An Integro-differential Operator Approach to Linear State-space Systems

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Abstract: In this paper, the algebraic analysis approach to linear state-space systems is further developed using rings of integro-differential operators. The module structure of linear state-space systems is investigated over these rings. The module associated with a linear state-space system is shown to be the direct sum of the stably free module defined by the linear system without inputs and the free module defined by the inputs of the system.

Keywords: Linear systems, continuous-time linear state-space models, polynomial methods, algebraic analysis, rings of integro-differential operators, system equivalence, behaviours

1. INTRODUCTION

Algebraic analysis is a mathematical theory, developed by Malgrange, Sato, Bernstein, Kashiwara, etc., which studies linear systems of ordinary or partial differential equations using rings of differential operators, module theory, homological algebra, complex analysis, sheaf theory, etc. See Kashiwara et al. (1971) and the references therein. Algebraic analysis nowadays plays an important role in different theories of fundamental mathematics.

In the nineties, algebraic analysis was introduced in control theory by Oberst, Fliess, Pommaret, etc. to study different classes of linear control systems such as ordinary or partial differential equations, differential time (dependent) delay equations, multidimensional systems, etc. (Oberst (1990); Fliess (1990); Fliess and al. (1998); Pommaret et al. (1999); Chyzak et al. (2005); Zerz (2006); Quadrat (2010, 2015)).

Within the algebraic analysis approach, a linear functional systems yields a *finitely presented left module* over a noncommutative polynomial ring a functional operators. Structural properties and equivalences of linear systems can be intrinsically reformulated within *module theory* and *homological algebra* (Rotman (2009)). Using computer algebra methods over noncommutative polynomial rings (e.g., *Gröbner bases*), dedicated packages can then be developed in standard computer algebra systems (Chyzak et al. (2005); Cluzeau et al. (2008); Quadrat (2010)).

The goal of the paper is to further develop the algebraic analysis approach to linear state-space systems using *rings of integro-differential operators*, initiated in Quadrat (2022). Rings of integro-differential operators have recently been studied in algebra (Bavula (2013)), in control theory (Quadrat (2015)), and in computer algebra (see Cluzeau et al. (2018) and the references therein). The use of these rings allows one to algebraize elementary calculus by combining the ordinary differential operator d/dt, the integral operator $\int_{t_0}^{t} \cdot \, d\tau$, and the evaluation at the initial time t_0. As shown in Quadrat (2022), the classical study of linear state-space systems (Kalman et al. (1969)) can then be reformulated within this functional operator algebra approach. In this paper, the module structure of linear state-space systems is investigated. We show that the module defined by a linear state-space system is the direct sum of the *stably free module* defined by with the linear system without inputs and the *free module* defined by the inputs of the system.

2. RINGS OF INTEGRO-DIFFERENTIAL OPERATORS

Let $\mathcal{A}$ be a $\Bbbk$-algebra, where $\Bbbk$ is a field of characteristic 0 (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$). Let $\mathrm{end}_{\Bbbk}(\mathcal{A})$ be the noncommutative ring formed by the all the $\Bbbk$-endomorphisms of $\mathcal{A}$, namely, the $\Bbbk$-linear maps from $\mathcal{A}$ to $\mathcal{A}$. A *derivation* ∂ of $\mathcal{A}$ is a $\Bbbk$-endomorphism of $\mathcal{A}$ satisfying the standard *Leibniz rule*:

$$\forall \, a_1, \, a_2 \in \mathcal{A}, \quad \partial(a_1 \, a_2) = \partial(a_1) \, a_2 + a_1 \, \partial(a_2). \tag{1}$$

Then, $(\mathcal{A}, \partial)$ is called a *differential ring*. The *subring of constants* of $\mathcal{A}$ is defined by $\mathcal{C}(\mathcal{A}) = \{a \in \mathcal{A} \mid \partial(a) = 0\}$.

Example 1. Standard examples of commutative differential rings for $\partial = d/dt$ are the ring $\Bbbk[t]$ of polynomials in t with coefficients in $\Bbbk$, the Laurent polynomial ring $\Bbbk[t, t^{-1}]$ of polynomials in t and t^{-1} with coefficients in $\Bbbk$, the ring of $\Bbbk$-valued smooth functions/distributions on an open interval of $\mathbb{R}$, where $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, the ring of analytic/holomorphic/meromorphic functions on an domain of $\mathbb{C}$. We then have $\mathcal{C}(\mathcal{A}) = \Bbbk$. Finally, if $(\mathcal{A}, \partial)$ is one of the above differential ring and $n \in \mathbb{Z}_{>0}$, then $(\mathcal{A}^{n \times n}, \partial \, \mathbb{1}_n)$ is a noncommutative differential ring with $\mathcal{C}(\mathcal{A}^{n \times n}) = \Bbbk^{n \times n}$, where $\mathbb{1}_n$ denotes the identity of the noncommutative ring $\mathcal{A}^{n \times n}$ formed by all the $n \times n$-matrices with entries in $\mathcal{A}$.

To simplify the exposition, in what follows, we shall simply assume that $\mathcal{A}$ is the ring of $\Bbbk$-valued smooth functions on an open subset $\mathcal{U}$ of $\mathbb{R}$, where $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, simply denoted by $\mathcal{A} = C^{\infty}(\mathcal{U})$. But the following construction can be extended to the case of a general differential ring $(\mathcal{A}, \partial)$ as explained in Cluzeau et al. (2018); Quadrat (2022).

Let $(\mathcal{A} = C^{\infty}(\mathcal{U}), \partial = d/dt)$ be the differential ring of real-valued smooth functions on $\mathcal{U}$. Fix $t_0 \in \mathcal{U}$ and let

$$\forall\, a \in \mathcal{A}, \quad \forall\, t \in \mathcal{U}, \quad I(a)(t) = \int_{t_0}^{t} a(\tau)\, d\tau$$

be the indefinite Riemann integral. Clearly, $I \in \mathrm{end}_{\Bbbk}(\mathcal{A})$. Let $\mathbb{1}$ be the identity of $\mathrm{end}_{\Bbbk}(\mathcal{A})$, i.e., $\mathbb{1}(a) = a$ for all $a \in \mathcal{A}$. Let $e \in \mathrm{end}_{\Bbbk}(\mathcal{A})$ be the *evaluation map* at $t = t_0$:

$$\forall\, a \in \mathcal{A}, \quad \forall\, t \in \mathcal{U}, \quad e(a)(t) = a(t_0).$$

Note that e is *multiplicative*, namely, $e(a_1\, a_2) = e(a_1)\, e(a_2)$ for all $a_1,\, a_2 \in \mathcal{A}$, and an *idempotent*, i.e., $e^2 = e$. In what follows, the composition of elements of $\mathrm{end}_{\Bbbk}(\mathcal{A})$ will be denoted multiplicatively, i.e.:

$$\forall\, d_1,\, d_2 \in \mathrm{end}_{\Bbbk}(\mathcal{A}), \quad d_1\, d_2 = d_1 \circ d_2.$$

Note that we have the following identities

$$\forall\, a \in \mathcal{A}, \quad \begin{cases} \dfrac{d}{dt} \displaystyle\int_{t_0}^{t} a(\tau)\, d\tau = a(t), \\ \displaystyle\int_{t_0}^{t} \dfrac{d}{d\tau}\, a(\tau)\, d\tau = a(t) - a(t_0) = a(t) - e(a)(t), \end{cases}$$

which yield the following identities in $\mathrm{end}_{\Bbbk}(\mathcal{A})$:

$$\partial\, I = \mathbb{1}, \quad I\, \partial = \mathbb{1} - e. \tag{2}$$

The structure (A, ∂, I, e) satisfying (2) is called an *integro-differential ring*. For the general definition of an integro-differential ring, see Cluzeau et al. (2018); Quadrat (2022).

