Almost Finite-Time Observers for a Family of Nonlinear Continuous-Time Systems

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***Abstract*— We provide a new class of observers for a class of nonlinear systems which are not required to be affine in the unmeasured states. The observers ensure exponential convergence of the observation errors to zero, under linear output measurements. The rate of exponential convergence converges to infinity, as the growth rate of the nonlinear state-dependent part of the dynamics converges to zero, so we call the observers almost finite-time. Under global Lipschitz conditions on the state-dependent part of the dynamics, our global result ensures convergence of the observers, for all initial states. For cases where the nonlinearity is of order two at the origin, we provide local results ensuring exponential convergence of the observation errors to zero, when the initial state is small enough. We apply the results to a model of a pendulum with friction, and to dynamics with Lotka-Volterra nonlinearities.**

I. Introduction

Observer design is a central area in controls, owing to its ability to construct estimators of states (called observers) based on output measurements, and because the estimators can often be used in place of state variables in controls for asymptotic stabilization; see, e.g., [1], [2], and [3].Although most available observers are limited to linear systems, there is considerable motivation to build estimators for systems with nonlinearities. It can also be important to obtain fixed time observers, i.e., finite-time observers whose convergence times are independent of the initial state of the dynamics.

For cases where the nonlinearity only depends on time and on the output measurements, several observer designs exist, e.g., those of [4] and [5]. However, the existing finite-time observers were limited to systems that are linear with respect to the unmeasured part of the state variable (as was the case in [5], [6], and [7]) or to systems which have a lower triangular structure [8]. They give global results. On the other hand, many systems do not belong to these families of systems and may not admit global observers. Hence, the preceding works left open the important challenge of observer design for more complex cases where the nonlinearity can depend on the entire state variable, including the design of observers for systems containing quadratic terms in the state variables.

Here, we help address the preceding challenge. We design observers that are almost finite-time, i.e., they ensure exponential convergence of the observation error to zero, with a rate of convergence that tends to infinity as a growth rate for the state-dependent nonlinearity converges to zero. Since the observation error is the difference between the unmeasured state and observer values, this ensures state estimation. When the state-dependent nonlinearity is globally Lipschitz, we obtain a global observer, whose observation error converges to zero from all initial states for the original system.

For dynamics that can have Lotka-Volterra nonlinearities (i.e., products of state components), we provide a second theorem that is instead local, meaning, it only applies when the norm of the initial state of the original system is small enough, and it provides estimates of the basin of attraction. This is an analog of [9], which solved delay compensation problems for systems whose nonlinearities may also have order two near the origin but which did not provide observers.

We use standard notation where the dimensions of the Euclidean spaces are arbitrary unless otherwise noted, $|\cdot|$ is the usual Euclidean vector and matrix norm, $|h|_J$ (resp., $|h|_\infty$) is the corresponding sup norm of a function h over a subset J of its domain (resp., its entire domain), and $M \leq N$ for square matrices M and N means $N-M$ is nonnegative definite. Also, we set $g_t(s) = g(t+s)$ for functions g and all $t \geq 0$ and $s \leq 0$ such that $t+s$ is in the domain of g. We use standard definitions of input-to-state stability (or ISS, which we also use to abbreviate input-to-state stable) [10].

II. Global Observer Design

A. Class of Systems and Theorem

We first consider the system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + f_1(t, y(t)) + f_2(t, x(t)) + \delta(t) \\ y(t) & = & Cx(t) \end{array} \right. \quad (1)$$

where x is valued in $\mathbb{R}^n$, y is valued in $\mathbb{R}^q$, and the unknown piecewise continuous locally bounded function δ represents uncertainty, where we have separate nonlinearity terms f_1 and f_2 because of the different roles these terms will play in the observer design. We first assume the following (but see Section III for local results under more general conditions), where the observability condition is motivated by the genericity and so also ubiquity of observable pairs (which follows from the proof of [11, Proposition 3.3.12]):

Assumption 1: The functions f_1 and f_2 are continuous, and f_1 (resp., f_2) is locally (resp., globally) Lipschitz in its second argument uniformly in its first variable. Also, the pair (A, C) is observable, and the uncertainty functions δ in (1) are such that (1) is forward complete. $\square$

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Assumption 1 provides a constant $k_f > 0$ such that

$$|f_2(t,a) - f_2(t,b)| \le k_f|a-b| \tag{2}$$

for all $t \ge 0$ and for all a and b in $\mathbb{R}^n$. We fix such a k_f in what follows. Choosing any constant $\tau > 0$, any constant $k_f > 0$ satisfying the condition from (2), and the matrices

$$C^\sharp = C^\top C \text{ and } M_\tau = \int_{-\tau}^{0} e^{A^\top s} C^\sharp e^{As}\mathrm{d}s, \tag{3}$$

and the constant

$$\bar\beta = |M_\tau^{-1}| \int_{-\tau}^{0} \left|e^{A^\top s} C^\sharp\right| \left(\int_s^0 \left|e^{Am}\right| \mathrm{d}m\right) \mathrm{d}s, \tag{4}$$

where the inverse M_τ^{-1} exists because (A,C) is observable (e.g., by [11, Section 3.5]), the final assumption is as follows, whose condition (5) will allow us to use the trajectory-based approach from [12, Lemma 1] to prove the first theorem:

