An Integro-differential-delay Operator Approach to Transformations of Linear Differential Time-delay Systems

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Abstract: In this paper, we further develop the study of rings of integro-differential-delay operators considered as noncommutative polynomial algebras satisfying standard calculus identities. Within the algebraic analysis approach, we show that transformations and reductions of linear differential time-delay systems can be interpreted as homomorphisms and isomorphisms of finitely presented left modules over an algebra of integro-differential-delay operators. In particular, we show how Fiagbedzi-Pearson's transformation can be found again and generalized. This transformation maps the solutions of a first-order differential linear system with state and input delays to the solutions of a purely state-space linear system. Fiagbedzi-Pearson's transformation reduces to the well-known Artstein's reduction when the system has no state delay and yields an isomorphism of the solution spaces.

Keywords: Linear systems, systems with time-delays, polynomial methods, delay compensation

1. INTRODUCTION

The goal of the paper is to further develop the study of transformations and reductions of linear differential time-delay systems within the *algebraic analysis approach* (Cluzeau et al. (2008, 2009); Quadrat (2015)).

Let us consider the linear differential time-delay system

$$\dot{x}(t) = A_0(t)\, x(t)+A_1(t)\, x(t-r)+B_0(t)\, u(t)+B_1(t)\, u(t-h), \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and A_0, A_1, B_0, and B_1 are matrices whose entries are regular functions of the time-variable t, and h and r are two fixed positive real numbers. An important issue for time-delay systems is to understand when (1) is equivalent to the linear state-space system

$$\dot{z}(t) = A(t)\, z(t) + B(t)\, v(t), \quad (2)$$

where $z(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}^m$, and A and B are also two matrices whose entries are regular functions of t.

In the case where the state x of the linear system (1) is not delayed, i.e., when $A_1 = 0$, then it is well-known that (1) and (2) are equivalent through invertible *integral transformations*. This latter equivalence is nowadays called *Artstein's reduction* (Artstein (1982)). The case of $A_1 \neq 0$ is more complicated. In Fiagbedzi et al. (1986), the study of integral transformations from (1) to (2) is initiated.

Algebraic analysis is a mathematical theory which studies general linear systems of partial differential equations using module theory, homological algebra, sheaf theory, etc. In the 90's, algebraic analysis was introduced in control theory by Oberst, Fliess, and Pommaret. Linear differential systems, multidimensional systems, differential (constant or time-dependent) time-delay systems were studied within this algebraic analysis approach. This approach also encompasses *linear systems over rings*. The *behavioural approach* can also be recasted within this approach. Finally, using symbolic computation methods (e.g., *Gröbner basis methods* over *Ore algebras* of *functional operators*), symbolic packages have been developed for the study of different classes of linear systems (Chyzak et al. (2005)).

Using *rings of integro-differential operators*, an algebraic analysis approach to linear differential systems has recently been initiated in Quadrat (2022b). An algebraic analysis approach to linear differential constant time-delay systems was proposed in Quadrat (2015) using *rings of integro-differential-delay operators*. Within this mathematical approach, it was shown that Artstein's reduction can be interpreted as an isomorphism between the *finitely presented modules* defined by (1), where $A_1 = 0$, and (2) over a ring of integro-differential-delay operators. Quadrat (2015) advocated for an effective study using recent symbolic computation methods. In Cluzeau et al. (2018), such an effective approach was initiated and the corresponding computations were reproduced and generalized using the computer algebra system `Mathematica`.

In this paper, after briefly stating again the main ideas of the algebraic analysis approach, we first further develop the study of rings of integro-differential-delay operators. We then explain that the standard integral representation of (1), used in the method of steps, defines another *presentation* of the finitely presented left module associated with (1) over the ring of integro-differential-delay operators. Finally, we study homomorphisms between the finitely presented modules defined by (1) and (2). We find again and generalize results obtained in Fiagbedzi et al. (1986).

2. ALGEBRAIC ANALYSIS APPROACH

If we consider the following *functional operators*

$$\partial\, y(t) = \dot{y}(t), \quad \delta\, y(t) = y(t-h), \quad \tau\, y(t) = y(t-r),$$

which act on functions y, then we can easily check that:

$$\partial \circ \delta = \delta \circ \partial, \; \partial \circ \tau = \tau \circ \partial, \; \delta \circ \tau = \tau \circ \delta. \qquad (3)$$

Hence, we can define the commutative polynomial ring $\mathcal{D} = \mathbb{R}[\partial, \delta, \tau]$ formed by all the differential time-delay operators in ∂, δ, and τ with coefficients in $\mathbb{R}$. Every element $P \in \mathcal{D}$ can be uniquely written as $P = \sum_{0 \leq |\nu| \leq r} a_\nu\, x^\nu$, where $\nu = (\nu_1 \;\, \nu_2 \;\, \nu_3) \in \mathbb{Z}^3_{\geq 0}$ is a multi-index of length $|\nu| = \nu_1 + \nu_2 + \nu_3$, $x^\nu = \partial^{\nu_1} \circ \delta^{\nu_2} \circ \tau^{\nu_3}$, $a_\nu \in \mathbb{R}$, and $r \in \mathbb{Z}_{\geq 0}$. This ring $\mathcal{D}$ can be used to develop a polynomial approach to linear differential constant time-delay systems with real coefficients. Since "linear algebra over a ring" corresponds to *module theory* in mathematics, module theory was naturally introduced and used in this study. This approach has largely been studied in the literature. See, e.g., Fliess and al. (1998) and the references therein.