We can extend the integro-differential ring structure from (A, ∂, I, e) to $(\mathcal{A}^{n \times n}, \partial\, \mathbb{1}_n, I\, \mathbb{1}_n, e\, \mathbb{1}_n)$ for all $n \in \mathbb{Z}_{>0}$.

We can now introduce the *ring of integro-differential operators* with coefficients in $\mathcal{A}$. This concept will play a fundamental role in what follows. The *ring of integro-differential operators*, denoted by $\mathcal{I}$, is the $\Bbbk$-sub-algebra of $\mathrm{end}_{\Bbbk}(\mathcal{A})$ generated by the following elements of $\mathrm{end}_{\Bbbk}(\mathcal{A})$:

$$\begin{aligned} & b : a(\cdot) \longmapsto a(\cdot)\, b(\cdot),\ b \in \mathcal{A}, \\ & \partial : a(\cdot) \longmapsto \dot{a}(\cdot),\ \partial(a)(t) = \frac{da(t)}{dt} = \dot{a}(t), \\ & I : a(\cdot) \longmapsto c(\cdot),\ c(t) = I(a)(t) = \int_{t_0}^{t} a(\tau)\, d\tau, \\ & e : a(\cdot) \longmapsto a(t_0). \end{aligned} \tag{3}$$

Thus, an element of $\mathcal{I}$ is a $\Bbbk$-endomorphism of $\mathcal{A}$ defined as a finite $\Bbbk$-linear combination of *words* defined by the *letters* ∂, I, e, and $b \in \mathcal{A}$. For instance, if $a_1,\, a_2 \in \mathcal{A}$, then $a_1\, I\, a_2 \in \mathcal{I}$ is the $\Bbbk$-endomorphism of $\mathcal{A}$ defined by:

$$a_1\, I\, a_2 : d(\cdot) \longmapsto c(\cdot),\ c(t) = a_1(t) \int_{t_0}^{t} a_2(\tau)\, d(\tau)\, d\tau.$$

Note that $\mathcal{I}$ is the noncommutative polynomial ring $\mathcal{A}\langle \partial, I, e \rangle$ formed by all the noncommutative polynomials in ∂, I, e, $a \in \mathcal{A}$ satisfying relations that we now study.

Let us now study the relations that satisfy the $\Bbbk$-endomorphisms ∂, I, e, and a for all $a \in A$. We first have

$$\begin{aligned} \forall\, b \in \mathcal{A}, \quad (\partial\, a)(b) &= \frac{d}{dt}(a\, b) = a \left(\frac{db}{dt}\right) + \left(\frac{da}{dt}\right) b \\ &= (a\, \partial + \partial(a))(b), \end{aligned}$$

i.e., $\partial\, a = a\, \partial + \partial(a)$ for all $a \in \mathcal{A}$. Moreover, we also have:

$$\forall\, b \in \mathcal{A}, \quad (\partial\, I)(b)(t) = \frac{d}{dt} \int_{t_0}^{t} b(\tau)\, d\tau = b(t),$$

$$(I\, \partial)(b)(t) = \int_{t_0}^{t} \dot{b}(\tau)\, d\tau = b(t) - b(t_0) = (\mathbb{1} - e)(b)(t),$$

$$\forall\, a,\, b \in \mathcal{A}, \quad (e\, a)(b) = e(a\, b) = a(t_0)\, b(t_0) = (e(a)\, e)(b).$$

Hence, we have the following relations in $\mathcal{I} = \mathcal{A}\langle \partial, I, e \rangle$:

$$\forall\, a \in \mathcal{A}, \quad \begin{cases} \partial\, a = a\, \partial + \partial(a), \\ \partial\, I = \mathbb{1}, \\ I\, \partial = \mathbb{1} - e, \\ e\, a = e(a)\, e. \end{cases} \tag{4}$$

For the general definition of a ring of integro-differential operators, see, e.g., Cluzeau et al. (2018); Quadrat (2022).

Note that ∂, I, e, and $a \in \mathcal{A}$ also satisfy the relations

$$\begin{aligned} & e^2 = e, \quad e\, I = 0, \quad \partial\, e = 0, \\ & \forall\, a \in \mathcal{A},\ \ I\, a\, \partial = -I\, \partial(a) + a - e(a)\, e, \\ & \forall\, a \in \mathcal{A},\ \ I\, a\, I = [I(a), I], \end{aligned} \tag{5}$$

where $[a, b] = a\, b - b\, a$ is the *commutator*. They are direct consequences of (4). Indeed, using (4), we first get:

$$e^2 = (\mathbb{1} - I\, \partial)\, (\mathbb{1} - I\, \partial) = \mathbb{1} - I\, \partial - I\, \partial + I\, (\partial\, I)\, \partial = \mathbb{1} - I\, \partial = e.$$

Moreover, using (4), we have:

$$e\, (I\, \partial) = e\, (\mathbb{1} - e) = e - e^2 = 0 \implies e\, I = (e\, I\, \partial)\, I = 0.$$

Using (4), we also obtain:

$$\partial\, e = \partial\, (\mathbb{1} - I\, \partial) = \partial - \partial = 0.$$

In particular, we have $e(a) \in \mathcal{C}(\mathcal{A})$ for all $a \in \mathcal{A}$, i.e., $e(\mathcal{A}) \subseteq \mathcal{C}(\mathcal{A}) = \Bbbk$. The last but one identity of (5) corresponds to the standard *integration by parts* since:

$$I\, a\, \partial = I\, (\partial\, a - \partial(a)) = (\mathbb{1} - e)\, a - I\, \partial(a) = a - e(a)\, e - I\, \partial(a).$$

Finally, using the first identity of (4) with $I(a)$, we first get $\partial\, I(a) = I(a)\, \partial + \partial(I(a)) = I(a)\, \partial + a$, and the last identity of (5) can be proved as follows:

$$\begin{aligned} I\, a\, I &= I\, (\partial\, I(a) - I(a)\, \partial)\, I = I\, \partial\, I(a)\, I - I\, I(a) \\ &= (\mathbb{1} - e)\, I(a)\, I - I\, I(a) = I(a)\, I - I\, I(a) = [I(a), I]. \end{aligned}$$

Setting $a = 1$ in this last identity of (5) and using $I(1) = t - t_0$, $I^2 = [t - t_0, I] = t\, I - I\, t = [t, I]$, which shows that I^2 is a polynomial of degree 1 in I. Note that $I^2(a) = I\, (t - \tau)\, a(\tau)\, d\tau$ is a convolution product. More generally, we have $I^n(a) = I\, \frac{(t-\tau)^{n-1}}{(n-1)!}\, a(\tau)\, d\tau$, i.e., I^n can be rewritten as a polynomial of degree 1 in I as follows:

$$I^n = \sum_{k=0}^{n-1} \frac{t^k}{k!}\, I\, \frac{(-t)^{n-1-k}}{(n-1-k)!}.$$

Using (4) and (5), we can show that an element d of $\mathcal{I}$ can be uniquely be written as a finite sum of terms of the form $a\, \partial^j$, $a\, I\, b$ and $a\, e\, \partial^j$, where $a,\, b \in \mathcal{A}$ and $j \in \mathbb{Z}_{\geq 0}$ (Cluzeau et al. (2018)). This form is called the *normal form* of d. Hence, $\mathcal{I}$ contains the ring $\mathcal{D} = \mathcal{A}\, \langle \partial \mid \partial\, a = a\, \partial + \partial(a) \rangle$ of differential operators. Moreover, $\mathcal{I}$ also contains the *Taylor algebra* $\mathcal{T} = \mathcal{A}\langle \partial,\, e \mid \partial\, a = a\, \partial + \partial(a),\, \partial\, e = 0 \rangle$ of differential evaluation operators, as well as the ring (without multiplicative identity) $\mathfrak{I}$ of integro operators:

$$\mathfrak{I} = \mathcal{A}\, \langle I \mid I\, a\, I = [I(a), I] \rangle = \left\{ \sum_{i=1}^{r} a_i\, I\, b_i \mid a_i,\, b_i \in \mathcal{A} \right\}.$$

