

Author Response to the Report on
From Mirrors to Wallpapers:
A Virtual Math Circle Module on Symmetry
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To the editors of Journal of Math Circles,

First and foremost, we would like to thank our reviewers for their favorable determination of our submission. We appreciate the time and energy that clearly went into their reviews. Furthermore, we would like to thank the editors for giving us this opportunity to improve our manuscript.

We have copied and enumerated all reviewer comments and suggestions from the report into this document and, after each note, we've responded in blue. In this unblinded, revised submission, we have included both a copy of our manuscript with all of our changes in red text (emailed), as well as a copy without any color alterations (via Bepress). At the suggestion of the reviewers, we have made significant changes to the structure and content of our manuscript:

- I. **New “Modifications” subsections were added at the end of every session description.** The reviewers made several thoughtful suggestions on both improving our activities, and making them more appropriate for other audiences. While we don't want to alter descriptions of what we did in Spring 2021, we do want to share these ideas with our audience. To do so, we've added a subsection at the end of every session which details how the session content can be modified. This approach allows us to still reflect on our specific implementation, but also give readers the opportunity to improve and/or modify our activities.
- II. **An appendix section added to discuss details of underlying mathematical topics.** Although we could not delve into all details about every topic, we have provided enough of an overview to initiate future session leaders.
- III. **An appendix section was added/expanded on to provide details on finding and navigating our Google Drive.** Both reviewers indicated an appreciation for our promise to share our materials. To make our materials even more accessible for future facilitators, we include a detailed explanation of the Google Drive materials in the Appendix.

Thank you again for your consideration of our manuscript. We are very excited to have it be accepted to the Journal of Math Circle.

Best,
Nicole Sullivant,
Christina Durón,
Douglas T. Pfeffer

1 Comments from Reviewer #1

In the following sections, we have listed the comments from the editor's report, along with our responses (in blue).

Suggestions from Part 2:

This paper presents a 7-part module on symmetry that was used in the City Math Circle (CMC) in Spring 2021. It was explicitly designed for an online setting and was used in Math Circle sessions for younger students aged 6–12. The paper provides a detailed account of the rationale and material for each session together with brief reflections on how the sessions went.

A lot of care was taken to carefully scaffold the activities and to make sure that students follow the steps. Activities consist of a nice mix of individual and group work and whole class discussions and aim at teaching and demonstrating reflective, rotational translational and glide symmetries. The ultimate goal of this module was that students could identify the intermediate and generating tile of different wallpapers.

Some of the sessions were too long to finish all the activities in one hour, but students could play with some of the material on their own. The activities utilized online tools well to get the ideas across. Maybe more varied activities could also be introduced in the module or as a follow-up — make some of the challenges into a game, or let students design their own wallpapers using different symmetries.

We have included a “Modifications” subsection to each session's description to give future facilitators some ways to alter our activities to suit their own students, or ways to craft follow-up activities for students to take home. We have also provided all of our materials in our Google Drive (see Appendix B) so future facilitators can easily craft their own activities without starting from scratch.

In summary, the paper is well-written and provides a good module for other people to use to teach a beautiful topic in math. There are a few typos in the paper listed below.

We thank the reviewer for their kind comments and suggestions.

I recommend publishing the paper after making these small corrections.

Typos:

- (1) line 342: Session 2 should probably be Session 1
We have made the suggested revision.
- (2) Caption to Figure 16: Warp-up should be Warm-up
We have made the suggested revision.

- (3) line 758: planer should be planar
We have made the suggested revision.

2 Comments from Reviewer #3

In the following sections, we have listed the comments from the editor's report, along with our responses (in blue).

Suggestions from Part 2:

- (1) Towards the end of this series of moduli on symmetries, students investigate the structure of wallpapers. A total of 6 wallpaper types is selected (out of the 17 existing wallpaper groups), and two examples are provided for each of these 6 wallpapers. The authors may consider handling students only one example for each type (instead of pairs) and then asking them to match the remaining 6 wallpapers with one of the examples they have previously investigate. To make the problem more interesting, they could also purposefully include wallpapers from the remaining 11 types which have no match.

