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Scattering in a partially open waveguide: the forward problem

Laurent Bourgeois, Sonia Fliss, Jean-François Fritsch,
Christophe Hazard and Arnaud Recoquillay

December 20, 2022

Abstract

This paper is dedicated to an acoustic scattering problem in a two-dimensional partially open waveguide, in the sense that the left part of the waveguide is closed, that is with a *bounded* cross-section, while the right part is bounded in the transverse direction by some Perfectly Matched Layers that mimic the situation of an open waveguide, that is with an *unbounded* cross-section. We prove well-posedness of such scattering problem in the Fredholm sense (uniqueness implies existence) and exhibit the asymptotic behaviour of the solution in the longitudinal direction with the help of the Kondratiev approach. Having in mind the numerical computation of the solution, we also propose some transparent boundary conditions in such longitudinal direction, based on Dirichlet-to-Neumann operators. After proving that such artificial conditions actually enable us to approximate the exact solution, some numerical experiments illustrate the quality of such approximation. keywords
2000 Math Subject Classification:

1 Introduction

Elongated structures such as cables, pipes and bars are widely used in the field of civil engineering. A typical configuration is the case of a cable which is partially embedded into another elastic medium, the cable being for example made of steel while the surrounding medium is made of concrete. It may happen that some defects appear in the inaccessible part of the cable, namely the embedded part of the cable or at the interface between the cable and the surrounding medium. Non Destructive Testing (NDT) is commonly used to image defects in such structures. NDT consists in solving an inverse problem: one produces several incident waves coming from the accessible part of the cable, measures the resulting scattered waves due to the presence of the defect, then try to find the defect by using those multistatic data. Applications of NDT in industry are for example given in [Leinov et al., 2015, Rose et al., 2009, Loveday, 2012]. Before addressing the inverse scattering problem, which will be the objective of a forthcoming article (see [Bourgeois et al., 2022]), it is important to analyze the forward problem, which consists for example in finding the scattering response of the partially embedded cable to an incident field coming from the part of the cable which is free. Such configuration can be seen as a junction between a closed waveguide and an open waveguide (i.e., with a *bounded* and an *unbounded* cross-section, respectively), this open waveguide being itself composed of a core and a sheath. We assume that the defect lies within the core of the open waveguide to simplify the presentation, having in mind that the case of a defect within the sheath would be treated the same way.

In order to simplify the analysis, we restrict ourselves to a scalar two-dimensional problem which models an isotropic antiplane shear situation, in other words we consider the sole SH waves. In addition, we restrict to the time-harmonic waves at a given frequency ω assuming a $e^{-i\omega t}$-dependance with respect to time t. The configuration that we consider is the following. The closed part of the waveguide occupies the domain $(-\infty,0)\times(-h,h)$, with $h>0$, while the open part of the waveguide occupies the domain $(0,+\infty)\times\mathbb{R}$. The shear modulus and the density are denoted μ and ρ, respectively, while the speed c and the wave number k are defined by

$$c:=\sqrt{\frac{\mu}{\rho}} \quad \text{and} \quad k:=\frac{\omega}{c}.$$

In the closed part of the waveguide, the shear modulus and the density are given by $(\mu,\rho)=(\mu_0,\rho_0)$, where μ_0 and ρ_0 are given positive constants. In the open part of the waveguide, the shear modulus

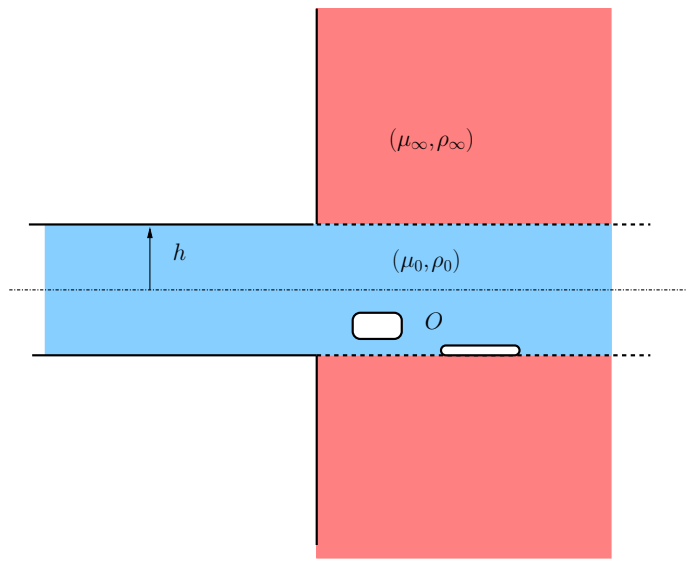

Figure 1: Original configuration

and the density are piecewise constants, more precisely $(\mu,\rho)=(\mu_0,\rho_0)$ in the domain $(0,+\infty)\times(-h,h)$ (the core) and $(\mu,\rho)=(\mu_\infty,\rho_\infty)$ in the domain $(0,+\infty)\times((-\infty,-h)\cup(h,+\infty))$ (the sheath), where μ_∞ and ρ_∞ are given positive constants. Denoting the defect O, we have represented such original configuration in Figure 1.

Studying well-posedness of a scattering problem in this configuration is a challenging task due to the definition of radiation conditions in the open part of the waveguide. In this vein, well-posedness for a similar case, that is a junction between two open waveguides, is established in [Bonnet-Ben Dhia et al., 2011]. Such result requires some complicated tools and lots of technicalities. More precisely, the behaviour at infinity is specified by means of modal radiation conditions (based on generalized Fourier transforms adapted to both half waveguides) which extend the classical conditions used for closed waveguides. The main difference is that they involve a continuum of modes. The main originality of [Bonnet-Ben Dhia et al., 2011] concerns the proof of uniqueness which relies on an argument of analyticity with respect to the generalized Fourier variable. Another angle of attack is chosen in the present paper. More precisely, the surrounding unbounded medium will be modeled by transverse Perfectly Matched Layers of finite depth, which is a classical way of replacing the true open waveguide by an equivalent closed one made of an artificial material (see [Berenger, 1994]). The justification of such transverse PMLs is not addressed in the present paper and is, to our best knowledge, an open question. Some arguments given in [Chandler-Wilde & Monk, 2009] for a similar problem could probably be reused in our configuration. The first step would be to adapt the proof of [Chandler-Wilde & Monk, 2009] to show the well-posedness of the original problem (without the PMLs), which might be attacked by using [Bonnet-Ben Dhia et al., 2011] for instance.

The physical medium will occupy the domain $(0,+\infty)\times(-h_{\rm in},h_{\rm in})$, with $h_{\rm in}\geq h$, while the PMLs will occupy the domain $(0,+\infty)\times((-h_{\rm out},-h_{\rm in})\cup(h_{\rm in},h_{\rm out}))$, with $h_{\rm out}>h_{\rm in}$. The PMLs involve a complex-valued function α such that $\alpha=1$ in the physical domain and α having well chosen real and imaginary parts in the PMLs in order to absorb waves (see section 2). Let us introduce the domains $\Omega^-:=(-\infty,0)\times(-h,h)$, $\Omega^+:=(0,+\infty)\times(-h_{\rm out},h_{\rm out})$ and $\Omega:=\Omega^-\cup\Sigma_0\cup\Omega^+$, with $\Sigma_0:=\{0\}\times(-h,h)$. On Figure 2, we have represented the configuration with PMLs, which will be the configuration of interest throughout the paper. The scattering problem we actually consider is the following: given an incident field u^i, which in practice is a mode coming from the left, find the total field u in the complete waveguide Ω such that

$$\begin{cases} Pu=0 & \text{in } \Omega\setminus\overline{O},\\ \partial_\nu u=0 & \text{on } \partial\Omega,\\ u=0 & \text{on } \partial O,\\ u-u^i \text{ is outgoing,} & \end{cases} \tag{1.1}$$

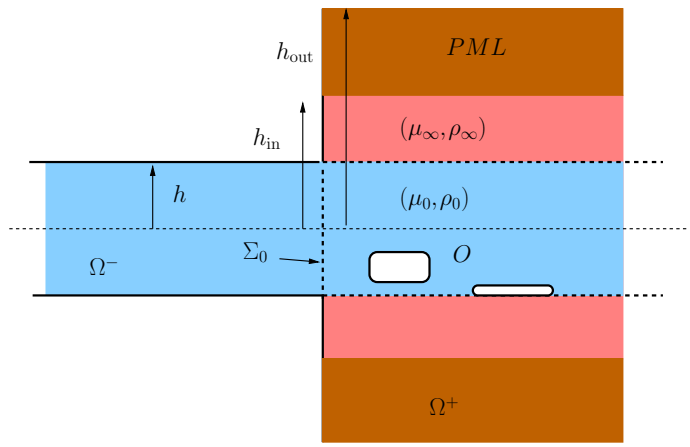

Figure 2: Configuration with PMLs

where the differential operator P is defined by

$$P := -\partial_y(\alpha\mu\partial_y\,\cdot) - \frac{\mu}{\alpha}\partial_{xx}\,\cdot - \frac{\mu}{\alpha}k^2 \tag{1.2}$$

and ν is the outward unit normal vector to $\partial\Omega$. The last line of the system (1.1) is a radiation condition applied to the scattered field $u^s := u - u^i$ that will be made precise in the following. Note that we have chosen in this paper to consider a Dirichlet obstacle. However, the case of a Neumann obstacle, for example, would be treated similarly without any additional technicalities.

Our paper has mainly three objectives: the first one is to establish the well-posedness in the Fredholm sense (uniqueness implies existence) of the scattering problem (1.1) in an appropriate functional space, the second one is to study the asymptotic behaviour of the solution at infinity, and the third one is to introduce some transparent boundary conditions along the axis of the waveguide and corresponding approximations in order to compute numerical solutions. In the case of a homogeneous closed waveguide, all these questions can be addressed using the fact that the underlying transverse operator is self-adjoint, and a complete orthonormal basis of eigenvectors can be used to derive a modal decomposition of all fields. In the presence of PMLs, the corresponding transverse operator is not self-adjoint any more, which prevents us from exploiting the same technique. As a result, our forthcoming analysis of the scattering problem relies on the theory exposed in [Kondratiev, 1967] (see also [Maz'ya & Plamenevskiĭ, 1977, Nazarov & Plamenevskiĭ, 1994, Kozlov et al., 1997, Kozlov et al., 2001]). In the first part of the analysis, we consider a straight closed waveguide with PMLs which is unperturbed, that is without any defect. The idea is to introduce weighted Sobolev spaces and to apply the Fourier transform in the unbounded direction in order to handle one dimensional problems in a bounded interval, those problems being naturally parametrized by the Fourier variable. It should be noted that in the waveguide with PMLs, the analysis is considerably simplified by the absence of propagating modes, which results from an assumption on k_0 and k_∞ (see section 2). In the second part, well-posedness of the scattering problem (1.1) in the junction of the two half-waveguides in the presence of the obstacle is obtained by using Dirichlet-to-Neumann operators in order to reduce the problem to an equivalent problem set in a bounded domain. The proper definition of these DtN operators relies on the first part of the analysis. With the help of the residue theorem, we additionnally obtain a precise asymptotic behaviour of the solution at infinity. The Kondratiev approach is used in an abstract framework in [Nazarov & Plamenevskiĭ, 1991], in [Nazarov, 2013] for an elastic waveguide, in [Bourgeois et al., 2019] for a 2D waveguide governed by a Kirchhoff-Love model, and in other different situations in [Nazarov, 1982, Nazarov & Taskinen, 2011, Bonnet-Ben Dhia et al., 2013, Bonnet-Ben Dhia & Chesnel, 2013].

Our article is organized as follows. In section 2, we give a brief review of the spectral theory of closed and open waveguides. Section 3 is dedicated to an analysis of the eigenvalues and eigenfunctions of the non-selfadjoint transverse operator in the presence of the PMLs which naturally comes into play in the Kondratiev approach. In section 4, we prove existence and uniqueness for the forward problem in a uniform unperturbed waveguide and show how the Kondratiev approach enables us to specify the behaviour at infinity of the solution. We prove well-posedness of the forward problem (1.1) of interest,

that is the partially embedded closed waveguide containing a defect, in section 5. The behaviour at infinity of the solution is specified as well. In section 6, we introduce some transparent boundary conditions in the direction of the waveguide and prove they enable us to approximate the true solution in the unbounded domain. The proof relies on the previous result giving the behaviour at infinity of such true solution. Some numerical experiments are presented in section 7. A brief concluding section completes the paper.

2 A short review of spectral theory of closed and open waveguides

This section essentially summarizes some results borrowed from [Goursaud, 2010]. Such section aims at describing the modes of closed and open waveguides, in particular to recall the effect of the infinite and truncated PMLs on the modes of open waveguides. It will help us to classify the computed modes in the numerical section.

We use here the concepts of *discrete* and *essential* spectra. The former is generally defined as the set of isolated eigenvalues of finite multiplicity, whereas there are various definitions of the essential spectrum, as shown for instance in [Edmunds & Evans, 1987], the simplest one being the part of the spectrum which complements the discrete spectrum. One can also define the essential spectrum as the part of the spectrum which remains unchanged under compact perturbations. Other definitions are based on the notion of Fredholm operators. All these possible definitions are equivalent for selfadjoint operators, but may differ for non-selfadjoint operators. Fortunately, in our case, all these definitions coincide (see Theorem 2.1 in [Goursaud, 2010]).

2.1 The closed waveguide

Let us denote $\tilde{\Omega} := \mathbb{R} \times I$, $I := (-h,h)$, a closed waveguide which is characterized by the constant material properties (μ_0, ρ_0), the celerity being $c_0 := \sqrt{\mu_0/\rho_0}$ and the wave number being $k_0 := \omega/c_0$. A generic point in the waveguide has coordinates (x,y), where x is the coordinate of the unbounded direction of the waveguide, while y is the coordinate in the transverse bounded section I. We consider the solutions u of the form $u(x,y) = e^{\lambda x}\varphi(y)$ for some $\lambda \in \mathbb{C}$ to the problem

$$\begin{cases} -\Delta u - k_0^2 u = 0 & \text{in } \tilde{\Omega} \\ \partial_\nu u = 0 & \text{on } \partial\tilde{\Omega}, \end{cases} \tag{2.3}$$

where ν is the outward unit normal vector to $\partial\tilde{\Omega}$. We are then naturally led to investigate the numbers $\gamma := \lambda^2$ and the corresponding functions φ which are respectively eigenvalues and eigenvectors of the transverse unbounded operator $\tilde{L} : D(\tilde{L}) \subset \mathrm{L}^2(I) \to \mathrm{L}^2(I)$ defined by

$$\begin{cases} \tilde{L}\varphi := -d_{yy}\varphi - k_0^2\varphi \\ D(\tilde{L}) := \{\varphi \in \mathrm{H}^2(I),\, d_y\varphi(-h) = d_y\varphi(h) = 0\}, \end{cases}$$

which is self-adjoint and has a compact resolvent. If we choose to normalize the eigenfunctions in $\mathrm{L}^2(I)$, it is readily seen that pairs (λ, φ) are given by

$$\pm\tilde{\lambda}_n, \quad \tilde{\lambda}_n := -{}^{i\mathbb{R}^+}\!\sqrt{\frac{n^2\pi^2}{4h^2} - k_0^2}, \quad n \in \mathbb{N}, \tag{2.4}$$

$$\tilde{\varphi}_0(y) := \frac{1}{\sqrt{2h}}, \quad \tilde{\varphi}_n(y) := \frac{1}{\sqrt{h}}\cos\Big(\frac{n\pi}{2h}(y+h)\Big), \quad n \in \mathbb{N}^*, \tag{2.5}$$

where $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and $\mathbb{R}^+ := [0, +\infty)$. The complex square root ${}^{i\mathbb{R}^+}\!\sqrt{\cdot}$ is defined for $z \in \mathbb{C} \setminus i\mathbb{R}^+$ by

$${}^{i\mathbb{R}^+}\!\sqrt{z} := \sqrt{|z|}e^{i\arg_{i\mathbb{R}^+}(z)/2}, \quad -\frac{3\pi}{2} < \arg_{i\mathbb{R}^+}(z) < \frac{\pi}{2}.$$

Such definition amounts to choose $i\mathbb{R}^+$ as the branch cut of the complex square root and coincides with the usual square root $\sqrt{x}$ for $x \in \mathbb{R}^+$.

It is important to note that the $\tilde{\varphi}_n$'s form a complete orthonormal basis of $\mathrm{L}^2(I)$. We assume from now on a restriction on the wave number k_0.

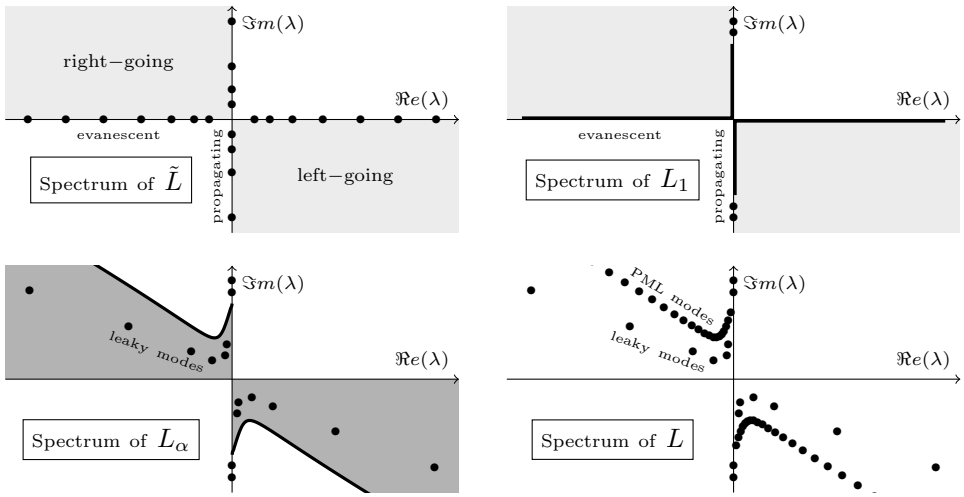

Figure 3: Location of the spectrum of $\tilde{L}$, L_1, L_α and L in the complex λ-plane. Dots represent the discrete spectrum, whereas thick lines stands for essential spectrum.

Assumption 2.1. *For all $n \in \mathbb{N}$, $n\pi/2h \neq k_0$.*

Assumption 2.1 amounts to saying that the $\tilde{\lambda}_n$'s never vanish. This implies in particular that there exists some $\tilde{N} \in \mathbb{N}^*$ such that for $n = 0, \cdots, \tilde{N}-1$, $\tilde{\lambda}_n \in i\mathbb{R}$ with $\Im m(\tilde{\lambda}_n) > 0$ while for $n = \tilde{N}, \cdots, +\infty$, $\tilde{\lambda}_n < 0$. The corresponding solutions $u(x,y) = e^{\lambda x}\varphi(y)$ to the problem (2.3) are given by

$$\tilde{w}_n^{\pm}(x,y) := e^{\pm\tilde{\lambda}_n x}\tilde{\varphi}_n(y) \tag{2.6}$$

and are called the *modes*. For $n = 0, \cdots, \tilde{N}-1$, the $\tilde{w}_n^+$ propagate from the left to the right, the $\tilde{w}_n^-$ propagate from the right to the left. They are called the *guided modes*. For $n = \tilde{N}, \cdots, +\infty$, the $\tilde{w}_n^+$ are exponentially decaying from the left to the right, the $\tilde{w}_n^-$ are exponentially decaying from the right to the left. They are called the *evanescent modes*.

The top-left part of Figure 3 shows the location of the spectrum of $\tilde{L}$ not in the natural spectral variable γ, but rather using its complex square root λ, which acts as a complex wavenumber in the x-direction. This square root, which yields two complex numbers opposite to each other, leads to a symmetry of the spectrum with respect to $\lambda = 0$, which reflects the symmetry of the waveguide with respect to $x = 0$. For us, the regions of interest in this complex λ-plane are the top-left and bottom-right quarter-planes, which correspond respectively to right-going and left-going modes in our conventions.

2.2 The open waveguide

Let us consider a stratified medium also called an open waveguide which occupies the whole space $\mathbb{R}^2$ and which is characterized by the piecewise constant material properties (μ, ρ), with $(\mu, \rho) = (\mu_0, \rho_0)$ in $\mathbb{R} \times (-h, h)$ (called the core of the waveguide) and $(\mu, \rho) = (\mu_\infty, \rho_\infty)$ in $\mathbb{R} \times ((-\infty, -h) \cup (h, +\infty))$ (called the sheath of the waveguide). Let us denote the associated celerities c_0 and c_∞ as well as the associated wave numbers k_0 and k_∞. In some sense, we have embedded the previous closed waveguide in an infinite surrounding medium. Again, we consider the solutions u of the form $u(x,y) = e^{\lambda x}\varphi(y)$ for some $\lambda \in \mathbb{C}$ to the following problem in $\mathbb{R}^2$:

$$-\partial_y(\mu\partial_y u) - \mu\partial_{xx}u - \omega^2\rho u = 0. \tag{2.7}$$

We then search the numbers $\gamma := \lambda^2$ and the functions φ as the eigenvalues and eigenvectors of the unbounded self-adjoint transverse operator $L_1 : D(L_1) \subset \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ defined by

$$\begin{cases} L_1\varphi := -\dfrac{1}{\mu}d_y(\mu\, d_y\varphi) - k^2\varphi \\ D(L_1) := \{\varphi \in \mathrm{H}^1(\mathbb{R}),\, \mu\, d_y\varphi \in \mathrm{H}^1(\mathbb{R})\}, \end{cases}$$

where $k^2 = \omega^2 \mu^{-1} \rho$. It can be shown (see Chapter 1 of [Goursaud, 2010]) that the spectrum of the operator L_1 is divided into a discrete spectrum composed of a finite number of eigenvalues $\gamma = \lambda^2$ in $(-k_{\max}^2, -k_\infty^2)$, where $k_{\max} := \max(k_0, k_\infty)$ and an essentiel spectrum, which is equal to $[-k_\infty^2, +\infty)$. For some $\gamma = \lambda^2$ in the discrete spectrum, by setting $\gamma = (i\beta)^2$ with $\beta \in (k_\infty, k_{\max})$, the corresponding function $w^\pm(x,y) = e^{\pm i\beta x}\varphi(y)$, where $\varphi \in D(L_1)$ is an eigenvector of L_1 associated with the eigenvalue γ, is called a *guided mode* as in the case of a closed waveguide. Indeed, it is localized in the core of the open waveguide (in the sense that it is exponentially decaying in the surrounding medium). For some $\gamma = \lambda^2$ in the essential spectrum, the modes are given by $w^\pm(x,y) = e^{\pm i\beta x}\Phi(y)$ (propagating) or $w^\pm(x,y) = e^{\mp \lambda x}\Phi(y)$ (evanescent), with $\beta, \lambda > 0$, where Φ is no more an eigenvector (it does not belong to $\mathrm{L}^2(\mathbb{R})$), but a so-called generalized eigenfunction. Such modes are called *radiation modes*, because they are oscillating at infinity in the tranverse direction.

In the complex λ-plane, the top-right part of Figure 3 shows the location of the spectrum of L_1. As for the closed waveguide, the part of the spectrum located on the imaginary axis corresponds to propagating modes in the x-direction. But now, it includes a discrete part (which may be empty) and a continuous one (which is never empty). On the other hand, on the real axis associated to evanescent modes, there is only essential spectrum.