We note that $\mathcal{I}$ has zero-divisors (e.g., e, I, ∂). The fact that ∂ is a left but not a right inverse of I yields $\mathcal{I}$ is not *Dedekind-finite* (Lam (1999)), and thus, not a *noetherian ring* due to a result of Jacobson (Jacobson (1950)). Hence, the ring structure of $\mathcal{I}$ and the module structure of $\mathcal{I}$ are more involved than its of $\mathcal{D}$, and still have to be investigated. If $\mathcal{A} = \Bbbk[t]$, Bavula (2013) proves that $\mathcal{I}$ is a *coherent ring* (Rotman (2009)), which means that an algorithmic approach to *the category of finitely presented* $\Bbbk[t]\langle\partial, I, e\rangle$*-modules* can then be developed and implemented in standard computer algebra systems.

3. ALGEBRAIC ANALYSIS APPROACH TO FIRST-ORDER LINEAR SYSTEMS

Let $\mathcal{I} = \mathcal{A}\langle\partial, I, e\rangle$ be the ring of integro-differential operators over the ring $\mathcal{A}$ defined in Section 2. Let us consider $A \in \mathcal{A}^{n\times n}$ and $R = \partial\, \mathbb{1}_n - A \in \mathcal{I}^{n\times n}$. We can consider the following respectively left and right $\mathcal{I}$-*homomorphisms* (i.e., left and right $\mathcal{I}$-linear maps):

$$\begin{array}{rclrcl} .R : \mathcal{I}^{1\times n} & \longrightarrow & \mathcal{I}^{1\times n} & R. : \mathcal{I}^{n\times 1} & \longrightarrow & \mathcal{I}^{n\times 1} \\ \mu = (\mu_1, \ldots, \mu_n) & \longmapsto & \mu\, R, & \eta = (\eta_1, \ldots, \eta_n)^T & \longmapsto & R\, \eta. \end{array}$$

We can consider the following canonical left $\mathcal{I}$-modules:

$$\begin{cases} \ker_{\mathcal{I}}(.R) = \left\{\mu \in \mathcal{I}^{1\times n} \mid \mu\, R = 0\right\}, \\ \operatorname{im}_{\mathcal{I}}(.R) = \left\{\lambda \in \mathcal{I}^{1\times n} \mid \exists\, \mu \in \mathcal{I}^{1\times n} : \lambda = \mu\, R\right\}, \\ \mathcal{M} = \operatorname{coker}_{\mathcal{I}}(.R) = \mathcal{I}^{1\times n}/\operatorname{im}_{\mathcal{I}}(.R). \end{cases}$$

Similar left $\mathcal{I}$-modules can be defined for $P \in \mathcal{I}^{q\times p}$.

Let $\{e_i\}_{i=1,\ldots,n}$ denote the *standard basis* of $\mathcal{I}^{1\times n}$, namely, e_i is the row vector of length n with 1 in the i^{th} entry and 0 elsewhere. In $\mathcal{M}$, we can identify the vectors $\lambda, \lambda' \in \mathcal{I}^{1\times n}$ satisfying $\lambda' - \lambda \in \operatorname{im}_{\mathcal{I}}(.R)$ by considering their *residue classes* $\pi(\lambda)$ and $\pi(\lambda')$. Hence, λ' and λ define the same residue class, i.e., $\pi(\lambda) = \pi(\lambda')$, if there exists $\mu \in \mathcal{I}^{1\times n}$ such that $\lambda' = \lambda + \mu\, R$. Let $\pi : \mathcal{I}^{1\times n} \longrightarrow \mathcal{M}$ be the canonical projection onto $\mathcal{M}$ defined by mapping $\lambda \in \mathcal{I}^{1\times n}$ onto $\pi(\lambda)$. Note that π is a left $\mathcal{I}$-homomorphism, i.e., $\pi(a_1\, \lambda_1 + a_2\, \lambda_2) = a_1\, \pi(\lambda_1) + a_2\, \pi(\lambda_2)$ for all a_1, $a_2 \in \mathcal{I}$ and λ_1, $\lambda_2 \in \mathcal{I}^{1\times n}$. If we set $x_i = \pi(e_i)$ for $i = 1, \ldots, n$, then the i^{th} row $R_{i\bullet} = (R_{i1}, \ldots, R_{in})$ of the matrix R satisfies $R_{i\bullet} = e_i\, R \in \operatorname{im}_{\mathcal{I}}(.R)$, yielding $\pi(R_{i\bullet}) = 0$ and

$$\sum_{i=1}^{n} R_{ij}\, y_i = \sum_{i=1}^{n} R_{ij}\, \pi(e_i) = \pi\left(\sum_{i=1}^{n} R_{ij}\, e_j\right) = \pi(R_{i\bullet}) = 0,$$

for $i = 1, \ldots, n$. Thus, $\{x_i\}_{i=1,\ldots,n}$ is a *set of generators* of the left $\mathcal{I}$-module $\mathcal{M}$ and the generators x_i's satisfy the *left* $\mathcal{I}$*-linear relations* $R\, x = 0$, where $x = (x_1, \ldots, x_n)^T$. We then say that $\mathcal{M}$ is a *finitely presented* left $\mathcal{I}$-module and R is a *presentation matrix* of $\mathcal{M}$ (Rotman (2009)).

Let us state again recent results on the module-theoretic interpretation of the *method of variation of constants*. For more details, see Cluzeau et al. (2018); Quadrat (2022). Let Φ be the *transition matrix* of the linear system:

$$\dot{x}(t) = A(t)\, x(t). \tag{6}$$

We shall suppose that $\Phi : t \in \mathcal{U} \longmapsto \Phi(t, t_0)$ belongs to $\mathcal{A}^{n\times n}$. Thus, Φ is an invertible matrix, i.e., $\Phi^{-1} \in \mathcal{A}^{n\times n}$, which satisfyies (6) and $\Phi(t_0, t_0) = \mathbb{1}_n$.