We agree that future Math Circle facilitators may be interested in having their students work with a larger variety of wallpapers. We have included additional wallpapers in our Google Drive, and have made a note of this in the "Session 5 Modifications" subsection. We have also added a reference to a free wallpaper generator website.

- (2) Because the audience is elementary school children, the activity is scaffolded, and wallpapers are introduced in an increasing order of difficulties (A and B, then C and D, and so on). Consider loosening up the structure of the activity for a more mature audience.

We have added this suggestion in the "Session 5 Modifications" subsection.

- (3) Also consider having multiple examples of each wallpaper and giving students more autonomy in choosing what wallpaper they want to explore.

We believe that this is a wonderful idea, and we have made additional wallpaper samples available in our Google Drive for facilitators who would rather have students work with many different types of wallpaper patterns. We have also referenced a free wallpaper generator website where future facilitators can create their own wallpaper patterns.

- (4) In this series of moduli on symmetries, students investigate symmetries of finite objects first, and then move on to (infinite) wallpapers. One suggestion would be to extract the intermediate tile from the wallpaper examples used later in the program, and introduce those tiles to students earlier in the program, simply as examples of objects with symmetry. This may result helpful when, later on, they try to understand the difference between intermediate tile and generating tile in a wallpaper. It may also help tie the modules together for students.

Our TMC students definitely struggled with the idea that the intermediate tile can be reduced to a generating tile. An earlier experience with analyzing intermediate tiles may very well have helped us avoid that confusion. Thank you for this observation; we have made a note of this for future facilitators in the “Session 3 Modifications” subsection.

- (5) Consider adding an angle grid as background for the star-shaped digital manipulative in Figure 14, to help students determine the angle for the smallest rotational symmetry. Also, before asking students to identify the smallest rotation and its order, it would be interested to ask them to find all the rotational symmetries of the figure.

This is an excellent suggestion that can help students measure the precise angle of rotation, and we have included a note of this for future facilitators in the “Session 1 Modifications” subsection.

- (6) Figure 22: The paw print used in Example 1, appears to have a reflective symmetry about the midline, hence the students may not be able to truly appreciate the effect of reflecting the paw print about a line parallel to the direction of translation. The authors may consider using a type of paw print which lacks this internal symmetry (like the footprint in example 2). I would also suggest giving student a single paw/footprint (as opposed to a bunch arranged in a column) and ask them to generate the full set of paw/footprints using iterations of reflections and translations.

The reviewer makes a good point about the internal symmetry of the paw print. While the actual paw print does not have internal symmetry, the distinguishing features between the left and right sides are lost when the images are so small. We have made some suggestions to alleviate this issue in the “Session 4 Modifications” subsection. See the next response for our comments on the paw-print-building activity.

- (7) Figure 23: It may be clearer for students if you started with a single paw print on the left-hand side (as opposed to 4 paw prints in a column) and used that single paw print to build the complete set of paw prints.

This is what we had originally planned when we were designing the session content, so we certainly agree with you! Unfortunately, while Miro is a wonderful tool that we love using for our TMC sessions, reflecting images was not an available feature on Miro when we designed our boards (and is still not a feature, as of Jan 2022); this served as part of the motivation to create the TMC Widget. We have given some alternative activities in the “Session 4 Modifications” subsection.

- (8) Prior to the start of activity 6, maybe as a warm up, you may give the students a single foot print and ask them to generate full set of footprints in 3 different scenarios: 1) if they were jumping on one foot, 2) if they were jumping on two feet, 3) if they were walking. This warm-up may help students appreciate the difference between translations, reflections and glide reflections.

We are really excited about this alternate introduction to glide symmetry, and have added these suggestions in the “Session 4 Modifications” subsection.

Suggestions from Part 3:

- (9) The authors provide ample documentation of their lesson plans, and kindly share the Math Circle’s Google Drive folder containing the worksheets and Miro Board templates. (This folder is not visible to the reviewers to facilitate a blind review.) Access information has been added to Appendix B. We have also added text in Appendix B to better outline the contents of the Google Drive.
- (10) The paper might be made stronger by proving additional evidence of student work, if available. For example, in Section 3.1.3 (Discussion 1), it would be nice to see students’ annotation to display translational symmetry. More generally, the authors may consider sharing the original pictures of the activities in the appendix or in the CMC Google Drive, and replacing them with pictures of students work, so the reader can see not only the images given to the students to showing how the students have annotated the images to display the symmetry.
We have added some screenshots and exposition to the “Reflections” subsections for Sessions 1, 2, and 3 to enhance our manuscript.