Assumption 2: The inequality

$$k_f \bar\beta < 1 \tag{5}$$

is satisfied. □

Assumption 2 can be viewed as the smallness condition $k_f < 1/\bar\beta$ on k_f when $\bar\beta \neq 0$. To provide the dynamics of the observer value $\hat x$, we introduce the dynamic extension

$$\begin{array}{rcl} \dot L_1(t) & = & -A^\top L_1(t) + C^\top y(t) \\ \dot L_2(t) & = & AL_2(t) + f_1(t,y(t)) \\ \dot L_3(t) & = & AL_3(t) + f_2(t,\hat x(t)) \\ \dot L_4(t) & = & -A^\top L_4(t) + C^\sharp L_2(t) \\ \dot L_5(t) & = & -A^\top L_5(t) + C^\sharp L_3(t) \end{array} \tag{6}$$

for all $t \ge 0$, where we assume that the initial times for the dynamics (6) are $t_0 = 0$. The global observer design is then:

Theorem 1: Let Assumptions 1-2 hold. Then

$$\begin{array}{rcl} \hat x(t) & = & M_\tau^{-1}[L_1(t) - e^{-A^\top\tau}L_1(t-\tau)] + L_2(t) \\ & & -M_\tau^{-1}[L_4(t) - e^{-A^\top\tau}L_4(t-\tau)] + L_3(t) \\ & & -M_\tau^{-1}[L_5(t) - e^{-A^\top\tau}L_5(t-\tau)] \end{array} \tag{7}$$

for all $t \ge \tau$ and $\hat x(t) = 0$ when $t \in [0,\tau]$ is such that: For all initial values for (1) and (6), the estimate

$$|\hat x(t) - x(t)| \le e^{\frac{\ln(k_f\bar\beta)}{\tau}(t-\tau)}|x|_{[0,\tau]} + \frac{\bar\beta}{(1-\bar\beta k_f)^2}|\delta|_{[0,t]} \tag{8}$$

holds for all $t \ge \tau$. □

Remark 1: We can write the observer value (7) as

$$\begin{array}{l} \hat x(t) = \Theta(t,y_t) \\ +M_\tau^{-1}\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\sharp \left(\int_s^t e^{A(s-\ell)} f_2(\ell,\hat x(\ell))\mathrm{d}\ell\right)\mathrm{d}s \end{array} \tag{9}$$

for all $t \ge \tau$ in terms of the operator

$$\begin{array}{l} \Theta(t,y_t) = M_\tau^{-1}\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\top y(s)\mathrm{d}s \\ +M_\tau^{-1}\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\sharp \left(\int_s^t e^{A(s-\ell)} f_1(\ell,y(\ell))\mathrm{d}\ell\right)\mathrm{d}s. \end{array} \tag{10}$$

This follows by applying the method of variation of parameters to dynamics of the form $\dot z(t) = Gz(t) + N(t)$ for suitable constant matrices G and functions $N(t)$, to obtain

$$z(t) - e^{G(t-r)}z(r) = \int_r^t e^{G(t-\ell)}N(\ell)\mathrm{d}\ell \tag{11}$$

when $t \ge r \ge 0$. We choose G, N, and r as follows. First, we write the single integral in (10) as $L_1(t) - e^{-A^\top\tau}L_1(t-\tau)$, by choosing $z = L_1$, $G = -A^\top$, $N(t) = C^\top y(t)$, and $r = t-\tau$ in (11). Next, we rewrite the double integral in (10) as

$$\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\sharp e^{A(s-t)} \left[\int_s^t e^{A(t-\ell)}f_1(\ell,y(\ell))\mathrm{d}\ell\right]\mathrm{d}s \tag{12}$$

whose inner integral can be written as $L_2(t) - e^{A(t-s)}L_2(s)$ (by choosing $z = L_2$, $G = A$, $N(t) = f_1(t,y(t))$, and $r = s$ in (11)), and then we pick $G = -A^\top$, $N(t) = C^\sharp L_2(t)$, and $r = t-\tau$ to write (12) as $M_\tau L_2(t) - (L_4(t) - e^{-A^\top\tau}L_4(t-\tau))$. We treat the double integral in (9) analogously, using $f_2(t,\hat x(t))$ instead of $f_1(t,y(t))$. Then (9) becomes (7). However, the expression (7) from Theorem 1 implies that we do not need to integrate to compute the observer. This may facilitate implementing the observer, because of the ease with which one can numerically solve systems of ordinary differential equations such as (6), e.g., using `NDSolve` in the Mathematica program. Also, we have the almost finite-time property that the rate $-\ln(k_f\bar\beta)/\tau$ of exponential convergence of the observation error to 0 from (8) converges to $+\infty$ as k_f converges to 0. □

B. Proof of Theorem 1

Using the simplifying notation

$$\varphi(t) = f_1(t,y(t)) + f_2(t,x(t)), \tag{13}$$

we can rewrite the dynamics in (1) as $\dot x(t) = Ax(t) + \varphi(t) + \delta(t)$. It follows from a variation of parameters argument that

$$x(t) = e^{A(t-s)}x(s) + \int_s^t e^{A(t-\ell)}(\varphi(\ell) + \delta(\ell))\mathrm{d}\ell \tag{14}$$

when $t \ge s \ge 0$. By left multiplying (14) through by $e^{A^\top(s-t)}C^\sharp e^{A(s-t)}$, then integrating the result over $s \in [t-\tau, t]$, and then left multiplying both sides of the result by M_τ^{-1} where M_τ is defined in (3), we obtain

$$\begin{array}{l} x(t) = M_\tau^{-1}\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\top y(s)\mathrm{d}s \\ +M_\tau^{-1}\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\sharp \left(\int_s^t e^{A(s-\ell)}(\varphi(\ell)+\delta(\ell))\mathrm{d}\ell\right)\mathrm{d}s. \end{array}$$