Assumption 2.2. *In our paper we will assume that $c_0 > c_\infty$ or in other words $k_0 < k_\infty$, which implies that $k_{\max} = k_\infty$. As a consequence, the discrete spectrum is empty: there are no guided modes in the open waveguide, only radiation modes.*

2.3 Adding infinite PMLs

Let us consider the previous open waveguide and let us introduce transverse infinite Perfectly Matched Layers. The PMLs play the role of a fictitious absorbing medium which replaces the physical medium outside a strip of finite thickness including the core of the waveguide. It exactly mimics the behaviour of the infinite non-dissipative sheath, in the sense that the interface between the physical medium and the PMLs does not reflect the outgoing waves. As a consequence, compared with the previous open waveguide (without PMLs), the acoustic field in this new artificial open waveguide is exactly the same in the physical medium (the bounded region between both PMLs), whereas *in the PMLs, the radiating field is transformed into an evanescent one.* The latter property makes PMLs a very powerful tool for both theory and numerics. Let us first consider infinite PMLs which occupy the region $|y| > h_{\mathrm{in}}$ with $h_{\mathrm{in}} \geq h$. It consists in transforming the partial derivative $\partial_y \cdot$ in equation (2.7) into the weighted partial derivative $\alpha \partial_y \cdot$, where $\alpha : \mathbb{R} \to \mathbb{C}$ is such that

$$\alpha(y) = \begin{cases} 1 & \text{if} \quad |y| < h_{\mathrm{in}}, \\ \alpha_\infty & \text{otherwise}, \end{cases} \tag{2.8}$$

and α_∞ is a complex constant such that

$$-\frac{\pi}{2} < \arg(\alpha_\infty) < 0. \tag{2.9}$$

As a consequence, let us now consider the solutions u of the form $u(x,y) = e^{\lambda x}\varphi(y)$ to the following problem in $\mathbb{R}^2$:

$$-\alpha \partial_y(\alpha \mu \partial_y u) - \mu \partial_{xx} u - \omega^2 \rho u = 0,$$

which amounts to searching the numbers $\gamma := \lambda^2$ and the functions φ respectively as eigenvalues and eigenvectors of the unbounded non self-adjoint transverse operator $L_\alpha : D(L_\alpha) \subset \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$ defined by

$$\begin{cases} L_\alpha \varphi := -\dfrac{\alpha}{\mu} d_y(\alpha \mu d_y \varphi) - k^2 \varphi \\ D(L_\alpha) := \{\varphi \in \mathrm{H}^1(\mathbb{R}),\, \alpha \mu\, d_y \varphi \in \mathrm{H}^1(\mathbb{R})\}. \end{cases}$$

Remark 2.1. *Note that if α is identically equal to 1, then the operator L_α coincides with the operator L_1.*

The following results about the spectrum of L_α are proved in Chapter 2 of [Goursaud, 2010]. They actually extend to waveguides the pionneering ideas of the so-called *analytic dilation* technique, also called *complex scaling* developed in the early seventies in theoretical physics (see, e.g., [Hilsop & Sigal, 1996]).

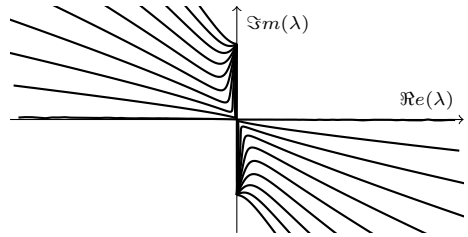

Figure 4: Successive locations of the essential spectrum of L_α as $\arg(\alpha_\infty)$ varies from 0 (two symmetric infinite "L"s, located on the real and imaginary axes) to $-\pi/2$.

As concerns the essential spectrum of L_α, it is given by $-k_\infty^2+\alpha_\infty^2\mathbb{R}^+$, that is the PMLs have the effect to rotate the essential spectrum of L_1 of an angle $2\arg(\alpha_\infty)$ in the complex plane with respect to the point $-k_\infty^2$. The effect of this rotation in the λ-plane is illustrated by Figure 4.

As concerns the discrete spectrum, it is composed of two parts. On the one hand, every λ in the discrete spectrum of L_1 also belongs to the discrete spectrum of L_α (recall that Assumption 2.2 implies that for us, there is no such λ). On the other hand, the discrete spectrum may contain a countable set of eigenvalues in the domain $\{\gamma\in\mathbb{C},\, 2\arg(\alpha_\infty)<\arg(k_\infty^2+\gamma)\leq 0\}$. The function $w(x,y)=e^{\lambda x}\varphi(y)$, where $\gamma=\lambda^2$ belongs to that set and φ is a corresponding eigenvector of L_α, is called a *leaky mode*. A coarse interpretation of those leaky modes is that they mimic the role of the radiation modes in the absence of the PMLs. They actually teach us about the preferred ways for the energy to radiate in the sheath of the waveguide: they are kinds of "harmonic states of leakage". According to [Oliner, 1984], the pioneering ideas about the concept of leaky modes date back to the 1930's in the context of electromagnetism, but the first sound theoretical basis is due to [Marcuvitz, 1956]. Since then, the concept has been widely used for the description of wave propagation in open waveguides, especially in optics (see, e.g., the survey paper by [Hu & Menyuk, 2009]), but also in other fields of application.

Coming back to Figure 4, we can understand that the motion of the essential spectrum of L_α as $\arg(\alpha_\infty)$ decreases from 0 *unveils* a "hidden" region of the complex plane (Riemann sheet) where the leaky modes live. This region is represented in gray in the bottom-left part of Figure 3, which also shows the unveiled leaky modes.

2.4 Truncating PMLs

For numerical purpose, we need to consider PMLs of finite thickness, which will from now on occupy the domain $\mathbb{R}\times((-h_{\rm out},-h_{\rm in})\cup(h_{\rm in},h_{\rm out}))$ for $h_{\rm out}>h_{\rm in}$. We have then obtained a closed waveguide $\Omega_{\rm out}:=\mathbb{R}\times I_{\rm out}$, with $I_{\rm out}:=(-h_{\rm out},h_{\rm out})$, on the boundary of which we arbitrarily impose a Neumann boundary condition (the leaky modes within the PMLs are exponentially decaying in the transverse direction). The shear modulus and the density are given in the truncated domain $\Omega_{\rm out}$ by

$$(\mu,\rho)(y):=\begin{cases}(\mu_0,\rho_0) & \text{if} \quad |y|<h,\\ (\mu_\infty,\rho_\infty) & \text{if} \quad h<|y|<h_{\rm out},\end{cases} \tag{2.10}$$

where the PML complex function α is defined in $\Omega_{\rm out}$ by

$$\alpha(y):=\begin{cases}1 & \text{if} \quad |y|<h_{\rm in},\\ \alpha_\infty & \text{if} \quad h_{\rm in}<|y|<h_{\rm out},\end{cases} \tag{2.11}$$

where α_∞ is a complex constant satisfying (2.9). We hence consider the solutions u of the form $u(x,y)=e^{\lambda x}\varphi(y)$ to the problem

$$\begin{cases}-\alpha\partial_y(\alpha\mu\partial_y u)-\mu\partial_{xx}u-\omega^2\rho u=0 & \text{in } \Omega_{\rm out}\\ \partial_\nu u=0 & \text{on } \partial\Omega_{\rm out},\end{cases} \tag{2.12}$$

where ν is the outward unit normal vector to $\partial\Omega_{\rm out}$. This amounts to searching the numbers $\gamma:=\lambda^2$ and the functions φ respectively as eigenvalues and eigenvectors of the unbounded transverse operator

$L: D(L) \subset \mathrm{L}^2(I_{\mathrm{out}}) \to \mathrm{L}^2(I_{\mathrm{out}})$ defined by

$$\begin{cases} L\varphi := -\dfrac{\alpha}{\mu} d_y(\alpha\mu d_y\varphi) - k^2\varphi \\ D(L) := \{\varphi \in \mathrm{H}^1(I_{\mathrm{out}}),\, \alpha\mu\, d_y\varphi \in \mathrm{H}^1(I_{\mathrm{out}}),\, d_y\varphi(-h_{\mathrm{out}}) = d_y\varphi(h_{\mathrm{out}}) = 0\}. \end{cases} \tag{2.13}$$

We easily check (for more details, see Theorem 3.1 in [Goursaud, 2010]) that L is a non self-adjoint operator if $\alpha_\infty \in \mathbb{C}\setminus\mathbb{R}$ and has a compact resolvent (this last property stems fom the compactness Rellich theorem). Therefore, it has complex discrete eigenvalues. We have to distinguish two kinds of modes: those which correspond to the approximation of the leaky modes due to the truncation of the PMLs, which will still be called the *leaky modes*, and those which correspond to the discretization of the continuous spectrum, which will be called the *PML modes*. The leaky modes are localized in the physical part of the waveguide and have an intrinsic meaning, while the PML modes are localized in the PMLs and depend on the parameters of such PMLs.

The bottom-right part of Figure 3 illustrates the two kinds of modes. The leaky modes are closed to those of the infinite PMLs (the larger the PMLs, the better the approximation), wheras the PML modes compose a discrete approximation of the continuous spectrum of the infinite PMLs (the larger the PMLs, the denser the approximation).

Remark 2.2. *The function $\alpha(y)$ in the PMLs given by (2.8) and (2.9) has a jump at $y = \pm h_{\mathrm{in}}$. It hence defines the so-called abrupt PMLs. An alternative choice for the function α, already used in [Nguyen et al., 2015], is given for $|y| \in (h_{\mathrm{in}}, h_{\mathrm{out}})$, by*

$$\alpha(y) = \frac{1}{1 + b(|y| - h_{\mathrm{in}})^2/h_{\mathrm{out}}^2}, \tag{2.14}$$

with $b \in \mathbb{C}$ satisfying $\Re e(b) > 0$ and $\Im m(b) > 0$. It is important to note that all the theoretical results of the paper still hold if we choose the smooth function α given by (2.14) instead of the one given by (2.11). The proofs of Lemma 1 and Lemma 5 are then slightly impacted but essentially rely on the signs of $\Re e(\alpha)$, $\Re e(1/\alpha)$, $\Im m(\alpha)$ and $\Im m(1/\alpha)$, which are unchanged. In the proof of Lemma 2, the second-order differential equations have variable coefficients instead of constant ones, but the useful well-posedness argument is conserved. The other proofs are unchanged.

3 The case of truncated PMLs: description of the modes

In this section, we specify some important properties of the pairs (λ, φ), $\varphi \neq 0$, which satisfy (2.12). Rather than working with the unbounded operator L defined by (2.13), we now introduce the symbol $\mathscr{L}(\lambda): \mathrm{H}^1(I_{\mathrm{out}}) \to \mathrm{H}^1(I_{\mathrm{out}})^*$, where $\mathrm{H}^1(I_{\mathrm{out}})^*$ is the topological dual of $\mathrm{H}^1(I_{\mathrm{out}})$, which is defined by

$$\langle \mathscr{L}(\lambda)\varphi, \psi\rangle_{I_{\mathrm{out}}} := \int_{I_{\mathrm{out}}} \Big(\alpha\mu d_y\varphi\, d_y\psi - \frac{\mu}{\alpha}(\lambda^2 + k^2)\varphi\psi\Big)\, dy, \qquad \forall \varphi, \psi \in \mathrm{H}^1(I_{\mathrm{out}}), \tag{3.15}$$

where the bracket $\langle \cdot, \cdot \rangle_{I_{\mathrm{out}}}$ means duality between $\mathrm{H}^1(I_{\mathrm{out}})$ and $\mathrm{H}^1(I_{\mathrm{out}})^*$. This choice is dictated by the Kondratiev approach we adopt in the next section. However, the link between the operators L and $\mathscr{L}$ is straightforward by using the theory of distributions: finding the pairs $(\lambda, \varphi) \in \mathbb{C} \times \mathrm{H}^1(I_{\mathrm{out}}) \setminus \{0\}$ such that

$$\mathscr{L}(\lambda)\varphi = 0 \tag{3.16}$$

is equivalent to finding the eigenvalues $\gamma = \lambda^2$ and eigenfunctions φ of the operator L defined by (2.13). We denote by Λ the set of $\lambda \in \mathbb{C}$ for which there exists a non zero $\varphi \in \mathrm{H}^1(I_{\mathrm{out}})$ such that (λ, φ) satisfies problem (3.16). Such set is discrete since L has a compact resolvent.

First of all, we establish that the spectrum Λ does not intersect the real and imaginary axes.

Lemma 1. *We have*

$$\Lambda \cap \{\lambda \in \mathbb{C},\, \Re e(\lambda)\Im m(\lambda) = 0\} = \emptyset.$$

Proof. Let us assume that $\lambda \in \Lambda$. There exists $\varphi \in \mathrm{H}^1(I_{\mathrm{out}})$, $\varphi \neq 0$, such that for all $\psi \in \mathrm{H}^1(I_{\mathrm{out}})$,

$$\langle \mathscr{L}(\lambda)\varphi, \psi\rangle_{I_{\mathrm{out}}} = 0.$$

Choosing $\psi=\overline{\varphi}$, we have in particular

$$\int_{I_{\text{out}}}\left(\alpha\mu|d_y\varphi|^2-\frac{\mu}{\alpha}(\lambda^2+k^2)|\varphi|^2\right)dy=0. \tag{3.17}$$

Let us show the result by contradiction. Assume first that $\lambda\in\mathbb{R}$. This implies that $\lambda^2+k^2>0$ in I_{out}. Since $\arg(\alpha)\in(-\pi/2,0]$ in I_{out} from (2.8) and (2.9), we have both $\Im m(\alpha)\leq 0$ and $\Im m(-1/\alpha)\leq 0$, so that by taking the imaginary part in (3.17), we get

$$\Im m(\alpha)\mu|d_y\varphi|^2+\mu(\lambda^2+k^2)\Im m\left(-\frac{1}{\alpha}\right)|\varphi|^2=0 \quad \text{in} \quad I_{\text{out}}.$$

For $h_{\text{in}}<|y|<h_{\text{out}}$ (see Figure 2 for notations), we have $\Im m(\alpha)<0$ and $\Im m(-1/\alpha)<0$, so that $\varphi=0$ in the domain $h_{\text{in}}<|y|<h_{\text{out}}$, and $\varphi=0$ in I_{out} by a unique continuation argument. We have obtained a contradiction.

Now let us assume that $\lambda\in i\mathbb{R}$, so that $\lambda^2=-|\lambda|^2$. We have to discuss several cases. We recall that $k_\infty>k_0$ (see Assumption 2.2).

- Case $|\lambda|<k_0$: we have $\lambda^2+k^2>0$ in I_{out}, so that we can conclude the same way as before.
- Case $k_0\leq|\lambda|<k_\infty$: we have from (3.17)

$$\begin{aligned}&\int_{-h_{\text{in}}}^{h_{\text{in}}}\left(\mu|d_y\varphi|^2-\mu(-|\lambda|^2+k^2)|\varphi|^2\right)dy\\&+\int_{I_{\text{out}}\cap\{|y|>h_{\text{in}}\}}\left(\alpha\mu_\infty|d_y\varphi|^2-\frac{\mu_\infty}{\alpha}(-|\lambda|^2+k_\infty^2)|\varphi|^2\right)dy=0.\end{aligned}$$

  The first integral is purely real, so that by taking the imaginary part of the above identity and using the fact that $-|\lambda|^2+k_\infty^2>0$, we conclude by the same reasoning as before.

- Case $|\lambda|\geq k_\infty$. Starting from (3.17) and taking the real part, we obtain

$$\begin{aligned}&\int_{-h}^{h}\left(\mu_0|d_y\varphi|^2-\mu_0(-|\lambda|^2+k_0^2)|\varphi|^2\right)dy\\&+\int_{I_{\text{out}}\cap\{|y|>h\}}\left(\Re e(\alpha)\mu_\infty|d_y\varphi|^2-\mu_\infty\Re e\left(\frac{1}{\alpha}\right)(-|\lambda|^2+k_\infty^2)|\varphi|^2\right)dy=0.\end{aligned}$$

  The two functions inside the two integrals are non negative since $|\lambda|\geq k_\infty$, $\Re e(\alpha)\geq 0$ and $\Re e(1/\alpha)\geq 0$. We hence obtain that $\varphi=0$ in $(-h,h)$, and then $\varphi=0$ in I_{out} by a unique continuation argument, which is a contradiction.

The proof is complete. □

We have the following result concerning the geometric multiplicity of the eigenvalues $\lambda\in\Lambda$, which is defined following [Kozlov et al., 1997] (see §5.1.1) by $\dim\text{Ker}(\mathscr{L}(\lambda))$, where $\text{Ker}(\mathscr{L}(\lambda))=\{\varphi\in \text{H}^1(I_{\text{out}}),\mathscr{L}(\lambda)(\varphi)=0\}$.

Lemma 2. *For all $\lambda\in\Lambda$, we have* $\dim\text{Ker}(\mathscr{L}(\lambda))=1$.

Proof. We observe that there is a unique function $\Phi\in\text{Ker}(\mathscr{L}(\lambda))$ such that $\Phi(-h_{\text{out}})=1$. Indeed, denoting $\Phi_1:=\Phi|_{(-h_{\text{out}},-h_{\text{in}})}$, Φ_1 satisfies a second-order ordinary differential equation in the interval $(-h_{\text{out}},-h_{\text{in}})$ with constant coefficients and initial conditions $\Phi_1(-h_{\text{out}})=1$ and $d_y\Phi_1(-h_{\text{out}})=0$. Hence Φ_1 is uniquely defined in $(-h_{\text{out}},-h_{\text{in}})$. Next, denoting $\Phi_2:=\Phi|_{(-h_{\text{in}},-h)}$, Φ_2 also satisfies a second-order ODE in the interval $(-h_{\text{in}},-h)$ with initial conditions which are given by the transmission conditions at $y=-h_{\text{in}}$ between Φ_1 and Φ_2 as well as $d_y\Phi_1$ and $d_y\Phi_2$. Hence Φ_2 is uniquely defined in $(-h_{\text{in}},-h)$. By reproducing this reasoning, we conclude that Φ is also uniquely defined in $(-h,h)$, (h,h_{in}) and $(h_{\text{in}},h_{\text{out}})$. That $d_y\Phi(h_{\text{out}})=0$ is a consequence of the fact that $\Phi\in\text{Ker}(\mathscr{L}(\lambda))$. Then for any $\varphi\in\text{Ker}(\mathscr{L}(\lambda))$, by linearity we have $\varphi=\varphi(-h_{\text{out}})\Phi$, which implies that $\dim\text{Ker}(\mathscr{L}(\lambda))=1$. □

In what follows, we refer to the notion of *Jordan chain* of the operator pencil $\mathscr{L}(\lambda)$ corresponding to a particular eigenvalue λ as well as the notion of *algebraic multiplicity* of λ. All these definitions are given in [Kozlov et al., 1997] (see §5.1.1). We have the following result concerning the algebraic multiplicity of the eigenvalues $\lambda \in \Lambda$.

Lemma 3. *Let us consider $\lambda \in \Lambda$ and $\varphi \in \mathrm{Ker}(\mathscr{L}(\lambda))$ such that*

$$\int_{I_{\mathrm{out}}} \frac{\mu}{\alpha} \varphi^2 \, dy \neq 0.$$

Then the algebraic multiplicity of λ is 1.

Proof. Let us consider $\lambda \in \Lambda$. Proving that the algebraic multiplicity of λ is 1 amounts to proving that the algebraic and geometric multiplicities of λ coincide, in other words that the length of any Jordan chain associated with λ is 1. Assume that such a Jordan chain is longer than 1. This implies that $\mathscr{L}(\lambda)\varphi = 0$ and there exists some $\hat{\varphi} \in \mathrm{H}^1(I_{\mathrm{out}})$, $\hat{\varphi} \neq 0$, such that

$$\mathscr{L}(\lambda)\hat{\varphi} + \frac{d\mathscr{L}}{d\lambda}(\lambda)\varphi = 0,$$

that is, for all $\psi \in \mathrm{H}^1(I_{\mathrm{out}})$,

$$\int_{I_{\mathrm{out}}} \left(\alpha\mu d_y \hat{\varphi} \, d_y \psi - \frac{\mu}{\alpha}(\lambda^2 + k^2)\hat{\varphi}\psi \right) dy - 2\lambda \int_{I_{\mathrm{out}}} \frac{\mu}{\alpha} \varphi \psi \, dy = 0.$$

By choosing $\psi = \varphi$, we notice that the first integral vanishes because it coincides with $\langle \mathscr{L}(\lambda)\varphi, \hat{\varphi} \rangle_{I_{\mathrm{out}}} = 0$. Since $\lambda \neq 0$ (see Lemma 1), we obtain

$$\int_{I_{\mathrm{out}}} \frac{\mu}{\alpha} \varphi^2 \, dy = 0,$$

which contradicts the assumption on φ. □

Since the spectrum Λ is discrete and in virtue of Lemma 1, we number the elements of Λ such that

$$\Lambda = \cup_{n=0}^{+\infty} \{-\lambda_n, \lambda_n\}, \quad \cdots \leq \Re e(\lambda_{n+1}) \leq \Re e(\lambda_n) \leq \cdots \leq \Re e(\lambda_0) < 0. \tag{3.18}$$

From Lemma 2, the geometric multiplicity of each element $\pm\lambda_n$ in Λ is equal to 1: the eigenfunction corresponding to both λ_n and $(-\lambda_n)$ is denoted φ_n and is defined up to a multiplicative constant. The modes are then given by

$$w_n^{\pm}(x,y) = e^{\pm\lambda_n x} \varphi_n(y). \tag{3.19}$$

Let us introduce the following fundamental assumption on the eigenfunctions φ_n.

Assumption 3.1. *The eigenvectors φ_n are such that for all $n \in \mathbb{N}$, we have*

$$\int_{I_{\mathrm{out}}} \frac{\mu}{\alpha} \varphi_n^2 \, dy \neq 0.$$

We now establish the biorthogonality relationship.

Proposition 4. *With Assumption 3.1, the eigenvectors φ_n may be rescaled such that, for $n, m \in \mathbb{N}$, we have*

$$\int_{I_{\mathrm{out}}} \frac{\mu}{\alpha} \varphi_n \varphi_m \, dy = \delta_{mn}, \tag{3.20}$$

where $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ otherwise.