Let us consider the following $n \times (n+m)$-matrices

$$\begin{cases} R = (\partial\, \mathbb{1}_n - A \quad - B)\,, \\ R' = (\partial\, \mathbb{1}_n - A_0 - A_1\, \tau \quad - B_0 - B_1\, \delta)\,, \end{cases} \qquad (4)$$

with entries in $\mathcal{D}$, i.e., $R,\, R' \in \mathcal{D}^{n \times (n+m)}$, where $\mathbb{1}_n$ denotes the $n \times n$ identity matrix. If we note $\eta = (z^T \quad v^T)^T$ and $\eta' = (x^T \quad u^T)^T$, then the above linear functional systems can respectively be rewritten as follows:

$$R\, \eta = 0, \quad R'\, \eta' = 0. \qquad (5)$$

If $\mathcal{F}$ is a $\mathcal{D}$-module, i.e., $d_1\, f_1 + d_2\, f_2 \in \mathcal{F}$ for all $d_1,\, d_2 \in \mathcal{D}$ and for all $f_1,\, f_2 \in \mathcal{F}$, then we can define the *behaviours*:

$$\ker_{\mathcal{F}}(R.) = \left\{ \eta \in \mathcal{F}^{(n+m) \times 1} \mid R\, \eta = 0 \right\},$$
$$\ker_{\mathcal{F}}(R'.) = \left\{ \eta' \in \mathcal{F}^{(n+m) \times 1} \mid R'\, \eta = 0 \right\}.$$

In this paper, the linear systems (1) and (2) will be studied within the *algebraic analysis approach to linear system theory*. Let us briefly state again the main ideals of this approach. Let $\mathcal{D}$ be a noncommutative ring and $R \in \mathcal{D}^{q \times p}$. Then, we can consider the following *left $\mathcal{D}$-homomorphism*

$$\begin{aligned} .R : \mathcal{D}^{1 \times q} &\longrightarrow \mathcal{D}^{1 \times p} \\ \mu = (\mu_1 \;\ldots\; \mu_q) &\longmapsto \mu\, R, \end{aligned}$$

i.e., $.R$ is a left $\mathcal{D}$-linear map. Let us consider the *image* of $.R$, i.e., $\mathrm{im}_{\mathcal{D}}(.R) = \{\lambda \in \mathcal{D}^{1 \times p} \mid \exists\, \mu \in \mathcal{D}^{1 \times q} :\; \lambda = \mu\, R\}$, simply denoted by $\mathcal{D}^{1 \times q}\, R$, i.e., the left $\mathcal{D}$-submodule of $\mathcal{D}^{1 \times p}$ formed by all the left $\mathcal{D}$-linear combinations of the rows of R. Moreover, we can introduce the *cokernel* of $.R$:

$$\mathcal{M} := \mathrm{coker}_{\mathcal{D}}(.R) = \mathcal{D}^{1 \times p} / \mathrm{im}_{\mathcal{D}}(.R) = \mathcal{D}^{1 \times p} / (\mathcal{D}^{1 \times q}\, R).$$

Let us give another description of $\mathcal{M}$. Let $\{e_i\}_{i=1,\ldots,p}$ be the *standard basis* of $\mathcal{D}^{1 \times p}$, namely, e_i is the row vector of length p with 1 at the i^{th}-entry and 0 elsewhere. Every element $\lambda = (\lambda_1 \;\ldots\; \lambda_p) \in \mathcal{D}^{1 \times p}$ can then be written as $\lambda = \sum_{i=1}^p \lambda_i\, e_i$. Let $\pi(\lambda)$ denote the *residue class* of $\lambda \in \mathcal{D}^{1 \times p}$ in $\mathcal{M}$, namely, $\pi(\lambda) = \{\lambda + \mu\, R \mid \mu \in \mathcal{D}^{1 \times q}\}$. We can define the following operations on residue classes

$$\pi(\lambda_1) + \pi(\lambda_2) := \pi(\lambda_1 + \lambda_2), \quad d\, \pi(\lambda) := \pi(d\, \lambda),$$