Let us determine when $S = a_0\, I\, a_1 + a_2$, where a_0, a_1, and $a_2 \in \mathcal{A}^{n\times n}$, is a right inverse of R. Using (4), we have:

$$\begin{aligned} R\, S &= (\partial\, \mathbb{1}_n - A)\,(a_0\, I\, a_1 + a_2) \\ &= (\dot{a}_0 - A\, a_0)\, I\, a_1 + a_2\, \partial + \dot{a}_2 - A\, a_2 + a_0\, a_1. \end{aligned}$$

If $a_1 = 0$, then $R\, S = \mathbb{1}_n$ has no solution. Let us suppose that $a_1 \neq 0$. Using the above normal form for $R\, S$, we have $R\, S = \mathbb{1}_n$ if and only if:

$$\begin{cases} \dot{a}_0 - A\, a_0 = 0, \\ a_2 = 0, \\ \dot{a}_2 - A\, a_2 + a_0\, a_1 = \mathbb{1}_n, \end{cases} \iff \begin{cases} \dot{a}_0 - A\, a_0 = 0, \\ a_2 = 0, \\ a_0\, a_1 = \mathbb{1}_n, \end{cases}$$

$$\iff \begin{cases} a_0(t) = \Phi(t, t_0)\, c_0, \ c_0 \in \Bbbk^{n\times n}, \\ a_2 = 0, \\ c_0\, a_1 = \Phi(t, t_0)^{-1} = \Phi(t_0, t). \end{cases}$$

Thus, the matrix R has the following right inverse:

$$S = \Phi(t, t_0)\, c_0\, I\, a_1 = \Phi(t, t_0)\, I\, c_0\, a_1 = \Phi(t, t_0)\, I\, \Phi(t_0, t).$$

Using $R\, S = \mathbb{1}_n$, we get $R\,(S\, f) = f$ for all $f \in \mathcal{A}^{n\times 1}$, i.e.,

$$\begin{aligned} (S\, f)(t) &= \Phi(t, t_0)\, I\, \Phi(t_0, t)\, f(t) = \Phi(t, t_0) \int_{t_0}^{t} \Phi(t_0, s)\, f(s)\, ds \\ &= \int_{t_0}^{t} \Phi(t, t_0)\, \Phi(t_0, s)\, f(s)\, ds = \int_{t_0}^{t} \Phi(t, s)\, f(s)\, ds \end{aligned} \tag{7}$$

is a particular solution of the inhomogeneous system:

$$\dot{x}(t) - A(t)\, x(t) = f(t). \tag{8}$$

Note that $R\, S = \mathbb{1}_n$ also yields $\lambda = (\lambda\, R)\, S$, showing that $\ker_{\mathcal{I}}(.R) = 0$, i.e., the left $\mathcal{I}$-homomorphism $.R$ is injective. Then, we have the *short exact sequence* of $\mathcal{I}$-modules

$$0 \longrightarrow \mathcal{I}^{1\times n} \xrightarrow{.R} \mathcal{I}^{1\times n} \xrightarrow{\pi} \mathcal{M} \longrightarrow 0, \tag{9}$$

i.e., $.R$ is injective, $\ker \pi = \operatorname{im}_{\mathcal{I}}(.R)$, and π is surjective (Rotman (2009)). The existence of a right inverse S of R implies that the short exact sequence (9) *splits*, i.e., the existence of a left $\mathcal{I}$-homomorphism $\rho : \mathcal{M} \longrightarrow \mathcal{I}^{1\times n}$ satisfying the identity $(.R)\circ(.S) + \rho\circ\pi = \operatorname{id}_{\mathcal{I}^{1\times n}}$. For more details, see, e.g., Rotman (2009). This split short exact sequence of left $\mathcal{I}$-modules can be displayed as follows:

$$0 \longrightarrow \mathcal{I}^{1\times n} \underset{.S}{\overset{.R}{\rightleftarrows}} \mathcal{I}^{1\times n} \underset{\rho}{\overset{\pi}{\rightleftarrows}} \mathcal{M} \longrightarrow 0. \tag{10}$$

We state again that $\Psi = \Phi^{-1}$ satisfies $\dot{\Psi} + \Psi^{-1}\, A = 0$. Using the integration by parts and $e(\Psi) = \mathbb{1}_n$, we get:

$$\Pi := S\, R = \Phi\, I\,(\partial\, \Psi - \dot{\Psi} - \Psi\, A) = \Phi\,(1 - e)\,\Psi = \mathbb{1}_n - \Phi\, e.$$

Clearly, we have $\Pi^2 = \Pi$, i.e., Π is an idempotent of $\mathcal{I}^{n\times n}$. For $m = \pi(\lambda) \in \mathcal{M}$, where $\lambda \in \mathcal{I}^{1\times n}$, we then have:

$$\rho(m) = \rho(\pi(\lambda)) = \lambda\,(\mathbb{1}_n - S\, R) = \lambda\, \Phi\, e. \tag{11}$$

A consequence of the splitting exact sequence (10) is then

$$\mathcal{M} \oplus \mathcal{I}^{1\times n} \cong \mathcal{I}^{1\times n}, \tag{12}$$

where $\cong$ denotes the existence of an *isomorphism*, namely, a bijective homomorphism. Thus, $\mathcal{M}$ is a *stably free* left $\mathcal{I}$-module. For more details, see (Lam (1999); Rotman (2009)). Note that $\operatorname{im}_{\mathcal{I}}(.R) = \operatorname{im}_{\mathcal{I}}(.(\mathbb{1}_n - \Phi\, e))$ since

$$\operatorname{im}_{\mathcal{I}}(.S\, R) \subseteq \operatorname{im}_{\mathcal{I}}(.R) = \operatorname{im}_{\mathcal{I}}(.R\, S\, R) \subseteq \operatorname{im}_{\mathcal{I}}(.S\, R),$$

and thus, $\mathcal{M} = \operatorname{coker}_{\mathcal{I}}(.(\mathbb{1}_n - \Phi\, e))$. Using the set of generators $\{x_i\}_{i=1,\ldots,n}$ of $\mathcal{M}$, we have $e\, x_i \in \mathcal{M}$ for $i = 1, \ldots, n$, which shows that the left $\mathcal{I}$-module generated by all the $e\, x_i$'s, i.e., $\sum_{i=1}^{n} \mathcal{I}\, e\, x_i$, is a sub-module of $\mathcal{M}$. Now, using $R\, x = 0$, where $x = (x_1\ \ldots\ x_n)^T$, $S\, R\, x = 0$, i.e., $x = \Phi\, e\, x$, which shows that $\mathcal{M} = \sum_{i=1}^{n} \mathcal{I}\, e\, x_i$, i.e., $\mathcal{M}$ is also finitely generated by the set $\{e\, x_i\}_{i=1,\ldots,n}$.

Let $\mathcal{F}$ be a left $\mathcal{I}$-module (e.g., $\mathcal{F} = \mathcal{A}$) and let us consider the following $\Bbbk$-linear map:

$$\begin{array}{rcl} R. : \mathcal{F}^{n\times 1} & \longrightarrow & \mathcal{F}^{n\times 1} \\ \eta & \longmapsto & R\, \eta. \end{array}$$

The $\mathcal{F}$-solutions of (6), called *behaviour*, are defined by:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^{n\times 1} \mid R\,\eta = 0\}.$$

Let us denote by $\hom_{\mathcal{I}}(\mathcal{M}, \mathcal{F})$ the $\Bbbk$-vector space formed by all the left $\mathcal{I}$-homomorphisms from $\mathcal{M}$ to $\mathcal{F}$ and let $f \in \hom_{\mathcal{I}}(\mathcal{M}, \mathcal{F})$. Using the fact that $\{x_i = \pi(e_i)\}_{i=1,\ldots,n}$ is a set of generators of $\mathcal{M}$, then f is determined by the knowledge of $f(x_i) = \eta_i \in \mathcal{F}$ for $i = 1, \ldots, n$. But, since the x_i's satisfy the linear relations $R\,x = 0$, where $x = (x_1, \ldots, x_n)^T$, and $f(0) = 0$, then $\eta = (\eta_1, \ldots, \eta_n)^T$ must satisfy the following condition

$$\sum_{i=1}^{n} R_{ij}\,\eta_j = \sum_{i=1}^{n} R_{ij}\,f(x_j) = f\left(\sum_{i=1}^{n} R_{ij}\,x_j\right) = f(0) = 0,$$

i.e., $\eta \in \mathcal{F}^{n\times 1}$ must satisfy $R\,\eta = 0$. Hence, we have

$$\ker_{\mathcal{F}}(R.) \cong \hom_{\mathcal{I}}(\mathcal{M}, \mathcal{F}),$$

which is a basic result of homological algebra (see, e.g., Rotman (2009)) at the core of algebraic analysis. See, e.g., Chyzak et al. (2005) and the references therein.