Suggestions from Part 4:

- (11) The exposition is very clear, with each section of the paper describing a different module in this series of Math Circle meetings. While I appreciate the detailed description of the activities and the reflections about whether the learning objectives for each module were achieved, at times the exposition could be made more concise. The authors could consider adding a table outlining the structure of each module. It is hard to keep track of the components of the various modules, especially because the number of the activities does not match the name of the session.
Thank you for this comment and for the suggested table. We agree that the readability is improved with the inclusion of a table that concisely outlines the session content. This addition may be found at the end of our first section. We note that the third column slightly differs from the reviewer’s suggestion, as we found listing all student learning outcomes for each session forced the table to exceed one page.
- (12) Note: the idea of a “stamp” is introduced for the first time in Line 469 to convey the idea of translational symmetry. In Line 688, the word “stamp” is again referring to translations. However, in Line 676 and in Lines 681-683, the word “stamp” seems to indicate a mix of reflective and translational symmetry (that is, a glide reflection) use to generate a complete set of full prints. This may generate some confusion. I suggest limiting the use of the word “stamp” to indicate translations. Another recommendation is to use more precise language when describing the difference

between intermediate tile and generating tile.

We thank the reviewer for this suggestion and have made the following revisions:

- “The purpose of this activity is to have students explore glide reflection symmetry. We also bring back the idea of ‘stamping’, this time using digital manipulatives.” has been changed to “The purpose of this activity is to have students explore glide reflection symmetry. We also bring back the idea of ‘stamping’ for translational symmetry, this time as part of a two-step procedure.”.
- “This is done by thinking of one single print as a *stamp* which is used to create all other prints” has been changed to “This is done by thinking of one single print as a building block which is used to create all other prints”
- “Then, demonstrate how to Duplicate the cutout and stamp it somewhere else.” has been changed to “Then, demonstrate how to Duplicate the cutout and drop it somewhere else.”
- The following sentence was added to the “Session 6 Modifications” section: “Students who are already familiarized with other wallpapers, such as Wallpapers C – H, could be encouraged to recreate those patterns in the widget as well. . . . To prime students for finding a generating tile in the upcoming activities, it may also be useful to point out that in some parallelograms, there is repeated information (so those parallelograms are not truly “generating tiles” as we will define in the next session). Facilitators with more advanced students should perhaps use the phrase *generating parallelogram* instead.”

Suggestions from Part 5:

- (13) The paper provides a clear exposition of a series of Math Circle moduli aimed at introducing elementary school students to the structure of wallpaper groups. Specifically designed for online instruction, these curriculum units leverage technology (e.g., the CMC Widget) to maximize student engagement. The use of Miro boards provides an excellent structure for an online Math Circle, independently of the topic at hand. Hence there is a lot of value in disseminating this tool.
We would like to thank the reviewer for these kind words.
- (14) In the paper, the authors appear to be speaking directly to Math Circle practitioners, sharing instructions and advice to replicate the lessons. This is one of the strengths of the paper. Because the audience is elementary school children, the language is kept simple and the mathematical depth of the exposition is intentionally “light”. The authors may consider adding an appendix at the end of the article with a more rigorous mathematical exposition of the topic. Consider, for example, the last paragraph of Section 4.1.1. describing the interplay of reflections and rotations (and the specific relation between the number of mirror lines and the order of rotation): the claims made are not obvious and require a deeper mathematical discussion. The

definition of isometry at the start of section 5.1 can also be made more precise. For the replica of these activities to be truly successful at another Math Circle, it is critical to provide future leaders access to the mathematical activities and to share insights about the pedagogy (which this paper does brilliantly), but this may not be enough. For the activities to be run successfully, the future Math Circle instructors must also have a clear understanding of the mathematics involved. That is why the paper may benefit from an appendix containing a rigorous treatment of the most complex mathematical ideas involved in the paper (like the ones at the end of section 4.1.1).

