as functions on $\mathbb{R}^n$, and vise versa. This leads naturally to the notions of test function classes $\mathcal{D}(G)$, $\mathcal{S}(G)$ and the distribution spaces $\mathcal{D}'(G)$, $\mathcal{S}'(G)$. For example, a function f is said to be in the Schwartz class $\mathcal{S}(G)$ if $f \circ \exp$ is a Schwartz function on $\mathbb{R}^n$. The coordinate function $G \ni x = (x_1, \cdots, x_n) \mapsto x_1 \in \mathbb{R}$ is denoted by x_1. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, we define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, as a function on G. Similarly, we set $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ in the universal enveloping Lie algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. We shall follow the usual custom of identifying each vector of $\mathfrak{g}$ with a left-invariant vector field on G and, more generally, we identify the universal enveloping Lie algebra of $\mathfrak{g}$ with the left-invariant differential operators. In what follows we keep the same notation for the vectors and the corresponding operators. By the Poincaré-Birkhoff-Witt theorem, the X^α's form a basis for the algebra of the left-invariant differential operators on G.

In a canonical way the dilations $\{\delta_r\}_{r>0}$ lead to the notions of homogeneity for functions and operators. For instance the degree of homogeneity of the function x^α and the differential operator X^α is

$$[\alpha] := \sum_{j=1}^{n} v_j \alpha_j.$$

A function $P : G \to \mathbb{C}$ is called a polynomial, if it is of the form

$$P(x) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha$$

where all but finitely many of the complex coefficients c_α vanish. The homogeneous degree of the polynomial P is defined as $\max\{[\alpha] : c_\alpha \neq 0\}$. For $M \in \mathbb{N}_0$, we set

$$\mathcal{P}_M := \{\text{all polynomials on } G \text{ with homogeneous degree } \leq M\}.\hat{}$$

We denote by $\widetilde{X}_1, \cdots, \widetilde{X}_n$ the corresponding basis for right-invariant vector fields, that is,

$$\widetilde{X}_j f(x) = \frac{d}{dt} f\big(\exp(tX_j)x\big)\big|_{t=0}, \quad j = 1, \cdots, n.$$

Also, for $\alpha \in \mathbb{N}_0^n$, we set $\widetilde{X}^\alpha = \widetilde{X}_1^{\alpha_1} \cdots \widetilde{X}_n^{\alpha_n}$.

If f and g are measurable functions on G, then their convulution is defined by

$$f * g(x) = \int_G f(y) g(y^{-1}x) d\mu(y) = \int_G f(xy^{-1}) g(y) d\mu(y),$$

provided that the integrals converge. For any multi-index $\alpha \in \mathbb{N}_0^n$ and sufficiently good functions f and g, we have (see [16, Chapter 1])

$$X^\alpha(f * g) = f * (X^\alpha g), \quad \widetilde{X}^\alpha(f * g) = (\widetilde{X}^\alpha f) * g, \quad (X^\alpha f) * g = f * (\widetilde{X}^\alpha g). \tag{2.5}$$

2.2. **Fourier analysis on graded Lie groups.** The general theory of representation of Lie groups may be found in [9]. Here we also refer to [14] for a description which is more adapted to our particular context.

A representation π of a Lie group G on a Hilbert space $\mathcal{H}_\pi \neq \{0\}$ is a homomorphism from G into the group of bounded linear operators on $\mathcal{H}_\pi$ with bounded inverse. More precisely,

- for every $x \in G$, the linear mapping $\pi(x) : \mathcal{H}_\pi \to \mathcal{H}_\pi$ is bounded and has bounded inverse;
- for every $x, y \in G$, we have $\pi(xy) = \pi(x)\pi(y)$.

A representation π of G is called irreducible if it has no closed invariant subspaces. π is called unitary if $\pi(x)$ is unitary for every $x \in G$, and is called strongly continuous if the mapping $\pi : G \to \mathscr{L}(\mathcal{H}_\pi)$ is continuous with respect to the strong operator topology in $\mathscr{L}(\mathcal{H}_\pi)$. Two representations π_1 and π_2 are said to be equivalent if there exists a bounded linear mapping $A : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ between their representation spaces with a bounded inverse such that the relation $A\pi_1(x) = \pi_2(x)A$ holds for all $x \in G$. In this case we write $\pi_1 \sim \pi_2$, and denote their equivalence class by $[\pi_1] = [\pi_2]$. The set of all equivalence classes of strongly continuous irreducible unitary representations of G is called the unitary dual of G and is denoted by $\widehat{G}$. In what follows, we will identify one representation π with its equivalent class $[\pi]$.

For a unitary representation of G, the corresponding infinitesimal representation which acts on the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is still denoted by π. This is characterized by its action on $\mathfrak{g}$:

$$\pi(X) = \partial_{t=0}\pi(e^{tX}), \quad X \in \mathfrak{g}.$$

The infinitesimal action acts on the space $\mathcal{H}_\pi^\infty$ of smooth vectors, that is, the space of vectors $v \in \mathcal{H}_\pi$ such that the function $G \ni x \mapsto \pi(x)v \in \mathcal{H}_\pi$ is of class C^∞.

The Fourier coefficients or group Fourier transform of a function $f \in L^1(G)$ at $\pi \in \widehat{G}$ is defined by

$$\mathcal{F}_G f(\pi) \equiv \widehat{f}(\pi) \equiv \pi(f) := \int_G f(x)\pi(x)^* d\mu(x).$$

It is readily seen that

$$\|\widehat{f}(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)} \leq \|f\|_{L^1(G)}.$$

For $f, g \in L^1(G)$, we also have

$$\widehat{f * g}(\pi) = \widehat{g}(\pi)\widehat{f}(\pi).$$

There exists a unique positive Borel measure $\widehat{\mu}$ on $\widehat{G}$, called the Plancherel measure, such that for any continuous function f on G with compact support, one has

$$\int_G |f(x)|^2 d\mu(x) = \int_{\widehat{G}} \|\mathcal{F}_G f(\pi)\|^2_{HS(\mathcal{H}_\pi)} d\widehat{\mu}(\pi),$$

where $\|\cdot\|_{HS(\mathcal{H}_\pi)}$ denotes the Hilbert-Schmidt norm on the space $HS(\mathcal{H}_\pi) \sim \mathcal{H}_\pi \otimes \mathcal{H}_\pi^*$ of Hilbert-Schmidt operators on the Hilbert space $\mathcal{H}_\pi$. Since $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$, the Fourier transform $\mathcal{F}_G$ extends to a unitary operator from $L^2(G)$ onto $L^2(\widehat{G})$.