Consequently,

$$\begin{array}{l} x(t) = \Theta(t,y_t) \\ +M_\tau^{-1}\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\sharp \left(\int_s^t e^{A(s-\ell)}f_2(\ell,x(\ell))\mathrm{d}\ell\right)\mathrm{d}s \\ +M_\tau^{-1}\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\sharp \left(\int_s^t e^{A(s-\ell)}\delta(\ell)\mathrm{d}\ell\right)\mathrm{d}s \end{array}$$

with $\Theta(t,y_t)$ defined in (10). Then (2) and (9) give

$$\begin{array}{l} |\hat x(t) - x(t)| \\ \le |M_\tau^{-1}|\int_{t-\tau}^{t} \left|e^{A^\top(s-t)}C^\sharp\right| \left(\int_s^t \left|e^{A(s-\ell)}\right|\Delta(\ell)\mathrm{d}\ell\right)\mathrm{d}s \\ +|M_\tau^{-1}|\int_{t-\tau}^{t} \left|e^{A^\top(s-t)}C^\sharp\right| \left(\int_s^t \left|e^{A(s-\ell)}\right||\delta(\ell)|\mathrm{d}\ell\right)\mathrm{d}s \end{array}$$

for all $t \ge \tau$, where $\Delta(\ell) = k_f|\hat x(\ell) - x(\ell)|$ was used to bound $|f_2(\ell,\hat x(\ell)) - f_2(\ell,x(\ell))|$. Hence,

$$\begin{array}{l} |\hat x(t) - x(t)| \\ \le |M_\tau^{-1}|\int_{t-\tau}^{t} \left|e^{A^\top(s-t)}C^\sharp\right| \left(\int_s^t \left|e^{A(s-\ell)}\right|\mathrm{d}\ell\right)\mathrm{d}s|\Delta|_{[t-\tau,t]} \\ +|M_\tau^{-1}|\int_{t-\tau}^{t} \left|e^{A^\top(s-t)}C^\sharp\right| \left(\int_s^t \left|e^{A(s-\ell)}\right||\delta(\ell)|\mathrm{d}\ell\right)\mathrm{d}s. \end{array} \tag{15}$$

and so also

$$|\hat{x}(t) - x(t)| \le k_f \bar{\beta} |\hat{x} - x|_{[t-\tau,t]} + \bar{\beta}|\delta|_{[t-\tau,t]} \qquad (16)$$

for all $t \ge \tau$ with $\bar{\beta}$ defined in (4). Since Assumption 2 gives $k_f \bar{\beta} \in (0,1)$, it follows from the trajectory based approach (e.g., [12, Lemma 1], applied with $w(\ell) = |x(\ell+\tau) - \hat{x}(\ell+\tau)|$) that the required ISS estimate (8) holds.

III. Local Observer and Feedback Design

A. Class of Systems and Theorem

We consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + f_1(t, y(t)) + f_2(t, x(t)) \\ y(t) = Cx(t) \end{cases} \qquad (17)$$

where x is valued in $\mathbb{R}^n$, y is valued in $\mathbb{R}^q$, and u is valued in $\mathbb{R}^p$. We assume that this system is forward complete for each locally bounded piecewise continuous choice of $u(t)$, which we will later specify as the control. The first assumption is:

Assumption 3: The pair (A, C) is observable. Also, there is a matrix $K \in \mathbb{R}^{p\times n}$ such that $H = A + BK$ is Hurwitz and $BK \neq 0$. □

Assumption 3 ensures that there are a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and constants $p > 0$, $\bar{q} > 0$ and $\underline{q} > 0$ such that

$$QH + H^\top Q \le -pQ \ \text{ and } \ \underline{q}I \le Q \le \bar{q}I \qquad (18)$$

hold, e.g., using the largest and smallest eigenvalues of Q, respectively. Using the positive definite quadratic function

$$V(\xi) = \xi^\top Q \xi, \qquad (19)$$

the last assumptions are then as follows:

Assumption 4: The functions f_1 and f_2 are continuous, f_1 is locally Lipschitz in its second argument uniformly in its first argument, and there is a C^1 function $\theta \in \mathcal{K}_\infty$ so that

$$|f_2(t,a) - f_2(t,b)| \le \sqrt{\theta\left(|a-b|^2 + V(b)\right)}|a-b| \qquad (20)$$

holds for all $t \ge 0$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$. □

Assumption 5: There is function Γ of class $\mathcal{K}_\infty$ such that

$$\left|2x^\top Q\left[f_1(t, Cx) + f_2(t,x)\right]\right| \le \Gamma(V(x))V(x) \qquad (21)$$

for all $t \ge 0$ and $x \in \mathbb{R}^n$. □

Assumptions 4-5 can be interpreted to mean that f_1 and f_2 are of order 2 at the origin. We use the dynamic extension

$$\begin{aligned} \dot{L}_6(t) &= AL_6(t) + BK\hat{x}(t) \\ \dot{L}_7(t) &= -A^\top L_7(t) + C^\top C L_6(t) \end{aligned} \qquad (22)$$

for all $t \ge 0$, the matrix M_τ from (3), the dynamic extension (6), and any constant $\tau > 0$ (which will again serve as the fixed almost finite convergence time), where the new choice of the observer value $\hat{x}(t)$ will be specified in the theorem below. Fixing Γ, K, θ, Q, p, τ, $\underline{q}$, and $\bar{q}$ satisfying the preceding requirements, we now introduce the constants

$$k_a = \tau^3 |M_\tau^{-1}|^2 \max_{r\in[-\tau,\tau]} |Ce^{Ar}|^4 \ \text{and}\ k_b = \tfrac{2|(BK)^\top QBK|}{p} \qquad (23)$$