Proof. Given Assumption 3.1, it is easy to rescale the functions φ_n such that, for all $n \in \mathbb{N}$, $\int_{I_{\mathrm{out}}} \frac{\mu}{\alpha} \varphi_n^2 \, dy = 1$. Now let us consider φ_n and φ_m for $n \neq m$. The corresponding eigenvalues are $\pm\lambda_n$ and $\pm\lambda_m$, respectively, with $\lambda_n^2 \neq \lambda_m^2$. By using $\mathscr{L}(\lambda_n)\varphi_n = 0$ and choosing $\varphi = \varphi_n$ and $\psi = \varphi_m$ in (3.15), we have

$$\int_{I_{\mathrm{out}}} \left(\alpha\mu d_y \varphi_n \, d_y \varphi_m - \frac{\mu}{\alpha}(\lambda_n^2 + k^2)\varphi_n \varphi_m \right) dy = 0.$$

By using $\mathscr{L}(\lambda_m)\varphi_m = 0$ and choosing $\varphi = \varphi_m$ and $\psi = \varphi_n$ in (3.15), we have

$$\int_{I_{\text{out}}} \left(\alpha\mu d_y\varphi_m \, d_y\varphi_n - \frac{\mu}{\alpha}(\lambda_m^2 + k^2)\varphi_m \varphi_n \right) dy = 0.$$

Subtracting the two previous identities implies that

$$(\lambda_n^2 - \lambda_m^2) \int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi_n \varphi_m \, dy = 0,$$

and since $\lambda_n^2 \neq \lambda_m^2$, we get

$$\int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi_n \varphi_m \, dy = 0,$$

which completes the proof. □

Remark 3.1. *It can be noted that for any profile function α, a dispersion relationship can easily be derived, since in each interval $(-h_{\text{out}}, -h_{\text{in}})$, $(-h_{\text{in}}, -h)$, $(-h, h)$, (h, h_{in}), $(h_{\text{in}}, h_{\text{out}})$, the eigenvector φ satisfies a second order ordinary differential equation. Denoting*

$$\beta_0 := {}^{i\mathbb{R}^+}\sqrt{k_0^2 + \lambda} \quad \text{and} \quad \beta_\infty := {}^{i\mathbb{R}^+}\sqrt{k_\infty^2 + \lambda},$$

the eigenvalues λ are solutions to the equation

$$\Big(\mu_\infty \beta_\infty \tan(\beta_\infty \kappa) + \mu_0 \beta_0 \tan(\beta_0 h)\Big) \left(\mu_\infty \beta_\infty \tan(\beta_\infty \kappa) - \mu_0 \beta_0 \frac{1}{\tan(\beta_0 h)} \right) = 0, \tag{3.21}$$

where the first factor above corresponds to the symmetric modes and the second one to the antisymmetric ones, with

$$\kappa := \int_h^{h_{\text{out}}} \frac{1}{\alpha(y)} \, dy. \tag{3.22}$$

In particular, the dispersion relationship (3.21) holds when the function α is given either by (2.11) for abrupt PMLs or by (2.14) for smooth PMLs.

4 Analysis of a straight unperturbed waveguide

In this section, we consider the following problem in the straight waveguide Ω_{out}: for a source term f given in the domain Ω_{out}, find a solution u in such domain Ω_{out} to the problem

$$\begin{cases} -\partial_y(\alpha\mu\partial_y u) - \dfrac{\mu}{\alpha}\partial_{xx}u - \dfrac{\mu}{\alpha}k^2 u = f & \text{in } \Omega_{\text{out}} \\ \partial_\nu u = 0 & \text{on } \partial\Omega_{\text{out}} \\ u \text{ is outgoing.} \end{cases} \tag{4.23}$$

The problem (4.23) is of course not the problem (1.1) set in the introduction. But we first study it for at least three reasons. The first one is that the unperturbed problem (4.23) has its own interest from the physical point of view. The second one is that it will enable us to first carry out the Kondratiev approach on a simpler problem than problem (1.1). More importantly, the third reason is that Theorem 8 in the particular case when $\beta = 0$ is a key step to properly define the right Dirichlet-to-Neumann operator used to analyze the problem (1.1).

Thanks to Assumption 2.2, which prevents the existence of propagating modes in Ω_{out}, the radiation condition which characterizes *outgoing* waves reduces here to a simple decay condition at infinity, which will be imposed by searching for u in the space $\mathrm{H}^1(\Omega_{\text{out}})$. In a view to study well-posedness of problem (4.23) using the Kondratiev theory (see [Kondratiev, 1967]), we introduce the weighted Sobolev spaces.

4.1 The weighted Sobolev spaces

For $\beta\in\mathbb{R}$, let us define the space

$$\mathrm{W}^1_\beta(\Omega_{\rm out}):=\{v\in\mathcal{D}'(\Omega_{\rm out}),\quad e^{\beta x}v,\, e^{\beta x}\partial_x v,\, e^{\beta x}\partial_y v\in\mathrm{L}^2(\Omega_{\rm out})\},\tag{4.24}$$

where $\mathcal{D}'(\Omega_{\rm out})$ is the usual space of distributions, the space $\mathrm{W}^1_\beta(\Omega_{\rm out})$ being endowed with its natural norm

$$\|v\|_{\mathrm{W}^1_\beta(\Omega_{\rm out})}:=\Big(\|e^{\beta x}v\|^2_{\mathrm{L}^2(\Omega_{\rm out})}+\|e^{\beta x}\partial_x v\|^2_{\mathrm{L}^2(\Omega_{\rm out})}+\|e^{\beta x}\partial_y v\|^2_{\mathrm{L}^2(\Omega_{\rm out})}\Big)^{1/2}.$$

Observe that for $\beta=0$, we have $\mathrm{W}^1_0(\Omega_{\rm out})=\mathrm{H}^1(\Omega_{\rm out})$. We denote $\mathrm{W}^1_\beta(\Omega_{\rm out})^*$ the topological dual space of $\mathrm{W}^1_\beta(\Omega_{\rm out})$ endowed with the norm

$$\|f\|_{\mathrm{W}^1_\beta(\Omega_{\rm out})^*}:=\sup_{v\in\mathrm{W}^1_\beta(\Omega_{\rm out})\setminus\{0\}}\frac{|\langle f,\overline{v}\rangle_{\Omega_{\rm out}}|}{\|v\|_{\mathrm{W}^1_\beta(\Omega_{\rm out})}}.$$

Here $\langle\cdot,\cdot\rangle_{\Omega_{\rm out}}$ refers to the bilinear duality pairing between $\mathrm{W}^1_\beta(\Omega_{\rm out})^*$ and $\mathrm{W}^1_\beta(\Omega_{\rm out})$. For $\beta\in\mathbb{R}$, in relation with problem (4.23) we define the linear and bounded operator $A_\beta:\mathrm{W}^1_\beta(\Omega_{\rm out})\to\mathrm{W}^1_{-\beta}(\Omega_{\rm out})^*$ such that for all $(u,v)\in\mathrm{W}^1_\beta(\Omega_{\rm out})\times\mathrm{W}^1_{-\beta}(\Omega_{\rm out})$,

$$\langle A_\beta u,v\rangle_{\Omega_{\rm out}}:=\int_{\Omega_{\rm out}}\Big(\alpha\mu\partial_y u\partial_y v+\frac{\mu}{\alpha}\partial_x u\partial_x v-\frac{\mu}{\alpha}k^2uv\Big)\,dxdy.\tag{4.25}$$

We observe that the boundedness of the operator A_β stems from the fact that all the functions α, α^{-1}, μ and k, which are piecewise constant in the transverse section, are bounded.

In what follows we will need the Fourier-Laplace transform $\mathcal{F}_{x\to\lambda}$ which is given, for well chosen $\lambda\in\mathbb{C}$, by

$$\hat{v}(\lambda)=(\mathcal{F}_{x\to\lambda}v)(\lambda):=\int_{-\infty}^{+\infty}e^{-\lambda x}v(x)\,dx.$$

Let us recall some basic properties of the Fourier-Laplace transform, which inherit from the properties of the Fourier and the Laplace transforms. Denoting $\mathcal{S}'(\mathbb{R})$ the usual space of tempered distributions, for any $\beta\in\mathbb{R}$ and $v\in\mathcal{S}'_\beta(\mathbb{R}):=\{v\in\mathcal{D}'(\mathbb{R}),e^{\beta x}v\in\mathcal{S}'(\mathbb{R})\}$, $\hat{v}(\lambda)$ is well-defined for $\lambda\in\ell_{-\beta}$, where $\ell_{-\beta}:=\{\lambda=-\beta+is,\,s\in\mathbb{R}\}$. In addition, the following formula applies to the derivative of $v\in\mathcal{S}'_\beta(\mathbb{R})$ on $\ell_{-\beta}$:

$$\widehat{v'}(\lambda)=\lambda\hat{v}(\lambda).\tag{4.26}$$

Furthermore, $\mathcal{F}_{x\to\lambda}$ is an isomorphism between $\mathrm{L}^2_\beta(\mathbb{R}):=\{v\in\mathcal{D}'(\mathbb{R}),e^{\beta x}v\in\mathrm{L}^2(\mathbb{R})\}$ and $\mathrm{L}^2(\ell_{-\beta})$, equipped with the norms

$$v\mapsto\|e^{\beta x}v\|_{\mathrm{L}^2(\mathbb{R})}\quad\text{and}\quad\hat{v}\mapsto\sqrt{\frac{1}{i}\int_{\ell_{-\beta}}|\hat{v}(\lambda)|^2\,d\lambda},$$

respectively, with the Plancherel identity

$$\|v\|^2_{\mathrm{L}^2_\beta(\mathbb{R})}=\frac{1}{2\pi i}\int_{\ell_{-\beta}}|\hat{v}(\lambda)|^2\,d\lambda.$$

The reader will note the presence of the complex number i in the Plancherel identity, having in mind that the path integral above is a purely imaginary number. Besides, the inverse $\mathcal{F}^{-1}_{x\to\lambda}$ is given by

$$\mathcal{F}^{-1}_{x\to\lambda}\hat{v}(x)=\frac{1}{2\pi i}\int_{\ell_{-\beta}}e^{\lambda x}\hat{v}(\lambda)\,d\lambda.$$

Lastly, for any $\beta_1<\beta_2\in\mathbb{R}$, for $v\in\mathcal{S}'_{\beta_1}(\mathbb{R})\cap\mathcal{S}'_{\beta_2}(\mathbb{R})$, the function $\hat{v}$ is holomorphic in the strip $\{\lambda\in\mathbb{C},-\beta_2<\Re e\lambda<-\beta_1\}$.

For functions of both variables x and y, we can define the partial Fourier-Laplace transform with respect to x, still denoted by $\mathcal{F}_{x\to\lambda}$, using the above lines with obvious changes. With the help of the above properties, we show that it is an isomorphism between

$$\mathrm{W}^1_\beta(\Omega_{\rm out})\quad\text{and}\quad\widehat{\mathrm{W}}^1_\beta:=\Big\{\hat{v}\in\mathrm{L}^2(\ell_{-\beta},\mathrm{H}^1(I_{\rm out})),\,\frac{1}{i}\int_{\ell_{-\beta}}\|\hat{v}(\lambda,\cdot)\|^2_{\mathrm{H}^1(I_{\rm out},|\lambda|)}\,d\lambda<+\infty\Big\},$$

for all $\beta\in\mathbb{R}$, where

$$\|\varphi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}:=\Big((1+|\lambda|^2)\|\varphi\|^2_{\mathrm{L}^2(I_{\mathrm{out}})}+\|d_y\varphi\|^2_{\mathrm{L}^2(I_{\mathrm{out}})}\Big)^{1/2},\qquad \forall\varphi\in\mathrm{H}^1(I_{\mathrm{out}}). \tag{4.27}$$

Note that for a fixed λ, the norms $\|\cdot\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}$ and $\|\cdot\|_{\mathrm{H}^1(I_{\mathrm{out}})}$ are equivalent on $\mathrm{H}^1(I_{\mathrm{out}})$. However the constants which characterize this equivalence depend on $|\lambda|$. The choice of the above norm is justified by the fact that thanks to (4.26), we have the following Plancherel formula:

$$\|v\|^2_{\mathrm{W}^1_\beta(\Omega_{\mathrm{out}})}=\frac{1}{2\pi i}\int_{\ell_{-\beta}}\|\hat{v}(\lambda,\cdot)\|^2_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}\,d\lambda=:\|\hat{v}\|^2_{\widehat{\mathrm{W}}^1_\beta}. \tag{4.28}$$

Let us denote $\widehat{\mathrm{W}}^{1*}_\beta$ the topological dual space of $\widehat{\mathrm{W}}^1_\beta$ which can be characterized as

$$\widehat{\mathrm{W}}^{1*}_\beta=\Big\{\hat{g}\in\mathrm{L}^2(\ell_\beta,\mathrm{H}^1(I_{\mathrm{out}})^*),\ \frac{1}{i}\int_{\ell_\beta}\|\hat{g}(\lambda,\cdot)\|^2_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}\,d\lambda<+\infty\Big\}$$

where

$$\|g\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}:=\sup_{\varphi\in\mathrm{H}^1(I_{\mathrm{out}})\setminus\{0\}}\frac{|\langle g,\overline{\varphi}\rangle_{I_{\mathrm{out}}}|}{\|\varphi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}},\qquad\forall g\in\mathrm{H}^1(I_{\mathrm{out}})^*. \tag{4.29}$$

The Fourier-Laplace Transform $\mathcal{F}_{x\to\lambda}$ can be defined by duality for functions in $\mathrm{W}^1_\beta(\Omega_{\mathrm{out}})^*$ as

$$\forall f\in\mathrm{W}^1_\beta(\Omega_{\mathrm{out}})^*,\forall\hat{v}\in\widehat{\mathrm{W}}^1_\beta,\quad\langle\mathcal{F}_{x\to\lambda}f,\hat{v}\rangle_{\hat{\Omega}_{\mathrm{out}}}:=\langle f,\mathcal{F}^{-1}_{x\to\lambda}\hat{v}\rangle_{\Omega_{\mathrm{out}}},$$

where $\langle\cdot,\cdot\rangle_{\hat{\Omega}_{\mathrm{out}}}$ refers to the duality pairing between $\widehat{\mathrm{W}}^{1*}_\beta$ and $\widehat{\mathrm{W}}^1_\beta$. Finally, we have also a Plancherel formula

$$\|f\|^2_{\mathrm{W}^1_\beta(\Omega_{\mathrm{out}})^*}=\frac{1}{2\pi i}\int_{\ell_\beta}\|\hat{f}(\lambda,\cdot)\|^2_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}\,d\lambda. \tag{4.30}$$

4.2 Reduction to problems in one dimension

By applying $\mathcal{F}_{x\to\lambda}$ to the equation $A_\beta u=f$ and with the help of the basic property (4.26), one obtains that $\mathscr{L}(\lambda)\hat{u}(\lambda,\cdot)=\hat{f}(\lambda,\cdot)$ for $\lambda\in\ell_{-\beta}$, so that one is naturally led to study the symbol $\mathscr{L}(\lambda)$ defined by (3.15). In the following lemma, we establish estimates in the λ dependent norms of $\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)$ and $\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*$.

Lemma 5. *There is $\tau_0>0$ such that for $\lambda=i\tau$, $\tau\in\mathbb{R}$ with $|\tau|\geq\tau_0$, $\mathscr{L}(\lambda):\mathrm{H}^1(I_{\mathrm{out}})\to\mathrm{H}^1(I_{\mathrm{out}})^*$ is an isomorphism. Moreover, if $\varphi\in\mathrm{H}^1(I_{\mathrm{out}})$ satisfies $\mathscr{L}(\lambda)\varphi=g\in\mathrm{H}^1(I_{\mathrm{out}})^*$, then there holds*

$$\|\varphi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}\leq C\,\|g\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}, \tag{4.31}$$

where $C>0$ is independent of g and λ.

Proof. For $\lambda=i\tau$, we have for all φ and ψ in $\mathrm{H}^1(I_{\mathrm{out}})$,

$$\langle\mathscr{L}(i\tau)\varphi,\overline{\psi}\rangle_{I_{\mathrm{out}}}=\int_{I_{\mathrm{out}}}\alpha\mu\,d_y\varphi\,d_y\overline{\psi}+\frac{\mu}{\alpha}(\tau^2-k^2)\varphi\overline{\psi}\,dy.$$

The proof relies on the Lax-Milgram Lemma. We know that in the waveguide Ω_{out}, we have $-\frac{\pi}{2}<\arg(\alpha)\leq 0$, and that α is bounded from above and bounded away from 0 in Ω_{out}. Writing that

$$\alpha=|\alpha|e^{i\theta}\quad\text{and}\quad\frac{1}{\alpha}=\frac{1}{|\alpha|}e^{-i\theta},$$

with $-\pi/2<\theta\leq 0$, there exists a constant $C>0$ such that

$$\Re e(\alpha)\geq C\quad\text{and}\quad\Re e(1/\alpha)\geq C.$$

If $\tau\geq k_\infty$, that $k_0<k_\infty$ implies that for all $\varphi\in\mathrm{H}^1(I_{\mathrm{out}})$,

$$\Re e(\langle\mathscr{L}(i\tau)\varphi,\overline{\varphi}\rangle_{I_{\mathrm{out}}})\geq C\min(\mu_0,\mu_\infty)\int_{I_{\mathrm{out}}}|d_y\varphi|^2+(\tau^2-k_\infty^2)|\varphi|^2\,dy.$$

For some given $\gamma\in(0,1)$, we have $\tau^2-k_\infty^2\geq\gamma(1+\tau^2)$ for τ large enough, more precisely if $\tau\geq\tau_0:=\sqrt{(k_\infty^2+\gamma)/(1-\gamma)}$. We obtain that for all $\varphi\in\mathrm{H}^1(I_{\mathrm{out}})$,

$$\Re e(\langle\mathscr{L}(i\tau)\varphi,\overline{\varphi}\rangle_{I_{\mathrm{out}}})\geq C\gamma\min(\mu_0,\mu_\infty)\int_{I_{\mathrm{out}}}|d_y\varphi|^2+(1+\tau^2)|\varphi|^2\,dy.$$

The estimate (4.31) follows from the above estimate, the fact that $\tau^2=|\lambda|^2$ and the very definitions of the norms of $\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)$ and $\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*$ given by (4.27) and (4.29), respectively. □

From Lemma 5, we deduce the following result with the help of the Fredholm alternative.

Corollary 6. *For all $\lambda\in\mathbb{C}$, $\mathscr{L}(\lambda):\mathrm{H}^1(I_{\mathrm{out}})\to\mathrm{H}^1(I_{\mathrm{out}})^*$ is an isomorphism if and only if $\lambda\notin\Lambda$.*

Remark 4.1. *We also retrieve from the analytic Fredholm theorem exposed in [Kozlov & Maz'ya, 1999] (see Proposition A.8.4) that the set Λ is discrete and does not have any accumulation point in $\mathbb{C}$.*

In order to apply the inverse Fourier-Laplace transform and use Plancherel formulas, we need estimates for $\mathscr{L}(\lambda)^{-1}$ on lines ℓ_β, $\beta\in\mathbb{R}$, in the parameter dependent norms (4.27), (4.29).

Lemma 7. *There are real positive constants ρ and δ such that for all $\lambda\in\mathbb{C}$ satisfying*

$$|\lambda|>\rho\qquad\text{and}\qquad|\Re e\,\lambda|<\delta|\Im m\,\lambda|,$$

$\mathscr{L}(\lambda):\mathrm{H}^1(I_{\mathrm{out}})\to\mathrm{H}^1(I_{\mathrm{out}})^*$ is an isomorphism. Moreover, if $\varphi\in\mathrm{H}^1(I_{\mathrm{out}})$ satisfies $\mathscr{L}(\lambda)\varphi=g\in\mathrm{H}^1(I_{\mathrm{out}})^*$, then (4.31) holds (with a different constant C).

Proof. Lemma 5 ensures that $\mathscr{L}(\lambda)$ is an isomorphism and that (4.31) holds for $\lambda\in i\mathbb{R}$ with $|\lambda|\geq\tau_0$, where τ_0 is defined in the proof of Lemma 5. Now let us consider the case $\lambda\notin i\mathbb{R}$. We write λ as $\lambda=\pm i|\lambda|e^{i\theta}$ with $\theta\in(-\pi/2,\pi/2)$. Set $\tilde{\lambda}:=\pm i|\lambda|$. Since $|\tilde{\lambda}|=|\lambda|$, by definition of the parameter dependent norm (4.27), for $\varphi\in\mathrm{H}^1(I_{\mathrm{out}})$, we have $\|\varphi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}=\|\varphi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\tilde{\lambda}|)}$. Define $\tilde{g}:=\mathscr{L}(\tilde{\lambda})\varphi$. Assume that $|\lambda|>\rho:=\tau_0$. In that case, according to the first step of the proof, we have

$$\|\varphi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}=\|\varphi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\tilde{\lambda}|)}\leq C\|\tilde{g}\|_{\mathrm{H}^1(I_{\mathrm{out}},|\tilde{\lambda}|)^*}.\tag{4.32}$$

Here and in what follows, $C>0$ is a constant which can change from one line to another but which is independent of λ, φ. Now we can write

$$\|\tilde{g}\|_{\mathrm{H}^1(I_{\mathrm{out}},|\tilde{\lambda}|)^*}=\|\tilde{g}\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}\leq\|g\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}+\|\tilde{g}-g\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}.\tag{4.33}$$

Besides, for all $\psi\in\mathrm{H}^1(I_{\mathrm{out}})$,

$$\langle\tilde{g}-g,\overline{\psi}\rangle_{I_{\mathrm{out}}}=\langle\mathscr{L}(\tilde{\lambda})\varphi-\mathscr{L}(\lambda)\varphi,\overline{\psi}\rangle_{I_{\mathrm{out}}}=(\lambda^2-\tilde{\lambda}^2)\int_{I_{\mathrm{out}}}\frac{\mu}{\alpha}\varphi\overline{\psi}\,dy.$$

In view of (4.29) and using that $|\lambda|\|\psi\|_{\mathrm{L}^2(I_{\mathrm{out}})}\leq\|\psi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}$, we deduce that

$$\|\tilde{g}-g\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}\leq C\frac{1}{|\lambda|}|\tilde{\lambda}^2-\lambda^2|\,\|\varphi\|_{\mathrm{L}^2(I_{\mathrm{out}})}\leq C|e^{2i\theta}-1|\,\|\varphi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}.$$

Thus for all $\varsigma>0$, there is δ small enough so that one has $\|\tilde{g}-g\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}\leq\varsigma\|\varphi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}$ for all $\lambda=\pm i|\lambda|e^{i\theta}$ such that $|\theta|<\delta$. Gathering the latter estimate, (4.32) and (4.33) leads to

$$\|\varphi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}\leq C\|g\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}+C\varsigma\|\varphi\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)}.$$

Taking ς sufficiently small ($\varsigma=1/(2C)$ for example), finally we obtain (4.31). □

Lemma 7 gives in particular information on the location of elements of the spectrum Λ, as shown in Figure 5.

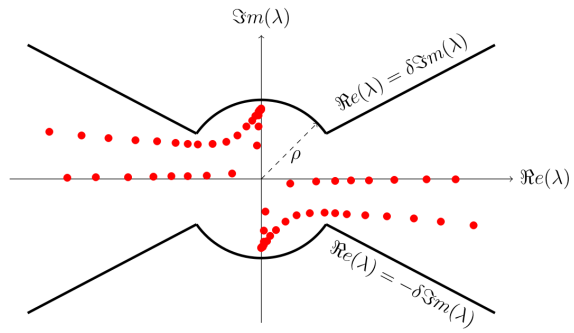

Figure 5: Location of the elements of Λ given by Lemma 7.

4.3 Well-posedness of problem (4.23)

From the previous lemma, we deduce the following result.

Theorem 8. *Let $\beta \in \mathbb{R}$ be such that Λ has no intersection with the line $-\beta + i\mathbb{R}$. Then the operator $A_\beta : \mathrm{W}^1_\beta(\Omega_{\mathrm{out}}) \to \mathrm{W}^1_{-\beta}(\Omega_{\mathrm{out}})^*$ defined in (4.25) is an isomorphism.*

Proof. Assume that $\mathscr{L}$ has no eigenvalue on the line $-\beta + i\mathbb{R}$. Let us first suppose that $u \in \mathrm{W}^1_\beta(\Omega_{\mathrm{out}})$ is such that $A_\beta u = 0$. Applying the partial Fourier-Laplace transform with respect to x, we obtain

$$\mathscr{L}(\lambda)\hat{u}(\lambda,\cdot) = 0, \qquad \text{for} \quad a.e. \quad \lambda \in \ell_{-\beta}.$$

From Corollary 6, we deduce that for almost every $\lambda \in \ell_{-\beta}$, $\hat{u}(\lambda,\cdot) = 0$. From the properties of the inverse Fourier-Laplace transform, we deduce that $u \equiv 0$. This shows that A_β is injective.