More generally, dualizing (10) with respect to $\mathcal{F}$, i.e., applying the *contravariant functor* $\hom_{\mathcal{I}}(\,\cdot\,, \mathcal{F})$ to split short exact sequence (10) and using the isomorphism $\ker_{\mathcal{F}}(R.) \cong \hom_{\mathcal{I}}(\mathcal{M}, \mathcal{F})$ of $\Bbbk$-vector spaces, we obtain the following split short exact sequence of $\Bbbk$-vector spaces

$$0 \longrightarrow \ker_{\mathcal{F}}(R.) \underset{\Phi\,e.}{\overset{i}{\rightleftarrows}} \mathcal{F}^{n\times 1} \underset{S.}{\overset{R.}{\rightleftarrows}} \mathcal{F}^{n\times 1} \longrightarrow 0, \tag{13}$$

where i is the canonical injection and $R. : \mathcal{F}^{n\times 1} \longrightarrow \mathcal{F}^{n\times 1}$ is defined by $(R.)(\eta) = R\,\eta$ for all $\eta \in \mathcal{F}^{n\times 1}$, and similarly for $S.$ and $\Phi\,e.$. See, e.g., Rotman (2009). Note that $\Phi\,e. : \mathcal{F}^{n\times 1} \longrightarrow \ker_{\mathcal{F}}(R.)$ is well-defined since:

$$\forall\,\eta \in \mathcal{F}^{n\times 1},\ (\partial\,\mathbb{1}_n - A)\,\Phi\,e\,\eta = (\Phi\,\partial + \dot{\Phi} - A\,\Phi)\,e\,\eta = 0.$$

In what follows, we shall simply denote $i \circ \Phi\,e.$ by $\Phi\,e.$

The splitting of (13) is equivalent to the following identity:

$$\Phi\,e + (\Phi\,I\,\Phi^{-1})\,(\partial\,\mathbb{1}_n - A) = \mathbb{1}_n. \tag{14}$$

Thus, for every $\xi \in \mathcal{A}^{n\times 1}$, if we set $\zeta(t) = \dot{\xi}(t) - A(t)\,\xi(t)$, then the operator identity (14) yields:

$$\begin{aligned} \xi(t) &= (\Phi\,e)\,\xi(t) + \Phi\,I\,\Phi^{-1}\,(\partial\,\mathbb{1}_n - A)\,\xi(t) \\ &= \Phi(t, t_0)\,\xi(t_0) + \Phi\,I\,\Phi^{-1}\,\zeta(t) \\ &= \Phi(t, t_0)\,\xi(t_0) + \int_{t_0}^{t} \Phi(t, s)\,\zeta(s)\,ds. \end{aligned} \tag{15}$$

Hence, the identity (14) is an integro-differential operator reformulation of the *method of the variation of constants*. For more details and more results, see Quadrat (2022).

4. ALGEBRAIC ANALYSIS APPROACH TO LINEAR STATE-SPACE SYSTEMS

We extend the algebraic analysis approach, developed in Section 3 for first-order linear systems, to linear state-space systems (Kalman et al. (1969)), namely:

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t). \tag{16}$$

Let us consider $A \in \mathcal{A}^{n\times n}$, $B \in \mathcal{A}^{n\times m}$, and:

$$P = (\partial\,\mathbb{1}_n - A \quad -B) \in \mathcal{I}^{n\times(n+m)}.$$

Then, (16) is defined by $P\,\eta = 0$, where $\eta = (x^T \quad u^T)^T$. Moreover, let $\mathcal{L} = \operatorname{coker}_{\mathcal{I}}(.P) = \mathcal{I}^{1\times(n+m)}/(\mathcal{I}^n\,P)$ be the left $\mathcal{I}$-module finitely presented by P. Extending what was done in Section 3, if $\{f_j\}_{j=1,\ldots,n+m}$ denotes the standard basis of $\mathcal{I}^{1\times(n+m)}$, $\kappa : \mathcal{I}^{1\times(n+m)} \longrightarrow \mathcal{L}$ is the canonical projection onto $\mathcal{L}$, $x_j = \kappa(f_j)$ for $j = 1, \ldots, n$, and $u_j = \kappa(f_j)$ for $j = n+1, \ldots, n+m$, then the set of generators $\{x_1, \ldots, x_n, u_1, \ldots, u_m\}$ of $\mathcal{L}$ satisfies the relations:

$$P\,(x^T \quad u^T)^T = 0,\ x = (x_1\,\ldots\,x_n)^T,\ u = (u_1\,\ldots\,u_m)^T.$$

See, e.g., Chyzak et al. (2005). Note that $R\,x = B\,u$ yields $S\,R\,x = S\,B\,u$, i.e., $x = \Phi\,e\,x + S\,B\,u$, which shows that $\mathcal{L}$ is also generated by the set $\{e\,x_i, u_j\}_{1\le i\le n, 1\le j\le m}$.

Moreover, if $\mathcal{F}$ is a left $\mathcal{I}$-module (e.g., $\mathcal{F} = \mathcal{A}$), then we have $\hom_{\mathcal{I}}(\mathcal{L}, \mathcal{F}) \cong \ker_{\mathcal{F}}(P.) = \{\eta \in \mathcal{F}^{(n+m)\times 1} \mid P\,\eta = 0\}$, where $\ker_{\mathcal{F}}(P.)$ is the behaviour associated with (16).

As in Section 3, let Φ be a transition matrix of (6). We assume again that $\Phi \in \mathcal{A}^{n\times n}$. As shown in Section 3, $S = \Phi\,I\,\Phi^{-1} \in \mathcal{I}^{n\times n}$ is a right inverse of $R = \partial\,\mathbb{1}_n - A$. Thus, $Q = (S^T \quad 0^T)^T \in \mathcal{I}^{(n+m)\times n}$ is a right inverse of P. It yields $\ker_{\mathcal{I}}(.P) = 0$ and the split exact sequence

$$0 \longrightarrow \mathcal{I}^{1\times n} \underset{.Q}{\overset{.P}{\rightleftarrows}} \mathcal{I}^{1\times(n+m)} \underset{\sigma}{\overset{\kappa}{\rightleftarrows}} \mathcal{L} \longrightarrow 0,$$

where $\sigma \in \hom_{\mathcal{I}}(\mathcal{L}, \mathcal{I}^{1\times(n+m)})$ is defined by

$$\forall\,\mu \in \mathcal{I}^{1\times(n+m)},\ \sigma(\kappa(\mu)) = \mu\,X,$$

where the matrix $X \in \mathcal{I}^{(n+m)\times(n+m)}$ is defined by:

$$X = \mathbb{1}_{n+m} - Q\,P = \begin{pmatrix} \Phi\,e & \Phi\,I\,\Phi^{-1}\,B \\ 0 & \mathbb{1}_m \end{pmatrix}. \tag{17}$$

Clearly X is an idempotent of $\mathcal{I}^{(n+m)\times(n+m)}$, which yields the following direct sum decomposition

$$\begin{aligned} \mathcal{I}^{1\times(n+m)} &= \ker_{\mathcal{I}}(.X) \oplus \operatorname{im}_{\mathcal{I}}(.X) \\ &= \operatorname{im}_{\mathcal{I}}(.Q\,P) \oplus \ker_{\mathcal{I}}(.Q\,P) \\ &= \underbrace{\operatorname{im}_{\mathcal{I}}(.P)}_{\cong\,\mathcal{I}^{1\times n}} \oplus \underbrace{\ker_{\mathcal{I}}(.Q)}_{\cong\,\mathcal{L}}, \end{aligned} \tag{18}$$

since $\operatorname{im}_{\mathcal{I}}(.Q) = \mathcal{I}^{1\times n}$ and $\ker_{\mathcal{I}}(.P) = 0$, which shows that $\mathcal{L}$ is a stably free left $\mathcal{I}$-module (Rotman (2009)).