for all $\lambda_1,\, \lambda_2 \in \mathcal{D}^{1 \times p}$ and for all $d \in \mathcal{D}$, which shows that $\mathcal{M}$ has a left $\mathcal{D}$-module structure. Moreover, we can define the left homomorphism $\pi : \mathcal{D}^{1 \times p} \longrightarrow \mathcal{M}$ which maps $\lambda \in \mathcal{D}^{1 \times p}$ onto its residue class $\pi(\lambda)$ in $\mathcal{M}$. Let us note $y_i = \pi(e_i)$ for $i = 1, \ldots, p$. Then, every element m of $\mathcal{M}$ is of the form $m = \pi(\lambda) = \sum_{i=1}^p \lambda_i\, \pi(e_i) = \sum_{i=1}^p \lambda_i\, y_i$ for a certain $\lambda \in \mathcal{D}^{1 \times p}$. Hence, $\{y_i\}_{i=1,\ldots,p}$ is a generating set of $\mathcal{M}$. Let us note $y = (y_1 \;\ldots\; y_p)^T$ so that $\pi(\lambda) = \lambda\, y$ for all $\lambda \in \mathcal{D}^{1 \times p}$. Now, let $\{f_j\}_{j=1,\ldots,q}$ be the standard basis of $\mathcal{D}^{1 \times q}$. Using $f_j\, R \in \mathrm{im}_{\mathcal{D}}(.R)$, $\pi(f_j\, R) = 0$ for $j = 1, \ldots, q$,

$$\pi(f_i\, R) = \pi((R_{i1} \;\ldots\; R_{ip})) = \sum_{j=1}^p R_{ij}\, \pi(e_j) = \sum_{j=1}^p R_{ij}\, y_j,$$

and thus, $\sum_{j=1}^p R_{ij}\, y_j$ for $j = 1, \ldots, q$, i.e., $R\, y = 0$. Hence, the generating set $\{y_i\}_{i=1,\ldots,p}$ of $\mathcal{M}$ satisfies the left $\mathcal{D}$-linear relations defined by $R\, y = 0$, i.e., by the system equations. The left $\mathcal{D}$-module $\mathcal{M} = \mathrm{coker}_{\mathcal{D}}(.R)$ is said to be *finitely presented* (see, e.g., Rotman (2009)).

Note that y is not a vector of functions, it is only the vector defined by a set of generators of $\mathcal{M}$. If we want y to be a proper vector of functions, let us consider a left $\mathcal{D}$-module $\mathcal{F}$ of functions. Denote by $\hom_D(\mathcal{M}, \mathcal{F})$ the *abelian group* (namely, the $\mathbb{Z}$-module) formed by all the homomorphisms (i.e., left D-linear maps) from $\mathcal{M}$ to $\mathcal{F}$. Let us consider $\phi \in \hom_{\mathcal{D}}(\mathcal{M}, \mathcal{F})$ and let us define $\psi(\phi) := (\phi(y_1) \;\ldots\; \phi(y_p))^T \in \mathcal{F}^{p \times 1}$. Then, we have

$$\sum_{j=1}^p R_{ij}\, \phi(y_j) = \phi\left(\sum_{j=1}^p R_{ij}\, y_j\right) = \phi(0) = 0,$$

i.e., $\psi(\phi) \in \ker_{\mathcal{F}}(R.)$. Conversely, if $\eta \in \ker_{\mathcal{F}}(R.)$, then we can define the map $\varphi_\eta : \mathcal{M} \longrightarrow \mathcal{F}$ by $\phi_\eta(\pi(\lambda)) = \lambda\, \eta \in \mathcal{F}$ for all $\lambda \in \mathcal{D}^{1 \times p}$. Note this map is well-defined since $\pi(\lambda') = \pi(\lambda)$ is equivalent to $\lambda' = \lambda + \mu\, R$ for a certain $\mu \in \mathcal{D}^{1 \times q}$, which yields $\lambda'\, \eta = \lambda\, \eta + \mu\, (R\, \eta) = \lambda\, \eta$. Clearly, we have $\varphi_\eta \in \hom_{\mathcal{D}}(\mathcal{M}, \mathcal{F})$. Moreover, $\psi(\varphi_\eta) = \eta$ for all $\eta \in \ker_{\mathcal{F}}(R.)$ and $\varphi_{\psi(\phi)} = \phi$ for all $\phi \in \hom_{\mathcal{D}}(\mathcal{M}, \mathcal{F})$, which shows that there exists an *isomorphism* (i.e., a bijective homomorphism) from $\ker_{\mathcal{F}}(R.)$ to $\hom_{\mathcal{D}}(\mathcal{M}, \mathcal{F})$, which is denoted by $\ker_{\mathcal{F}}(R.) \cong \hom_{\mathcal{D}}(\mathcal{M}, \mathcal{F})$. Hence, the behaviours can be seen as the dual with value in $\mathcal{F}$ to the finitely presented left $\mathcal{D}$-modules. The behaviour $\ker_{\mathcal{F}}(R.)$ can be studied by means of the left $\mathcal{D}$-modules $\mathcal{M}$ and $\mathcal{F}$.