We prove now that A_β is onto. Let $f \in \mathrm{W}^1_{-\beta}(\Omega_{\mathrm{out}})^*$. First of all, the operator $\mathscr{L}(\lambda)$ is invertible for all $\lambda \in \ell_{-\beta}$ according to Corollary 6. In addition, Lemma 7 guarantees that there exist $\nu_\beta > 0$ (which only depends on β) and $C > 0$ such that for $\lambda \in \mathbb{C}$ such that $\Re e\,\lambda = -\beta$ and $|\Im m\,\lambda| \geq \nu_\beta$, we have the estimate

$$\|\mathscr{L}(\lambda)^{-1}\hat{f}(\lambda,\cdot)\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)} \leq C\,\|\hat{f}(\lambda,\cdot)\|_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}. \tag{4.34}$$

For $\lambda \in -\beta + i[-\nu_\beta, \nu_\beta]$, the continuity of $\lambda \mapsto \mathscr{L}(\lambda)^{-1}$, ensured by the analytic Fredholm theorem, guarantees that the estimate (4.34) also holds for λ in the compact set $-\beta + i[-\nu_\beta, \nu_\beta]$. Therefore (4.34) is valid for all $\lambda \in \ell_{-\beta}$ with a constant $C > 0$ independent of λ.
By definition

$$f \in \mathrm{W}^1_{-\beta}(\Omega_{\mathrm{out}})^* \quad \Rightarrow \quad \frac{1}{2\pi i}\int_{\ell_{-\beta}} \|\hat{f}(\lambda,\cdot)\|^2_{\mathrm{H}^1(I_{\mathrm{out}},|\lambda|)^*}\,d\lambda < +\infty.$$

We deduce that

$$u(x,y) = \frac{1}{2\pi i}\int_{\ell_{-\beta}} e^{\lambda x}\mathscr{L}(\lambda)^{-1}\hat{f}(\lambda,y)\,d\lambda \quad \in \quad \mathrm{W}^1_\beta(\Omega_{\mathrm{out}}) \tag{4.35}$$

is solution of $A_\beta u = f$ with, by the Plancherel formulas (4.28) and (4.30),

$$\|u\|_{\mathrm{W}^1_\beta(\Omega_{\mathrm{out}})} \leq C\|f\|_{\mathrm{W}^1_{-\beta}(\Omega_{\mathrm{out}})^*},$$

which completes the proof. □

We have proved that $\Lambda \cap i\mathbb{R} = \emptyset$ (see Lemma 1). From Theorem 8, we deduce the following corollary.

Corollary 9. *The operator A_0 is an isomorphism from $\mathrm{H}^1(\Omega_{\mathrm{out}})$ to $\mathrm{H}^1(\Omega_{\mathrm{out}})^*$. In other words, for $f \in \mathrm{H}^1(\Omega_{\mathrm{out}})^*$, the problem (4.23) has a unique solution u in $\mathrm{H}^1(\Omega_{\mathrm{out}})$.*

4.4 Behaviour at infinity of the solutions to problem (4.23)

In the following proposition, we compare two solutions u_1 and u_2 which are associated with two different operators A_{β_1} and A_{β_2}, with $\beta_1<\beta_2$. From Lemma 7 and Remark 4.1, the intersection of Λ with the strip $\{\lambda\in\mathbb{C},\, -\beta_2<\Re e\lambda<-\beta_1\}$ is a finite set.

Proposition 10. *Let us consider $\beta_1<\beta_2\in\mathbb{R}$ such that $\Lambda\cap(\ell_{-\beta_1}\cup\ell_{-\beta_2})=\emptyset$. The intersection of Λ with the strip $\{\lambda\in\mathbb{C},\, -\beta_2<\Re e\lambda<-\beta_1\}$ is supposed to be non empty and is denoted $\{\mu_1,\mu_2,\dots,\mu_N\}$.*

Let $f\in \mathrm{W}^1_{-\beta_1}(\Omega_{\mathrm{out}})^*\cap \mathrm{W}^1_{-\beta_2}(\Omega_{\mathrm{out}})^*$ and let us denote $u_1:=A_{\beta_1}^{-1}f\in \mathrm{W}^1_{\beta_1}(\Omega_{\mathrm{out}})$ and $u_2:=A_{\beta_2}^{-1}f\in \mathrm{W}^1_{\beta_2}(\Omega_{\mathrm{out}})$. There exist complex numbers $\{c_j\}_{j=1}^N$ such that

$$u_2=\sum_{j=1}^N c_j w_j+u_1,$$

where $w_j(x,y):=e^{\mu_j x}\psi_j(y)$, with $\psi_j\in\mathrm{Ker}(\mathscr{L}(\mu_j))$.

Proof. As $f\in \mathrm{W}^1_{-\beta_1}(\Omega_{\mathrm{out}})^*\cap \mathrm{W}^1_{-\beta_2}(\Omega_{\mathrm{out}})^*$, one can show that $\hat f(\lambda,\cdot)$ is holomorphic in the strip $\{-\beta_2<\Re e\lambda<-\beta_1\}$. The only singularities of the function $e^{\lambda x}\mathscr{L}(\lambda)^{-1}\hat f(\lambda,\cdot)$ in $\{-\beta_2<\Re e\lambda<-\beta_1\}$ are the poles of $\mathscr{L}(\lambda)^{-1}$, *i.e.* the elements of Λ.
Let ρ be sufficiently large so that

$$\Lambda\cap\{\lambda\in\mathbb{C},\, -\beta_2<\Re e\lambda<-\beta_1\}\subset\{\lambda\in\mathbb{C},\, -\beta_2<\Re e\lambda<-\beta_1,\, |\Im m\,\lambda|<\rho\}.$$

Using the Residue Theorem, we get from (4.35) that

$$\begin{aligned}u_2(x,\cdot)&=\frac{1}{2i\pi}\lim_{\rho\to+\infty}\int_{-\beta_2-i\rho}^{-\beta_2+i\rho}e^{\lambda x}\mathscr{L}(\lambda)^{-1}\hat f(\lambda,\cdot)\,d\lambda\\&=\frac{1}{2i\pi}\lim_{\rho\to+\infty}\left[\int_{-\beta_1-i\rho}^{-\beta_1+i\rho}\dots d\lambda+\int_{-\beta_2-i\rho}^{-\beta_1-i\rho}\dots d\lambda-\int_{-\beta_2+i\rho}^{-\beta_1+i\rho}\dots d\lambda\right]\\&\qquad+\sum_{j=1}^N\mathrm{Res}\big(e^{\lambda x}\mathscr{L}(\lambda)^{-1}\hat f(\lambda,\cdot),\mu_j\big).\end{aligned}$$

The first integral tends to u_1 in $\mathrm{W}^1_{\beta_1}(\Omega_{\mathrm{out}})$ from the very definition of the Fourier-Laplace transform on $\ell_{-\beta_1}$. For the second and the third ones, it suffices to extend [Kozlov et al., 1997] (see Lemma 5.4.1) to state that for all M

$$\left\|\int_{-\beta_2+i\rho}^{-\beta_1+i\rho}e^{\lambda x}\mathscr{L}(\lambda)^{-1}\hat f(\lambda,\cdot)d\lambda\right\|_{\mathrm{L}^2(\Omega_{\mathrm{out}}^M)}\underset{\rho\to+\infty}{\longrightarrow}0,$$

where $\Omega_{\mathrm{out}}^M:=\Omega_{\mathrm{out}}\cap\{-M\le x\le M\}$. We observe that $\mathrm{L}^2(\Omega_{\mathrm{out}}^M)$ contains the restrictions to Ω_{out}^M of functions in $\mathrm{W}^1_{\beta_1}(\Omega_{\mathrm{out}})$ and functions in $\mathrm{W}^1_{\beta_2}(\Omega_{\mathrm{out}})$. As a consequence, the sum of the three integrals converges to u_1 in $\mathrm{L}^2(\Omega_{\mathrm{out}}^M)$. We conclude that the equality holds in $\mathrm{L}^2(\Omega_{\mathrm{out}}^M)$ and then in $\mathrm{W}^1_{\beta_2}(\Omega_{\mathrm{out}})$ since $u_2\in\mathrm{W}^1_{\beta_2}(\Omega_{\mathrm{out}})$. To compute the residues, we apply [Kozlov et al., 1997] (see Theorem 5.1.1). Since the geometric and algebraic multiplicities of all eigenvalues μ_j are equal to 1 (see Lemma 3), we get that in the neighborhood of μ_j,

$$\mathscr{L}(\lambda)^{-1}=\frac{P_j}{\lambda-\mu_j}+\mathcal{U}_j(\lambda)$$

where $P_j:\mathrm{H}^1(I_{\mathrm{out}})^*\to\mathrm{H}^1(I_{\mathrm{out}})$ is the projector on ψ_j and $\mathcal{U}_j:\mathrm{H}^1(I_{\mathrm{out}})^*\to\mathrm{H}^1(I_{\mathrm{out}})$ is an holomorphic operator function in a neighborhood of μ_j. □

In order to quantify the exponentially growing or decaying behaviours as $x\to\pm\infty$, for $\beta\in\mathbb{R}$, let us introduce the weighted Sobolev space

$$\mathcal{W}^1_\beta(\Omega_{\mathrm{out}}):=\{v\in\mathcal{D}'(\Omega_{\mathrm{out}}),\quad e^{\beta|x|}v,\, e^{\beta|x|}\partial_x v,\, e^{\beta|x|}\partial_y v\in\mathrm{L}^2(\Omega_{\mathrm{out}})\},\tag{4.36}$$

endowed with its natural norm

$$\|v\|_{\mathcal{W}^1_\beta(\Omega_{\rm out})} := \Big(\|e^{\beta|x|}v\|^2_{\mathrm{L}^2(\Omega_{\rm out})} + \|e^{\beta|x|}\partial_x v\|^2_{\mathrm{L}^2(\Omega_{\rm out})} + \|e^{\beta|x|}\partial_y v\|^2_{\mathrm{L}^2(\Omega_{\rm out})} \Big)^{1/2}.$$

Remark the absolute value in the weight $e^{\beta|x|}$, which implies that

$$\beta_1 \leq \beta_2 \qquad \Rightarrow \qquad \mathcal{W}^1_{\beta_2}(\Omega_{\rm out}) \subset \mathcal{W}^1_{\beta_1}(\Omega_{\rm out}). \tag{4.37}$$

Note that this property is not true for the spaces $\mathrm{W}^1_\beta(\Omega_{\rm out})$ introduced in (4.24). Observe also that we have $\mathcal{W}^1_0(\Omega_{\rm out}) = \mathrm{H}^1(\Omega_{\rm out})$. Let $\langle \cdot,\cdot \rangle_{\Omega_{\rm out}}$ stand for the bilinear duality pairing between $\mathcal{W}^1_\beta(\Omega_{\rm out})^*$ and $\mathcal{W}^1_\beta(\Omega_{\rm out})$, where $\mathcal{W}^1_\beta(\Omega_{\rm out})^*$ is the topological dual space of $\mathcal{W}^1_\beta(\Omega_{\rm out})$ endowed with the norm

$$\|f\|_{\mathcal{W}^1_\beta(\Omega_{\rm out})^*} = \sup_{v \in \mathcal{W}^1_\beta(\Omega_{\rm out})\setminus\{0\}} \frac{|\langle f, \overline{v} \rangle_{\Omega_{\rm out}}|}{\|v\|_{\mathcal{W}^1_\beta(\Omega_{\rm out})}}. \tag{4.38}$$

Due to (4.37), we have

$$\beta_1 \leq \beta_2 \qquad \Rightarrow \qquad \mathcal{W}^1_{\beta_1}(\Omega_{\rm out})^* \subset \mathcal{W}^1_{\beta_2}(\Omega_{\rm out})^*. \tag{4.39}$$

In what follows, we introduce the cut-off functions $\chi^\pm \in \mathscr{C}^\infty(\mathbb{R}^2)$ which are equal to one for $\pm x \geq 2L$ and to zero for $\pm x \leq L$, for a given $L > 0$. We have the following theorem.

Theorem 11. *Let us consider $\beta > 0$ such that $\Lambda \cap \ell_{-\beta} = \emptyset$ and let us assume that the set $\Lambda \cap \{\lambda \in \mathbb{C}, -\beta < \Re e\lambda < 0\}$ is non empty and denoted $\{\lambda_0, \lambda_1, \ldots, \lambda_{N_\beta - 1}\}$, where the λ_n's are defined by (3.18).*

For $f \in \mathcal{W}^1_{-\beta}(\Omega_{\rm out})^* \subset \mathrm{H}^1(\Omega_{\rm out})^*$, let us denote $u \in \mathrm{H}^1(\Omega_{\rm out})$ the unique solution to the problem $A_0 u = f$ given by Corollary 9. There exist some complex numbers a_n^+, a_n^- and a function $\tilde{u} \in \mathcal{W}^1_\beta(\Omega_{\rm out})$ such that

$$u = \chi^+ \sum_{n=0}^{N_\beta - 1} a_n^+ w_n^+ + \chi^- \sum_{n=0}^{N_\beta - 1} a_n^- w_n^- + \tilde{u},$$

where we recall that $w_n^\pm(x,y) = e^{\pm\lambda_n x}\varphi_n(y)$ with $\varphi_n \in \mathrm{Ker}(\mathscr{L}(\pm\lambda_n))$.

Proof. Since $f \in \mathcal{W}^1_{-\beta}(\Omega_{\rm out})^* \subset \mathrm{W}^1_{-\beta}(\Omega_{\rm out})^*$, by using Theorem 8, there exists a unique $u_\beta \in \mathrm{W}^1_\beta(\Omega_{\rm out})$ such that $A_\beta u_\beta = f$. By Proposition 10, we have that there exist some complex numbers a_n^+ such that

$$u = \sum_{n=0}^{N_\beta - 1} a_n^+ w_n^+ + u_\beta.$$

Similarly, since $f \in \mathcal{W}^1_{-\beta}(\Omega_{\rm out})^* \subset \mathrm{W}^1_\beta(\Omega_{\rm out})^*$, there exists a unique $u_{-\beta} \in \mathrm{W}^1_{-\beta}(\Omega_{\rm out})$ such that $A_{-\beta} u_{-\beta} = f$. The set Λ is symmetrical with respect to 0, we have $\Lambda \cap \{\lambda \in \mathbb{C}, 0 < \Re e\lambda < \beta\} = \{-\lambda_0, -\lambda_1, \ldots, -\lambda_{N_\beta - 1}\}$, hence there exist some complex numbers a_n^- such that

$$u = \sum_{n=0}^{N_\beta - 1} a_n^- w_n^- + u_{-\beta}.$$

It remains to write

$$\begin{aligned} u &= \chi^+ u + \chi^- u + (1 - \chi^+ - \chi^-) u \\ &= \chi^+ \sum_{n=0}^{N_\beta - 1} a_n^+ w_n^+ + \chi^- \sum_{n=0}^{N_\beta - 1} a_n^- w_n^- + \tilde{u}, \end{aligned}$$

where

$$\tilde{u} := \chi^+ u_\beta + \chi^- u_{-\beta} + (1 - \chi^+ - \chi^-) u.$$

We easily check that the three terms of the above sum belong to $\mathcal{W}^1_\beta(\Omega_{\rm out})$. □

Remark 4.2. *In particular, if λ_0 has the strictly largest real part among $\{\lambda_0, \lambda_1, \ldots, \lambda_{N_\beta - 1}\}$ (which have negative real parts), the solution u behaves asymptotically as $a_0^\pm e^{\pm\lambda_0 x}\varphi_0(y)$ for $x \to \pm\infty$. This result could seem surprising because we expect that the physical solution of the original problem (without the PMLs) decays only algebraically for $x \to \pm\infty$. This exponential decay is caused by the PMLs truncation and in view of [Chandler-Wilde & Monk, 2009] we expect that when the PMLs thickness increases, the value of $|\lambda_0|$ decreases, that is the exponential decay becomes slower. This is coherent with the fact that the solution obtained with PMLs is an approximation of the physical solution.*

5 Well-posedness of the forward problem (1.1)

Let us come back to the problem (1.1) set in the introduction and illustrated by Figure 2, that is a junction between a half-closed waveguide and a half-open waveguide which is closed by finite PMLs. We recall that $\Omega^- := (-\infty,0)\times(-h,h)$, $\Omega^+ := (0,+\infty)\times(-h_{\rm out},h_{\rm out})$, $\Sigma_0 := \{0\}\times(-h,h)$ and $\Omega := \Omega^-\cup\Sigma_0\cup\Omega^+$. We also denote $\Sigma_L := \{L\}\times(-h_{\rm out},h_{\rm out})$. We assume now that the obstacle O is a Lipschitz bounded domain. It lies within the core of the open half-waveguide with PMLs, between the transverse sections Σ_0 and Σ_L, so that $O\subset(0,L)\times(-h,h)$. Let us denote $D := \Omega\setminus\overline{O}$ and $D^+ := \Omega^+\setminus\overline{O}$. We hereafter study the scattering response of the structure D to an incident wave which is sent from the left. Let us hence define the incident wave $u^i = \tilde{w}^+_{n,0}$ in Ω as

$$\tilde{w}^+_{n,0} := \begin{cases} \tilde{w}^+_n & \text{in} \quad \Omega^-, \\ 0 & \text{in} \quad \Omega^+, \end{cases} \tag{5.40}$$

where $\tilde{w}^+_n$ for $n\in\mathbb{N}$ is a mode coming from the left closed waveguide and defined by (2.6). Let us introduce the space $\tilde{\rm H}^1_{\rm loc}(D)$ as the set of distributions v in D such that $\chi v\in {\rm H}^1(D)$, for all $\chi\in\mathscr{C}^\infty(\mathbb{R}^2)$ vanishing for sufficiently large $(-x)$ i.e., as $x\to-\infty$. Problem (1.1) can be rewritten as: find $u\in\tilde{\rm H}^1_{\rm loc}(D)$ such that

$$\begin{cases} -\Delta u - k_0^2 u = 0 & \text{in } \Omega^-, \\ -\partial_y(\alpha\mu\partial_y u) - \dfrac{\mu}{\alpha}\partial_{xx}u - \dfrac{\mu}{\alpha}k^2 u = 0 & \text{in } D^+, \\ [\![u]\!] = 0 & \text{on } \Sigma_0, \\ [\![\partial_x u]\!] = 0 & \text{on } \Sigma_0, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial O, \\ u - u^i \text{ is outgoing}, \end{cases} \tag{5.41}$$

where $u^i = \tilde{w}^+_{n,0}$ and $[\![\cdot]\!]$ denotes the jump of the solution at the interface Σ_0. As for problem (4.23), the radiation condition on the right side of D consists in the choice of a ${\rm H}^1$-type functional space in view of the absence of propagating modes. On the left-side of D, such radiation condition will consist in selecting the scattered waves which either propagate or exponentially decrease from the right to the left. In order to analyze well-posedness of problem (5.41), let us find an equivalent problem set in a bounded domain. In this view we introduce Dirichlet-to-Neumann maps on the two sections Σ_0 and Σ_L.

Firstly, let us define the Dirichlet-to-Neumann operator T_0 on Σ_0. Let us denote ${\rm H}^1_{\rm loc}(\Omega^-)$ the set of distributions v in Ω^- such that $\chi v\in{\rm H}^1(\Omega^-)$, for all $\chi\in\mathscr{C}^\infty(\mathbb{R}^2)$ vanishing for sufficiently large $(-x)$. For $\varphi\in{\rm H}^{1/2}(\Sigma_0)$, the problem set in the half-waveguide Ω^-: find $u^-(\varphi)\in{\rm H}^1_{\rm loc}(\Omega^-)$ such that

$$\begin{cases} -\Delta u^- - k_0^2 u^- = 0 & \text{in } \Omega^-, \\ \partial_\nu u^- = 0 & \text{on } \partial\Omega^-\setminus\Sigma_0, \\ u^- = \varphi & \text{on } \Sigma_0, \\ u^- \text{ is outgoing}, \end{cases} \tag{5.42}$$

is well-posed, so that we can define

$$\begin{aligned} T_0 : {\rm H}^{1/2}(\Sigma_0) \to \quad & \tilde{\rm H}^{-1/2}(\Sigma_0) \\ \varphi \mapsto \quad & T_0\varphi = -\mu_0\partial_x u^-(\varphi)|_{\Sigma_0}, \end{aligned}$$

where $u^-(\varphi)$ is the solution to problem (5.42), or equivalently

$$\varphi\mapsto T_0\varphi = \mu_0\sum_{n\in\mathbb{N}}\tilde{\lambda}_n(\varphi,\tilde{\varphi}_n)_{{\rm L}^2(\Sigma_0)}\tilde{\varphi}_n, \tag{5.43}$$

where the complex numbers $\tilde{\lambda}_n$ and the functions $\tilde{\varphi}_n$ are defined by (2.4) and (2.5), respectively. The space $\tilde{\rm H}^{-1/2}(\Sigma_0)$ here denotes the topological dual space of ${\rm H}^{1/2}(\Sigma_0)$ and coincides with the set of distributions in ${\rm H}^{-1/2}(\{0\}\times\mathbb{R})$ the support of which belongs to the closure of Σ_0. Following [Lenoir & Tounsi, 1988], well-posedness of (5.42) is easily obtained by projecting $\varphi = u|_{\Sigma_0}$ on the functions $\tilde{\varphi}_n$, which form a complete basis of ${\rm L}^2(\Sigma_0)$ (see also [Bourgeois & Lunéville, 2008]).

Secondly, we define the Dirichlet-to-Neumann operator T_L on Σ_L. Since it is not known that the functions φ_n form a complete basis of $\mathrm{L}^2(\Sigma_L)$, the operator T_L cannot be defined as was T_0. In this view, we need the following Lemma, which is a consequence of section 4.

Lemma 12. *For $\varphi \in \mathrm{H}^{1/2}(\Sigma_L)$, the problem set in the half-waveguide $\Omega_L^+ := (L,+\infty)\times(-h_{\mathrm{out}},h_{\mathrm{out}})$: find $u^+(\varphi)\in \mathrm{H}^1(\Omega_L^+)$ such that*

$$\begin{cases} -\partial_y(\alpha\mu\partial_y u^+) - \dfrac{\mu}{\alpha}\partial_{xx}u^+ - \dfrac{\mu}{\alpha}k^2 u^+ = 0 & \text{in } \Omega_L^+, \\ \partial_\nu u^+ = 0 & \text{on } \partial\Omega_L^+\setminus\Sigma_L, \\ u^+ = \varphi & \text{on } \Sigma_L \end{cases} \tag{5.44}$$

has a unique solution and there exists a constant $C>0$ independent on φ such that

$$\|u^+(\varphi)\|_{\mathrm{H}^1(\Omega_L^+)} \le C\,\|\varphi\|_{\mathrm{H}^{1/2}(\Sigma_L)}, \quad \forall \varphi \in \mathrm{H}^{1/2}(\Sigma_L).$$

Proof. The idea is to deduce the statement of the Lemma from the well-posedness of problem (4.23) using a symmetry argument. To do so, we first have to notice that for any $\varphi\in\mathrm{H}^{1/2}(\Sigma_L)$, there exists a function $U_\varphi^+\in\mathrm{H}^1(\Omega_L^+)$ such that $U_\varphi^+|_{\Sigma_L}=\varphi$ and $\partial_\nu U_\varphi^+|_{\partial\Omega_L^+\setminus\Sigma_L}=0$. Hence, instead of solving (5.44), we can equivalently search for a function $u_0^+ := u^+ - U_\varphi^+ \in \mathrm{H}^1(\Omega_L^+)$ which satisfies

$$\begin{cases} Pu_0^+ = -PU_\varphi^+ & \text{in } \Omega_L^+, \\ \partial_\nu u_0^+ = 0 & \text{on } \partial\Omega_L^+\setminus\Sigma_L, \\ u_0^+ = 0 & \text{on } \Sigma_L, \end{cases} \tag{5.45}$$

where we recall that P is the differential operator given by (1.2). We show below that thanks to the homogeneous boundary condition on Σ_L, this problem amounts to searching for solutions to (4.23) which are antisymmetric with respect to Σ_L.