We agree that including additional details about the underlying mathematics will make our lesson plans more accessible for a wider variety of readers. We have added Appendix A to explain the underlying mathematics with greater rigor. While it is possible to dig into the very fine details of all of the mathematics touched on in this symmetry module, we did not feel that such an exposition would be appropriate. Consequently, we have included a reference to our Google Drive that contains a list of excellent online resources for any readers who would like to learn more.

Suggestions from Part 6:

- (15) Line 38, consider replacing “reflective” with either “reflective symmetry” or “reflection”. Similarly, replace “rotational” with either “rotational symmetry” or “rotation”, and “translational” with either “translational symmetry” or “translation”.
We have made the suggested revision.
- (16) Line 52, consider replacing “will also discuss” with “also discusses”.
We have made the suggested revision.
- (17) Line 52, consider replacing “transition to online” with “transition online”.
We have made the suggested revision.
- (18) Line 53, consider replacing “will close” with “closes”.
We have made the suggested revision.
- (19) Line 68, extra space between St. and Mark’s.
We have made the suggested revision.
- (20) Line 100, consider replacing “the audience of students” with “attendance”.
We have made the suggested revision.
- (21) Line 101, remove the article “the” before “student participation”.
We have made the suggested revision.
- (22) Line 106, consider replacing “website” with “platform”.
We have made the suggested revision.
- (23) Line 110, consider replacing “It is this setting for which the module on symmetry was designed.” with “This is the setting for which the module on symmetry was

designed.”

[We have made the suggested revision.](#)

- (24) Line 132, consider replacing “understanding” with “understanding of symmetry”.
[We have made the suggested revision.](#)
- (25) Lines 133-4, consider replacing “reflective” with either “reflective symmetry” or “reflection”. Similarly, replace “rotational” with either “rotational symmetry” or “rotation”, and “translational” with either “translational symmetry” or “translation”.
[We have made the suggested revision.](#)
- (26) Line 138, consider replacing “of the line” with “of the image”.
[We have made the suggested revision.](#)
- (27) Lines 140-143, consider making the exposition more compact. You may eliminate Figure 1. And say: “For example, consider a square cut into quadrants, where each pair of diagonally opposed quadrants are colored light and dark blue, respectively. As shown in Figure xx, reflecting the shape across each of the two diagonal lines yield the exact same shape. These two diagonal lines are known formally as the lines of symmetry of the shape.”
[We have made the suggested revision.](#)
- (28) Caption of Figure 2: Consider rephrasing: “The lines of symmetry of a square with diagonally opposed quadrants of matching color.”
[We have made the suggested revision.](#)
- (29) Line 144, consider replacing “Other common examples are” with “Other common examples of shapes with reflective symmetry include”.
[We have made the suggested revision.](#)
- (30) Line 151, consider replacing “The number of partial rotations necessary, n , to achieve the full rotation” with “The number of times, n , that a partial rotation must be iterated to achieve the full rotation”.
[We have made the suggested revision.](#)
- (31) Line 153, full stop is missing.
[We have made the suggested revision.](#)
- (32) Line 154, consider replacing “For example, reconsider the shape from Figure 1. We can choose the center of the shape and then rotate exactly 180 degree to yield the same shape. This occurs two times before we have completed a full rotation; therefore the shape has an order of rotation of 2.” With “For example, revisit the shape from Figure 1. A rotation by 180 degree around the center of the shape will yield the same shape. We need to perform this rotation twice before we get to a full 360 degree rotation; therefore the shape has a rotational symmetry of order 2.”
[We have made the suggested revision.](#)