Within the algebraic analysis approach, if $R' \in \mathcal{D}^{q' \times p'}$, $\mathcal{M}' = \mathrm{coker}_{\mathcal{D}}(.R')$, and $\ker_{\mathcal{F}}(R'.)$, then homomorphisms from $\ker_{\mathcal{F}}(R'.)$ to $\ker_{\mathcal{F}}(R.)$ can be studied by homomorphisms from $\mathcal{M}$ to $\mathcal{M}'$. Let us briefly state again these results. Let $f \in \hom_{\mathcal{D}}(\mathcal{M}, \mathcal{M}')$. Then, it can be shown that f is defined by a pair of matrices $P \in \mathcal{D}^{p \times p'}$ and $Q \in \mathcal{D}^{q \times q'}$ satisfying $R\, P = Q\, R'$. Then, we have $f(\pi(\lambda)) = \pi'(\lambda\, P)$ for all $\lambda \in \mathcal{D}^{1 \times p}$, where $\pi' : \mathcal{D}^{1 \times p'} \longrightarrow \mathcal{M}'$ is the left $\mathcal{D}$-homomorphism which sends $\lambda' \in \mathcal{D}^{1 \times p'}$ onto its residue class $\pi'(\lambda')$ in $\mathcal{M}'$. For more details, see Cluzeau et al. (2008). Using the above identity, we get the $\mathbb{Z}$-linear map $f^\star : \ker_{\mathcal{F}}(R'.) \longrightarrow \ker_{\mathcal{F}}(R.)$ defined by $f^\star(\eta') = P\, \eta'$ for all $\eta' \in \ker_{\mathcal{F}}(R'.)$. The left $\mathcal{D}$-modules $\ker f$, $\mathrm{im}\, f$, and $\mathrm{coker}\, f$ can be explicitly characterized (Cluzeau et al. (2008)). In particular, f is isomorphism if it is both injective and surjective, i.e., $\ker f = 0$ and $\mathrm{coker}\, f = 0$. For the algorithmic aspects of the computation of matrices P and Q for given matrices R and R', see Cluzeau et al. (2008) and the OreMorphisms package (Cluzeau et al. (2009)).

Using the above algebraic analysis approach to time-invariant linear differential constant time-delay systems, we can study the homomorphisms between the finitely presented $\mathcal{D} = \mathbb{R}[\partial, \delta, \tau]$-modules $\mathcal{M} = \mathrm{coker}_{\mathcal{D}}(.R)$ to $\mathcal{M}' = \mathrm{coker}_{\mathcal{D}}(.R')$, where the matrices R and R' are defined by (4), and thus, the corresponding homomorphisms between the behaviours/linear systems $\ker_{\mathcal{F}}(R.)$ and $\ker_{\mathcal{F}}(R'.)$. Since $\mathcal{D}$ is commutative, $\hom_{\mathcal{D}}(\mathcal{M}, \mathcal{M}')$ inherits a $\mathcal{D}$-module structure and it can be proved that $\hom_{\mathcal{D}}(\mathcal{M}, \mathcal{M}')$ is a finitely presented $\mathcal{D}$-module, and thus, it can be defined by a finite set of generators and a finite set of $\mathcal{D}$-linear relations. Finally, using Gröbner basis methods, these generators and relations can be computed.

But, in the literature of delay systems, it is well-known that differential time-delay transformations are usually not enough to study interesting problems such as *Artstein's reduction*, *finite spectrum assignment*, *stabilizability*, etc., where integral transformations are required (e.g., distributed delays). For the study of the transformations from (1) to (2), we thus have to enlarge the ring $\mathcal{D}$ to include integral operators. We are naturally led to introduce *rings of integro-differential constant time-delay operators* as shown in Quadrat (2015); Cluzeau et al. (2018).

3. RINGS OF INTEGRO-DIFFERENTIAL-DELAY OPERATORS

As explained in Section 2, to study linear differential time-delay systems, it is natural to consider rings of integro-differential-delay operators. Let us first briefly introduce the *rings of integro-differential operators*. For more details, see Quadrat (2022b). To simplify, we shall consider here the $\mathbb{R}$-algebra $\mathcal{A} = C^\infty(\mathbb{R})$ of real-valued smooth functions on $\mathbb{R}$, but more algebras can also be used. Let us fix $t_0 \in \mathbb{R}$. The *ring of integro-differential operators* $\mathcal{I}$ is then the $\mathbb{R}$-subalgebra of the ring $\mathrm{end}_{\mathbb{R}}(\mathcal{A})$ of all the $\mathbb{R}$-endomorphisms of $\mathcal{A}$ generated by the following $\mathbb{R}$-endomorphisms of $\mathcal{A}$:

$$\begin{aligned}
&\overline{a} : b(\cdot) \longmapsto a(\cdot)\, b(\cdot),\ a \in \mathcal{A},\\
&\partial : a(\cdot) \longmapsto \dot{a}(\cdot),\ \dot{a}(t) = \frac{da(t)}{dt},\\
&I : a(\cdot) \longmapsto b(\cdot),\ b(t) = \int_{t_0}^t a(s)\, ds,\\
&e : a(\cdot) \longmapsto a(t_0).
\end{aligned}$$