Let us introduce some notations. We denote by Ω_L^- the half-waveguide which is symmetric to Ω_L^+ with respect to section Σ_L. We thus have $\Omega_L^-\cup\Sigma_L\cap\Omega_L^+=\Omega_{\mathrm{out}}$. For any $w\in\mathrm{H}^1(\Omega_{\mathrm{out}})$, we denote respectively by $\mathcal{S}w$ and $\mathcal{A}w$ the symmetric and antisymmetric parts of w, namely,

$$\mathcal{S}w := \frac{1}{2}(w+\check{w}) \quad \text{and} \quad \mathcal{A}w := \frac{1}{2}(w-\check{w}) \quad \text{where} \quad \check{w}(x,y) := w(2L-x,y),$$

which both belong to $\mathrm{H}^1(\Omega_{\mathrm{out}})$. By transposition, $\mathcal{S}$ and $\mathcal{A}$ can be extended to any $f\in\mathrm{H}^1(\Omega_{\mathrm{out}})^*$, for example

$$\langle \mathcal{S}f, w\rangle_{\Omega_{\mathrm{out}}} := \langle f, \mathcal{S}w\rangle_{\Omega_{\mathrm{out}}}, \quad \forall w\in\mathrm{H}^1(\Omega_{\mathrm{out}}),$$

where $\langle\cdot,\cdot\rangle_{\Omega_{\mathrm{out}}}$ refers to the duality between $\mathrm{H}^1(\Omega_{\mathrm{out}})^*$ and $\mathrm{H}^1(\Omega_{\mathrm{out}})$. Finally $\mathrm{H}^1_{0,L}(\Omega_L^+)$ stands for the subspace of functions in $\mathrm{H}^1(\Omega_L^+)$ which vanish on Σ_L.

Consider $f_\varphi^+ := -PU_\varphi^+$ the right-hand side of the first equation of (5.45). As $\partial_\nu U_\varphi^+|_{\partial\Omega_L^+\setminus\Sigma_L}=0$, we remark that f_φ^+ belongs to the dual space $\mathrm{H}^1_{0,L}(\Omega_L^+)^*$ of $\mathrm{H}^1_{0,L}(\Omega_L^+)$. We can then define $f_\varphi\in\mathrm{H}^1(\Omega_{\mathrm{out}})^*$ by setting

$$\langle f_\varphi, w\rangle_{\Omega_{\mathrm{out}}} := 2\Big\langle f_\varphi^+, (\mathcal{A}w)|_{\Omega_L^+}\Big\rangle_{\Omega_L^+}, \quad \forall w\in\mathrm{H}^1(\Omega_{\mathrm{out}}),$$

where $\langle\cdot,\cdot\rangle_{\Omega_L^+}$ refers to the duality between $\mathrm{H}^1_{0,L}(\Omega_L^+)^*$ and $\mathrm{H}^1_{0,L}(\Omega_L^+)$ (note that this definition makes sense since $(\mathcal{A}w)|_{\Omega_L^+}\in\mathrm{H}^1_{0,L}(\Omega_L^+)$). The distribution f_φ is nothing but the antisymmetric extension of f_φ^+. Indeed, on the one hand, it is clear that f_φ is antisymmetric, since for any $w\in\mathrm{H}^1(\Omega_{\mathrm{out}})$,

$$\langle \mathcal{S}f_\varphi, w\rangle_{\Omega_{\mathrm{out}}} = \langle f_\varphi, \mathcal{S}w\rangle_{\Omega_{\mathrm{out}}} = 2\Big\langle f_\varphi^+, (\mathcal{A}\mathcal{S}w)|_{\Omega_L^+}\Big\rangle_{\Omega_L^+} = 0.$$

On the other hand, for any $w^+\in\mathrm{H}^1_{0,L}(\Omega_L^+)$, denoting by $w\in\mathrm{H}^1(\Omega_{\mathrm{out}})$ its extension by 0 in Ω_L^-, we have

$$\langle f_\varphi|_{\Omega_L^+}, w^+\rangle_{\Omega_L^+} = \langle f_\varphi, w\rangle_{\Omega_{\mathrm{out}}} = 2\Big\langle f_\varphi^+, (\mathcal{A}w)|_{\Omega_L^+}\Big\rangle_{\Omega_L^+} = \big\langle f_\varphi^+, w^+\big\rangle_{\Omega_L^+},$$

since $(\mathcal{A}w)|_{\Omega_L^+} = w^+/2$. This shows that $f_\varphi|_{\Omega_L^+} = f_\varphi^+$.

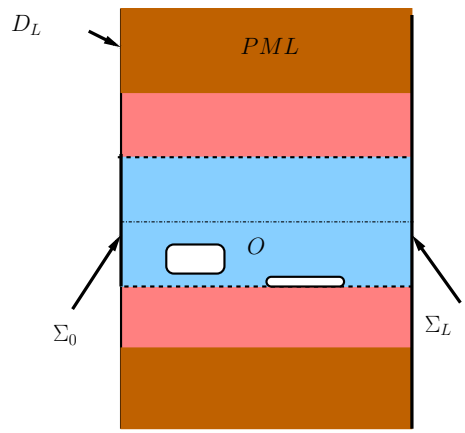

Figure 6: Domain D_L

In virtue of Corollary 9, the unique solution to (4.23) for $f=f_\varphi$ is $u_0:=A_0^{-1}f_\varphi\in \mathrm{H}^1(\Omega_{\text{out}})$. Let us verify that $u_0^+:=u_0|_{\Omega_L^+}$ is a solution to (5.45). As $f_\varphi|_{\Omega_L^+}=f_\varphi^+$, the first two equations of (5.45) are simply the restrictions of the first two equations of (4.23) to Ω_L^+ and $\partial\Omega_L^+\setminus\Sigma_L$, respectively. For the third one, just notice that as f_φ is antisymmetric, u_0 is antisymmetric as well (this follows from the injectivity of A_0 and the fact that $A_0\mathcal{S}=\mathcal{S}A_0$), so that $u_0|_{\Sigma_L}=0$.

Finally, the uniqueness of the solution to (5.45) is straightforward by using again a symmetry argument and the injectivity of operator A_0. □

With the help of Lemma 12, we can define the operator

$$\begin{aligned} T_L:\mathrm{H}^{1/2}(\Sigma_L)\to &\quad \tilde{\mathrm{H}}^{-1/2}(\Sigma_L)\\ \varphi\mapsto &\quad T_L\varphi=\frac{\mu}{\alpha}\partial_x u^+(\varphi)|_{\Sigma_L}, \end{aligned} \tag{5.46}$$

where $u^+(\varphi)$ is the solution to problem (5.44). Note that an explicit expression of T_L such as (5.43) is not available any longer.

That the scattered field $u-\tilde{w}_{n,0}^+$ is outgoing in problem (5.41) is equivalent to the identities $(\mu/\alpha)\partial_x u|_{\Sigma_L}=T_L u|_{\Sigma_L}$ and $-\mu_0\partial_x(u-\tilde{w}_n^+)|_{\Sigma_0}=T_0(u|_{\Sigma_0}-\tilde{w}_n^+|_{\Sigma_0})$, which is itself equivalent to

$$-\mu_0\partial_x u|_{\Sigma_0}=T_0(u|_{\Sigma_0})+g,\quad g:=-2\mu_0\tilde{\lambda}_n\tilde{w}_n^+|_{\Sigma_0},\quad n\in\mathbb{N}. \tag{5.47}$$

As a result, the problem (5.41) set in D is equivalent to the following problem set in the bounded domain D_L of D which is delimited by the sections Σ_0 and Σ_L: find $u\in\mathrm{H}^1(D_L)$ such that

$$\begin{cases} -\partial_y(\alpha\mu\partial_y u)-\dfrac{\mu}{\alpha}\partial_{xx}u-\dfrac{\mu}{\alpha}k^2u=0 & \text{in } D_L,\\ \partial_\nu u=0 & \text{on } \partial D_L\setminus(\Sigma_0\cup\Sigma_L\cup\partial O),\\ u=0 & \text{on } \partial O,\\ -\mu_0\partial_x u=T_0(u|_{\Sigma_0})+g & \text{on } \Sigma_0,\\ \dfrac{\mu}{\alpha}\partial_x u=T_L(u|_{\Sigma_L}) & \text{on } \Sigma_L. \end{cases} \tag{5.48}$$

The configuration of problem (5.48) is represented in Figure 6. A weak formulation which is equivalent to problem (5.48) is the following, with $\mathrm{H}^1_{0,\partial O}(D_L)$ denoting the subspace of functions in $\mathrm{H}^1(D_L)$ which vanish on ∂O: find $u\in\mathrm{H}^1_{0,\partial O}(D_L)$ such that

$$a_L(u,v)=\ell(v),\quad \forall v\in\mathrm{H}^1_{0,\partial O}(D_L), \tag{5.49}$$

where the sesquilinear form a_L is given by

$$a_L(u,v)=\int_{D_L}\left(\alpha\mu\partial_y u\partial_y\overline{v}+\frac{\mu}{\alpha}\partial_x u\partial_x\overline{v}-\frac{\mu}{\alpha}k^2u\overline{v}\right)dxdy-\langle T_0u,\overline{v}\rangle_{\Sigma_0}-\langle T_Lu,\overline{v}\rangle_{\Sigma_L}, \tag{5.50}$$

$\langle\cdot,\cdot\rangle_{\Sigma_M}$ refering to the duality pairing between $\tilde{\mathrm{H}}^{-1/2}(\Sigma_M)$ and $\mathrm{H}^{1/2}(\Sigma_M)$ for $M=0$ and $M=L$, and the antilinear form ℓ is given by

$$\ell(v):=\int_{\Sigma_0} g\overline{v}\,ds, \tag{5.51}$$

where g is defined by (5.47). To simplify notations, in (5.50) we have replaced $T_M(u|_{\Sigma_0})$ simply by $T_M u$ for $M=0$ and $M=L$. We will adopt this slight abuse of notation whenever there is no ambiguity. We are in a position to prove well-posedness of problem (5.41).

Theorem 13. *Assume that for $u^i=0$, the problem (5.41) has only the trivial solution $u=0$. Then the problem (5.41) is well-posed for $u^i=\tilde{w}^+_{n,0}$.*

Proof. Since we have uniqueness for problem (5.41), we also have uniqueness for problem (5.48). In order to prove existence for both problems, it suffices to prove that the problem (5.48) is of Fredholm type. Let us introduce a smooth real non-negative function χ in Ω^+_L which depends only on x and such that χ vanishes for $x\geq 2L$ and $\chi=1$ for $L\leq x\leq 3L/2$. We remark that for $\varphi,\psi\in\mathrm{H}^{1/2}(\Sigma_L)$,

$$\langle T_L\varphi,\overline{\psi}\rangle_{\Sigma_L}=\langle\frac{\mu}{\alpha}\partial_x u^+(\varphi),\overline{u^+(\psi)}\rangle_{\Sigma_L},$$

where $u^+(\varphi)$ and $u^+(\psi)$ are the solutions in Ω^+_L to the problem (5.44) which correspond to data φ and ψ, respectively. Multiplying the equation satisfied by $u^+(\varphi)$ by $\chi\overline{u^+(\psi)}$ and using the Green Formula, we obtain

$$\begin{aligned}-\langle\frac{\mu}{\alpha}\partial_x u^+(\varphi),\overline{u^+(\psi)}\rangle_{\Sigma_L}=&\int_{\Omega^+_L}\Big(\alpha\mu\chi\,\partial_y u^+(\varphi)\partial_y\overline{u^+(\psi)}+\frac{\mu}{\alpha}\chi\,\partial_x u^+(\varphi)\,\partial_x\overline{u^+(\psi)}\Big)\,dxdy\\&+\int_{\Omega^+_L}\Big(\frac{\mu}{\alpha}(\partial_x\chi)\partial_x u^+(\varphi)\overline{u^+(\psi)}-\frac{\mu}{\alpha}k^2\chi\,u^+(\varphi)\overline{u^+(\psi)}\Big)\,dxdy.\end{aligned}$$

Applying that identity to $\varphi=u|_{\Sigma_L}$ and $\psi=v|_{\Sigma_L}$, it enables us to decompose the sesquilinear form a_L as $a_L=c_L+k_L$, with for $u,v\in\mathrm{H}^1_{0,\partial O}(D_L)$,

$$\begin{aligned}c_L(u,v):=&\int_{D_L}\Big(\alpha\mu\partial_y u\,\partial_y\overline{v}+\frac{\mu}{\alpha}\partial_x u\partial_x\overline{v}\Big)\,dxdy\\&+\int_{\Omega^+_L}\Big(\alpha\mu\chi\,\partial_y u^+(u|_{\Sigma_L})\partial_y\overline{u^+(v|_{\Sigma_L})}+\frac{\mu}{\alpha}\chi\,\partial_x u^+(u|_{\Sigma_L})\,\partial_x\overline{u^+(v|_{\Sigma_L})}\Big)\,dxdy\\&-\langle T_0 u,\overline{v}\rangle_{\Sigma_0}\end{aligned}$$

and

$$\begin{aligned}k_L(u,v):=&-\int_{D_L}\frac{\mu}{\alpha}k^2u\overline{v}\,dxdy\\&+\int_{\Omega^+_L}\Big(\frac{\mu}{\alpha}(\partial_x\chi)\partial_x u^+(u|_{\Sigma_L})\overline{u^+(v|_{\Sigma_L})}-\frac{\mu}{\alpha}k^2\chi\,u^+(u|_{\Sigma_L})\overline{u^+(v|_{\Sigma_L})}\Big)\,dxdy.\end{aligned}$$

With the help of the Riesz theorem, we can define two operators C_L and K_L mapping $\mathrm{H}^1_{0,\partial O}(D_L)$ to itself and which are associated with the sesquilinear forms c_L and k_L, respectively. It remains to prove that C_L is an isomorphism and K_L is a compact operator. Let us prove that c_L is coercive. Taking $v=u$, we get

$$\begin{aligned}c_L(u,u)=&\int_{D_L}\Big(\alpha\mu|\partial_y u|^2+\frac{\mu}{\alpha}|\partial_x u|^2\Big)\,dxdy\\&+\int_{\Omega^+_L}\Big(\alpha\mu\chi|\partial_y u^+(u|_{\Sigma_L})|^2+\frac{\mu}{\alpha}\chi|\partial_x u^+(u|_{\Sigma_L})|^2\Big)\,dxdy-\langle T_0 u,\overline{u}\rangle_{\Sigma_0}.\end{aligned}$$

Using (5.43), we have

$$\begin{aligned}-\langle T_0 u,\overline{u}\rangle_{\Sigma_0}&=-\mu_0\sum_{n\in\mathbb{N}}\tilde{\lambda}_n|(u|_{\Sigma_0},\tilde{\varphi}_n)_{\mathrm{L}^2(\Sigma_0)}|^2\\&=-i\mu_0\sum_{n=0}^{\tilde{N}-1}\tilde{\beta}_n|(u|_{\Sigma_0},\tilde{\varphi}_n)_{\mathrm{L}^2(\Sigma_0)}|^2-\mu_0\sum_{n=\tilde{N}}^{+\infty}\tilde{\lambda}_n|(u|_{\Sigma_0},\tilde{\varphi}_n)_{\mathrm{L}^2(\Sigma_0)}|^2,\end{aligned}$$

with $\tilde{\beta}_n > 0$ for $n = 0, \cdots, \tilde{N} - 1$ and $\tilde{\lambda}_n < 0$ for $n \geq \tilde{N}$. Hence

$$\Re e(-\langle T_0 u, \overline{u} \rangle_{\Sigma_0}) = -\mu_0 \sum_{n=\tilde{N}}^{+\infty} \tilde{\lambda}_n |(u|_{\Sigma_0}, \tilde{\varphi}_n)_{\mathrm{L}^2(\Sigma_0)}|^2 \geq 0.$$

Since there exists some $C > 0$ such that $\Re e(\alpha) \geq C$ and $\Re e(1/\alpha) \geq C$, in view of $\chi \geq 0$ and using the Poincaré inequality, there exists some $c > 0$ such that for all $u \in \mathrm{H}^1_{0,\partial O}(D_L)$,

$$\Re e(c_L(u,u)) \geq c \|u\|^2_{\mathrm{H}^1(D_L)}.$$

We have proved that c_L is coercive, thus C_L is an isomorphism. Now let us prove that K_L is compact. It suffices to prove that if $(u_m, v_m)_{m \in \mathbb{N}}$ is a sequence which weakly converges to (u,v) in $\mathrm{H}^1_{0,\partial O}(D_L) \times \mathrm{H}^1_{0,\partial O}(D_L)$, then there exists a subsequence of (u_m, v_m), still denoted (u_m, v_m), such that $k_L(u_m, v_m)$ converges to $k_L(u,v)$. Indeed, this result would have the consequence that if $u_m \rightharpoonup u$ in $\mathrm{H}^1_{0,\partial O}(D_L)$, which implies that $K_L u_m \rightharpoonup K_L u$ in $\mathrm{H}^1_{0,\partial O}(D_L)$, then

$$\|K_L(u_m - u)\|^2_{\mathrm{H}^1(D_L)} = k_L(u_m - u, K_L(u_m - u)) \to 0.$$

Since the sequences (u_m) and (v_m) are bounded in $\mathrm{H}^1(D_L)$ and D_L is a bounded domain, by the Rellich theorem there exist subsequences (u_m) and (v_m), still denoted (u_m) and (v_m), such that $u_m \to u$ and $v_m \to v$ in $\mathrm{L}^2(D_L)$, which implies that

$$\int_{D_L} \frac{\mu}{\alpha} k^2 u_m \overline{v_m} \, dxdy \to \int_{D_L} \frac{\mu}{\alpha} k^2 u \overline{v} \, dxdy.$$

Using the continuity of the trace, another consequence of $u_m \rightharpoonup u$ in $\mathrm{H}^1_{0,\partial O}(D_L)$ is that $u_m|_{\Sigma_L} \rightharpoonup u|_{\Sigma_L}$ in $\mathrm{H}^{1/2}(\Sigma_L)$. Since the mapping $h \to u^+(h)$ given by Lemma 12 is continuous from $\mathrm{H}^{1/2}(\Sigma_L)$ to $\mathrm{H}^1(\Omega_L^+)$, we have that $u^+(u_n|_{\Sigma_L}) \rightharpoonup u^+(u|_{\Sigma_L})$ and $u^+(v_n|_{\Sigma_L}) \rightharpoonup u^+(v|_{\Sigma_L})$ in $\mathrm{H}^1(\Omega_L^+)$, so that $u^+(u_n|_{\Sigma_L}) \to u^+(u|_{\Sigma_L})$ and $u^+(v_n|_{\Sigma_L}) \to u^+(v|_{\Sigma_L})$ in $\mathrm{L}^2(D_{L,2L})$, where $D_{L,2L} = (L, 2L) \times (-h_{\mathrm{out}}, h_{\mathrm{out}})$. That the function χ vanishes for $x \geq 2L$ leads to

$$\int_{\Omega_L^+} \Big(\frac{\mu}{\alpha} (\partial_x \chi) \partial_x u^+(u_n|_{\Sigma_L}) \overline{u^+(v_n|_{\Sigma_L})} \Big) \, dxdy \to \int_{\Omega_L^+} \Big(\frac{\mu}{\alpha} (\partial_x \chi) \partial_x u^+(u|_{\Sigma_L}) \overline{u^+(v|_{\Sigma_L})} \Big) \, dxdy$$

and

$$\int_{\Omega_L^+} \Big(\frac{\mu}{\alpha} k^2 \chi u^+(u_n|_{\Sigma_L}) \overline{u^+(v_n|_{\Sigma_L})} \Big) \, dxdy \to \int_{\Omega_L^+} \Big(\frac{\mu}{\alpha} k^2 \chi u^+(u|_{\Sigma_L}) \overline{u^+(v|_{\Sigma_L})} \Big) \, dxdy.$$

As a conclusion we have $k_L(u_m, v_m) \to k_L(u,v)$ and we have proved that K_L is compact. The problem (5.48) is thus of Fredhlom type, which completes the proof. □

In order to specify the behaviour of solutions for $x \to +\infty$, let us introduce, for $\beta \in \mathbb{R}$, the space $\mathring{\mathcal{W}}^1_\beta(D^+) = \{v \in \mathcal{W}^1_\beta(D^+), v|_{\partial O} = 0\}$, where $\mathcal{W}^1_\beta(D^+)$ is defined by (4.36) for Ω_{out} replaced by D^+. Note that for $\beta = 0$, the space $\mathring{\mathcal{W}}^1_0(D^+)$ coincides with the subspace of functions in $\mathrm{H}^1(D^+)$ which vanish on ∂O. Let us introduce a function $\chi^+ \in \mathscr{C}^\infty(\mathbb{R}^2)$ such that $\chi^+ = 1$ for $x \geq 2L$ and $\chi^+ = 0$ for $x \leq L$.