By the general theory on locally compact unimodular groups of type I (see e.g. [9]), if T is an L^2-bounded operator on G which commutes with left-translations, then there exists a field of bounded operators $\widehat{T}(\pi)$ such that for all $f \in L^2(G)$,

$$\mathcal{F}_G(Tf)(\pi) = \widehat{T}(\pi)\widehat{f}(\pi) \quad \text{a.e. } \pi \in \widehat{G}.$$

Moreover, we have

$$\|T\|_{\mathscr{L}(L^2(G))} = \sup_{\pi \in \widehat{G}} \|\widehat{T}(\pi)\|_{\mathscr{L}(\mathcal{H}_\pi)},$$

where the supremum here is understood as the essential supremum with respect to the Plancherel measure μ. Conversely, given any $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\} \in L^\infty(\widehat{G})$, there is a corresponding operator T_σ given by

$$\mathcal{F}_G(T_\sigma f)(\pi) = \sigma(\pi)\widehat{f}(\pi), \quad f \in L^2(G).$$

By the Plancherel theorem, T_σ is bounded on $L^2(G)$ with $\|T_\sigma\|_{\mathscr{L}(L^2(G))} = \|\sigma\|_{L^\infty(\widehat{G})}$.

If π is a unitary irreducible representation of G and $r > 0$, we define $r \cdot \pi$ to be the unitary irreducible representation such that

$$r \cdot \pi(x) = \pi(\delta_r x), \quad x \in G.$$

2.3. **Rockland operators.** Let G be a graded Lie group. A left-invariant differential operator $\mathcal{R}$ on G is called a Rockland operator if it is homogeneous of positive degree and for each unitary irreducible non-trivial representation π of G, the operator $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$. Rockland operators may be defined on any homogeneous group, however it turns out that the existence of a Rockland operator on a homogeneous group implies that (the Lie algebra of) the group admits a gradation. This is the reason why we and the authors in [13] consider the setting of graded Lie groups. On any graded Lie group G, the operator

$$\sum_{j=1}^{n} (-1)^{\frac{\nu_0}{\nu_j}} c_j X_j^{2\frac{\nu_0}{\nu_j}}$$

with $c_j > 0$ is a Rockland operator of homogeneous degree $2\nu_0$ if ν_0 is any common multiple of $v_1, \cdots, v_n$.

We will mainly consider positive Rockland operators. A Rockland operator $\mathcal{R}$ is said to be positive, if

$$\int_G \mathcal{R}f(x)\overline{f(x)}d\mu(x) \geq 0$$

for all $f \in \mathcal{S}(G)$. If a Rockland operator $\mathcal{R}$ is positive then $\mathcal{R}$ and $\pi(\mathcal{R})$ admit self-adjoint extensions on $L^2(G)$ and $\mathcal{H}_\pi$, respectively. We use the same notation for their self-adjoint extensions. By the spectral theory, we have

$$\mathcal{R} = \int_0^\infty \lambda dE_{\mathcal{R}}(\lambda) \quad \text{and} \quad \pi(\mathcal{R}) = \int_0^\infty \lambda dE_{\pi(\mathcal{R})}(\lambda),$$

where $E_{\mathcal{R}}(\lambda)$ (resp. $E_{\pi(\mathcal{R})}(\lambda)$) is the resolution of the identity associated to $\mathcal{R}$ (resp. $\pi(\mathcal{R})$).

For any bounded Borel function φ on $[0, \infty)$, the operator

$$\varphi(\mathcal{R}) = \int_0^\infty \varphi(\lambda) dE_{\mathcal{R}}(\lambda)$$

is bounded on $L^2(G)$, and commutes with left translations. Thus, by the Schwartz kernel theorem, there exists a distribution $K_{\varphi(\mathcal{R})} \in \mathcal{S}'(G)$ such that

$$\varphi(\mathcal{R})f = f * K_{\varphi(\mathcal{R})}, \quad \forall f \in \mathcal{S}(G).$$

Note that the point $\lambda = 0$ may be neglected in the spectral resolution, since the projection measure of $\{0\}$ is zero (see [21] or [14, Remark 4.2.8]). Consequently we should regard φ as a function on $(0, \infty)$ rather than on $[0, \infty)$.

We now recall Hulanicki's theorem, which will play an important role in the proof of our main result.

Theorem 2.1 (Hulanicki [20])**.** *Let G be a graded Lie groups and let $\mathcal{R}$ be a positive Rockland operator on G. For any $M_1 \in \mathbb{N}$ and $M_2 \geq 0$, there exist $C = C(M_1, M_2) > 0$ and $k = k(M_1, M_2), k' = k'(M_1, M_2) \in \mathbb{N}_0$ such that, for any $\varphi \in C^k(0, \infty)$, the convolution kernel $K_{\varphi(\mathcal{R})}$ of $\varphi(\mathcal{R})$ satisfies*

$$\sum_{[\alpha] \leq M_1} \int_G |X^\alpha K_{\varphi(\mathcal{R})}(x)|(1 + |x|_\kappa)^{M_2} d\mu(x) \leq C \sup_{\substack{\lambda \in (0,\infty) \\ \ell \in \{0,1,\cdots,k\} \\ \ell' \in \{0,1,\cdots,k'\}}} (1+\lambda)^{\ell'} \left| \frac{d^\ell}{d\lambda^\ell} \varphi(\lambda) \right|.$$

The same result with the right-invariant vector fields $\widetilde{X}_j$'s instead of the left-invariant vector fields X_j's also holds.

Corollary 2.2. *Let $\mathcal{R}$ be a positive Rockland operator on a stratified Lie group G. If φ is a function on $(0, \infty)$ such that $\varphi = \widetilde{\varphi}|_{(0,\infty)}$ for some $\widetilde{\varphi} \in \mathcal{S}(\mathbb{R})$, then $K_{\varphi(\mathcal{R})} \in \mathcal{S}(G)$.*

2.4. **Difference operators.** The Mihlin condition (1.1) is formulated in terms of the derivatives with respect to the Fourier variable ξ. However, for a field $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ of operators, there is no direct way to define an analogue of derivatives with respect to the Fourier variable π. To generalize the symbolic conditions to the setting of graded Lie groups, Fischer and Ruzhansky [14] introduced the so-called difference operators, whose definition we now recall.