and we choose any positive constants δ_1 and δ_2 such that

$$\delta_1 < \Gamma^{-1}\left(\tfrac{p}{4}\right), \qquad (24)$$

$$\delta_2 = \tfrac{p}{4k_b}\delta_1, \ \text{and} \qquad (25)$$

$$\tau k_a \theta(\delta_1 + \delta_2) \in (0,1). \qquad (26)$$

We also set $\delta_3 = \min\left\{\delta_1, \delta_2 \underline{q}\right\}$, and we use the function

$$\rho(m) = \tfrac{2|QA|}{\underline{q}} + \Gamma(m). \qquad (27)$$

The constants δ_i for $1 \le i \le 3$ and $\underline{q}$ and $\bar{q}$ and the function ρ are essential for computing the basin of attraction

$$\mathcal{B}_* = \left\{ x \in \mathbb{R}^n : |x| \le \sqrt{\frac{\delta_3}{\bar{q}e^{\rho(\delta_3)\tau}}} \right\} \qquad (28)$$

which will be the set of all initial states for (17) for which the observer design will be effective for the given constant $\tau > 0$. We will use the observer values

$$\begin{aligned} \hat{x}(t) = \ & M_\tau^{-1}[L_1(t) - e^{-A^\top\tau}L_1(t-\tau)] + L_2(t) \\ & -M_\tau^{-1}[L_4(t) - e^{-A^\top\tau}L_4(t-\tau)] + L_3(t) \\ & -M_\tau^{-1}[L_5(t) - e^{-A^\top\tau}L_5(t-\tau)] \\ & +L_6(t) - M_\tau^{-1}[L_7(t) - e^{-A^\top\tau}L_7(t-\tau)] \end{aligned} \qquad (29)$$

for all $t \ge \tau$, where the L_i's are the states of (6) and (22) for $1 \le i \le 7$ and $\tau > 0$ is the arbitrary constant, and where we assume that $\hat{x}(t) = 0$ for all $t \in [0, \tau]$ in accordance with the convention that the observer is initialized at 0 when the state is unknown (so the values $x(s)$ of the state of (17) and of the initial observation error $x(s) - \hat{x}(s)$ are equal for all $s \in [0, \tau]$). The constants are also needed to express the rate of convergence $-\bar{r}$ of the observer design, where

$$\bar{r} = \tfrac{1}{2\tau}\ln\left(\tau k_a \theta(\delta_1 + \delta_2)\right) \qquad (30)$$

is negative, because of (26). The main result is:

Theorem 2: Consider the system (17) in closed loop with

$$u(\hat{x}(t)) = K\hat{x}(t). \qquad (31)$$

Let Assumptions 3-5 hold. Then the observer values given by (29) for all $t \ge \tau$ and $\hat{x}(t) = 0$ when $t \in [0, \tau]$ and the negative constant (30) are such that: For all initial states $x(0) \in \mathcal{B}_*$ for the preceding closed loop system, we have

$$|x(t) - \hat{x}(t)| \le e^{\bar{r}(t-\tau)}|x|_{[0,\tau]} \ \text{for all}\ t \ge \tau. \qquad (32)$$

Also, (17) in closed loop with (31) is locally exponentially stable to zero, with the domain of attraction of this closed loop system including all initial states $x(0) \in \mathcal{B}_*$. □

Remark 2: Analogously to Remark 1 above, we can write the observer value $\hat{x}$ from Theorem 2 as

$$\begin{aligned} &\hat{x}(t) = \Theta(t, y_t) + J(t, \hat{x}_t) \\ &+M_\tau^{-1}\textstyle\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\sharp\left(\int_s^t e^{A(s-\ell)}f_2(\ell, \hat{x}(\ell))\mathrm{d}\ell\right)\mathrm{d}s \end{aligned} \qquad (33)$$

in terms of the operator Θ from (10), $C^\sharp = C^\top C$, and

$$\begin{aligned} &J(t, \hat{x}_t) = \\ &M_\tau^{-1}\textstyle\int_{t-\tau}^{t} e^{A^\top(s-t)}C^\sharp\left(\int_s^t e^{A(s-\ell)}BK\hat{x}(\ell)\mathrm{d}\ell\right)\mathrm{d}s. \end{aligned} \qquad (34)$$

However, the $\hat{x}$ formula (29) again shows that we do not need to integrate to compute the observer, which can again facilitate implementing the observer. Also, the convergence rate $-\bar{r}$ with the choice $\bar{r}$ from (30) converges to $+\infty$ as the growth rate function θ converges to 0, which is an analog of the almost finite-time convergence from Theorem 1. □

B. Proof of Theorem 2

The proof has two parts. First, we use the structure of the observer to obtain a growth rate estimate for the squared norm $|\tilde{x}|^2$ of the observation error $\tilde{x} = \hat{x} - x$, and a decay estimate for $V(x(t))$ with an overshoot depending on $|\tilde{x}|^2$. In the second step, we use a contractivity argument to ensure that $|\tilde{x}(t)|$ exponentially converges to zero, which in conjunction with a small-gain argument, will imply that $x(t)$ also exponentially converges to 0. The proof is as follows.