Theorem 14. *Assume that for $u^i = 0$, the problem (5.41) has only the trivial solution $u = 0$. Then the solution u to the problem (5.41) for $u^i = \tilde{w}^+_{n,0}$ is such that for any $\beta > 0$ with $\Lambda \cap \ell_{-\beta} = \emptyset$, if*

$$\Lambda \cap \{\lambda \in \mathbb{C}, -\beta < \Re e \lambda < 0\} = \{\lambda_0, \lambda_1, \ldots, \lambda_{N_\beta - 1}\},$$

there exist some complex numbers a^+_n and a function $\tilde{u} \in \mathring{\mathcal{W}}^1_\beta(D^+)$ such that u satisfies

$$u = \chi^+ \sum_{n=0}^{N_\beta - 1} a^+_n w^+_n + \tilde{u}.$$

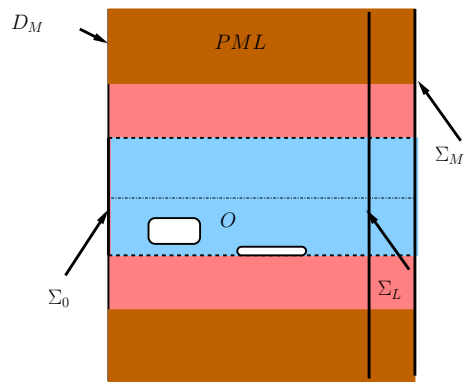

Figure 7: Domain D_M

Proof. We observe that $\chi^+ u \in \mathrm{H}^1(\Omega_{\text{out}})$ and $f := A_0(\chi^+ u) \in \mathrm{H}^1(\Omega_{\text{out}})^*$ (in fact, it could be noted that $f \in \mathrm{L}^2(\Omega_{\text{out}})$). Since f is compactly supported, we have $f \in W^1_{-\beta}(\Omega_{\text{out}})^*$. Then, let us denote $u_\beta := A_\beta^{-1} f \in W^1_\beta(\Omega_{\text{out}})$. By Proposition 10, there exist some complex numbers a_n^+ such that

$$\chi^+ u = \sum_{n=0}^{N_\beta - 1} a_n^+ w_n^+ + u_\beta.$$

We have

$$u = (\chi^+)^2 u + (1-(\chi^+)^2) u = \chi^+ \left(\sum_{n=0}^{N_\beta - 1} a_n^+ w_n^+ + u_\beta \right) + (1-(\chi^+)^2) u.$$

We notice that the function

$$\tilde{u} := \chi^+ u_\beta + (1-(\chi^+)^2) u,$$

belongs to $\mathring{\mathcal{W}}^1_\beta(D^+)$, and the proof is complete. □

In order to provide a numerical approximation of problem (5.41), we introduce a DtN map with an overlap and an equivalent problem to problem (5.41) with the help of such map. We hence introduce, for $M \geq L$, the operator

$$\begin{aligned} T_{L,M} : \mathrm{H}^{1/2}(\Sigma_L) \to & \quad \tilde{\mathrm{H}}^{-1/2}(\Sigma_M) \\ \varphi \mapsto & \quad T_{L,M}\varphi = \frac{\mu}{\alpha} \partial_x u^+|_{\Sigma_M}, \end{aligned} \tag{5.52}$$

where u^+ is the solution to problem (5.44). Then we introduce the problem in D_M: find $u \in \mathrm{H}^1_{0,\partial O}(D_M)$ such that

$$\begin{cases} -\partial_y(\alpha\mu\partial_y u) - \dfrac{\mu}{\alpha}\partial_{xx} u - \dfrac{\mu}{\alpha} k^2 u = 0 & \text{in } D_M, \\ \partial_\nu u = 0 & \text{on } \partial D_M \setminus (\Sigma_0 \cup \Sigma_M \cup \partial O), \\ u = 0 & \text{on } \partial O, \\ -\mu_0 \partial_x u = T_0(u|_{\Sigma_0}) + g & \text{on } \Sigma_0, \\ \dfrac{\mu}{\alpha}\partial_x u = T_{L,M}(u|_{\Sigma_L}) & \text{on } \Sigma_M. \end{cases} \tag{5.53}$$

The configuration of problem (5.53) is represented in Figure 7. Problem (5.53) is equivalent to the following weak formulation: find $u \in \mathrm{H}^1_{0,\partial O}(D_M)$ such that

$$a_{L,M}(u,v) = \ell(v), \quad \forall v \in \mathrm{H}^1_{0,\partial O}(D_M), \tag{5.54}$$

where the sequilinear form $a_{L,M}$ and the antilinear form ℓ are given by

$$a_{L,M}(u,v) := \int_{D_M} \left(\alpha\mu \partial_y u \partial_y \overline{v} + \frac{\mu}{\alpha}\partial_x u \partial_x \overline{v} - \frac{\mu}{\alpha} k^2 u \overline{v} \right) dxdy - \langle T_0 u, \overline{v} \rangle_{\Sigma_0} - \langle T_{L,M} u, \overline{v} \rangle_{\Sigma_M} \tag{5.55}$$

and (5.51), respectively.

We now prove that the initial problem (5.41) is equivalent to the problem (5.53) in the following sense.

Proposition 15. *If u satisfies the problem (5.41) in D, then $u|_{D_M}$ satisfies the problem (5.53) in D_M. Conversely, assume that u_M satisfies the problem (5.53) in D_M. If u^- (resp. u^+) denotes the solution to the problem (5.42) (resp. (5.44)) in Ω^- (resp. Ω_L^+) for $\varphi = u_M|_{\Sigma_0}$ (resp. $\psi = u_M|_{\Sigma_L}$), then the functions u_M and u^+ coincide in $D_M \setminus D_L$ and the function u defined by $u = u^-$ in Ω^-, $u = u_M$ in D_M and $u = u^+$ in Ω_L^+ satisfies problem (5.41).*

Proof. The first part of the proof is straightforward. As concerns the second part, we just have to check that the two functions u_M and u^+ coincide in $D_{L,M} := (L,M) \times (-h_{\text{out}}, h_{\text{out}})$. Let us denote $v := u_M - u^+$ in $D_{L,M}$. The very definition of u^+ implies that $v|_{\Sigma_L} = 0$. Moreover, we have on Σ_M:

$$\frac{\mu}{\alpha}\partial_x u_M = T_{L,M}(u_M|_{\Sigma_L}) = T_{L,M}(\psi) = \frac{\mu}{\alpha}\partial_x u^+,$$

which implies that $(\mu/\alpha)\partial_x v = 0$ on Σ_M. As a result, the function $v \in \mathrm{H}^1(D_{L,M})$ satisfies in $D_{L,M}$ the problem

$$\begin{cases} Pv = 0 & \text{in } D_{L,M}, \\ \partial_\nu v = 0 & \text{on } \partial D_{L,M} \setminus (\Sigma_L \cup \Sigma_M), \\ v = 0 & \text{on } \Sigma_L, \\ \dfrac{\mu}{\alpha}\partial_x v = 0 & \text{on } \Sigma_M. \end{cases} \tag{5.56}$$

It remains to prove that any function v in $\mathrm{H}^1(D_{L,M})$ which satisfies (5.56) actually vanishes. We observe that the space $\mathrm{H}^1_{0,L}(D_{L,M})$, which denotes the set of functions in $\mathrm{H}^1(D_{L,M})$ which vanish on Σ_L, coincides with the set of functions

$$v(x,y) = \sum_{n=0}^{+\infty} v_n(y)\sin(\mu_n(x-L)), \quad \mu_n := (2n+1)\frac{\pi}{2}\frac{1}{M-L}, \tag{5.57}$$

such that

$$\sum_{n=0}^{+\infty}(1+\mu_n^2)\|v_n\|^2_{\mathrm{L}^2(I_{\text{out}})} + \|d_y v_n\|^2_{\mathrm{L}^2(I_{\text{out}})} < +\infty.$$

Hence, any function $v \in \mathrm{H}^1(D_{L,M})$ satisfying (5.56) has the previous decomposition, where the $v_n \in \mathrm{H}^1(I_{\text{out}})$ satisfy in addition

$$\mathscr{L}(i\mu_n)(v_n) = 0, \quad \forall n \in \mathbb{N},$$

where $\mathscr{L}(\lambda)$ is defined by (3.15). In view of Corollary 6 and Lemma 1, we observe that $\mathscr{L}(i\mu_n)$ is invertible for all $n \in \mathbb{N}$. We conclude that $v_n = 0$ for all $n \in \mathbb{N}$, hence $v = 0$. The proof is complete. □

Remark 5.1. *It can be remarked that Proposition 15 holds whether or not uniqueness is satisfied for problem (5.41).*

6 Approximation of the DtN map

We first introduce the truncated DtN maps with or without an overlap and prove that they enable us to set problems of Fredholm type. For the DtN map with an overlap, we secondly prove a convergence result of the approximated solution obtained by a "truncation" of such DtN map to the true solution with respect to the size of the overlap.

6.1 The truncated DtN map with or without an overlap

For computational purpose, we need to approximate the DtN maps given by (5.46) and (5.52). Let us denote for $M \in \mathbb{R}$ and $\varphi, \psi \in \mathrm{L}^2(\Sigma_M)$,

$$\langle \varphi, \psi \rangle_{\alpha,M} := \int_{\Sigma_M} \frac{\mu}{\alpha}\varphi\psi\, dy.$$

Drawing inspiration from the expression (5.43) for the closed waveguide, for a fixed $\beta > 0$, for $M \geq L$, let us introduce the operator $T^{\beta}_{L,M}$ defined by

$$\begin{aligned} T^{\beta}_{L,M} : \mathrm{H}^{1/2}(\Sigma_L) \to & \quad \tilde{\mathrm{H}}^{-1/2}(\Sigma_M) \\ \varphi \mapsto & \quad T^{\beta}_{L,M}\varphi = \sum_{n=0}^{N_\beta - 1} \frac{\mu}{\alpha} \lambda_n e^{\lambda_n (M-L)} \langle \varphi, \varphi_n \rangle_{\alpha,L} \varphi_n. \end{aligned} \tag{6.58}$$

Note that for $M = L$, the operator $T^{\beta}_{L,L}$ is expected to be an approximation of the DtN map T_L given by (5.46). Making use of this approximate operator, we introduce a new problem set in the bounded domain D_M, which is the analogous of problem (5.53) with operator $T_{L,M}$ replaced by operator $T^{\beta}_{L,M}$. This problem corresponds to the weak formulation: find $u \in \mathrm{H}^1_{0,\partial O}(D_M)$ such that for all $v \in \mathrm{H}^1_{0,\partial O}(D_M)$,

$$a^{\beta}_{L,M}(u,v) = \ell(v), \tag{6.59}$$

where the sesquilinear form $a^{\beta}_{L,M}$ is given by

$$a^{\beta}_{L,M}(u,v) := b_M(u,v) - \int_{D_M} \frac{\mu}{\alpha} k^2 u \overline{v} \, dx dy - \langle T^{\beta}_{L,M} u, \overline{v} \rangle_{\Sigma_M}, \tag{6.60}$$

the sesquilinear form b_M is defined by

$$b_M(u,v) := \int_{D_M} \left(\alpha\mu \partial_y u \partial_y \overline{v} + \frac{\mu}{\alpha} \partial_x u \partial_x \overline{v} \right) dx dy - \langle T_0 u, \overline{v} \rangle_{\Sigma_0}, \tag{6.61}$$

while the antilinear form ℓ is given by (5.51). We have the following proposition.

Proposition 16. *For $M \geq L$, assume that the problem (6.59) has at most one solution. Then the problem (6.59) is well-posed.*

Proof. By using the same arguments as in the proof of Theorem 13, the sesquilinar form b_M is associated by the Riesz theorem with an invertible operator, while the sequilinear form $a^{\beta}_{L,M} - b_M$ is associated with a compact operator. In particular, the sesquilinear form

$$(u,v) \in \mathrm{H}^1_{0,\partial O}(D_M) \times \mathrm{H}^1_{0,\partial O}(D_M) \mapsto \langle T^{\beta}_{L,M} u, \overline{v} \rangle_{\Sigma_M}$$

is associated with a finite rank operator in view of (6.58). This shows that the problem (6.59) is of Fredholm type, which completes the proof. □

6.2 An error estimate for the DtN map with an overlap

We wish now to prove that the solution $u^{\beta}_{L,M}$ to the problem (6.59) is a good approximation of the solution u to the problem (5.41) in the domain D_L when those two problems are well-posed. In the simple case of a homogeneous closed waveguide, the convergence of the solution obtained with the DtN operator without an overlap to the true solution can be proved by using that the eigenvectors of the transverse operator form a complete basis. More precisely, the convergence rate is of exponential type, both with respect to the distance between the artificial boundary and the perturbation and with respect to the number of terms in the truncated series. The proof is very similar to the one used in [Bécache et al., 2004] to justify the approximation of the true solution with Perfectly Matched Layers. In our case, since we do not know whether the eigenfunctions φ_n form a complete basis, the convergence estimate for the DtN operator without an overlap is an open question. By exploiting the description of the solution in terms of the modes up to an exponentially decaying remainder, we were able to derive an estimate for the DtN operator with an overlap. Moreover, the convergence rate is exponential with respect to the overlap $(M-L)$, but the convergence rate with respect to β, which determines the number of terms in the series (6.58), is not specified. Note that in the case of a periodic waveguide, an error estimate in the presence of an artificial boundary condition of the same type as our thick DtN boundary condition is derived in [Nazarov, 2018].

We begin with the following lemma, which specifies the properties of solutions to the half-waveguide problem (5.44). For $\beta\in\mathbb{R}$, we introduce the space $\mathcal{W}^1_\beta(\Omega^+_L)$ defined by (4.36) for $\Omega_{\rm out}$ replaced by Ω^+_L, equipped with the norm

$$\|v\|_{\mathcal{W}^1_\beta(\Omega^+_L)}:=\Big(\|e^{\beta(x-L)}v\|^2_{\mathrm{L}^2(\Omega^+_L)}+\|e^{\beta(x-L)}\partial_x v\|^2_{\mathrm{L}^2(\Omega^+_L)}+\|e^{\beta(x-L)}\partial_y v\|^2_{\mathrm{L}^2(\Omega^+_L)}\Big)^{1/2}.$$

Lemma 17. *For $\varphi\in\mathrm{H}^{1/2}(\Sigma_L)$, let us denote by $u^+(\varphi)$ the solution in $\mathrm{H}^1(\Omega^+_L)$ to the problem (5.44) in the half-waveguide Ω^+_L given by Lemma 12. For any $\beta>0$ such that $\Lambda\cap\ell_{-\beta}=\emptyset$, if*

$$\Lambda\cap\{\lambda\in\mathbb{C},\,-\beta<\Re e\lambda<0\}=\{\lambda_0,\lambda_1,\ldots,\lambda_{N_\beta-1}\},$$

there exist some unique complex numbers $a^+_n(\varphi)$ and a unique function $u_\beta(\varphi)\in\mathcal{W}^1_\beta(\Omega^+_L)$ such that $u^+(\varphi)$ satisfies

$$u^+(\varphi)=\sum_{n=0}^{N_\beta-1}a^+_n(\varphi)w^+_n+u_\beta(\varphi). \tag{6.62}$$

Moreover, the coefficients a^+_n are given by

$$a^+_n(\varphi)=e^{-\lambda_n L}\langle\varphi,\varphi_n\rangle_{\alpha,L} \tag{6.63}$$

and there exists a constant C_β such that for all $\varphi\in\mathrm{H}^{1/2}(\Sigma_L)$,

$$\sqrt{\sum_{n=0}^{N_\beta-1}|a^+_n(\varphi)|^2+\|u_\beta(\varphi)\|^2_{\mathcal{W}^1_\beta(\Omega^+_L)}}\le C_\beta\,\|\varphi\|_{\mathrm{H}^{1/2}(\Sigma_L)}. \tag{6.64}$$

Proof. Existence of coefficients $a^+_n(\varphi)$ and the function $u_\beta(\varphi)\in\mathcal{W}^1_\beta(\Omega^+_L)$ in the decomposition (6.62) follows the same lines as the proof of Theorem 14. For $M\ge L$, let us introduce the bilinear form

$$q_M(u,v):=\langle\frac{\mu}{\alpha}\partial_x u,v\rangle_{\Sigma_M}-\langle\frac{\mu}{\alpha}\partial_x v,u\rangle_{\Sigma_M},$$

whenever the two duality brackets are well defined. It should be remarked that if u and v in $\mathrm{H}^1(\Omega^+_L)$ both satisfy the equations

$$\begin{cases}-\partial_y(\alpha\mu\partial_y w)-\dfrac{\mu}{\alpha}\partial_{xx}w-\dfrac{\mu}{\alpha}k^2w=0 & \text{in } \Omega^+_L\\ \partial_\nu w=0 & \text{on } \partial\Omega^+_L\setminus\Sigma_L,\end{cases} \tag{6.65}$$

then by using the Green Formula in $(L,M)\times I_{\rm out}$, the quantity $q_M(u,v)$ is well-defined for $M\ge L$ and does not depend on M. Let us prove that

$$a^+_n(\varphi)=\frac{q_L(u^+(\varphi),w^-_n)}{2\lambda_n},\quad\text{and}\quad q_L(u^+(\varphi),w^+_n)=0. \tag{6.66}$$

Using the decomposition of $u^+(\varphi)$ given by (6.62), we get

$$q_L(u^+(\varphi),w^\pm_n)=\sum_{m=0}^{N_\beta-1}a^+_m(\varphi)q_L(w^+_m,w^\pm_n)+q_L(u_\beta(\varphi),w^\pm_n).$$

A straightforward computation shows that

$$q_L(w^+_m,w^+_n)=0\quad\text{and}\quad q_L(w^+_m,w^-_n)=(\lambda_m+\lambda_n)\delta_{m,n},$$

where we have used the biorthogonality relationship $\langle\varphi_m,\varphi_n\rangle_{\alpha,L}=\delta_{m,n}$. Next, let us prove that $q_L(u_\beta(\varphi),w^\pm_n)=0$. The functions $u_\beta(\varphi)$ and $w^\pm_n$ both satisfy (6.65), so that $q_M(u_\beta(\varphi),w^\pm_n)$ does not depend on $M\ge L$. We have for $M>L$,

$$\begin{aligned}q_L(u_\beta(\varphi),w^-_n)&=q_M(u_\beta(\varphi),w^-_n)\\&=e^{-\lambda_n M}\langle\frac{\mu}{\alpha}\partial_x u_\beta,\varphi_n\rangle_{\Sigma_M}+\lambda_n e^{-\lambda_n M}\langle\frac{\mu}{\alpha}\varphi_n,u_\beta\rangle_{\Sigma_M}\\&=e^{-\lambda_n M}e^{-\beta(M-L)}\Big(\langle e^{\beta(M-L)}\frac{\mu}{\alpha}\partial_x u_\beta,\varphi_n\rangle_{\Sigma_M}+\lambda_n\langle\frac{\mu}{\alpha}\varphi_n,e^{\beta(M-L)}u_\beta\rangle_{\Sigma_M}\Big).\end{aligned}$$

The solution in $\mathrm{H}^1(\Omega_M^+)$ to the problem (5.44) corresponding to $\varphi=\varphi_n$ is nothing but $v_{M,n}(x,y):=w_n^+(x-M,y)$, then by the Green formula in Ω_M^+,

$$\begin{aligned}&-\langle e^{\beta(M-L)}\frac{\mu}{\alpha}\partial_x u_\beta,\varphi_n\rangle_{\Sigma_M}\\&=\int_{\Omega_M^+}\Big(\alpha\mu\partial_y u_\beta\partial_y v_{M,n}e^{\beta(x-L)}+\frac{\mu}{\alpha}\partial_x u_\beta\partial_x(e^{\beta(x-L)}v_{M,n})-\frac{\mu}{\alpha}k^2u_\beta v_{M,n}e^{\beta(x-L)}\Big)\,dxdy.\end{aligned}$$

We then obtain the estimate

$$|\langle e^{\beta(M-L)}\frac{\mu}{\alpha}\partial_x u_\beta,\varphi_n\rangle_{\Sigma_M}|\le C\max(1,\beta)\|u_\beta(\varphi)\|_{\mathcal{W}^1_\beta(\Omega_L^+)}\|v_{M,n}\|_{\mathrm{H}^1(\Omega_M^+)}.$$

Similarly, we obtain

$$\begin{aligned}&-\lambda_n\langle\frac{\mu}{\alpha}\varphi_n,e^{\beta(M-L)}u_\beta\rangle_{\Sigma_M}=-\langle\frac{\mu}{\alpha}\partial_x v_{M,n},e^{\beta(M-L)}u_\beta\rangle_{\Sigma_M}\\&=\int_{\Omega_M^+}\Big(\alpha\mu\partial_y v_{M,n}\partial_y u_\beta e^{\beta(x-L)}+\frac{\mu}{\alpha}\partial_x v_{M,n}\partial_x(e^{\beta(x-L)}u_\beta)-\frac{\mu}{\alpha}k^2v_{M,n}u_\beta e^{\beta(x-L)}\Big)\,dxdy,\end{aligned}$$

which implies the estimate

$$|\lambda_n\langle\frac{\mu}{\alpha}\varphi_n,e^{\beta(M-L)}u_\beta\rangle_{\Sigma_M}|\le C\max(1,\beta)\|u_\beta(\varphi)\|_{\mathcal{W}^1_\beta(\Omega_L^+)}\|v_{M,n}\|_{\mathrm{H}^1(\Omega_M^+)}.$$

Gathering the previous estimates, and using that $\|v_{M,n}\|_{\mathrm{H}^1(\Omega_M^+)}=\|w_n^+\|_{\mathrm{H}^1(\Omega^+)}$ does not depend on M, provides

$$|q_L(u_\beta(\varphi),w_n^-)|\le C\,e^{\beta L}e^{-(\Re e(\lambda_n)+\beta)M}\max(1,\beta)\|u_\beta(\varphi)\|_{\mathcal{W}^1_\beta(\Omega_L^+)}.$$

Since $\Re e(\lambda_n)>-\beta$ for all $n=0,\cdots,N_\beta-1$, passing to the limit $M\to+\infty$, we finally obtain $q_L(u_\beta(\varphi),w_n^-)=0$. We would prove with the same arguments that $q_L(u_\beta(\varphi),w_n^+)=0$. This implies the identities (6.66). From (6.66), we observe that by linearity

$$a_n^+=\frac{q_L(u^+(\varphi),w_n^--zw_n^+)}{2\lambda_n},\quad\forall z\in\mathbb{C}.$$

Choosing z such that $w_n^--zw_n^+=0$ on Σ_L, that is $z=e^{-2\lambda_nL}$, implies that

$$a_n^+=-\frac{1}{2\lambda_n}\langle\frac{\mu}{\alpha}\partial_x(w_n^--e^{-2\lambda_nL}w_n^+),u^+(\varphi)\rangle_{\Sigma_L}=e^{-\lambda_nL}\langle\frac{\mu}{\alpha}\varphi_n,\varphi\rangle_{\Sigma_L},$$

which is (6.63). This in particular implies the uniqueness of the coefficients a_n^+ and of the function u_β in the decomposition (6.62).

In order to prove the continuity estimate (6.64), let us consider, for a fixed $\beta>0$, the Hilbert space $V_{\mathrm{aug}}:=\mathbb{C}^{N_\beta}\times V$, with

$$V:=\{v\in\mathcal{W}^1_\beta(\Omega_L^+),\quad Pv=0\quad\text{in}\quad\Omega_L^+,\quad\partial_\nu v=0\quad\text{on}\quad\partial\Omega_L^+\setminus\Sigma_L\},$$

equipped with the norm $\|\cdot\|_{V_{\mathrm{aug}}}$ defined by

$$\|(b_0,b_1,\cdots,b_{N_\beta-1},v)\|^2_{V_{\mathrm{aug}}}:=\sum_{n=0}^{N_\beta-1}|b_n|^2+\|v\|^2_{\mathcal{W}^1_\beta(\Omega_L^+)}$$

and the operator $T_{\mathrm{aug}}:V_{\mathrm{aug}}\to\mathrm{H}^{1/2}(\Sigma_L)$ such that

$$(b_0,b_1,\cdots,b_{N_\beta-1},v)\mapsto w|_{\Sigma_L},\quad w:=\sum_{n=0}^{N_\beta-1}b_nw_n^++v.$$

From the first part of the proof, the operator T_{aug} is invertible. Its inverse is continuous from the Banach theorem, which implies (6.64). □

We are now in a position to state the main theorem of this section. It requires two lemmas which are proved at the end of this section.

Theorem 18. *For a fixed L and for any M large enough, if problem (5.41) is well-posed, then problem (6.59) is well-posed as well. Moreover, if u is the solution to problem (5.41) and $u^{\beta}_{L,M}$ is the solution to problem (6.59), there exists a constant $C>0$ which is independent of β and M such that*

$$\|u-u^{\beta}_{L,M}\|_{\mathrm{H}^1(D_L)} \leq C C_{\beta} \max(1,\beta) e^{-\beta(M-L)} \|u\|_{\mathrm{H}^1(D^+)}, \tag{6.67}$$

where C_{β} is the constant (independent of M) given in (6.64).