Using $\Pi = \mathbb{1}_n - \Phi\,e$, let us consider the following matrix:

$$P' = S\,P = (\Pi \quad -S\,B) = \begin{pmatrix} \mathbb{1}_n - \Phi\,e & -\Phi\,I\,\Phi^{-1}\,B \end{pmatrix}. \tag{19}$$

Note that $R\,P' = (R\,S)\,P = P$, which shows that $\operatorname{im}_{\mathcal{I}}(.P) = \operatorname{im}_{\mathcal{I}}(.P')$, and thus, $\mathcal{L} = \operatorname{coker}_{\mathcal{I}}(.P')$. The matrix P' is thus another presentation matrix of $\mathcal{L}$, i.e., an equivalent representation of the linear state-space system (16), i.e., $P\,\eta = 0$ is equivalent to $P'\,\eta = 0$, i.e.:

$$\begin{aligned} &\begin{pmatrix} \mathbb{1}_n - \Phi\,e & -\Phi\,I\,\Phi^{-1}\,B \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = 0 \\ \iff\ & x(t) = \Phi(t, t_0)\,x(t_0) + \int_{t_0}^{t} \Phi(t, s)\,B(s)\,u(s)\,ds. \end{aligned}$$

The last equality corresponds to (15) with $f = B\,u$.

Let us now state the main result of the paper.

Theorem 1. Let us consider the finitely presented left $\mathcal{I}$-modules $\mathcal{M} = \operatorname{coker}_{\mathcal{I}}(.(\partial\,\mathbb{1}_n - A))$, $\mathcal{M}' = \operatorname{coker}_{\mathcal{I}}(.\partial\,\mathbb{1}_n)$, and $\mathcal{L} = \operatorname{coker}_{\mathcal{I}}(.(\partial\,\mathbb{1}_n - A \quad -B))$, respectively associated with (6), $\dot{x} = 0$, and (16). Then, we have:

$$\mathcal{L} \cong \mathcal{M} \oplus \mathcal{I}^{1\times m} \cong \mathcal{M}' \oplus \mathcal{I}^{1\times m}. \tag{20}$$

Thus, we have the isomorphisms of $\Bbbk$-vector spaces:

$$\begin{aligned}\ker_{\mathcal{A}}((\partial\,\mathbb{1}_n - A \quad -B).) &\cong \ker_{\mathcal{A}}((\partial\,\mathbb{1}_n - A).) \oplus \mathcal{A}^{m\times 1}\\ &\cong \ker_{\mathcal{A}}(\partial\,\mathbb{1}_n.) \oplus \mathcal{A}^{m\times 1}\\ &= \mathbb{k}^{n\times 1} \oplus \mathcal{A}^{m\times 1}.\end{aligned} \tag{21}$$

Proof. Let us consider $.B \in \hom_{\mathcal{I}}(\mathcal{I}^{1\times n}, \mathcal{I}^{1\times m})$ defined by $(.B)(\nu) = \nu\, B$ for all $\nu \in \mathcal{I}^{1\times n}$. Using (9), we have the following commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}^{1\times n} & \xrightarrow{.R} & \mathcal{I}^{1\times n} & \xrightarrow{\pi} & \mathcal{M} & \longrightarrow & 0\\ & & \downarrow{\scriptstyle .B} & & \downarrow{\scriptstyle \kappa\circ i_1} & & \| & & \\ 0 & \longrightarrow & \mathcal{I}^{1\times m} & \xrightarrow{\kappa\circ i_2} & \mathcal{L} & \xrightarrow{\delta} & \mathcal{M} & \longrightarrow & 0, \end{array}$$

where i_1 (resp. i_2) is the first (resp. second) inclusion

$$\begin{array}{rcl} i_1 : \mathcal{I}^{1\times n} & \longrightarrow & \mathcal{I}^{1\times (n+m)}\\ \mu_1 & \longmapsto & (\mu_1 \quad 0), \end{array} \qquad \begin{array}{rcl} i_2 : \mathcal{I}^{1\times m} & \longrightarrow & \mathcal{I}^{1\times (n+m)}\\ \mu_2 & \longmapsto & (0 \quad \mu_2), \end{array} \tag{22}$$

and $\delta(\kappa((\mu_1 \quad \mu_2))) = \pi(\mu_1)$ for all $\mu_1 \in \mathcal{I}^{1\times n}$ and for all $\mu_2 \in \mathcal{I}^{1\times m}$. For more details, see the next Remark 2.

Let us now show that the second horizontal sequence of the above commutative exact diagram splits due to the splitting of (9) (see (10)). To do that, we first consider $\varpi = \kappa \circ i_1 \circ \rho \in \hom_{\mathcal{I}}(\mathcal{M}, \mathcal{L})$. Using (11), we have

$$\varpi(\pi(\lambda)) = (\kappa\circ i_1\circ\rho\circ\pi)(\lambda) = (\kappa\circ i_1)(\lambda\,\Phi\, e) = \kappa(\lambda\,(\Phi\, e \quad 0)),$$

for all $\lambda \in \mathcal{I}^{1\times n}$. Then, $(\delta\circ\varpi)(\pi(\lambda)) = \delta(\kappa(\lambda\,(\Phi\, e \quad 0)))$. Now, using $\mathcal{L} = \operatorname{coker}_{\mathcal{I}}(.P')$, where P' is defined by (19), $\kappa(\lambda\,(\Phi\, e \quad 0)) = \kappa(\lambda\,(\mathbb{1}_n \quad -S\,B))$, which yields $\delta(\kappa(\lambda\,(\mathbb{1}_n \quad -S\,B))) = \pi(\lambda)$ and shows that $\delta\circ\varpi = \mathrm{id}_{\mathcal{M}}$. Hence, we have $\mathcal{L} \cong \mathcal{M} \oplus \mathcal{I}^{1\times m}$ (see, e.g., Rotman (2009)). Now, according to 5 of Theorem 7 of Quadrat (2022), we have $\mathcal{M}' \cong \mathcal{M}$, where $\mathcal{M}' = \operatorname{coker}_{\mathcal{I}}(.\partial\,\mathbb{1}_n)$, and thus, $\mathcal{L} \cong \mathcal{M}' \oplus \mathcal{I}^{1\times m}$. Using $\ker_{\mathcal{A}}(\partial\,\mathbb{1}_n.) = \mathbb{k}^{n\times 1}$ and applying the functor $\hom_{\mathcal{I}}(\,\cdot\,, \mathcal{F})$ to (20), we finally obtain (21).