First Part. Applying a variation of parameters to (17) gives

$$\begin{aligned}&e^{-At}x(t) = e^{-As}x(s)\\&+\textstyle\int_s^t e^{-Am}\left[Bu(m)+f_1(m,y(m))+f_2(m,x(m))\right]\mathrm{d}m\end{aligned} \tag{35}$$

for all $t \geq s$ and $s \geq 0$. By left multiplying this equality by $e^{A^\top(s-t)}C^\sharp e^{As}$ where $C^\sharp = C^\top C$ as before, and then integrating the result over $[t-\tau, t]$ with $t \geq \tau$, we obtain

$$\begin{aligned}&M_\tau x(t) = \textstyle\int_{t-\tau}^t \big[e^{A^\top(s-t)}C^\top y(s)\\&+e^{A^\top(s-t)}C^\sharp \textstyle\int_s^t e^{A(s-m)}\left[Bu(m)+f_1(m,y(m))\right.\\&\left.+f_2(m,x(m))\right]\mathrm{d}m\big]\mathrm{d}s.\end{aligned} \tag{36}$$

By (36) and the control (31), it follows that with the choice $G_\tau = M_\tau^{-1}$ and the operators (10) and (34), we have

$$\begin{aligned}&x(t) = \theta(t, y_t) + J(t, \hat{x}_t)\\&+G_\tau \textstyle\int_{t-\tau}^t e^{A^\top(s-t)}C^\sharp\big(\int_s^t e^{A(s-m)}f_2(m,x(m))\mathrm{d}m\big)\mathrm{d}s\end{aligned} \tag{37}$$

for all $t \geq \tau$. Then, $\tilde{x}(t) = -x(t)$ for all $t \in [0, \tau]$, and (33) and (37) give

$$\begin{aligned}&|\tilde{x}(t)|\\&\leq \left|G_\tau \textstyle\int_{t-\tau}^t e^{A^\top(s-t)}C^\sharp\big(\int_s^t e^{A(s-m)}f_2^\Delta(m)\mathrm{d}m\big)\mathrm{d}s\right|\\&\leq c_*|G_\tau|\textstyle\int_{t-\tau}^t\big(\int_s^t\left|f_2^\Delta(m)\right|\mathrm{d}m\big)\mathrm{d}s\end{aligned} \tag{38}$$

for all $t \geq \tau$, where $c_* = \max_{r\in[-\tau,\tau]}|Ce^{Ar}|^2$ and $f_2^\Delta(m) = f_2(m, \hat{x}(m)) - f_2(m, x(m))$. From Assumption 4, we obtain

$$\begin{aligned}|\tilde{x}(t)| &\leq c_*|G_\tau|\textstyle\int_{t-\tau}^t\big(\int_s^t V^\sharp(m)|\tilde{x}(m)|\mathrm{d}m\big)\mathrm{d}s\\&\leq \tau c_*|G_\tau|\textstyle\int_{t-\tau}^t V^\sharp(m)|\tilde{x}(m)|\mathrm{d}m\end{aligned} \tag{39}$$

for all $t \geq \tau$, where $V^\sharp(m) = \sqrt{\theta\left(|\tilde{x}(m)|^2 + V(x(m))\right)}$. Hence, it follows from the choice of k_a in (23) that

$$|\tilde{x}(t)|^2 \leq k_a \textstyle\int_{t-\tau}^t \theta\left(|\tilde{x}(m)|^2 + V(x(m))\right)|\tilde{x}(m)|^2\mathrm{d}m \tag{40}$$

for all $t \geq \tau$, by Jensen's inequality. Also, (31) gives

$$\dot{x}(t) = Hx(t) + BK\tilde{x}(t) + f_1(t, y(t)) + f_2(t, x(t)). \tag{41}$$

Then the time derivative of V along solutions of (41) satisfies

$$\begin{aligned}\dot{V}(t) &\leq -pV(x(t)) + 2x(t)^\top Q[BK\tilde{x}(t)\\&\quad+f_1(t,y(t)) + f_2(t,x(t))]\\&\leq -pV(x(t)) + \Gamma(V(x(t)))V(x(t))\\&\quad+2x(t)^\top QBK\tilde{x}(t) \text{ for all t} \geq 0,\end{aligned} \tag{42}$$

by Assumption 5. Hence, by the choice of k_b in (23),

$$\begin{aligned}\dot{V}(t) &\leq -\tfrac{p}{2}V(x(t)) + \Gamma(V(x(t)))V(x(t))\\&\quad+\tfrac{2}{p}\tilde{x}(t)^\top(BK)^\top QBK\tilde{x}(t)\\&\leq -\tfrac{p}{2}V(x(t)) + \Gamma(V(x(t)))V(x(t)) + k_b|\tilde{x}(t)|^2,\end{aligned} \tag{43}$$

where we applied Cauchy's inequality to upper bound the product $2\{|x(t)^\top\sqrt{Q}|\sqrt{p/2}\}\{|\sqrt{Q}BK\tilde{x}(t)|\sqrt{2/p}\}$ to bound the last term in (42). For $t \geq \tau$, (40) and (43) give

$$\begin{cases}|\tilde{x}(t)|^2 \leq k_a\int_{t-\tau}^t \theta\left(|\tilde{x}(m)|^2 + V(x(m))\right)|\tilde{x}(m)|^2\mathrm{d}m\\ \dot{V}(t) \leq -\frac{p}{2}V(x(t)) + \Gamma(V(x(t)))V(x(t)) + k_b|\tilde{x}(t)|^2.\end{cases} \tag{44}$$

Second Part. The basin of attraction (28) and the $\bar{q}$ from (18) give $V(x(0)) \leq \delta_3/e^{\rho(\delta_3)\tau}$ when $x(0) \in \mathcal{B}_*$. By Lemma A.1 below, this gives $V(x(t)) < \delta_3$ for all $t \in [0, \tau)$. Hence,

$$V(x(\ell)) < \delta_1 \text{ and } |\tilde{x}(\ell)|^2 = |x(\ell)|^2 < \delta_2 \tag{45}$$

for all $\ell \in [0, \tau)$, by the choice $\delta_3 = \min\{\delta_1, \delta_2\underline{q}\}$. Hence, $|\tilde{x}(\tau)|^2 \leq \tau k_a\theta(\delta_2 + \delta_1)\delta_2$, by (44). Since x is continuous, we proceed by contradiction. Suppose there were a $t_c \geq \tau$ such that

$$V(x(t)) < \delta_1 \text{ and } |\tilde{x}(t)|^2 < \delta_2 \tag{46}$$

for all $t \in [0, t_c)$ and either $V(x(t_c)) = \delta_1$ or $|\tilde{x}(t_c)|^2 = \delta_2$.