Proof. We assume that the function $u^{\beta}_{L,M}$ satisfies the problem (6.59) in D_M. The basic idea consists in finding the problem set in the fixed domain D_L which is solved by $u^{\beta}_{L,M}$. In this view we now introduce two operators $N_{L,M}$ and $R^{\beta}_{L,M}$ defined as follows. We first consider the Neumann-to-Neumannn operator

$$\begin{aligned} N_{L,M} : \tilde{\mathrm{H}}^{-1/2}(\Sigma_M) \to & \quad \tilde{\mathrm{H}}^{-1/2}(\Sigma_L) \\ \psi \mapsto & \quad N_{L,M}\psi = \frac{\mu}{\alpha}\partial_x v_{L,M}|_{\Sigma_L}, \end{aligned} \tag{6.68}$$

where $v_{L,M}$ is the solution to the following problem: find $v \in \mathrm{H}^1(D_{L,M})$ such that

$$\begin{cases} Pv = 0 & \text{in } D_{L,M}, \\ \partial_\nu v = 0 & \text{on } \partial D_{L,M} \setminus (\Sigma_L \cup \Sigma_M), \\ v = 0 & \text{on } \Sigma_L, \\ \dfrac{\mu}{\alpha}\partial_x v = \psi & \text{on } \Sigma_M. \end{cases} \tag{6.69}$$

From Lemma 19 hereafter, the problem (6.69) is well-posed, so that the operator $N_{L,M}$ is well-defined. We secondly consider the operator $R^{\beta}_{L,M} : \mathrm{H}^{1/2}(\Sigma_L) \to \tilde{\mathrm{H}}^{-1/2}(\Sigma_M)$ given by $R^{\beta}_{L,M} := T_{L,M} - T^{\beta}_{L,M}$, where the operators $T_{L,M}$ and $T^{\beta}_{L,M}$ are defined by (5.52) and (6.58), respectively.
Let us consider the restriction of the function $e^{\beta}_{LM} := u^+(u^{\beta}_{L,M}|_{\Sigma_L}) - u^{\beta}_{L,M}$ in the domain $D_{L,M}$, where $u^+(u^{\beta}_{L,M}|_{\Sigma_L})$ is defined by (5.44) for φ replaced by $u^{\beta}_{L,M}|_{\Sigma_L}$. By definition of the operators $T_{L,M}$ and $T^{\beta}_{L,M}$, we observe that

$$\frac{\mu}{\alpha}\partial_x u^+(u^{\beta}_{L,M}|_{\Sigma_L})|_{\Sigma_M} = T_{L,M}(u^{\beta}_{L,M}|_{\Sigma_L}), \quad \frac{\mu}{\alpha}\partial_x u^{\beta}_{L,M}|_{\Sigma_M} = T^{\beta}_{L,M}(u^{\beta}_{L,M}|_{\Sigma_L}),$$

hence the function e^{β}_{LM} satisfies the problem (6.69) for $\psi = R^{\beta}_{L,M}(u^{\beta}_{L,M}|_{\Sigma_L})$. Writing $u^{\beta}_{L,M} = u^+(u^{\beta}_{L,M}|_{\Sigma_L}) - e^{\beta}_{LM}$, we obtain by definition of the operators T_L, $N_{L,M}$ and $R^{\beta}_{L,M}$ that

$$\frac{\mu}{\alpha}\partial_x u^{\beta}_{L,M}|_{\Sigma_L} = T_L(u^{\beta}_{L,M}|_{\Sigma_L}) - N_{L,M}R^{\beta}_{L,M}(u^{\beta}_{L,M}|_{\Sigma_L}),$$

so that the function $u^{\beta}_{L,M}$ satisfies the following problem in D_L: find $v \in \mathrm{H}^1_{0,\partial O}(D_L)$ such that

$$\begin{cases} Pv = 0 & \text{in } D_L, \\ \partial_\nu v = 0 & \text{on } \partial D_L \setminus (\Sigma_0 \cup \Sigma_L \cup \partial O), \\ v = 0 & \text{on } \partial O, \\ -\mu_0 \partial_x v = T_0(v|_{\Sigma_0}) + g & \text{on } \Sigma_0, \\ \dfrac{\mu}{\alpha}\partial_x v = (T_L - N_{L,M}R^{\beta}_{L,M})(v|_{\Sigma_L}) & \text{on } \Sigma_L. \end{cases} \tag{6.70}$$

Let us define $G \in \mathrm{H}^1_{0,\partial O}(D_L)^*$ by

$$\langle G, \overline{v} \rangle_{D_L} := \int_{\Sigma_0} g\,\overline{v}\,dy, \quad \forall v \in \mathrm{H}^1_{0,\partial O}(D_L), \tag{6.71}$$

where $\langle \cdot,\cdot \rangle_{D_L}$ refers to the duality pairing between $\mathrm{H}^1_{0,\partial O}(D_L)^*$ and $\mathrm{H}^1_{0,\partial O}(D_L)$. Let us also introduce the operators A_L and $A^{\beta}_{L,M}$ which map the space $\mathrm{H}^1_{0,\partial O}(D_L)$ to its dual space $\mathrm{H}^1_{0,\partial O}(D_L)^*$ and such that for all $(v,w) \in \mathrm{H}^1_{0,\partial O}(D_L) \times \mathrm{H}^1_{0,\partial O}(D_L)$,

$$\langle A_L v, \overline{w} \rangle_{D_L} := a_L(v,w), \quad \langle A^{\beta}_{L,M} v, \overline{w} \rangle_{D_L} := a_L(v,w) + \langle N_{L,M}R^{\beta}_{L,M} v, \overline{w} \rangle_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_L), \mathrm{H}^{1/2}(\Sigma_L)}, \tag{6.72}$$

where the sesquilinear form a_L is defined by (5.50). In view of (5.48) and (6.70), considering (6.71) and (6.72), we have that

$$A_L u = A^{\beta}_{L,M} u^{\beta}_{L,M} = G. \tag{6.73}$$

Since the operator A_L is invertible, we have

$$A^{\beta}_{L,M} = A_L(I + A_L^{-1}(A^{\beta}_{L,M} - A_L)). \tag{6.74}$$

From (6.72), we obtain that for all $(v,w) \in \mathrm{H}^1_{0,\partial O}(D_L) \times \mathrm{H}^1_{0,\partial O}(D_L)$,

$$\langle A^{\beta}_{L,M} v, \overline{w} \rangle_{D_L} = \langle A_L v, \overline{w} \rangle_{D_L} + \langle N_{L,M} R^{\beta}_{L,M} v, \overline{w} \rangle_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_L), \mathrm{H}^{1/2}(\Sigma_L)},$$

which implies that there exists a constant C (independent of β and M) such that

$$\|A^{\beta}_{L,M} - A_L\| \leq C \|N_{L,M}\| \|R^{\beta}_{L,M}\|.$$

By using Lemmas 19 and 20, we obtain that for sufficiently large M, there exists a constant C (independent of β and M) such that

$$\|A^{\beta}_{L,M} - A_L\| \leq C C_{\beta} \max(1,\beta) e^{-\beta(M-L)}, \tag{6.75}$$

where C_{β} is the constant of (6.64). We infer from (6.74) and (6.75) that for sufficiently large M the operator $A^{\beta}_{L,M}$ is invertible and

$$\|(A^{\beta}_{L,M})^{-1}\| \leq 2 \|A_L^{-1}\|. \tag{6.76}$$

It remains to remark that from (6.73) we get

$$A^{\beta}_{L,M}(u - u^{\beta}_{L,M}) = (A^{\beta}_{L,M} - A_L) u,$$

so that by using (6.75) and (6.76), we obtain that for sufficiently large M,

$$\|u - u^{\beta}_{L,M}\|_{\mathrm{H}^1_{0,\partial O}(D_L)} \leq C C_{\beta} \max(1,\beta) e^{-\beta(M-L)} \|u\|_{\mathrm{H}^1_{0,\partial O}(D_L)}.$$

The proof is complete. □

Remark 6.1. *The estimate (6.67) shows that for a fixed β the error $\|u - u^{\beta}_{L,M}\|_{\mathrm{H}^1(\Omega_L)}$ tends exponentially to 0 when $M - L$ tends to $+\infty$, with an exponential rate equal to β. However, it would be desirable to specify how the constant C_{β} in (6.67), which coincides with the constant C_{β} in (6.64), depends on β. For example, if one could prove that C_{β} has a polynomial behaviour with respect to β, then for a fixed $M > L$, one could also deduce that the error $\|u - u^{\beta}_{L,M}\|_{\mathrm{H}^1(\Omega_L)}$ tends exponentially to 0 when β tends to $+\infty$. Once again, the major obstacle is the fact that we have no idea whether the eigenfunctions φ_n of the transverse operator L given by (2.13) form a complete basis or not. Such non-selfadjoint operator, due to the jump of the coefficients α and μ in the principal part, cannot be seen as a first order perturbation of the Laplace operator, therefore is not a particular case of the second-order Sturm-Liouville operators studied in [Naimark, 1967], for example.*

We complete this section by stating and proving the two Lemmas 19 and 20 which are used in the proof of the above theorem.

Lemma 19. *For $\psi \in \tilde{\mathrm{H}}^{-1/2}(\Sigma_M)$, the problem (6.69) is well-posed and the norm of the operator $N_{L,M}$ defined by (6.68) satisfies: for all $\delta > 0$, there exists a constant C_{δ} such that*

$$\|N_{L,M}\| \leq C_{\delta}, \quad \forall M \geq L + \delta.$$

Proof. Recalling that $\mathrm{H}^1_{0,L}(D_{L,M})$ is the space of functions in $\mathrm{H}^1(D_{L,M})$ which vanish on Σ_L, a weak formulation of problem (6.69) is: find $v \in \mathrm{H}^1_{0,L}(D_{L,M})$ such that

$$a(v,w) := \int_{D_{L,M}} \left(\alpha\mu \partial_y v \partial_y \overline{w} + \frac{\mu}{\alpha} \partial_x v \partial_x \overline{w} - \frac{\mu}{\alpha} k^2 v \overline{w} \right) dxdy = \langle \psi, \overline{w} \rangle_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_M), \mathrm{H}^{1/2}(\Sigma_M)}, \quad \forall w \in \mathrm{H}^1_{0,L}(D_{L,M}).$$

Such weak formulation is equivalent to the equation $Av=F$, where the operator $A:\mathrm{H}^1_{0,L}(D_{L,M})\mapsto \mathrm{H}^1_{0,L}(D_{L,M})^*$ and the source term $F\in\mathrm{H}^1_{0,L}(D_{L,M})^*$ are defined, for all $v,w\in\mathrm{H}^1_{0,L}(D_{L,M})$, by

$$\langle Av,\overline{w}\rangle_{\mathrm{H}^1_{0,L}(D_{L,M})^*,\mathrm{H}^1_{0,L}(D_{L,M})}:=a(v,w),\quad \langle F,\overline{w}\rangle_{\mathrm{H}^1_{0,L}(D_{L,M})^*,\mathrm{H}^1_{0,L}(D_{L,M})}:=\langle\psi,\overline{w}\rangle_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_M),\mathrm{H}^{1/2}(\Sigma_M)}.$$

The proof of Proposition 15 already establishes uniqueness for problem (6.69). Well-posedness of such problem comes from the fact that the operator A is Fredholm of index 0 (see for instance the proof of Theorem 13 in a similar situation). Let us use the decomposition of v given by (5.57) and a similar decomposition for F, that is

$$F(x,y)=\sum_{n=0}^{+\infty}f_n(y)\sin(\mu_n(x-L)),\quad \mu_n:=(2n+1)\frac{\pi}{2}\frac{1}{M-L}.$$

By a direct computation, the usual norms of $\mathrm{H}^1_{0,L}(D_{L,M})$ and of its dual space can be expressed in terms of the following series involving the weighted norms defined in section 4:

$$\|v_{L,M}\|^2_{\mathrm{H}^1_{0,L}(D_{L,M})}=\frac{1}{2}(M-L)\sum_{n=0}^{+\infty}\|v_n\|^2_{\mathrm{H}^1(I_{\mathrm{out}},\mu_n)} \tag{6.77}$$

and

$$\|F\|^2_{\mathrm{H}^1_{0,L}(D_{L,M})^*}=\frac{1}{2}(M-L)\sum_{n=0}^{+\infty}\|f_n\|^2_{\mathrm{H}^1(I_{\mathrm{out}},\mu_n)^*}. \tag{6.78}$$

In addition, $Av_{L,M}=F$ is equivalent to $\mathscr{L}(i\mu_n)v_n=f_n$, for all $n\in\mathbb{N}$. From the proof of Theorem 8, we know that the operator $\mathscr{L}(i\mu_n):\mathrm{H}^1(I_{\mathrm{out}},\mu_n)\to\mathrm{H}^1(I_{\mathrm{out}},\mu_n)^*$ is invertible and its inverse is uniformly bounded with respect to n, that is there exists a constant $c>0$ (independent of n) such that

$$\|v_n\|_{\mathrm{H}^1(I_{\mathrm{out}},\mu_n)}\le c\|f_n\|_{\mathrm{H}^1(I_{\mathrm{out}},\mu_n)^*},\quad \forall n\in\mathbb{N}.$$

In view of (6.77) and (6.78), this yields

$$\|v_{L,M}\|_{\mathrm{H}^1_{0,L}(D_{L,M})}\le c\|F\|_{\mathrm{H}^1_{0,L}(D_{L,M})^*}, \tag{6.79}$$

where we note that the constant c does not depend on M. To complete the proof, let us remark that for all $\delta>0$, there exists a constant C_δ (independent of M) such that we have the trace inequality

$$\|w\|_{\mathrm{H}^{1/2}(\Sigma_M)}\le C_\delta\|w\|_{\mathrm{H}^1(D_{M-\delta,M})},\quad \forall w\in\mathrm{H}^1_{0,L}(D_{L,M}),$$

which implies that for $M\ge L+\delta$,

$$\|w\|_{\mathrm{H}^{1/2}(\Sigma_M)}\le C_\delta\|w\|_{\mathrm{H}^1_{0,L}(D_{L,M})}.$$

Then

$$\begin{aligned}\|F\|_{\mathrm{H}^1_{0,L}(D_{L,M})^*}&=\sup_{w\in\mathrm{H}^1_{0,L}(D_{L,M})\setminus\{0\}}\frac{\left|\langle F,\overline{w}\rangle_{\mathrm{H}^1_{0,L}(D_{L,M})^*,\mathrm{H}^1_{0,L}(D_{L,M})}\right|}{\|w\|_{\mathrm{H}^1_{0,L}(D_{L,M})}}=\sup_{w\in\mathrm{H}^1_{0,L}(D_{L,M})\setminus\{0\}}\frac{\left|\langle\psi,\overline{w}\rangle_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_M),\mathrm{H}^{1/2}(\Sigma_M)}\right|}{\|w\|_{\mathrm{H}^1_{0,L}(D_{L,M})}}\\&\le\|\psi\|_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_M)}\sup_{w\in\mathrm{H}^1_{0,L}(D_{L,M})\setminus\{0\}}\frac{\|w\|_{\mathrm{H}^{1/2}(\Sigma_M)}}{\|w\|_{\mathrm{H}^1_{0,L}(D_{L,M})}}\le C_\delta\|\psi\|_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_M)}.\end{aligned} \tag{6.80}$$

On the other hand, there exists a constant C_δ (independent of M) such that for $M\ge L+\delta$,

$$\|\frac{\mu}{\alpha}\partial_x v_{L,M}\|_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_L)}\le C_\delta\|v_{L,M}\|_{\mathrm{H}^1_{0,L}(D_{L,M})}. \tag{6.81}$$

Indeed, we have by definition

$$\|\frac{\mu}{\alpha}\partial_x v_{L,M}\|_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_L)}=\sup_{\varphi\in\mathrm{H}^{1/2}(\Sigma_L)\setminus\{0\}}\frac{\left|\langle\frac{\mu}{\alpha}\partial_x v_{L,M},\overline{\varphi}\rangle_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_L),\mathrm{H}^{1/2}(\Sigma_L)}\right|}{\|\varphi\|_{\mathrm{H}^{1/2}(\Sigma_L)}}.$$

There exists a continuous map $\varphi\in \mathrm{H}^{1/2}(\Sigma_L)\mapsto w\in \mathrm{H}^1(D_{L,L+\delta})$ such that $w|_{\Sigma_L}=\varphi$ and $w|_{\Sigma_{L+\delta}}=0$, in particular there exists a constant C'_δ (independent of M) such that for all $\varphi\in \mathrm{H}^{1/2}(\Sigma_L)$,

$$\|w\|_{\mathrm{H}^1(D_{L,L+\delta})}\leq C'_\delta\|\varphi\|_{\mathrm{H}^{1/2}(\Sigma_L)}.$$

For any $\varphi\in \mathrm{H}^{1/2}(\Sigma_L)$, since $v_{L,M}$ satisfies the problem (6.69), we have using the Green formula

$$-\langle\frac{\mu}{\alpha}\partial_x v_{L,M},\overline{\varphi}\rangle_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_L),\mathrm{H}^{1/2}(\Sigma_L)}=\int_{D_{L,L+\delta}}\Big(\alpha\mu\partial_y v_{L,M}\partial_y\overline{w}+\frac{\mu}{\alpha}\partial_x v_{L,M}\partial_x\overline{w}-\frac{\mu}{\alpha}k^2 v\overline{w}\Big)\,dxdy,$$

hence there exists a constant C which depends neither on M nor on δ such that

$$\Big|\langle\frac{\mu}{\alpha}\partial_x v_{L,M},\overline{\varphi}\rangle_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_L),\mathrm{H}^{1/2}(\Sigma_L)}\Big|\leq C\|v_{L,M}\|_{\mathrm{H}^1(D_{L,L+\delta})}\|w\|_{\mathrm{H}^1(D_{L,L+\delta})}\leq CC'_\delta\|v_{L,M}\|_{\mathrm{H}^1_{0,L}(D_{L,M})}\|\varphi\|_{\mathrm{H}^{1/2}(\Sigma_L)},$$

which yields (6.81) with $C_\delta=CC'_\delta$.
Gathering the estimates (6.79), (6.80) and (6.81) implies that there exists a new constant C_δ (independent of M) such that for all $\psi\in\tilde{\mathrm{H}}^{-1/2}(\Sigma_M)$,

$$\|\frac{\mu}{\alpha}\partial_x v_{L,M}\|_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_L)}\leq C_\delta\,\|\psi\|_{\tilde{\mathrm{H}}^{-1/2}(\Sigma_M)},$$

where $v_{L,M}$ is the solution to problem (6.69), which completes the proof. □

Lemma 20. *Considering the operator $R^\beta_{L,M}:=T_{L,M}-T^\beta_{L,M}:\mathrm{H}^{1/2}(\Sigma_L)\to\tilde{\mathrm{H}}^{-1/2}(\Sigma_M)$, where the operators $T_{L,M}$ and $T^\beta_{L,M}$ are defined by (5.52) and (6.58), respectively, there exists a constant C (independent of β and M) such that*

$$\|R^\beta_{L,M}\|\leq CC_\beta\max(1,\beta)e^{-\beta(M-L)},$$

where C_β is the continuity constant in (6.64).

Proof. Let us denote $u^+(\varphi)$ the solution to the problem (5.44) in Ω^+_L with datum φ on Σ_L. Lemma 17 implies that $u^+(\varphi)$ satisfies on Σ_M the identity

$$\frac{\mu}{\alpha}\partial_x u^+(\varphi)|_{\Sigma_M}=\sum_{n=0}^{N_\beta-1}a^+_n(\varphi)\frac{\mu}{\alpha}\lambda_n e^{\lambda_n M}\varphi_n+\frac{\mu}{\alpha}\partial_x u_\beta(\varphi)|_{\Sigma_M}.$$

Using (6.63), we obtain that

$$R^\beta_{L,M}\varphi=T_{L,M}\varphi-T^\beta_{L,M}\varphi=\frac{\mu}{\alpha}\partial_x u_\beta(\varphi)|_{\Sigma_M}.$$

For any $\varphi\in\mathrm{H}^{1/2}(\Sigma_L)$ and any $\psi\in\mathrm{H}^{1/2}(\Sigma_M)$, since $u_\beta(\varphi)$ satisfies $Pu_\beta=0$ in Ω^+_M and $\partial_\nu u_\beta=0$ on $\partial\Omega^+_M\setminus\Sigma_M$, we have

$$\begin{aligned}&-\langle e^{\beta(M-L)}R^\beta_{L,M}\varphi,\psi\rangle_{\Sigma_M}\\&=\int_{\Omega^+_M}\Big(\alpha\mu\partial_y u_\beta(\varphi)\partial_y v e^{\beta(x-L)}+\frac{\mu}{\alpha}\partial_x u_\beta(\varphi)\partial_x(e^{\beta(x-L)}v)-\frac{\mu}{\alpha}k^2u_\beta(\varphi)e^{\beta(x-L)}v\Big)\,dxdy,\end{aligned}$$

where $v\in\mathrm{H}^1(\Omega^+_M)$ is the unique solution to the half-guide problem (5.44) with L replaced by M and data φ replaced by ψ. In addition, $\|v\|_{\mathrm{H}^1(\Omega^+_M)}\leq C\|\psi\|_{\mathrm{H}^{1/2}(\Sigma_M)}$, for a constant C which does not depend on M. Since $u_\beta(\varphi)\in\mathcal{W}^1_\beta(\Omega^+_L)$, we have that there exists a constant C (independent of β and M) such that

$$\Big|\langle e^{\beta(M-L)}R^\beta_{L,M}\varphi,\psi\rangle_{\Sigma_M}\Big|\leq C\max(1,\beta)\|u_\beta(\varphi)\|_{\mathcal{W}^1_\beta(\Omega^+_L)}\|\psi\|_{\mathrm{H}^{1/2}(\Sigma_M)}.$$

In view of (6.64), we conclude that

$$\|R^\beta_{L,M}\|\leq CC_\beta\max(1,\beta)e^{-\beta(M-L)},$$

which is the result. □

7 Numerical experiments

In the following numerical computations, we set $h=5\times10^{-2}m$, $h_{\text{in}}=7.5\times10^{-2}m$ and $h_{\text{out}}=12.5\times10^{-2}m$. The core is made of steel, with $\mu_0=84.298\times10^9\,Pa$ and $\rho_0=7932\,kg.m^{-3}$, while the sheath is made of concrete, with $\mu_\infty=15.908\times10^9\,Pa$ and $\rho_\infty=2300\,kg.m^{-3}$. Using those values, it is easy to check that $c_0>c_\infty$, which is consistent with Assumption 2.2. We also take $\omega=2\pi\times10^5\,Hz$. All the numerical experiments have been implemented using the FEM-BEM library XLiFE++ (see [Kielbasiewicz & Lunéville, 2019]).