As in Section 2, the composition of endomorphisms is written multiplicatively and $\overline{a} \in \mathrm{end}_{\mathbb{R}}(\mathcal{A})$ is simply denoted by a. E.g., $a_1\, I\, a_2 \in \mathcal{I}$ is the endomorphism of $\mathcal{A}$ defined by:

$$a_1\, I\, a_2 : b(\cdot) \longmapsto c(\cdot),\ c(t) = a_1(t) \int_{t_0}^t a_2(\tau)\, b(\tau)\, d\tau.$$

The endomorphisms ∂, I, e, and a for all $a \in \mathcal{A}$ satisfy standard relations from calculus. For instance, we have

$$\forall\, b \in \mathcal{A},\ (\partial\, a)(b) = \partial(a\, b) = \partial(a)\, b + a\, \partial(b) = (a\, \partial + \partial(a))(b),$$

i.e., the identity $\partial\, a = a\, \partial + \partial(a)$ holds in $\mathcal{I}$. Let $\mathbb{1}$ denote the identity endomorphism of $\mathcal{A}$. Similarly, we have:

$$\forall\, b \in \mathcal{A}, \quad (\partial\, I)(b) = \frac{d}{dt} \int_{t_0}^t b(\tau)\, d\tau = b,$$

$$\forall\, b \in \mathcal{A},\ (I\, \partial)(b) = \int_{t_0}^t \dot{b}(\tau)\, d\tau = b(t) - b(t_0) = (\mathbb{1} - e)(b),$$

$$\forall\, a,\, b \in \mathcal{A}, \quad (e\, a)(b) = e(a\, b) = a(t_0)\, b(t_0) = (e(a)\, e)(b).$$

Hence, we have the following identities in $\mathcal{I}$:

$$\forall\, a \in \mathcal{A}, \quad \begin{cases} \partial\, a = a\, \partial + \partial(a),\\ \partial\, I = \mathbb{1},\\ I\, \partial = \mathbb{1} - e,\\ e\, a = e(a)\, e. \end{cases} \tag{6}$$

The following identities in $\mathcal{I}$ can be derived from (6)

$$\begin{aligned}
&e^2 = e, \quad e\, I = 0, \quad \partial\, e = 0,\\
&\forall\, a \in \mathcal{A},\ \ I\, a\, \partial = -I\, \partial(a) + a - e(a)\, e,\\
&\forall\, a \in \mathcal{A},\ \ I\, a\, I = [I(a), I],
\end{aligned} \tag{7}$$

where $[a, b] = a\, b - b\, a$ denotes the *commutator* of a and b. For instance, the fourth identity of (7) corresponds to the integration by parts. For more details, see Quadrat (2022b). Using (6) and (7), it is clear that $\mathcal{I}$ is a non-commutative polynomial ring in ∂, I, and e with coefficients in $\mathcal{A}$, denoted by $\mathcal{A}\langle \partial, I, e\rangle$. One can show that every element P of $\mathcal{I}$ can be written uniquely as $P = P_1 + P_2 + P_3$, where $P_1 = \sum_{i=0}^r a_i\, \partial^i \in \mathcal{A}\langle\partial\rangle$, where $\mathcal{A}\langle\partial\rangle$ is the *ring of differential operators* with coefficients in $\mathcal{A}$, namely, the non-commutative polynomials in ∂ and $a \in \mathcal{A}$ satisfying the first identity of (6), P_2 belongs to the *ring (without multiplicative identity) of integral operators*

$$\mathfrak{I} = \mathcal{A}\, \langle I \mid I\, a\, I = [I(a), I]\rangle = \left\{ \sum_{i=1}^r a_i\, I\, b_i \mid a_i,\, b_i \in \mathcal{A} \right\},$$

and P_3 belongs to the only two-sided ideal $\langle e\rangle = \mathcal{I}\, e\, \mathcal{I}$ of $\mathcal{I}$ defined by e. Using the identities in $\mathcal{I}$, we can check again that P_3 is of the form $P_3 = \sum_{i=0}^r a_i\, e\, \partial^i$, where $a_i \in \mathcal{A}$.

To define *rings of integro-differential-delay operators*, we add the new $\mathbb{R}$-endomorphism δ of $\mathcal{A}$ defined by:

$$\forall\, a \in \mathcal{A}, \quad \delta(a)(t) = a(t - h).$$

In this paper, we shall only consider the case where h is a fixed positive real number. Thus, δ is a $\mathbb{R}$-automorphism of $\mathcal{A}$ and $\delta^{-1}(a)(t) = a(t + h)$ for all $a \in \mathcal{A}$. We also consider the $\mathbb{R}$-endomorphism e_h of $\mathcal{A}$ defined by:

$$\forall\, a \in \mathcal{A}, \quad e_h(a)(t) = a(t_0 + h).$$

Note that we then have

$$\forall\, a \in \mathcal{A}, \quad (e_h\, \delta)(a)(t) = e_h(a(t - h)) = a(t_0) = e(a)(t),$$