We observe that (44) and (46) give

$$\begin{aligned}|\tilde{x}(t_c)|^2 &\leq k_a\textstyle\int_{t_c-\tau}^{t_c}\theta(\delta_2+\delta_1)\delta_2\mathrm{d}m\\&= \tau k_a\theta(\delta_2+\delta_1)\delta_2.\end{aligned} \tag{47}$$

From (26), we deduce that $|\tilde{x}(t_c)|^2 < \delta_2$. It follows that

$$V(x(t_c)) = \delta_1. \tag{48}$$

Also, for all $t \in [\tau, t_c)$, we have

$$\begin{aligned}&V(x(t)) < \delta_1 \text{ and}\\&\dot{V}(t) \leq -\tfrac{p}{2}V(x(t)) + \Gamma(\delta_1)\delta_1 + k_b\delta_2,\end{aligned} \tag{49}$$

by (44). By the preceding inequality and (24)-(25), it follows that

$$\begin{aligned}\dot{V}(t) &< -\tfrac{p}{2}V(x(t)) + \tfrac{p}{4}\delta_1 + k_b\delta_2\\&= -\tfrac{p}{2}\left[V(x(t)) - \delta_1\right] \leq 0\end{aligned} \tag{50}$$

for all $t \in [\tau, t_c]$. Hence, $\dot{V}(t_c) < 0$, so the function $\mathcal{H}(\ell) = V(x(\ell))$ is strictly decreasing in a neighborhood of $\ell = t_c$. This contradicts the first inequality in (46) and (48). Therefore, for all $\ell \in (0, +\infty)$, we have

$$V(x(\ell)) < \delta_1 \text{ and } |\tilde{x}(\ell)|^2 < \delta_2. \tag{51}$$

Next, bearing in mind (44), we deduce that we have

$$\begin{aligned}|\tilde{x}(t)|^2 &\leq k_a\theta(\delta_1+\delta_2)\textstyle\int_{t-\tau}^t|\tilde{x}(m)|^2\mathrm{d}m\\&\leq \tau k_a\theta(\delta_1+\delta_2)|\tilde{x}|^2_{[t-\tau,t]}\end{aligned} \tag{52}$$

for all $t \geq \tau$. Also, (24), (44), and (51) give

$$\dot{V}(t) \leq -\tfrac{p}{4} V(x(t)) + k_b |\tilde{x}(t)|^2 \text{ for all } t \geq 0. \tag{53}$$

Then (26), (52), and [12, Lemma 1] applied to the function $w(t) = |\tilde{x}(t+\tau)|^2$ (and then taking square roots of both sides of the result) provide positive constants c_a and c_b such that $|\tilde{x}(t)| \leq c_a e^{-c_b t} |\tilde{x}|_{[0,\tau]}$ for all $t \geq \tau$. Also, applying variation of parameters to (53) and using the quadratic structure of V (and then using the subadditivity of the square root) provide positive constants c_1, c_2, and c_3 such that $|x(t)| \leq c_1 e^{-c_2 t}|x(0)| + c_3 |\tilde{x}|_{[0,t]}$ for all $t \geq \tau$. By combining the preceding two upper bounds for $|\tilde{x}(t)|$ and $|x(t)|$, it now follows from standard small-gain arguments (e.g., from [13]) that $x(t)$ exponentially converges to 0.

IV. Illustrations

We illustrate both theorems. Going beyond only providing dynamics satisfying the assumptions, the examples show trade-offs between parameter values and robustness, and how Theorem 2 can lead to arbitrarily large basins of attraction.

A. Applying Theorem 1

We consider the dynamics of an unforced pendulum with friction, namely, for any constant $c > 0$,

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) \\ \dot{x}_2(t) & = & -k(t)x_2(t) - c\sin(x_1(t)) + \delta_2(t) \\ y(t) & = & x_1(t) \end{array} \right. \tag{54}$$

with the friction function $k(t) = 0.1 + \Delta_k(t)$ for a known bounded piecewise continuous function Δ_k; see [13, p.542]. The unknown piecewise continuous locally bounded function δ_2 represents uncertainty. Then we write

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + f_1(y(t)) + f_2(t, x(t)) + \delta(t) \\ y(t) & = & Cx(t) \end{array} \right. \tag{55}$$

with $\delta(t) = [0, \delta_2(t)]^\top$ and the following matrices:

$$\begin{array}{l} A = \left[\begin{array}{cc} 0 & 1 \\ 0 & -0.1 \end{array} \right], \; C = [1 \;\; 0], \\ f_1(y) = \left[\begin{array}{c} 0 \\ -c\sin(y) \end{array} \right], \; f_2(t,x) = \left[\begin{array}{c} 0 \\ -\Delta_k(t) x_2 \end{array} \right]. \end{array} \tag{56}$$

Then the assumptions of Theorem 1 are satisfied with $k_f = |\Delta_k|_{[0,\infty)}$, and the sufficient condition from Theorem 1 is

$$|\Delta_k|_{[0,\infty)} < 1/\bar{\beta}, \tag{57}$$

where $\bar{\beta}$ is defined in (4) and depends on the fixed convergence time τ. Equation (57) exhibits the trade-off that τ values that lead to larger $\bar{\beta}$ values will result in smaller allowable upper bounds for the perturbation Δ_k of the friction. This is illustrated in Fig. 1, where we used (56)-(57) and Mathematica to plot the allowable upper bounds $1/\bar{\beta}$ for $|\Delta_k|_{[0,\infty)}$ on the vertical axis, as a function of τ.