- (33) Caption of Figure 5: Consider changing it to: “The parallel lines method applied to a pair of grey microphones.”
[We have made the suggested revision.](#)
- (34) Line 186, consider replacing “Starting with a single set of left footprints, you first reflect them across a horizontal line, and then translate them to the right.” With “Starting with a single footprint, you first reflect it across a horizontal line, and then translate it to the right.”
[We have made the suggested revision.](#)
- (35) Line 189, consider replacing “These four types of symmetry” with “Reflection, rotation, translation and glide reflection”. Consider eliminating Figure 7. Change caption to Figure 8 to read: A sample wallpaper, with an intermediate tile outlined in orange.
[We have made the suggested revision.](#)
- (36) Line 204, consider replacing “the generating tile” and “the intermediate tile” with “a generating tile” and “an intermediate tile”.
[We have made the suggested revision.](#)
- (37) Line 207, consider replacing “the intermediate tile has an order of rotation of 4” with “the intermediate tile has a rotation of order 4 about its center”.
[We have made the suggested revision.](#)
- (38) Line 208, replace “as the intermediate tile” and “as a generating tile”.
[We have made the suggested revision.](#)
- (39) Line 212, consider replacing “it was rotated 90 degrees, replicated 4 times about one of its corners” with “it was rotated by 90, 180 and 270 degrees, about one of its corners”.
[We have made the suggested revision.](#)
- (40) Line 213, consider replacing “translated in all directions” with “translated in the two directions parallel to the sides of the square”.
[We have made the suggested revision.](#)
- (41) Line 214, consider eliminating the article “the” before “wallpapers”.
[We have made the suggested revision.](#)
- (42) Line 239, consider replacing “and to move” with “or to move”.
[We have made the suggested revision.](#)
- (43) Line 245, consider replacing “was doing” with “is doing”.
[We have made the suggested revision.](#)
- (44) Line 256, consider replacing “to explore the new ideas” with “for the new ideas”.
[We have made the suggested revision.](#)

- (45) Line 257, consider replacing “we built an activity board (Figure 11)” with “we built an activity board (Figure 11), following a similar template for each meeting.”
[We have made the suggested revision.](#)
- (46) Line 262, consider replacing “During the meetings, each student claimed” with “Each student can claim”.
[We have made the suggested revision.](#)
- (47) Line 264, consider replacing “We kept” with “we keep”.
[We have made the suggested revision.](#)
- (48) Line 265, consider replacing “could view” with “view”.
[We have made the suggested revision.](#)
- (49) Line 267, consider replacing “would write” with “will write”.
[We have made the suggested revision.](#)
- (50) Line 270, consider replacing “might have” with “may have had”.
[We have made the suggested revision.](#)
- (51) Line 284, consider replacing “We often used convenient images that we found with an image search engine” with “We often utilized an image search engine to find convenient images to construct a set of examples on Miro.”
[We have made the suggested revision.](#)
- (52) Line 294, consider replacing “of” with “in”.
[We have made the suggested revision.](#)
- (53) Line 296, consider replacing “the order of rotation” with “the center of rotation and the order of rotation”.
[We have made the suggested revision.](#)
- (54) Caption for Figure 12. Replace “A collection of the eight shapes containing various amounts of reflective symmetry; used in Warm-Up 1” with “A collection of the eight shapes each containing a single mirror line, used in Warm-Up 1”.
[We have made the suggested revision.](#)
- (55) Line 363. Consider replacing “Share your Zoom screen and have the students explain their work” with “Share your Zoom screen, showcase students’ mats and ask them to explain their work.”
[We have made the suggested revision.](#)
- (56) Line 387, consider replacing “twisting” with “rotating”.
[We have made the suggested revision.](#)
- (57) Line 389, consider replacing “attend” with “join the meeting”.
[We have made the suggested revision.](#)

- (58) Line 397, consider replacing “the order of rotation” with “the center and the order of rotation”.
We have made the suggested revision.
- (59) Figure 17, you may consider coloring the two pairs differently (e.g., the left pair green and the right pair red) to make it clear which pairs the students are comparing. Although the instructions are very clear, students may get confused and think of reflectional symmetry between pairs of figures across the line.
In the “Session 2 Modifications” subsection, we have warned future session leaders that this activity’s layout may cause confusion and have offered possible solutions.
- (60) Line 621, consider deleting the word “operations”.
We have made the suggested revision.
- (61) Lines 626-628, consider replacing “demonstrate rotational symmetry from one angle and reflective symmetry from another angle.” with “exhibit both rotational and reflective symmetry.”
We have made the suggested revision.
- (62) Lines 637-638, consider replacing “the three types of symmetry” with “reflective, translational and rotational symmetry”.
We have made the suggested revision.
- (63) Line 640, consider replacing “repeat” with “revisit”.
We have made the suggested revision.
- (64) Line 648, consider replacing “all of three symmetries” with “reflective, translational and rotational symmetry”.
We have made the suggested revision.
- (65) Line 676 should the idea of stamping be used for translational symmetry only? It may be confusing because now they need to reflect and translate
We have revised the wording of several sentences (bulleted in our response to comment #12).
- (66) Line 681, consider replacing the word “stamp” with “model”.
We have revised the wording of several sentences (bulleted in our response to comment #12).
- (67) Line 725, consider replacing “order” by “the order in which we compose transformations”
We have made the suggested revision.
- (68) Line 733, consider replacing “a shame” with “disappointing”.
We have made the suggested revision.
- (69) Line 751, consider replacing “be displayed” with “occur”.
We have made the suggested revision.