which shows the following identity in $\mathrm{end}_{\mathbb{R}}(\mathcal{A})$:

$$e = e_h\, \delta. \tag{8}$$

Thus, e is a consequence of δ and e_h. More generally, we have $(e_h\, \delta^{r+1})(a)(t) = a(t_0 - r\, h)$ for all $r \in \mathbb{N}$, i.e., $e_h\, \delta^{r+1} = e_{t_0 - r\, h}$ is the evaluation at $t_0 - r\, h$ for $r \in \mathbb{N}$. Similarly, we can easily check the following identities

$$\begin{aligned}
&\partial\, \delta = \delta\, \partial, \quad \partial\, e_h = 0, \quad e\, I = e_h\, \delta\, I = 0, \quad \delta\, e_h = e_h,\\
&e_h^2 = e_h, \quad I\, e_h = (t - t_0)\, e_h.
\end{aligned}$$

Using the change of variables $s = \tau + h$, we get

$$\begin{aligned}
(\delta\, I)(a)(t) &= \int_{t_0}^{t-h} a(\tau)\, d\tau = \int_{t_0+h}^t a(s - h)\, ds\\
&= \int_{t_0}^t a(s - h)\, ds - \int_{t_0}^{t_0+h} a(s - h)\, ds,
\end{aligned}$$

which yields the following identity:

$$\delta\, I = I\, \delta - e_h\, I\, \delta = (\mathbb{1} - e_h)\, I\, \delta. \tag{9}$$

Using the change of variables $s = \tau - h$, we also have

$$\begin{aligned}
(I\, \delta)(a)(t) &= \int_{t_0}^t a(\tau - h)\, d\tau = \int_{t_0-h}^{t-h} a(s)\, ds\\
&= \int_{t_0}^{t-h} a(s)\, ds - \int_{t_0}^{t_0-h} a(s)\, ds,
\end{aligned}$$

which shows that the identity holds in $\mathrm{end}_{\mathbb{R}}(\mathcal{A})$:

$$I\, \delta = \delta\, I - e_{t_0-h}\, I = \delta\, I - e_h\, \delta^2\, I = (\mathbb{1} - e_h\, \delta)\, \delta\, I = (\mathbb{1} - e)\, \delta\, I. \tag{10}$$

Let us check that (9) and (10) are equivalent. Multiply (9) by $\mathbb{1}-e_h\,\delta$ and using the identities $\delta\,e_h=e_h$, $e_h^2=e_h$, and $e\,I=e_h\,\delta\,I=0$, we first have

$$\begin{aligned}(\mathbb{1}-e_h\,\delta)\,\delta\,I&=(\mathbb{1}-e_h\,\delta)\,(\mathbb{1}-e_h)\,I\,\delta\\&=(\mathbb{1}-e_h\,\delta-e_h+e_h\,\delta\,e_h)\,I\,\delta\\&=(\mathbb{1}-e_h\,\delta)\,I\,\delta=I\,\delta-e\,I\,\delta=I\,\delta,\end{aligned}$$

i.e., (10). Similarly, multiplying (10) by $\mathbb{1}-e_h$ and using $e_h^2=e_h$ and $e\,I=e_h\,\delta\,I=0$, we then have

$$\begin{aligned}(\mathbb{1}-e_h)\,I\,\delta&=(\mathbb{1}-e_h)\,(\mathbb{1}-e_h\,\delta)\,\delta\,I\\&=(\mathbb{1}-e_h-e_h\,\delta+e_h\,\delta)\,\delta\,I=(\mathbb{1}-e_h)\,\delta\,I\\&=\delta\,I-e_h\,\delta\,I=\delta\,I-e\,I=\delta\,I,\end{aligned}$$

i.e., (9), which proves that (9) and (10) are equivalent.

We sum up the above identities in the following table, where $c\,r$ denotes the product of an element in the first column by an element in the first row.

$c\,r$	∂	I	δ	e_h
∂	∂^2	$\mathbb{1}$	$\delta\,\partial$	0
I	$\mathbb{1}-e_h\,\delta$	$t\,I-I\,t$	$(\mathbb{1}-e_h\,\delta)\,\delta\,I$	$(t-t_0)\,e_h$
δ	$\partial\,\delta$	$(\mathbb{1}-e_h)\,I\,\delta$	δ^2	e_h
e_h	$e_h\,\partial$	$e_h\,I$	$e_h\,\delta=e$	e_h

(11)

Moreover, we can check that the following identities:

$$\begin{aligned}\forall\,a\in\mathcal{A},\quad &\partial\,a=a\,\partial+\partial(a),\\&\delta\,a=\delta(a)\,\delta,\\&e_h\,a=e_h(a)\,e_h,\\&I\,a\,\partial=-I\,\partial(a)+(1-e_h\,\delta)\,a,\\&I\,a\,\delta=(\mathbb{1}-e_h\,\delta)\,\delta\,I\,\delta^{-1}(a),\\&I\,a\,I=[I(a),I],\\&I\,a\,e_h=I(a)\,e_h.\end{aligned}\tag{12}$$