7.1 Computation of the modes

First of all, we wish to compute the transverse eigenvalues $\pm\lambda_n$ in Λ by using one-dimensional Lagrange $P1$ finite elements and a regular mesh in the transverse section $(-h_{\text{out}},h_{\text{out}})$ of size $10^{-3}\,cm$. These approximate eigenvalues are plotted on the Figure 8 for the abrupt PMLs (α is defined by (2.11) with $\alpha_\infty=e^{-i\pi/3}$) and for the smooth PMLs (α is defined by (2.14) with $b=3(3+4i)$). We observe three kinds of computed eigenvalues: those which correspond to leaky modes (the red circles), those which correspond to PML modes (the blue squares) and those which correspond to spurious modes (the green triangles), that is modes which are due to the discretization and not to the truncation of the PMLs (see for example [Kim & Pasciak, 2009, Nannen & Wess, 2018] for discussions about spurious modes in the presence of PMLs). We now discuss how we distinguish the different modes. The spurious modes are identified by using the dispersion relationship (3.21), which is not satisfied by the spurious eigenvalues, but is by the other modes (up to the discretization error). We observe that within our computation window, spurious modes only appear in the case of the abrupt PMLs. Since the support of a leaky mode is mostly included in the physical part of the medium while the support of a PML mode is mostly included in the PMLs, we adopt the following criterium to separate the computed leaky modes from the computed PML modes. Following [Nguyen et al., 2015], we associate to sections $S^{\text{tot}}:=(-h_{\text{out}},h_{\text{out}})$ and $S^{\text{PML}}:=(-h_{\text{out}},-h_{\text{in}})\cup(h_{\text{in}},h_{\text{out}})$ the generalized kinetic energy of the mode φ_n defined by

$$E_n^{\text{tot}}:=\frac{\omega^2}{4}\int_{S^{\text{tot}}}\frac{\rho}{\alpha}|\varphi_n|^2\,dy,\quad E_n^{\text{PML}}:=\frac{\omega^2}{4}\int_{S^{\text{PML}}}\frac{\rho}{\alpha}|\varphi_n|^2\,dy.$$

For some prescribed $\eta\in(0,1)$, if $|E_n^{\text{PML}}|/|E_n^{\text{tot}}|<\eta$, the mode φ_n is considered as a leaky mode. If not, it is considered as a PML mode. In practice, we choose $\eta=0.4$ for abrupt PMLs and $\eta=0.9$ for smooth PMLs. Indeed, the numerical experiments have shown that both PML and spurious modes are almost totally confined in the PML for the smooth profile, whereas it is not as sharp for the abrupt profile. As a consequence, the choice of η is easier for the smooth profile.

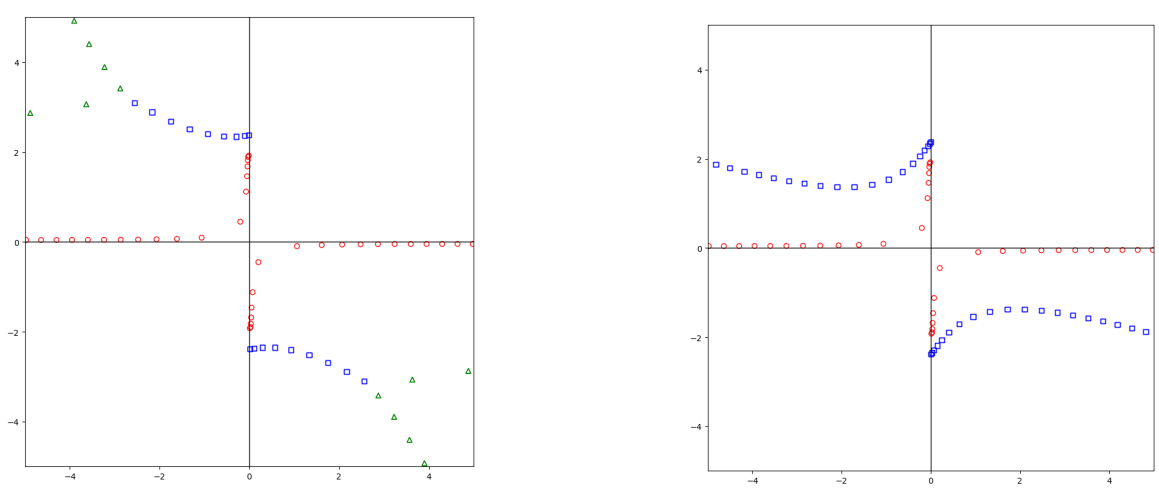

Figure 8: Discretized transverse eigenvalues related to the open waveguide closed by truncated PMLs (red circles correspond to leaky modes, blue squares correspond to PML modes, green triangles correspond to spurious modes). Left: abrupt PMLs. Right: smooth PMLs.

In order to check numerically that the modes φ_n satisfy Assumption 3.1, for the abrupt PMLs we have represented on the Figure 9 the normalized quantity

$$j_n := \left| \frac{\int_{I_{\text{out}}} \frac{\mu}{\alpha} \varphi_n^2 \, dy}{\int_{I_{\text{out}}} \mu |\varphi_n|^2 \, dy} \right|^{\frac{1}{2}} \tag{7.82}$$

with respect to n, with $n = 0, \cdots, 18$ (it includes both leaky and PML modes). Two different mesh sizes were used, either $10^{-2} cm$ (red circles) or $10^{-3} cm$ (blue crosses). It can be seen on that example that the j_n are bounded away from 0.

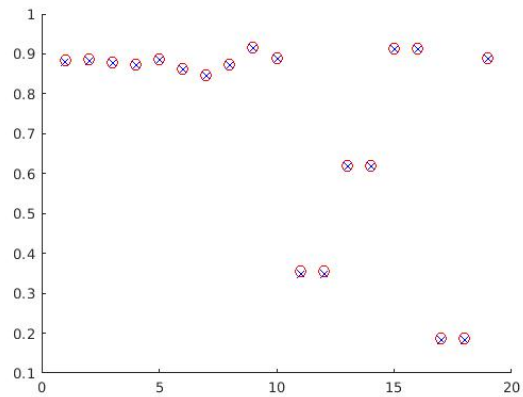

Figure 9: The quantities j_n defined by (7.82) are represented as a function of n. Red circles (resp. blue crosses) correspond to the mesh interval $10^{-2} cm$ (resp. $10^{-3} cm$).

7.2 The role of PMLs

In order to emphasize the role of PMLs, we consider the approximation of the total solution u of the problem (5.41) without any obstacle in the whole domain Ω, for $u^i = \tilde{w}_{0,0}^+$, where $\tilde{w}_{0,0}^+$ is defined by (5.40) and $\tilde{w}_0^+ = e^{ik_0 x}/\sqrt{2h}$. In other words, we send the first propagating mode of the closed waveguide coming from the left, which happens to be a plane wave. In Figure 10, where the real part of the solution $u_{L,M}^\beta$ to the problem (6.59) is represented, we compare the abrupt PMLs given by (2.11) with $\alpha_\infty = e^{-i\pi/3}$ (left picture) and the smooth PMLs given by (2.14) with $b = 3(3+4i)$ (right picture). The computation is based on a two-dimensional Lagrange $P2$ finite element method, the mesh size being $0.15 cm$. The left Dirichlet-to-Neumann operator T_0 given by (5.43) is discretized by truncating the series by the first 22 terms corresponding to 7 propagating modes and 15 evanescent modes. The Dirichlet-to-Neumann operator with an overlap $T_{L,M}^\beta$ given by (6.58) is used on the right with $\beta = 1.07$, $L = 15 cm$ and $M = 25 cm$. For the particular functions α that we chose, we remark that in the abrupt PMLs the solution $u_{L,M}^\beta$ vanishes almost everywhere while in the smooth ones it vanishes at the outer boundary of the PMLs. The advantage of smooth PMLs is that they are continuous at the boundary between the physical and the artificial media (at $y = \pm h_{\text{in}}$), which at the discrete level reduces spurious reflexions (see for example [Joly, 2012]).

7.3 The role of the overlap in the right DtN operator

We now emphasize the role of the overlap in the right Dirichlet-to-Neumann operator $T_{L,M}^\beta$ for the problem (5.53) in the presence of a Dirichlet obstacle formed by a disk centered at point $(6 cm, 1.75 cm)$ and of radius $1.25 cm$. The incident field u^i is the same as above, that is we send the first propagating mode of the left closed waveguide. We represent on Figure 11 the real part of the total solution u when either a thin DtN operator ($M = L = 10 cm$) or a thick DtN operator ($L = 10 cm$ and $M = 20 cm$) is used. Here we have chosen the smooth PMLs and $\beta = 0.04$. The computation is again based on a two-dimensional Lagrange $P2$ finite element method, the mesh size being $0.15 cm$ and the approximation of the operator T_0 is the same as previously. On Figure 11, it can be seen that the two solutions

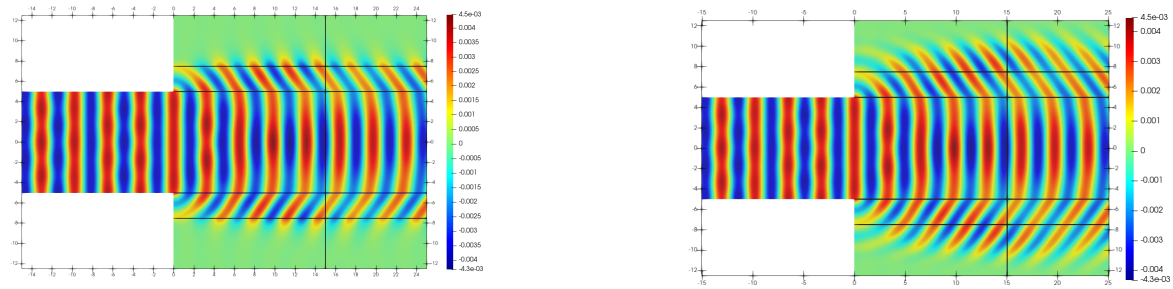

Figure 10: Left: abrupt PMLs. Right: smooth PMLs.

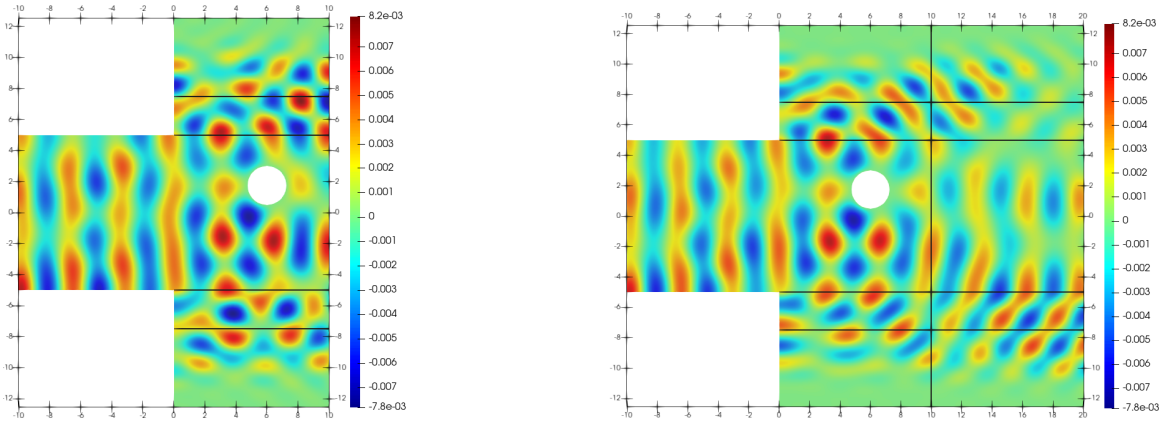

Figure 11: Left: thin DtN. Right: DtN with an overlap.

are "qualitatively close" in the physical domain. To refine our study, we compare the solution $u^{\beta}_{L,M}$ obtained for different values of the overlap and $\beta = 0.21$, precisely $L = 4$ and $M = 4$ (thin DtN), $M = 5$, $M = 6$, $M = 7$ and $M = 8$, with a reference solution $u_{\text{ref}} := u^{\beta_{\text{ref}}}_{L,M_{\text{ref}}}$ obtained with a large value of β, that is $\beta_{\text{ref}} = 1.62$, and a large overlap, that is $M_{\text{ref}} = 14$. The computation is again based on a two-dimensional Lagrange $P2$ finite element method, but with a refined mesh of size $0.05\,cm$, while the truncated series defining the left Dirichlet-to-Neumann operator T_0 contains 37 terms, that is 7 propagating modes and 30 evanescent modes. We have plotted the function $|u^{\beta}_{L,M} - u_{\text{ref}}|/\|u_{\text{ref}}\|_{\infty}$ on Figure 12. In view of the estimate (6.67), we expect that the error between the exact solution and the solution approached with the DtN operator improves with respect to the length of the overlap $(M - L)$, which is exactly what Figure 12 illustrates.

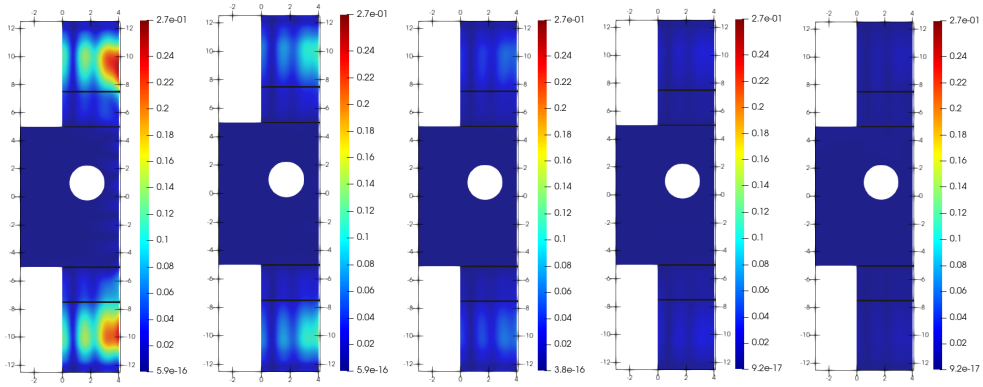

Figure 12: Impact of the overlap on the numerical error. From left to right: $M = L = 4$, $M = 5$, $M = 6$, $M = 7$, $M = 8$.

In order to be more quantitative, we complete this numerical section by testing the estimate (6.67)

numerically, that is we plug the graph of the function f such that

$$\log\left(\frac{\|u^{\beta}_{L,M}-u_{\mathrm{ref}}\|_{H^1(D_L)}}{\|u_{\mathrm{ref}}\|_{H^1(D_L)}}\right)=f(M-L), \tag{7.83}$$

which is supposed to be at worse a line of slope $-\beta$. On the top left picture of Figure 13, we have drawn this function in the case of the abrupt PMLs for different increasing values of β. Increasing β amounts to increase the number of modes N_β taken in the series (6.58). In the Figure 1, we have specified, for each value of β, the number N_{leaky} of leaky modes and the number N_{PML} of PML modes. We observe that when β is not too large, the graph of f looks like a line, the slope of which increases with β, which is consistent with the estimate (6.67). However, the absolute value of these slopes are much larger than β, which is hence a pessimistic value of the slope. In addition, it seems that, except for $\beta=1.07$ and $\beta=1.62$, the larger is β, the better is the approximation. The swap between the curves corresponding to $\beta=1.07$ and $\beta=1.62$ for small values of the overlap $(M-L)$ is due to a bad discrete approximation of the PML modes φ_n which correspond to large absolute values of the real parts $\Re e(\lambda_n)$: the solver does not really separate those PML modes from the spurious modes. A precise explanation of that phenomenon for abrupt PMLs (based on the notion of pseudo-spectrum) can be found in [Goursaud, 2010], where it is shown that when the distance between the PMLs and the core increases, the approximation of both leaky and PML modes worsens and the spurious modes appear for smaller $|\Re e(\lambda)|$. The consequence is that it deteriorates the approximation of the Dirichlet-to-Neumann operator, and therefore the approximation of the solution. The top right and bottom pictures of Figure 13 illustrate two different ways to fix this problem. The first one consists in using our smooth PMLs instead of the abrupt ones. The bad approximation of high order PML modes seems not to happen with the smooth PMLs that we chose, which is a consequence of the fact that, as already emphasized on Figure 8, spurious modes appear for larger values of $|\Re e(\lambda_n)|$ than for abrupt PMLs. The second way consists in sticking the abrupt PMLs to the core of the waveguide, that is $h_{\mathrm{in}}=h$, the thickness $(h_{\mathrm{out}}-h_{\mathrm{in}})$ of the PML being unchanged, which improves the computation of the PML modes. However, it should be noted that such technique is not satisfactory if one wants to compute the solution in the sheath and not only in the core of the waveguide.

Lastly, we note that the curves which correspond to $\beta=1.07$ and $\beta=1.62$ almost coincide for large $(M-L)$, whatever the PMLs, which is a saturation effect due to the fact that the mesh is too coarse for such a number of modes.

Remark 7.1. *The reader should not conclude from our numerical results that smooth PMLs give better results than abrupt PMLs in general. Our experience is that those results depend in a complex way on the properties of the function α together with the mesh size. In particular, even by selecting the parameters of the smooth and abrupt PMLs so that κ defined by (3.22) is the same in both cases, the conclusion seemed not clear. A comprehensive comparison between these two kinds of profiles is not the objective of the present paper. We only claim that taking into account spurious modes worsens the quality of the numerical approximation of the DtN operators.*

	Abrupt PMLs		Smooth PMLs		Stuck abrupt PMLs	
β	N_{leaky}	N_{PML}	N_{leaky}	N_{PML}	N_{leaky}	N_{PML}
0.038	3	2	3	4	3	2
0.073	6	2	6	6	5	2
0.21	7	4	7	8	7	4
1.07	8	10	8	12	8	8
1.62	9	12	9	14	9	12

Table 1: Number of leaky and PML modes for the different values of β in Figure 13.

8 Conclusions and perspectives

We begin this section with a few comments on the assumptions made throughout this paper. Firstly, Assumption 2.1 amounts to suppose that none of the modes in the left half-waveguide has a vanishing

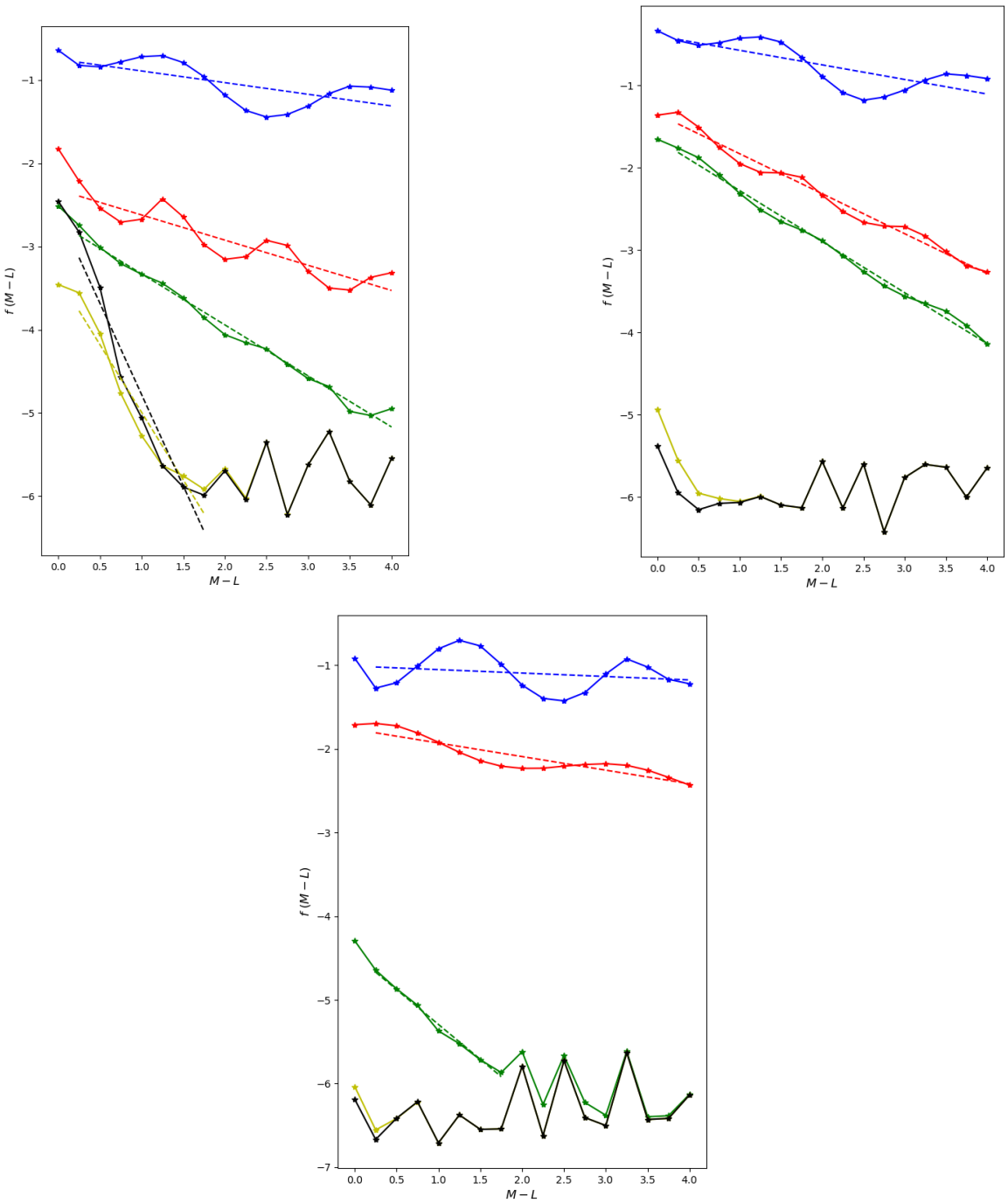

Figure 13: Graph of the function f defined by (7.83) showing the error estimate with respect to $(M-L)$. Blue: $\beta=0.038$. Red: $\beta=0.073$. Green: $\beta=0.21$. Yellow: $\beta=1.07$. Black: $\beta=1.62$. Top left: abrupt PMLs. Top right: smooth PMLs. Bottom: abrupt PMLs stuck to the core.

group velocity. This is a reasonable assumption in the frequency domain. However, if we envision a real NDT experiment in the time domain, it is not easy to control the incident wave in order to avoid those cut-off frequencies. As shown numerically in [Baronian et al., 2016], if the support of the Fourier transform of the incident field meets the cut-off frequencies, the corresponding scattered field is slowly decaying with respect to time, which requires to measure it during a long time interval to solve the inverse problem and constitutes a real drawback in practice.

Assumption 2.2 (the celerity in the core is larger than the celerity in the sheath) was made because

it is satisfied in the practical NDT applications we have in mind (see the beginning of the introduction). The case when the celerity in the sheath is larger than the celerity in the core is more complicated from the point of view of the analysis because some guided modes might exist in the right half-waveguide, which implies the specification of a radiation condition. However, incorporating those guided modes in the Kondratiev approach, which requires some new functional spaces and introduces additional technicalities, is doable (see for example [Nazarov & Plamenevskiĭ, 1991], [Nazarov, 2013] or [Bourgeois et al., 2019]).

Concerning Assumption 3.1, which corresponds to the absence of Jordan blocks, we think it holds generically. We have numerically illustrated its validity on a particular example in section 7 (see Figure 9). However, as can be seen in [Kozlov et al., 1997], the Kondratiev analysis can be carried out in the presence of Jordan blocks. In that case, the residues in Proposition 10 have a more complicated expression.

We complete this section by highlighting a few challenging theoretical problems which are not addressed in this paper. The first one is the question of completeness of the transverse eigenfunctions φ_n of the non-selfadjoint operator L. The second one is the uniqueness question in problem (5.41), even in the absence of an obstacle. The last one is of course the well-posedness of the problem set in the original configuration of Figure 1.

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