For $a, b \in \mathbb{R}$, we denote by $\mathscr{L}_L(L^2_a(G), L^2_b(G))$ the subspace of all $T \in \mathscr{L}(L^2_a(G), L^2_b(G))$ which are left-invariant. Here $L^2_a(G)$ is the Bessel potential space (fractional Sobolev space) defined in [15]. Define

$$\mathcal{K}_{a,b}(G) := \Big\{ K \in \mathcal{S}'(G) : \text{the operator } \mathcal{S}(G) \ni f \mapsto f * K \text{ extends to a bounded operator from } L^2_a(G) \text{ to } L^2_b(G) \Big\}$$

and define $L^\infty_{a,b}(\widehat{G})$ to be the space of all fields $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ such that

$$\|\sigma\|_{L^\infty_{a,b}(\widehat{G})} := \sup_{\pi \in \widehat{G}} \|\pi(I+\mathcal{R})^{\frac{b}{\nu}} \sigma(\pi) \pi(I+\mathcal{R})^{-\frac{a}{\nu}}\|_{\mathscr{L}(\mathcal{H}_\pi)} < \infty.$$

From [14, Proposition 5.1.24] we see that, if $\sigma \in L^\infty_{a,b}(\widehat{G})$ then the Fourier multiplier operator T_σ corresponding to σ belongs to $\mathscr{L}_L(L^2_a(G), L^2_b(G))$ with

$$\|T_\sigma\|_{\mathscr{L}(L^2_a(G), L^2_b(G))} = \|\sigma\|_{L^\infty_{a,b}(\widehat{G})}.$$

Conversely, if $T \in \mathscr{L}_L(L^2_a(G), L^2_b(G))$, then there exists a unique $\sigma \in L^\infty_{a,b}(\widehat{G})$ such that

$$\mathcal{F}_G(Tf)(\pi) = \sigma(\pi)\widehat{f}(\pi), \quad f \in L^2(G).$$

In this case, denoting by $K \in \mathcal{K}_{a,b}(G)$ the convolution kernel of T, we define

$$\mathcal{F}_G K = \sigma \quad \text{and} \quad \mathcal{F}_G^{-1}\sigma = K.$$

This extends the definition of Fourier transform to the space $\mathcal{K}_{a,b}(G)$. See [14, Definition 5.1.25].

For $\alpha \in \mathbb{N}_0^n$ and $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\} \in L^\infty_{a,b}(\widehat{G})$, the difference operator Δ^α acting on σ is defined according to the formula (see [14, Definition 5.2.1])

$$\Delta^\alpha \sigma(\pi) = \mathcal{F}_G(q_\alpha \mathcal{F}_G^{-1}\sigma)(\pi) \quad \text{for a.e. } \pi \in \widehat{G},$$

where $q_\alpha(x) = x^\alpha$. Analogously to the derivatives in the Euclidean setting, the operator Δ^α satisfies the Lebnitz rule [14, Section 5.2.2]:

$$\Delta^\alpha(\sigma\tau) = \sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1,\alpha_2} \Delta^{\alpha_1}(\sigma)\Delta^{\alpha_2}(\tau), \quad \sigma, \tau \in L^\infty_{a,b}(\widehat{G}). \tag{2.6}$$

3. Hardy spaces on graded Lie groups

A comprehensive theory of Hardy spaces on general homogeneous groups was built by Folland and Stein [16]. Since all graded Lie groups are homogeneous, the theory in [16] applies to our setting.

In what follows, G is always a graded Lie group with homogeneous dimension Q.

3.1. **Definition of Hardy spaces on G.** We first introduce some maximal functions. Given a function $\Phi \in \mathcal{S}(G)$, we defined the nontangential maximal function $M_\Phi f$ and the radial maximal function $M^0_\Phi f$ of $f \in \mathcal{S}'(G)$ by

$$M_\Phi f(x) := \sup_{|x^{-1}y|<t} |f * \Phi_t(y)| \quad \text{and} \quad M^0_\Phi f(x) := \sup_{t>0} |f * \Phi_t(x)|,$$

respectively, where $\Phi_t(x) := t^{-Q}\Phi(\delta_{t^{-1}}x)$. We then define the grand maximal function $M_{(N)}f$ for each $N \in \mathbb{N}$ by

$$M_{(N)}f(x) := \sup_{\Phi\in\mathcal{S}(G),\|\Phi\|_{(N)}\le 1} M_\Phi f(x),$$

where

$$\|\Phi\|_{(N)} := \sup_{|\alpha|\le N, x\in G} (1+|x|_\kappa)^{(N+1)(Q+1)}|\widetilde{X}^\alpha\Phi(x)|.$$

Moreover, given $N \in \mathbb{N}$, we define the grand maximal function $M_{(N)}f$ of $f \in \mathcal{S}'(G)$ by

Definition 3.1. For $0 < p < \infty$, the Hardy space $H^p(G)$ is defined as

$$H^p(G) := \big\{f \in \mathcal{S}'(G) : M_{(N_p)}f < \infty\big\},$$

where

$$N_p := \min\{[\alpha] : \alpha \in \mathbb{N}_0^n \text{ with } [\alpha] > Q(1/p-1)\}.$$

The quasi-norm on $H^p(G)$ is defined by

$$\|f\|_{H^p(G)} := \|M_{(N_p)}f\|_{L^p(G)}.$$

The Hardy spaces $H^p(G)$, initially defined via grand maximal function, can be characterized by radial maximal function and nontangential maximal function equivalently. To recall these maximal characterizations, we need the notion of commutative approximate identities introduced in [16]. A commutative approximate identity on G is a function $\Phi \in \mathcal{S}(G)$ such that $\int_G \Phi(x)dx = 1$ and $\Phi_s * \Phi_t = \Phi_t * \Phi_s$ for all $s, t > 0$. On a graded Lie group it is easy to construct a commutative approximate identity. Indeed, if $\mathcal{R}$ is a positive Rockland operator on G, and $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\varphi(0) = 1$, then the convolution kernel of the operator $\varphi(\mathcal{R})$ is a commutative approximate identity.