Definition 1. We call *ring of integro-differential-delay operators* over $\mathcal{A}=C^\infty(\mathbb{R})$, denoted by $\mathcal{H}=\mathcal{A}\langle\partial,I,\delta,e_h\rangle$, the $\mathbb{R}$-subalgebra of $\mathrm{end}_{\mathbb{R}}(\mathcal{A})$ generated by ∂, I, e_h, δ, and all the multiplication by elements of $\mathcal{A}$. Equivalently, $\mathcal{H}$ is the noncommutative polynomial ring $\mathcal{A}\langle\partial,I,\delta,e_h\rangle$ defined by ∂, I, δ, and e_h which satisfy (11), (12), and $e_h\,\delta\,I=0$.

Every element h of $\mathcal{H}$ can be written uniquely as

$$h=a(t,\delta,\partial)+b(t,\delta)\,I\,c(t)+d(t)\,e_h\,f(\delta)\,I\,g(t)+i(t)\,e_h\,j(\partial,\delta),$$

where $a(t,\delta,\partial)$ is a polynomial in ∂ and δ with coefficients in $\mathcal{A}$, $b(t,\delta)$ a polynomial in δ with coefficients in $\mathcal{A}$, c, d, g, $i\in\mathcal{A}$, f a polynomial in δ with coefficients in $\mathbb{R}$ and j a polynomial in ∂ and δ with coefficients in $\mathbb{R}$.

Consider the $\mathbb{R}$-endomorphisms τ and e_r of $\mathcal{A}$ defined by

$$\forall\,a\in\mathcal{A},\quad\tau(a)(t)=a(t-r),\;e_r(a)(t)=a(t_0+r),$$

where r is a fixed positive real number. Then, similar identities as (11) and (12) hold for τ and e_r instead of δ and e_h, as well as $e_r\,\tau\,I=0$. Similarly as above, we can define the $\mathbb{R}$-subalgebra of $\mathrm{end}_{\mathbb{R}}(\mathcal{A})$ defined by ∂, I, δ, e_h, τ, and e_r satisfying the corresponding identities as well as:

$$\delta\,\tau=\tau\,\delta,\quad e_r\,e_h=e_h,\quad e_h\,e_r=e_r.$$

4. ALGEBRAIC ANALYSIS OVER RINGS OF INTEGRO-DIFFERENTIAL-DELAY OPERATORS

Following the algebraic analysis approach briefly introduced in Section 2, let $\mathcal{H}=\mathcal{A}\langle\partial,I,\delta,e_h,\tau,e_r\rangle$ be the ring of integro-differential-delay operators defined in Section 3, R, $R'\in\mathcal{H}^{n\times(n+m)}$ the matrices defined by (4), and $\mathcal{M}=\mathrm{coker}_{\mathcal{H}}(.R)$ (resp., $\mathcal{M}'=\mathrm{coker}_{\mathcal{H}}(.R')$) the left $\mathcal{H}$ finitely presented by R (resp., R').

Now, let $P=\partial\,\mathbb{1}_n-A_0\in\mathcal{H}^{n\times n}$, Θ be the *transition matrix* of the linear differential system $\dot{x}(t)=A_0\,x(t)$, and $Q=\Theta\,I\,\Theta^{-1}\in\mathcal{H}^{n\times n}$. Then, using (12), we have

$$\begin{aligned}P\,Q&=(\partial\,\mathbb{1}_n-A_0)\,(\Theta\,I\,\Theta^{-1})\\&=(\Theta\,\partial+\dot{\Theta})\,I\,\Theta^{-1}-A_0\,\Theta\,I\,\Theta^{-1}\\&=\Theta\,\partial\,I\,\Theta^{-1}+(\dot{\Theta}-A_0\,\Theta)\,I\,\Theta^{-1}=\mathbb{1}_n,\end{aligned}$$

which shows that Q is a right inverse of P. Using the fact that $\Psi:=\Theta^{-1}$ is a transition matrix of $\dot{\Psi}+\Psi\,A_0=0$ and the integration by parts, we get the *idempotent* of $\mathcal{H}^{n\times n}$:

$$\begin{aligned}Q\,P&=(\Theta\,I\,\Theta^{-1})\,(\partial\,\mathbb{1}_n-A_0)\\&=\Theta\,I\,(\partial\,\Theta^{-1}-\partial(\Theta^{-1}))-\Theta\,I\,\Theta^{-1}\,A_0\\&=\Theta\,(\mathbb{1}-e)\,\Phi^{-1}-\Theta\,I\,(\partial(\Phi^{-1})+\Theta^{-1}\,A_0)\\&=\mathbb{1}_n-\Theta\,e\,\Theta^{-1}=\mathbb{1}_n-\Theta\,e(\Theta^{-1})\,e=\mathbb{1}_n-\Theta\,e.\end{aligned}$$