Fig. 1 illustrates that by choosing a τ value close to 0.4, we can allow a bound of 0.015 on the allowable perturbation Δ_k, i.e., about 15% of the constant part 0.1 of the friction.

Remark 3: If the friction $k(t)$ is instead piecewise continuous, bounded, and unknown but has a known positive lower bound, then for each compact set $\mathcal{C} \subseteq \mathbb{R}$, we can find a constant $B_{\mathcal{C}} > 0$ such that all solutions of the unperturbed system $\dot{x}_1(t) = x_2(t)$, $\dot{x}_2 = -k(t)x_2(t) - c\sin(x_1(t))$ for all initial states $x(0) \in \mathbb{R} \times \mathcal{C}$ satisfy $\sup_{t \geq 0} |x_2(t)| \leq B_{\mathcal{C}}$. This follows by viewing $x_2(t)$ as a solution of $\dot{z}(t) + k(t)z(t) = -c\sin(x_1(t))$. Therefore, by writing the friction as $k(t) = 0.1 + \Delta_k(t) + \delta_k(t)$ for a known Δ_k and an unknown function δ_k, we can write the dynamics (54) as $\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = -(0.1 + \Delta_k(t))x_2(t) - c\sin(x_1(t)) + \delta_2(t)$ where $\delta_2(t) = -\delta_k(t)x_2(t)$. Then the preceding example provides an exponential ISS estimate with respect to δ_2 when (57) holds, and so also semi-global exponential ISS with respect to δ_k for initial states $x(0) \in \mathbb{R} \times \mathcal{C}$ (because $|\delta_2(t)| = |x_2(t)\delta_k(t)| \leq B_{\mathcal{C}}|\delta_k(t)|$ for all $t \geq 0$). The uncertainty δ_k is motivated by the fact that friction is commonly regarded as the most uncertain quantity in mechanical systems. □

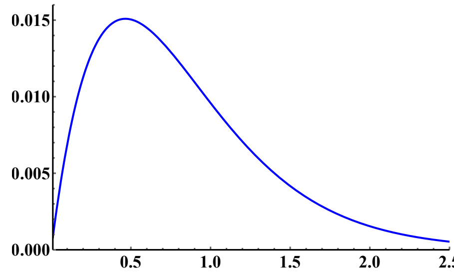

Fig. 1: Bounds $1/\bar{\beta}$ on allowable Δ_k as function of τ

B. Applying Theorem 2

Theorem 1 does not in general cover systems of the form

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) + \sum_{i=1}^{q} y_i(t) E_i(t) y(t) \\ & & + \sum_{i=1}^{n} x_i(t) D_i(t) x(t) \\ y(t) & = & Cx(t) \end{array} \tag{58}$$

for continuous bounded coefficient matrices D_i and E_i. The quadratics in (58) are called Lotka-Volterra nonlinearities, since they naturally arise in predator-prey systems. We next show how Theorem 2 can cover (58), assuming that the coefficient matrices A, B, and C satisfy Assumption 3. To this end, we choose the matrices K and Q and the positive constants p, $\underline{q}$, and $\bar{q}$ to satisfy the requirements from Section III-A. It remains to find functions θ and Γ satisfying the requirements of Assumptions 4-5, where $f_1(t, y(t))$ and $f_2(t, x(t))$ are the first and second sums in (58), respectively.

To find the needed functions θ and Γ, first note that for any $i \in \{1, \ldots, n\}$ and a and b in $\mathbb{R}^n$ and $t \geq 0$, we have

$$\begin{array}{l} |a_i D_i(t) a - b_i D_i(t) b| \\ \leq |(a_i - b_i) D_i(t) a| + |b_i D_i(t)(a-b)| \\ \leq |a-b|(|D_i|_\infty |a-b| + 2|D_i|_\infty |b|) \\ \leq |a-b|\sqrt{2}\sqrt{|D_i|_\infty^2 |a-b|^2 + (4/\underline{q})|D_i|_\infty^2 \underline{q}|b|^2} \\ \leq |a-b|\sqrt{\theta_i(|a-b|^2 + V(b))}, \\ \text{where } \theta_i(s) = 2|D_i|_\infty^2 \max\{1, 4/\underline{q}\}s, \end{array} \tag{59}$$

where we used the fact that $s + r \leq \sqrt{2(s^2 + r^2)}$ for all $s \geq 0$ and $r \geq 0$. Hence, Assumption 4 holds with

$\theta(s) = 2n^2 \max_i |D_i|_\infty^2 \max\{1, 4/\underline{q}\}s$. Also, if we set $\bar{D} = \max_i |D_i|_\infty$ and $\bar{E} = \max_i |E_i|_\infty$ (which we assume are nonzero), then we can upper bound the left side of (21) by

$$\begin{aligned} &2|x||Q|(q\bar{E}|C|^2|x|^2 + n\bar{D}|x|^2) \\ &\leq \left\{2|Q|(q\bar{E}|C|^2 + n\bar{D})\tfrac{1}{\underline{q}^{3/2}}\right\}(\sqrt{\underline{q}}|x|)(\underline{q}|x|^2), \end{aligned} \tag{60}$$

so we can take $\Gamma(s) = g_* \sqrt{s}$, where g_* is the constant in curly braces in (60). Then Theorem 2 provides the local almost finite-time observer and a basin of attraction estimate.