Proposition 3.2. ([16, Corollary 4.17]) *Suppose $0 < p < \infty$ and Φ is a commutative approximate identity. Then for $f \in \mathcal{S}'(G)$, the following are equivalent:*

(i) $M^0_\Phi f \in L^p(G)$;
(ii) $M_\Phi f \in L^p(G)$;
(iii) $M_{(N_p)}f \in L^p(G)$.

Moreover, we have

$$\|M^0_\Phi f\|_{L^p(G)} \sim \|M_\Phi f\|_{L^p(G)} \sim \|M_{(N_p)}f\|_{L^p(G)}$$

with the implicit constants depending only on Φ and p.

Remark 3.3. If $1 < p < \infty$, the spaces $H^p(G)$ and $L^p(G)$ coincide with equivalent norms. See [16, p. 75].

3.2. **Atomic decomposition.** Atomic decomposition is a very useful tool for the study of boundedness of operators on Hardy spaces. Analogously to the Euclidean case, Hardy spaces on graded Lie groups also admit an atomic decomposition, which we now recall. See [16] for more details.

A triplet (p, q, M) is said to be admissible, if $0 < p \le 1 \le q \le \infty$, $p \ne q$ and $M \in \mathbb{N}_0$ with $M \ge \max\big\{[\alpha] : \alpha \in \mathbb{N}_0^n \text{ with } [\alpha] \le Q(1/p-1)\big\}$.

Definition 3.4. Given an admissible triplet (p, q, M), we say that a function a on G is a (p, q, M)-atom, if it is a compactly supported L^q function such that

(i) there is a ball B such that $\operatorname{supp} a \subset \overline{B}$ and $\|a\|_{L^q} \leq \mu(B)^{1/q-1/p}$;
(ii) for every $P \in \mathcal{P}_M$, $\int_G a(x)P(x)d\mu(x) = 0$.

The atomic decomposition of $H^p(G)$ can be stated as follows.

Proposition 3.5 ([16]). *Let (p, q, M) be an admissible triplet. Then there is a constant $c_1 > 0$ such that for all any (p, q, M)-atom a, one has*

$$\|a\|_{H^p(G)} \leq c_1.$$

Conversely, given any $f \in H^p(G)$, there exist a sequence $\{a_j\}_{j=1}^{\infty}$ of (p, q, M)-atoms and a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of complex numbers such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ with convergence in $\mathcal{S}'(G)$ and

$$\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} \leq c_2 \|f\|_{H^p(G)},$$

where c_2 is a constant independent of f.

4. Proof of main result

We need the following Taylor's formula with integral remainder on homogeneous groups, due to Bonfiglioli (see [3, Theorem 2]). Note that in [3] it is assumed that $v_1 = 1$, in which case one has $\lceil M \rfloor = M$.

Lemma 4.1. *Suppose $f \in C^{M+1}(G)$ for some $M \in \mathbb{N}_0$. Let $y \mapsto P^f_{x,M}(y)$ denote the right Taylor polynomial of f at x of homogeneous degree M. Then there exists a positive group constant C_M such that*

$$\begin{aligned} f(yx) - P^f_{x,M}(y) = &\sum_{\substack{|\alpha| \leq \lceil M \rfloor \\ [\alpha] \geq M+1}} \widetilde{X}^\alpha f(x) \left(\sum_{\beta:[\beta]=[\alpha]} C_{\alpha,\beta} y^\beta \right) \\ &+ \sum_{\substack{|\alpha| \leq \lceil M \rfloor + 1 \\ [\alpha] \geq M+1}} \left(\sum_{\beta:[\beta]=[\alpha]} C'_{\alpha,\beta} y^\beta \right) \int_0^1 (\widetilde{X}^\alpha f)(y_{(t)}x) \frac{(1-t)^M}{M!} dt, \end{aligned}$$

where

$$y_{(t)} := \exp\left(\sum_{j=1}^{n} t y_j X_j\right) \equiv (ty_1, \cdots, ty_n),$$

$\lceil M \rfloor := \max\{|\alpha| : \alpha \in \mathbb{N}_0^n \text{ with } [\alpha] \leq M\}$, and $C_{\alpha,\beta}$, $C'_{\alpha,\beta}$ are constants.

Lemma 4.2. *Suppose that $\sigma = \{\sigma(\pi), \pi \in \widehat{G}\}$ is a measurable field of operators, $\mathcal{R}$ is positive Rockland operator (of homogeneous degree ν), and N is an integer, all of which satisfy the hypothesis of Theorem 1.1. Let $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp} \varphi \subset [2^{-\nu}, 2^{\nu}]$ and*

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-\nu j}\lambda) = 1 \quad \forall \lambda \in (0, \infty).$$

Let $\sigma_j(\pi) := \sigma(2^j \cdot \pi)\varphi(\pi(\mathcal{R}))$ and $K_j := \mathcal{F}_G^{-1}\sigma_j$ for $j \in \mathbb{Z}$. Then for any $\alpha \in \mathbb{N}_0^n$, there exits a constant C (depending on α) such that

$$\int_G (1+|x|)^{2N} |\widetilde{X}^\alpha K_j(x)|^2 d\mu(x) \leq C. \tag{4.1}$$

Proof. Let $\psi \in \mathcal{S}(\mathbb{R})$ such that $\psi = 1$ on $[2^{-\nu}, 2^\nu]$. Then $\varphi(\lambda) = \varphi(\lambda)\psi(\lambda)$ for all $\lambda \in \mathbb{R}$. Consequently

$$\sigma_j(\pi) = \sigma_j(\pi)\psi(\pi(\mathcal{R})).$$

Letting Ψ be the convolution kernel of $\psi(\mathcal{R})$, it follows that

$$\widetilde{X}^\alpha K_j(x) = \widetilde{X}^\alpha\big(\Psi * K_j\big)(x) = (\widetilde{X}^\alpha \Psi) * K_j(x),$$

where we used (2.5). From the quasi-triangle inequality (2.3) we have

$$(1+|x|)^N \lesssim (1+|xy^{-1}|)^N (1+|y|)^N,$$

which yields

$$(1+|x|)^N \big|(\widetilde{X}^\alpha \Psi) * K_j(x)\big| \lesssim \big[(1+|\cdot|)^N |\widetilde{X}^\alpha \Psi|\big] * \big[(1+|\cdot|)^N |K_j|\big](x)$$

By Hulanicki's theorem (see Corollary 2.2), we have $\Psi \in \mathcal{S}(G)$, which implies

$$(1+|\cdot|)^N |\widetilde{X}^\alpha \Psi| \in L^1(G).$$

Hence by Young's inequality,

$$\begin{aligned}
&\int_G (1+|x|)^{2N} |\widetilde{X}^\alpha K_j(x)|^2 d\mu(x) \\
&\qquad \lesssim \Big\| \big[(1+|\cdot|)^N |\widetilde{X}^\alpha \Psi|\big] * \big[(1+|\cdot|)^N |K_j|\big] \Big\|_{L^2(G)}^2 \\
&\qquad \lesssim \int_G (1+|x|)^{2N} |K_j(x)|^2 d\mu(y) \\
&\qquad \sim \int_G (1+|x|_\kappa^{2\kappa})^{N/\kappa} |K_j(x)|^2 d\mu(y),
\end{aligned} \tag{4.2}$$

where $|\cdot|_\kappa$ is the homogeneous quasi-norm defined by (2.2).