- (70) Line 771, consider replacing “translational symmetry and reflective symmetry” with “translational symmetry, reflective symmetry and glide reflection.”
We have made the suggested revision.
- (71) Figure 25: I find these images a little confusing because only a portion of the image exhibits some symmetry (e.g., only the wall or only the wrapping paper). Overall, the pictures used lack symmetry and do not form a wallpaper. I think introducing students to pictures of wallpapers would be more effective. To increase student familiarity with the pictures presented, you could select real-life examples like walls or wrapping papers, but they should constitute the whole picture.
To make the purpose of these images more clear, we have altered the phrasing of the instructions in the Discussion 5 description: “...this discussion should focus on infinitely repeating planar patterns. Have students imagine real-world examples of repeating patterns, such as fabric, floor tilings, gift wrapping paper, and literal wallpaper. If time permits, ask students to find and display examples. To help students understand what to think about or find, provide photos of examples (Figure ...”
- (72) Line 818, consider replacing “a subset of a wallpaper” with “a subset of a wallpaper (the “intermediate tile”)”.
We have made the suggested revision.
- (73) Line 843, consider replacing “the wallpaper” with “the wallpaper under translational symmetry”.
We have made the suggested revision.
- (74) Caption of Figure 28: consider replacing “The wallpaper from Figure 26a, but with the intermediate tile outlined in yellow; used in Warm-Up 4.” with “An intermediate tile for the wallpaper from Figure 26a, used in Warm-Up 4.”
We have made the suggested revision.
- (75) Line 855, consider saying “a generating tile” (instead of “the generating tile”).
We have made the suggested revision.
- (76) Line 861, consider using “functionality” (instead of “operation”).
We have made the suggested revision.
- (77) Line 877, consider saying “a generating tile” (instead of “the generating tile”).
We have made the suggested revision.
- (78) Line 888, the word “wallpaper” should be replaced with “generating tile”.
We have made the suggested revision.
- (79) Line 924, consider replacing “into” with “to identify”.
We have made the suggested revision.
- (80) Line 935, consider replacing “review” with “the review”.
We have made the suggested revision.

- (81) Line 946, consider replacing “wallpaper” with “wallpaper under translation”.
[We have made the suggested revision.](#)
- (82) Line 952, consider replacing “the tile” with “a tile”.
[We have made the suggested revision.](#)
- (83) Line 961, consider replacing comma by colon.
[We have made the suggested revision.](#)
- (84) Line 962, missing full stop.
[We have made the suggested revision.](#)
- (85) Line 972, consider replacing with “the intermediate” with “an intermediate”.
[We have made the suggested revision.](#)
- (86) Line 992, consider replacing “a tile” (as opposed to “the tile”).
[We have made the suggested revision.](#)
- (87) Line 995, consider replacing with “the generating” with “a generating”.
[We have made the suggested revision.](#)
- (88) Figure 30: consider replacing “Widget, where each row” by “Widget. Each row”.
[We have made the suggested revision.](#)
- (89) Line 1052, consider replacing “reflective and rotational” with “reflection and rotation”.
[We have made the suggested revision.](#)
- (90) Line 1053, consider replacing “translational” with “translation”.
[We have made the suggested revision.](#)
- (91) Line 1075, consider replacing “have” with “had”.
[We have made the suggested revision.](#)
- (92) Line 1104, consider deleting “from abstract algebra”.
[We have made the suggested revision.](#)
- (93) Line 1206, the word “board” is repeated twice.
[We have made the suggested revision.](#)