With the notations of the proof of Theorem 1, we have that $\mathrm{id}_{\mathcal{L}} - \varpi\circ\delta \in \hom_{\mathcal{I}}(\mathcal{L}, \mathcal{L})$ is defined by

$$\begin{aligned}(\mathrm{id}_{\mathcal{L}} - \varpi\circ\delta)(\kappa(\mu)) &= \kappa(\mu) - \varpi(\pi(\mu_1))\\ &= \kappa(\mu) - \kappa((\mu_1\,\Phi\, e \quad 0))\\ &= \kappa((\mu_1\,(\mathbb{1}_n - \Phi\, e) \quad \mu_2))\\ &= \kappa((0 \quad \mu_2 + \mu_1\, S\, B))\\ &= ((\kappa\circ i_2)\circ\varepsilon)(\kappa(\mu)),\end{aligned}$$

where $\varepsilon \in \hom_{\mathcal{I}}(\mathcal{L}, \mathcal{I}^{1\times m})$ is defined by

$$\forall\, \mu = (\mu_1 \quad \mu_2) \in \mathcal{I}^{1\times(n+m)},\ \varepsilon(\kappa(\mu)) = \mu \begin{pmatrix} S\,B \\ \mathbb{1}_n \end{pmatrix},$$

i.e., $\varpi\circ\delta + \kappa\circ i_2\circ\varepsilon = \mathrm{id}_{\mathcal{L}}$, and the split exact sequence:

$$0 \longrightarrow \mathcal{I}^{1\times m} \underset{\varepsilon}{\overset{\kappa\circ i_2}{\rightleftarrows}} \mathcal{L} \underset{\varpi}{\overset{\delta}{\rightleftarrows}} \mathcal{M} \longrightarrow 0. \tag{23}$$

Applying the functor $\hom_{\mathcal{I}}(\,\cdot\,, \mathcal{A})$ to (23), we get the following split exact sequence of $\mathbb{k}$-vector spaces

$$0 \longrightarrow \mathcal{A}^{m\times 1} \underset{(\kappa\circ i_2)^\star}{\overset{\varepsilon^\star}{\rightleftarrows}} \ker_{\mathcal{A}}(P.) \underset{\delta^\star}{\overset{\varpi^\star}{\rightleftarrows}} \ker_{\mathcal{A}}(R.) \longrightarrow 0,$$

where we have

$$\begin{aligned}&(\kappa \circ i_2)^\star\left((x^T \quad u^T)^T\right) = u,\\ &\forall\, z \in \ker_{\mathcal{A}}(R.),\ \delta^\star(z) = (z^T \quad 0^T)^T,\\ &\varpi^\star((x^T \quad u^T)^T) = (\Phi\, e \quad 0)\,(x^T \quad u^T)^T = \Phi\, e\, x,\\ &\forall\, u \in \mathcal{A}^{m\times 1},\ \varepsilon^\star(v) = \begin{pmatrix} S\,B \\ \mathbb{1}_n \end{pmatrix} v = \begin{pmatrix} \Phi\, I\, \Phi^{-1}\, B\, v \\ v \end{pmatrix},\end{aligned}$$

for all $(x^T \quad u^T)^T \in \ker_{\mathcal{A}}(P.)$. Thus, $\ker_{\mathcal{A}}(\varpi^\star)$ corresponds to the $\mathcal{A}$-solutions of (16) defined by the initial condition $e(x) = x(t_0) = 0$. These solutions are then defined by $\operatorname{im}_{\mathcal{A}}(\varepsilon^\star)$, i.e., they are of the form $\varepsilon^\star(v)$ for all $v \in \mathcal{A}^{m\times 1}$. Thus, they are *parametrized* by $\varepsilon^\star$.

Remark 2. In module theory, $\mathcal{L} = \operatorname{coker}_{\mathcal{I}}(.P)$ is the so-called *pushout* of $.R$ and $.B$ (see, e.g., Rotman (2009)). This pushout yields the second horizontal exact sequence of the commutative exact diagram (22), i.e., (23). See, e.g., Rotman (2009). Within homological algebra (see, e.g., Rotman (2009)), the short exact sequence (9) defines a so-called *extension* of $\mathcal{M}$ by $\mathcal{I}^{1\times n}$, denoted by ϵ. The extension ϵ is said to be a *trivial* because (9) splits. The extension (23) of $\mathcal{M}$ by $\mathcal{I}^{1\times m}$, obtained by pushout, denoted by $(.B)_\star(\epsilon)$, is known to split (Rotman (2009)).

Remark 3. Note that (18) is a direct consequence of (20) and (12) because $\mathcal{L} \cong \mathcal{M} \oplus \mathcal{I}^{1\times m}$ yields:

$$\mathcal{L}\oplus\mathcal{I}^{1\times n} \cong \mathcal{M}\oplus\mathcal{I}^{1\times n}\oplus\mathcal{I}^{1\times m} \cong \mathcal{I}^{1\times n}\oplus\mathcal{I}^{1\times m} \cong \mathcal{I}^{1\times(n+m)}.$$

Remark 4. Note that (20) does not hold if we consider the modules $\mathcal{M}$ and $\mathcal{L}$ over the ring $\mathcal{D}$ of ordinary differential operators (as done classically in the literature). This isomorphism emerges from the use of the ring $\mathcal{I}$.

The isomorphism of Theorem 1 can be made explicit.

Corollary 5. With the notations of Theorem 1, the isomorphism g defining the isomorphism (20) and its inverse $g^{-1} = h$ are explicitly defined by the commutative exact diagrams (24) and (27) defined below in the proof.

Proof. Let $\mathcal{L}' = \operatorname{coker}_{\mathcal{I}}(.P')$ be the left $\mathcal{I}$-module finitely presented by $P' = (\partial\,\mathbb{1}_n - A \quad 0) \in \mathcal{I}^{n\times(n+m)}$. We have $\mathcal{L}' = \mathcal{M} \oplus \mathcal{I}^{1\times m}$. Let X be defined by (17). Note that

$$P\,X = (R \quad -B)\begin{pmatrix} \Phi\, e & S\,B \\ 0 & \mathbb{1}_m \end{pmatrix} = (0 \quad 0),$$

since $(\partial\,\mathbb{1}_n - A)\,\Phi\, e = \Phi\,\partial\, e + (\dot{\Phi} - A\,\Phi)\, e = 0$ because $\partial\, e = 0$ and $\dot{\Phi} = A\,\Phi$, and $R\,S = \mathbb{1}_n$ yields $R\,S\,B = B$. Hence, if $Y = 0 \in \mathcal{I}^{n\times n}$, we have the identity $P\,X = Y\,P'$, which yields the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}^{1\times n} & \xrightarrow{.P} & \mathcal{I}^{1\times(n+m)} & \xrightarrow{\kappa} & \mathcal{L} & \longrightarrow & 0\\ & & \downarrow{\scriptstyle .Y} & & \downarrow{\scriptstyle .X} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & \mathcal{I}^{1\times n} & \xrightarrow{.P'} & \mathcal{I}^{1\times(n+m)} & \xrightarrow{\kappa'} & \mathcal{L}' & \longrightarrow & 0, \end{array} \tag{24}$$

where $g \in \hom_{\mathcal{I}}(\mathcal{L}, \mathcal{L}')$ is defined by $g(\kappa(\lambda)) = \kappa'(\lambda\, X)$ for all $\lambda \in \mathcal{I}^{1\times(n+m)}$. Let us prove that g is an isomorphism. Using Cluzeau et al. (2008), we first know that:

$$\operatorname{coker}(g) = \operatorname{coker}_{\mathcal{I}}\left(.\left(X^T \quad {P'}^T\right)^T\right). \tag{25}$$

Using (25), $\operatorname{coker}(g)$ is defined by the following equations:

$$\begin{cases} \Phi\, e\, y_1 + S\,B\, y_2 = 0,\\ y_2 = 0,\\ R\, y_1 = 0, \end{cases} \iff \begin{cases} \Phi\, e\, y_1 = 0,\\ R\, y_1 = 0,\\ y_2 = 0. \end{cases}$$