Since N is a common multiple of the dilation weights $v_1, \cdots, v_n$ and κ is the smallest such common multiple, $(1+|x|_\kappa^{2\kappa})^{N/\kappa}$ of the form

$$(1+|x|_\kappa^{2\kappa})^{N/\kappa} = \sum_{[\beta] \leq N} c_\beta (x^\beta)^2.$$

Inserting this into (4.2), and using the Plancherel theorem and the Lebniz rule (2.6), we have

$$\begin{aligned}
&\int_G (1+|x|)^{2N} |\widetilde{X}^\alpha K_j(x)|^2 d\mu(x) \\
&\qquad \lesssim \sum_{[\beta] \leq N} \big\| \Delta^\beta \big[\sigma_j(\pi)\varphi(\pi(\mathcal{R}))\big] \big\|_{L^2(\widehat{G})} \\
&\qquad \lesssim \sum_{[\beta']+[\beta''] \leq N} \Big\| (\Delta^{\beta'} \sigma_j)(\pi) \Delta^{\beta''} \big[\varphi(\pi(\mathcal{R}))\big] \Big\|_{L^2(\widehat{G})}.
\end{aligned} \tag{4.3}$$

Inserting the powers $\pi(\mathcal{R})^{[\beta']/\nu}$, each term in the above sum can be estimated as follows:

$$\begin{aligned}
&\Big\|(\Delta^{\beta'}\sigma_j)(\pi)\Delta^{\beta''}\big[\varphi(\pi(\mathcal{R}))\big]\Big\|_{L^2(\widehat{G})} \\
&\qquad\le \Big\|(\Delta^{\beta'}\sigma_j)(\pi)\pi(\mathcal{R})^{[\beta']/\nu}\Big\|_{L^\infty(\widehat{G})}\Big\|\pi(\mathcal{R})^{-[\beta']/\nu}\Delta^{\beta''}\big[\varphi(\pi(\mathcal{R}))\big]\Big\|_{L^2(\widehat{G})} \qquad (4.4)\\
&\qquad=: E_1\cdot E_2.
\end{aligned}$$

For the factor E_1, we have

$$\begin{aligned}
E_1 &= \sup_{\pi\in\widehat{G}}\Big\|(\Delta^{\beta'}\sigma_j)(\pi)\pi(\mathcal{R})^{[\beta']/\nu}\Big\|_{\mathscr{L}(\mathcal{H}_\pi)}\\
&= \sup_{\pi\in\widehat{G}}\Big\|2^{j[\beta']}(\Delta^{\beta'}\sigma)\big(2^j\cdot\pi\big)\pi(\mathcal{R})^{[\beta']/\nu}\Big\|_{\mathscr{L}(\mathcal{H}_\pi)}\\
&= \sup_{\pi\in\widehat{G}}\Big\|(\Delta^{\beta'}\sigma)\big(2^j\cdot\pi\big)\big[(2^j\cdot\pi)(\mathcal{R})\big]^{[\beta']/\nu}\Big\|_{\mathscr{L}(\mathcal{H}_\pi)} \qquad (4.5)\\
&= \sup_{\pi\in\widehat{G}}\Big\|(\Delta^{\beta'}\sigma)(\pi)\big[\pi(\mathcal{R})\big]^{[\beta']/\nu}\Big\|_{\mathscr{L}(\mathcal{H}_\pi)}\\
&\le C_{\beta'},
\end{aligned}$$

where the last inequality follows from the Mihlin-type condition (1.3).

Using Hulanicki's theorem (2.2) and the fact that φ vanishes near the origin, one can show that (see the proof of Proposition 4.9 in [13] for details)

$$E_2 = \Big\|\pi(\mathcal{R})^{-[\beta']/\nu}\Delta^{\beta''}\big[\varphi(\pi(\mathcal{R}))\big]\Big\|_{L^2(\widehat{G})} \le C_{\beta',\beta''}. \qquad (4.6)$$

Combining (4.3) through (4.6) yields the desired estimate (4.1). □

Now we give the proof of our main result.

Proof of Theorem 1.1. Let $0 < p \le 1$ and let σ, $\mathcal{R}$ and N satisfy the hypothesis of Theorem 1.1. We fix an integer M such that

$$M \ge \max\big\{[\alpha] : \alpha\in\mathbb{N}_0^n \text{ with } [\alpha]\le Q(1/p-1)\big\} \qquad (4.7)$$

and

$$\frac{Q}{2} + (M+1) - N > 0. \qquad (4.8)$$

The condition (4.7) means that $(p,2,M)$ is an admissible triplet. Hence, to prove that T_σ is bounded from $H^p(G)$ to $L^p(G)$, by Proposition 3.5 it suffices to show that there exists a constant C such that for an arbitrary $(p,2,M)$-atom a,

$$\|T_\sigma a\|_{L^p(G)} \lesssim 1. \qquad (4.9)$$

Suppose a is a $(p,2,M)$-atom associated to a ball $B = B(x_0,r)$. Since T_σ commutes with left translations, we may assume without loss of generality that $x_0 = e$, i.e., the ball B is centered at the group identity e. Let c be a sufficient large positive constant. We write

$$\begin{aligned}
\|T_\sigma a\|_{L^p(G)}^p &= \int_{B(e,cr)}|T_\sigma a(x)|^p d\mu(x) + \int_{B(e,cr)^c}|T_\sigma a(x)|^p d\mu(x)\\
&=: I_1 + I_2.
\end{aligned}$$