Also, with the notation from Theorem 2, for each constant $\bar{B} > 0$, we can find a constant $\epsilon_{\bar{B}} > 0$ such that if we choose $\delta_1 = \lambda\Gamma^{-1}(p/4)$ in (24) for any constant $\lambda \in (0,1)$, and if $\max\{\bar{D}/\bar{E}, \bar{D}, \bar{E}\} \leq \epsilon_{\bar{B}}$, then (26) is satisfied and

$$\tfrac{\delta_3}{\bar{q}e^{\rho(\delta_3)\tau}} \geq \bar{B}. \tag{61}$$

To see why, first note that $\mathcal{M}_1 = 8|(BK)^\top QBK|$, the function $\mathcal{M}(\bar{D}, \bar{E}) = 64|Q|^2(\underline{q}\bar{E}|C|^2 + n\bar{D})^2$, and (24)-(25) give $\delta_1 = \lambda p^2 \underline{q}^3/\mathcal{M}(\bar{D}, \bar{E})$ and $\delta_2 = \lambda p^4 \underline{q}^3/(\mathcal{M}_1\mathcal{M}(\bar{D}, \bar{E}))$. Hence, $\delta_3 = \delta_1\mathcal{M}_0$, where $\mathcal{M}_0 = \min\{1, p^2\underline{q}/\mathcal{M}_1\}$. Also, by the choice of θ, (26) can be written as

$$\tfrac{2\lambda\bar{D}^2 \max\{1,4/\underline{q}\} n^2 \tau^4 |M_\tau^{-1}|^2 c_*^2 p^2 \underline{q}^3}{\mathcal{M}(\bar{D},\bar{E})}\left(1 + \tfrac{p^2}{\mathcal{M}_1}\right) \in (0,1), \tag{62}$$

where $c_* = \max_{r\in[-\tau,\tau]}|Ce^{Ar}|^2$. Hence, since the left side of (62) converges to 0 as $\bar{D}/\bar{E} \to 0$ (because $\bar{D}^2/\mathcal{M}(\bar{D}, \bar{E}) \to 0$ as $\bar{D}/\bar{E} \to 0$), we can satisfy (26) when $\bar{D}/\bar{E} \leq \epsilon_{\bar{B}}$ for a small enough $\epsilon_{\bar{B}}$. Also, (27) gives

$$\rho(\delta_3) = \tfrac{2|QA|}{\underline{q}} + \tfrac{p\sqrt{\lambda\mathcal{M}_0}}{4} \tag{63}$$

which does not depend on $\bar{D}$ or $\bar{E}$. Hence,

$$\frac{\delta_3}{\bar{q}e^{\rho(\delta_3)\tau}} = \frac{\delta_1\mathcal{M}_0}{\bar{q}e^{\tau(2|QA|/\underline{q} + p\sqrt{\lambda\mathcal{M}_0}/4)}} \to +\infty \tag{64}$$

as $\max\{\bar{D}, \bar{E}\} \to 0$, by the δ_1 formula. This proves the assertion. This can be summarized by saying that Theorem 2 provides arbitrarily large basins of attraction for the system (58) when the bounds $\bar{D}$ and $\bar{E}$ on the D_i's and E_i's are small enough, if $\bar{D}$ is also small enough relative to $\bar{E}$.

V. Conclusions

We provided new observer designs for a large class of nonlinear systems. When the nonlinearities are of order two near the origin (e.g., Lotka-Volterra nonlinearities), we found local observers and estimates of the basins of attraction, which led to local feedback stabilization. When one of the nonlinearities satisfies a global Lipschitzness condition, we obtain global convergence. The examples illustrate trade-offs between convergence and growth rates of the dynamics. We aim to develop event-triggered analogs.

Appendix. Technical Result

We used the following in the proof of Theorem 2, where we continue the notation that from the proof of Theorem 2:

Lemma A.1: Let $\tau > 0$ be a constant and $x(0)$ satisfy $V(x(0)) \leq \delta_3/e^{\rho(\delta_3)\tau}$. Then the corresponding solution x of (17) in closed-loop with (31) satisfies $V(x(t)) < \delta_3$ for all $t \in [0,\tau)$. □

Proof: Let us observe that for all $t \in [0,\tau]$, we have

$$\dot{x}(t) = Ax(t) + f_1(t, Cx(t)) + f_2(t, x(t)), \tag{A.1}$$

since $\hat{x} = 0$ on $[0,\tau]$. Hence, by the choice (27) of ρ, the time derivative of $V(x)$ along the trajectories of (A.1) satisfies

$$\begin{aligned} \dot{V}(t) &= 2x(t)^\top QAx(t) \\ &\quad + 2x(t)^\top Q[f_1(t, Cx(t)) + f_2(t, x(t))] \\ &\leq \rho(V(x(t)))V(x(t)) \end{aligned} \tag{A.2}$$

by Assumption 5. We next use the auxiliary system

$$\dot{v}(t) = \rho(v(t))v(t). \tag{A.3}$$

Let $v(0) = \frac{\delta_3}{e^{\rho(\delta_3)\tau}}$ and $t_c > 0$ be such that $v(t) < \delta_3$ for all $t \in [0, t_c)$ and $v(t_c) = \delta_3$. Then $\dot{v}(t) \leq \rho(\delta_3)v(t)$ for all $t \in [0, t_c]$. Hence,

$$v(t_c) \leq e^{\rho(\delta_3)t_c}v(0) = e^{\rho(\delta_3)t_c}\frac{\delta_3}{e^{\rho(\delta_3)\tau}}. \tag{A.4}$$

Since $v(t_c) = \delta_3$, we deduce that $e^{\rho(\delta_3)\tau} \leq e^{\rho(\delta_3)t_c}$. Hence, $t_c \geq \tau$. We deduce from the comparison lemma (applied to (A.2) and (A.3)) that if $V(x(0)) \leq v(0)$, then $V(x(t)) < \delta_3$ for all $t \in [0,\tau)$. This allows us to conclude. ■

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