Now, using (14), we obtain $S\,R\,y_1 = (\mathbb{1}_n - \Phi\, e)\, y_1 = 0$, which, combining with $\Phi\, e\, y_1 = 0$, yields $y_1 = 0$ and shows that $\operatorname{coker}(g) = 0$, and thus, g is surjective. In particular, it shows that a left inverse of $\left(X^T \quad {P'}^T\right)^T$ is defined by:

$$\begin{pmatrix} \mathbb{1}_n & -S\,B & S \\ 0 & \mathbb{1}_m & 0 \end{pmatrix}. \tag{26}$$

Now, let us consider $(\lambda_1\ \lambda_2\ \lambda_3) \in \ker_{\mathcal{I}}\left(.\left(X^T \quad {P'}^T\right)^T\right)$, where $\lambda_1 \in \mathcal{I}^{1\times n}$, $\lambda_2 \in \mathcal{I}^{1\times m}$, and $\lambda_3 \in \mathcal{I}^{1\times n}$, i.e.:

$$\begin{cases} \lambda_1\, \Phi\, e + \lambda_3\, R = 0, \\ \lambda_1\, S\, B + \lambda_2 = 0. \end{cases}$$

Using the identity $R\, S = \mathbb{1}_n$, the first above equation yields $\lambda_3 = -\lambda_1\, \Phi\, e\, (\Phi\, I\, \Phi^{-1}) = 0$, and thus, $\lambda_1\, \Phi\, e = 0$. Now, we have $\mathrm{im}_{\mathcal{I}}(.\Pi) = \mathrm{im}_{\mathcal{I}}(.R)$ since $.S$ is surjective. Moreover, we have $\mathrm{im}_{\mathcal{I}}(.\Pi) = \ker_{\mathcal{I}}(.(\mathbb{1}_n - \Pi)) = \ker_{\mathcal{I}}(.\Phi\, e)$ since Π is an idempotent of $\mathcal{I}^{n\times n}$, which yields $\ker_{\mathcal{I}}(.\Phi\, e) = \mathrm{im}_{\mathcal{I}}(.R)$. Hence, $\lambda_1 = \mu\, R$ for a certain $\mu \in \mathcal{I}^{1\times n}$, and thus:

$$\lambda_2 = -\lambda_1\, S\, B = -\mu\, R\, S\, B = -\mu\, B.$$

Thus, $(\lambda_1\ \lambda_2\ \lambda_3) = \mu\,(R \quad -B \quad 0)$ for all $\mu \in \mathcal{I}^{1\times n}$, i.e.:

$$\ker_{\mathcal{I}}\left(.\left(X^T \quad {P'}^T\right)^T\right) = \mathrm{im}_{\mathcal{I}}(.(P \quad 0)).$$

Hence, we have $\ker(g) = \mathrm{im}_{\mathcal{I}}(.P)/\mathrm{im}_{\mathcal{I}}(.P) = 0$ (see Cluzeau et al. (2008)), which shows that g is injective, and thus, defines an isomorphism, which proves again (20). Finally, using Cluzeau et al. (2008), g^{-1} is defined by the first two columns of the matrix (26), namely:

$$U = \begin{pmatrix} \mathbb{1}_n & -S\,B \\ 0 & \mathbb{1}_m \end{pmatrix} \in \mathcal{I}^{(n+m)\times(n+m)}.$$

More precisely, we can check again the identity $P'\, U = P$, which yields the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}^{1\times n} & \xrightarrow{.P} & \mathcal{I}^{1\times(n+m)} & \xrightarrow{\kappa} & \mathcal{L} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle .\mathbb{1}_n} & & \uparrow{\scriptstyle .U} & & \uparrow{\scriptstyle h} & & \\ 0 & \longrightarrow & \mathcal{I}^{1\times n} & \xrightarrow{.P'} & \mathcal{I}^{1\times(n+m)} & \xrightarrow{\kappa'} & \mathcal{L}' & \longrightarrow & 0, \end{array} \quad (27)$$

where the left $\mathcal{I}$-homomorphism h is then defined by:

$$\forall\, \lambda \in \mathcal{I}^{1\times(n+m)},\ h(\kappa'(\lambda)) = \kappa(\lambda\, U).$$

Noting $Z = -(S^T \quad 0^T)^T \in \mathcal{I}^{(n+m)\times n}$, then (14) yields:

$$X\, U = \begin{pmatrix} \Phi\, e & S\, B \\ 0 & \mathbb{1}_m \end{pmatrix} = \mathbb{1}_{n+m} + Z\, P,$$

$$U\, X = \begin{pmatrix} \Phi\, e & 0 \\ 0 & \mathbb{1}_m \end{pmatrix} = \mathbb{1}_{n+m} + Z\, P'.$$

Therefore, we have

$$(h \circ g)(\kappa(\lambda)) = \kappa(\lambda\, X\, U) = \kappa(\lambda + (\lambda\, Z)\, P) = \kappa(\lambda),$$
$$(g \circ h)(\kappa'(\lambda)) = \kappa'(\lambda\, U\, X) = \kappa'(\lambda + (\lambda\, Z)\, P') = \kappa'(\lambda),$$

i.e., $h \circ g = \mathrm{id}_{\mathcal{L}}$ and $g \circ h = \mathrm{id}_{\mathcal{L}'}$, which yields $h = g^{-1}$.

Remark 6. Applying $\hom_{\mathcal{I}}(\cdot\,, \mathcal{F})$ to (24) and (27), we get the commutative exact diagrams of $\Bbbk$-vector spaces

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathcal{F}^{n\times 1} & \xleftarrow{P.} & \mathcal{F}^{(n+m)\times 1} & \longleftarrow & \ker_{\mathcal{F}}(P.) & \longleftarrow & 0 \\ & & \uparrow{\scriptstyle Y.} & & \uparrow{\scriptstyle X.} & & \uparrow{\scriptstyle g^\star} & & \\ 0 & \longleftarrow & \mathcal{F}^{n\times 1} & \xleftarrow{P'.} & \mathcal{F}^{(n+m)\times 1} & \longleftarrow & \ker_{\mathcal{F}}(P'.) & \longleftarrow & 0, \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathcal{F}^{n\times 1} & \xleftarrow{P.} & \mathcal{F}^{(n+m)\times 1} & \longleftarrow & \ker_{\mathcal{F}}(P.) & \longleftarrow & 0 \\ & & \downarrow{\scriptstyle \mathbb{1}_m.} & & \downarrow{\scriptstyle U.} & & \downarrow{\scriptstyle h^\star} & & \\ 0 & \longleftarrow & \mathcal{F}^{n\times 1} & \xleftarrow{P'.} & \mathcal{F}^{(n+m)\times 1} & \longleftarrow & \ker_{\mathcal{F}}(P'.) & \longleftarrow & 0, \end{array}$$

$$\forall\, (z^T \quad v^T)^T \in \ker_{\mathcal{F}}(P'.) = \ker_{\mathcal{F}}(R.) \oplus \mathcal{F}^{m\times 1}:$$

$$g^\star\left(\begin{pmatrix} z \\ v \end{pmatrix}\right) = \begin{pmatrix} \Phi\, e & S\, B \\ 0 & \mathbb{1}_m \end{pmatrix} \begin{pmatrix} z \\ v \end{pmatrix} \in \ker_{\mathcal{F}}(P.),$$

$$\forall\, (x^T \quad u^T)^T \in \ker_{\mathcal{F}}(P.):$$

$$h^\star\left(\begin{pmatrix} x \\ u \end{pmatrix}\right) = \begin{pmatrix} \mathbb{1}_n & -S\, B \\ 0 & \mathbb{1}_m \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \in \ker_{\mathcal{F}}(P'.).$$

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