First we estimate I_1. Indeed, by the Plancherel theorem and Hölder's inequality,

$$I_1 \lesssim \|T_\sigma a\|_{L^2(G)}^p |B(e,cr)|^{1-\frac{p}{2}} \lesssim \|\sigma\|_{L^\infty(\widehat{G})}^p \|a\|_{L^2(G)}^p |B(e,cr)|^{1-\frac{p}{2}} \lesssim 1. \tag{4.10}$$

Next we estimate I_2. Choose a function $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset [2^{-\nu}, 2^\nu]$ and

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-\nu j}\lambda) = 1 \quad \forall \lambda \in (0,\infty).$$

By the spectral theorem (recalling that 0 can be neglected in the spectral resolution),

$$a = \sum_{j\in\mathbb{Z}} \varphi(2^{-\nu j}\mathcal{R})a \quad \text{in } L^2(G).$$

Consequently (using the L^2-boundedness of T_σ) we have

$$T_\sigma a(x) = \sum_{j\in\mathbb{Z}} T_\sigma \varphi(2^{-\nu j}\mathcal{R})a(x), \quad \text{a.e. } x \in G. \tag{4.11}$$

We set

$$\begin{aligned}
\sigma_j(\pi) &:= \sigma(2^j\cdot\pi)\varphi(\pi(\mathcal{R})),\\
\widetilde{\sigma}_j(\pi) &:= \sigma_j(2^{-j}\cdot\pi) = \sigma(\pi)\varphi(2^{-j}\cdot\pi(\mathcal{R})),\\
K_j &:= \mathcal{F}_G^{-1}\sigma_j,\\
\widetilde{K}_j &:= \mathcal{F}_G^{-1}\widetilde{\sigma}_j.
\end{aligned}$$

Then (4.11) can be rewritten as

$$T_\sigma a(x) = \sum_{j\in\mathbb{Z}} T_{\widetilde{\sigma}_j} a(x) = \sum_{j\in\mathbb{Z}} \int_{B(e,r)} \widetilde{K}_j(y^{-1}x)a(y)d\mu(y) = \sum_{j\in\mathbb{Z}} F_j(x),$$

where

$$F_j(x) := \int_{B(e,r)} \widetilde{K}_j(y^{-1}x)a(y)d\mu(y).$$

Using $(\sum_j u_j)^p \le \sum_j |u_j|^p$ $(0<p\le 1)$ and Hölder's inequality, it follows that

$$\begin{aligned}
\int_{B(e,cr)^c} |T_\sigma a(x)|^p d\mu(x) &= \int_{B(e,cr)^c} \left|\sum_{j\in\mathbb{Z}} F_j(x)\right|^p d\mu(x)\\
&\le \sum_{j\in\mathbb{Z}} \int_{B(e,cr)^c} |F_j(x)|^p \, d\mu(x)\\
&= \sum_{j\in\mathbb{Z}} \int_{B(e,cr)^c} |x|^{-pN}|x|^{pN} \, |F_j(x)|^p \, d\mu(x)\\
&\le \sum_{j\in\mathbb{Z}} \left(\int_{B(e,cr)^c} |x|^{-\frac{2pN}{2-p}} d\mu(x)\right)^{1-\frac{p}{2}} \left(\int_{B(e,cr)^c} |x|^{2N}|F_j(x)|^2 d\mu(x)\right)^{\frac{p}{2}}\\
&\sim \sum_{j\in\mathbb{Z}} r^{\frac{(2-p)Q}{2}-pN} \left(\int_{B(e,cr)^c} |x|^{2N}|F_j(x)|^2 d\mu(x)\right)^{\frac{p}{2}}.
\end{aligned} \tag{4.12}$$

Here we also used the assumption that $N > Q(1/p-1/2)$, which implies $\frac{2pN}{2-p} > Q$ and hence the function $|\cdot|^{\frac{2pN}{2-N}}$ is integrable on $B(e,cr)^c$.

Let j_0 be the unique integer such that $2^{-j_0} \le r < 2^{-j_0+1}$ (i.e., $r \sim 2^{-j_0}$). To estimate the last integral in (4.12), we shall consider two cases: $j \ge j_0$ and $j < j_0$.

Case 1: $j \ge j_0$. Observe that $|x| \sim |y^{-1}x|$ whenever $x \in B(e,cr)^c$ and $y \in B(e,r)$. Hence for all $x \in B(e,cr)^c$,

$$\begin{aligned}
|x|^N |F_j(x)| &= |x|^N \left| \int_{B(e,r)} \widetilde{K}_j(y^{-1}x) a(y) d\mu(y) \right| \\
&\sim \int_{B(e,r)} |y^{-1}x|^N \widetilde{K}_j(y^{-1}x) |a(y)| d\mu(y) \\
&\le \|a\|_{L^2(G)} \left(\int_{B(e,r)} |y^{-1}x|^{2N} |\widetilde{K}_j(y^{-1}x)|^2 d\mu(y) \right)^{\frac{1}{2}} \\
&\lesssim |B(e,r)|^{\frac{1}{2}-\frac{1}{p}} \left(\int_{B(e,r)} |y^{-1}x|^{2N} |\widetilde{K}_j(y^{-1}x)|^2 d\mu(y) \right)^{\frac{1}{2}}.
\end{aligned}$$

It follows by Fubini's theorem, the fact that $\widetilde{K}_j(x) = 2^{jQ} K_j(\delta_{2^j} x)$, and Lemma 4.2 that

$$\begin{aligned}
&\int_{B(e,cr)^c} |x|^{2N} |F_j(x)|^2 d\mu(x) \\
&\quad\lesssim |B(e,r)|^{1-\frac{2}{p}} \int_{B(e,cr)^c} \left(\int_{B(e,r)} |y^{-1}x|^{2N} |\widetilde{K}_j(y^{-1}x)|^2 d\mu(y) \right) d\mu(x) \\
&\quad= |B(e,r)|^{1-\frac{2}{p}} \int_{B(e,r)} \left(\int_{B(e,cr)^c} |y^{-1}x|^{2N} |\widetilde{K}_j(y^{-1}x)|^2 d\mu(x) \right) d\mu(y) \\
&\quad\le |B(e,r)|^{1-\frac{2}{p}} \int_{B(e,r)} \left(\int_{G} |y^{-1}x|^{2N} |\widetilde{K}_j(y^{-1}x)|^2 d\mu(x) \right) d\mu(y) \\
&\quad= |B(e,r)|^{1-\frac{2}{p}} \int_{B(e,r)} \left(\int_{G} |x|^{2N} |\widetilde{K}_j(x)|^2 d\mu(x) \right) d\mu(y) \\
&\quad= |B(e,r)|^{2-\frac{2}{p}} \int_{G} |x|^{2N} |\widetilde{K}_j(x)|^2 d\mu(x) \\
&\quad= |B(e,r)|^{2-\frac{2}{p}} 2^{j(Q-2N)} \int_{G} |x|^{2N} |K_j(x)|^2 d\mu(x) \\
&\quad\lesssim r^{(2-\frac{2}{p})Q} 2^{j(Q-2N)}.
\end{aligned} \tag{4.13}$$

Case 2: $j < j_0$. Let $P^{\widetilde{K}_j}_{x,M}$ be the right Taylor polynomial of $\widetilde{K}_j$ at x of homogeneous degree M. By the vanishing moments of a, we have, for each $x \in B(e,cr)^c$,

$$F_j(x) = \int_{B(e,r)} \left[\widetilde{K}_j(y^{-1}x) - P^{\widetilde{K}_j}_{x,M}(y^{-1}) \right] a(y) d\mu(y) \tag{4.14}$$

Using Lemma 4.1 we can write

$$\begin{aligned}\widetilde{K}_j(y^{-1}x) - P^{\widetilde{K}_j}_{x,M}(y^{-1}) = &\sum_{\substack{|\alpha|\le\lceil M\rfloor\\ [\alpha]\ge M+1}} \widetilde{X}^\alpha \widetilde{K}_j(x)P_\alpha(y)\\ &+ \sum_{\substack{|\alpha|\le\lceil M\rfloor+1\\ [\alpha]\ge M+1}} P'_\alpha(y)\int_0^1 (\widetilde{X}^\alpha\widetilde{K}_j)\big((y^{-1})_{(t)}x\big)\frac{(1-t)^M}{M!}dt,\end{aligned} \tag{4.15}$$

where both P_α and P'_α are polynomials on G of homogeneous degree $[\alpha]$, and

$$(y^{-1})_{(t)} := \exp\left(-\sum_{j=1}^n y_jX_j\right) \equiv (-ty_1,\cdots,-ty_n).$$

Inserting (4.15) into (4.14), and using the Cauchy-Schwarz inequality and the size condition of atoms, we have

$$\begin{aligned}|F_j(x)| \le &\sum_{\substack{|\alpha|\le\lceil M\rfloor\\ [\alpha]\ge M+1}} \int_{B(e,r)} \left|\widetilde{X}^\alpha\widetilde{K}_j(x)P_\alpha(y)a(y)\right| d\mu(y)\\ &+ \sum_{\substack{|\alpha|\le\lceil M\rfloor+1\\ [\alpha]\ge M+1}} \int_{B(e,r)} \left|P'_\alpha(y)\int_0^1 (\widetilde{X}^\alpha\widetilde{K}_j)\big((y^{-1})_{(t)}x\big)\frac{(1-t)^M}{M!}dt\right| |a(y)|d\mu(y)\\ \lesssim &\sum_{\substack{|\alpha|\le\lceil M\rfloor\\ [\alpha]\ge M+1}} r^{(\frac12-\frac1p)Q}\left(\int_{B(e,r)}\left|\widetilde{X}^\alpha\widetilde{K}_j(x)P_\alpha(y)\right|^2 d\mu(y)\right)^{1/2}\\ &+ \sum_{\substack{|\alpha|\le\lceil M\rfloor+1\\ [\alpha]\ge M+1}} r^{(\frac12-\frac1p)Q}\left(\int_{B(e,r)}\left|\int_0^1 P'_\alpha(y)(\widetilde{X}^\alpha\widetilde{K}_j)\big((y^{-1})_{(t)}x\big)\frac{(1-t)^M}{M!}dt\right|^2 d\mu(y)\right)^{1/2}\\ =: &\, F_j^{(1)}(x) + F_j^{(2)}(x).\end{aligned}$$

Thus,

$$\int_{B(e,cr)^c}|x|^{2N}|F_j(x)|^2d\mu(x) \lesssim \int_{B(e,cr)^c}|x|^{2N}|F_j^{(1)}(x)|^2d\mu(x) + \int_{B(e,cr)^c}|x|^{2N}|F_j^{(2)}(x)|^2d\mu(x).$$

We first estimate $\int_{B(e,cr)^c}|x|^{2N}|F_j^{(2)}(x)|^2d\mu(x)$. Indeed, by Minkowski's inequality and Fubini's theorem,

$$\begin{aligned}&\int_{B(e,cr)^c}|x|^{2N}|F_j^{(2)}(x)|^2d\mu(x)\\ &\lesssim \sum_{\substack{|\alpha|\le\lceil M\rfloor+1\\ [\alpha]\ge M+1}} r^{(1-\frac2p)Q}\int_{B(e,cr)^c}|x|^{2N}\left(\int_{B(e,r)}\left|\int_0^1 P'_\alpha(y)(\widetilde{X}^\alpha\widetilde{K}_j)\big((y^{-1})_{(t)}x\big)\frac{(1-t)^M}{M!}dt\right|^2 d\mu(y)\right)d\mu(x)\\ &\le \sum_{\substack{|\alpha|\le\lceil M\rfloor+1\\ [\alpha]\ge M+1}} r^{(1-\frac2p)Q}\int_{B(e,cr)^c}|x|^{2N}\left[\int_0^1\left(\int_{B(e,r)}\left|P'_\alpha(y)(\widetilde{X}^\alpha\widetilde{K}_j)\big((y^{-1})_{(t)}x\big)\right|^2 d\mu(y)\right)^{1/2}dt\right]^2 d\mu(x)\end{aligned}$$

$$\leq \sum_{\substack{|\alpha|\leq\lceil M\rfloor+1\\ [\alpha]\geq M+1}} r^{(1-\frac{2}{p})Q}\left[\int_0^1\left(\int_{B(e,cr)^c}|x|^{2N}\int_{B(e,r)}\left|P'_\alpha(y)(\widetilde{X}^\alpha\widetilde{K}_j)\big((y^{-1})_{(t)}x\big)\right|^2 d\mu(y)d\mu(x)\right)^{1/2}dt\right]^2$$

$$= \sum_{\substack{|\alpha|\leq\lceil M\rfloor+1\\ [\alpha]\geq M+1}} r^{(1-\frac{2}{p})Q}\left[\int_0^1\left(\int_{B(e,r)}|P'_\alpha(y)|^2\int_{B(e,cr)^c}|x|^{2N}\left|(\widetilde{X}^\alpha\widetilde{K}_j)\big((y^{-1})_{(t)}x\big)\right|^2 d\mu(x)d\mu(y)\right)^{1/2}dt\right]^2.$$

Note that if $y\in B(e,r)$, $x\in B(e,cr)^c$ and $t\in[0,1]$, then $|(y^{-1})_{(t)}x|\sim|x|$. Thus, for every $y\in B(e,r)$ and $t\in[0,1]$,

$$\begin{aligned}
&\int_{B(e,cr)^c}|x|^{2N}\left|(\widetilde{X}^\alpha\widetilde{K}_j)\big((y^{-1})_{(t)}x\big)\right|^2 d\mu(x)\\
&\quad\sim\int_{B(e,cr)^c}|(y^{-1})_{(t)}x|^{2N}\left|(\widetilde{X}^\alpha\widetilde{K}_j)\big((y^{-1})_{(t)}x\big)\right|^2 d\mu(x)\\
&\quad\leq\int_G|x|^{2N}\left|(\widetilde{X}^\alpha\widetilde{K}_j)(x)\right|^2 d\mu(x)\\
&\quad=2^{j(Q+2[\alpha]-2N)}\int_G|x|^{2N}\left|(\widetilde{X}^\alpha K_j)(x)\right|^2 d\mu(x)\\
&\quad\lesssim 2^{j(Q+2[\alpha]-2N)},
\end{aligned}$$

where for the last line we used Lemma 4.2. Inserting this estimate yields

$$\begin{aligned}
&\int_{B(e,cr)^c}|x|^{2N}|F_j^{(2)}(x)|^2 d\mu(x)\\
&\quad\lesssim\sum_{\substack{|\alpha|\leq\lceil M\rfloor+1\\ [\alpha]\geq M+1}} r^{(1-\frac{2}{p})Q}2^{j(Q+2[\alpha]-2N)}\left[\int_0^1\left(\int_{B(e,r)}|P'_\alpha(y)|^2 d\mu(y)\right)^{1/2}dt\right]^2\\
&\quad\leq\sum_{\substack{|\alpha|\leq\lceil M\rfloor+1\\ [\alpha]\geq M+1}} r^{(1-\frac{2}{p})Q}2^{j(Q+2[\alpha]-2N)}\int_{B(e,r)}|y|^{2[\alpha]}d\mu(y)\\
&\quad\sim\sum_{\substack{|\alpha|\leq\lceil M\rfloor+1\\ [\alpha]\geq M+1}} r^{(2-\frac{2}{p})Q+2[\alpha]}2^{j(Q+2[\alpha]-2N)}.
\end{aligned}$$

With a similar but easier argument, we get the analogous estimate

$$\int_{B(e,cr)^c}|x|^{2N}|F_j^{(1)}(x)|^2 d\mu(x)\lesssim\sum_{\substack{|\alpha|\leq\lceil M\rfloor\\ [\alpha]\geq M+1}} r^{(2-\frac{2}{p})Q+2[\alpha]}2^{j(Q+2[\alpha]-2N)}.$$

The details are left to the interested reader.

Therefore, we have

$$\int_{B(e,cr)^c}|x|^{2N}|F_j(x)|^2 d\mu(x)\lesssim\sum_{\substack{|\alpha|\leq\lceil M\rfloor+1\\ [\alpha]\geq M+1}} r^{(2-\frac{2}{p})Q+2[\alpha]}2^{j(Q+2[\alpha]-2N)}. \tag{4.16}$$

Inserting (4.13) and (4.16) into (4.12), and using that $r \sim 2^{-j_0}$, we obtain

$$\begin{aligned}
&\int_{B(e,cr)^c} |T_\sigma a(x)|^p d\mu(x) \\
&\lesssim \sum_{j\in\mathbb{Z}} r^{\frac{(2-p)Q}{2}-pN} \left(\int_{B(e,cr)^c} |x|^{2N} |F_j(x)|^2 d\mu(x) \right)^{\frac{p}{2}} \\
&\lesssim \sum_{j\geq j_0} r^{\frac{(2-p)Q}{2}-pN} \left[r^{(2-\frac{2}{p})Q} 2^{j(Q-2N)} \right]^{\frac{p}{2}} + \sum_{j<j_0} \sum_{\substack{|\alpha|\leq \lceil M\rfloor+1 \\ [\alpha]\geq M+1}} r^{\frac{(2-p)Q}{2}-pN} \left[r^{(2-\frac{2}{p})Q+2[\alpha]} 2^{j(Q+2[\alpha]-2N)} \right]^{\frac{p}{2}} \\
&\sim \sum_{j\geq j_0} 2^{(j-j_0)p(\frac{Q}{2}-N)} + \sum_{j<j_0} \sum_{\substack{|\alpha|\leq \lceil M\rfloor+1 \\ [\alpha]\geq M+1}} 2^{(j-j_0)p(\frac{Q}{2}+[\alpha]-N)}.
\end{aligned}$$

Since $N > Q(1/p - 1/2) > Q/2$, we have

$$\sum_{j\geq j_0} 2^{(j-j_0)p(Q/2-N)} \lesssim 1.$$

Note that the index set $\mathcal{I} := \{\alpha \in \mathbb{N}_0^n : |\alpha| \leq \lceil M\rfloor+1,\ [\alpha] \geq M+1\}$ has finite elements. Moreover, the condition (4.8) implies that $\frac{Q}{2} + [\alpha] - N > 0$ whenever $\alpha \in \mathcal{I}$. Thus

$$\sum_{j<j_0} \sum_{\substack{|\alpha|\leq \lceil M\rfloor+1 \\ [\alpha]\geq M+1}} 2^{(j-j_0)p(\frac{Q}{2}+[\alpha]-N)} \lesssim 1.$$

Therefore,

$$I_2 = \int_{B(e,cr)^c} |T_\sigma a(x)|^p d\mu(x) \lesssim 1, \tag{4.17}$$

Combining (4.10) and (4.17), we obtain

$$\|T_\sigma a\|_{L^2(G)} \lesssim 1.$$

This completes the proof of Theorem 1.1. □

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