Let us now introduce the following matrix:

$$\begin{aligned}R''&=Q\,R'=Q\;(\partial\,\mathbb{1}_n-A_0-A_1\,\tau\quad-B_0-B_1\,\delta)\\&=(\mathbb{1}_n-\Theta\,e-\Theta\,I\,\Theta^{-1}\,(A_0+A_1\,\tau)\;-\Theta\,I\,\Theta^{-1}\,(B_0+B_1\,\delta)).\end{aligned}$$

Note that $R''=Q\,R'$ yields $\mathrm{im}_{\mathcal{H}}(.R'')\subseteq\mathrm{im}_{\mathcal{H}}(.R')$. $R'=(P\,Q)\,R'=P\,R''$ yields $\mathrm{im}_{\mathcal{H}}(.R')\subseteq\mathrm{im}_{\mathcal{H}}(.R'')$, which proves $\mathrm{im}_{\mathcal{H}}(.R'')=\mathrm{im}_{\mathcal{H}}(.R')$. Therefore, we have $\mathcal{M}'=\mathrm{coker}_{\mathcal{H}}(.R')=\mathrm{coker}_{\mathcal{H}}(.R'')$, which proves that $\mathcal{M}'$ is equivalently defined by the following representation:

$$x=\Theta\,e\,x+\Theta\,I\,\Theta^{-1}\,(A_0+A_1\,\tau)\,x+\Theta\,I\,\Theta^{-1}\,(B_0+B_1\,\delta)\,u.\tag{13}$$

Using the properties of the transition matrix Θ, namely, $\Theta^{-1}(t,t_0)=\Theta(t_0,t)$, $\Theta(t_0,t_1)\,\Theta(t_1,t_2)=\Theta(t_0,t_2)$, we get:

$$Q(y)(t)=\Theta(t,t_0)\int_{t_0}^{t}\Theta(t_0,\tau)\,y(\tau)\,d\tau=\int_{t_0}^{t}\Theta(t,\tau)\,y(\tau)\,d\tau.$$

Thus, the linear system (1) is equivalent to:

$$\begin{aligned}x(t)&=(\Theta\,e\,x+\Theta\,I\,\Theta^{-1}\,(A_0+A_1\,\tau))\,x(t)\\&\quad+(\Theta\,I\,\Theta^{-1})\,(B_0+B_1\,\delta)\,u(t)\\&=\Theta(t,t_0)\,x(t_0)+\int_{t_0}^{t}\Theta(t,\tau)\,A_0(\tau)\,x(\tau)\,d\tau\\&\quad+\int_{t_0}^{t}\Theta(t,\tau)\,A_1(\tau)\,x(\tau-r)\,d\tau+\int_{t_0}^{t}\Theta(t,\tau)\,B_0(\tau)\,u(\tau)\,d\tau\\&\quad+\int_{t_0}^{t}\Theta(t,\tau)\,B_1(\tau)\,u(\tau-h)\,d\tau.\end{aligned}$$

Using (12) and (10) with τ instead of δ, we have:

$$\begin{aligned}I\,\Theta^{-1}\,A_1\,\tau&=I\,\tau\,((\Theta^{-1}\,A_1)(\,\cdot+r))\\&=(\mathbb{1}-e)\,\tau\,I\,((\Theta^{-1}\,A_1)(\,\cdot+r)),\\I\,\Theta^{-1}\,B_1\,\delta&=I\,\delta\,((\Theta^{-1}\,B_1)(\,\cdot+h))\\&=(\mathbb{1}-e)\,\delta\,I\,((\Theta^{-1}\,B_1)(\,\cdot+h)).\end{aligned}$$

Therefore, (13) becomes:

$$\begin{aligned}x&=\Theta\,e\,x+\Theta\,I\,\Theta^{-1}\,A_0\,x\\&\quad+\Theta\,(\mathbb{1}-e)\,\tau\,I\,((\Theta^{-1}\,A_1)(\,\cdot+r))\,x\\&\quad+\Theta\,I\,\Theta^{-1}\,B_0\,u+\Theta\,(\mathbb{1}-e)\,\delta\,I\,((\Theta^{-1}\,B_1)(\,\cdot+h))\,u.\end{aligned}$$

The linear system (1) with the following initial condition

$$\forall\,t\in[t_0-h,t_0],\quad x(t)=\phi(t),$$

can then equivalently be represented as follows:

$$
\begin{aligned}
x(t) &= \Theta(t,t_0)\,\phi(t_0) + \int_{t_0}^{t} \Theta(t,\tau)\,A_0(\tau)\,x(\tau)\,d\tau \\
&+ \int_{t_0-r}^{t-r} \Theta(t,\tau+r)\,A_1(\tau+r)\,x(\tau)\,d\tau \\
&+ \int_{t_0}^{t} \Theta(t,\tau)\,B_0(\tau)\,u(\tau)\,d\tau \\
&+ \int_{t_0-h}^{t-h} \Theta(t,\tau+h)\,B_1(\tau+h)\,u(\tau)\,d\tau.
\end{aligned}
$$

The last representation of (1) is at the core to the well-known *method of steps* in time-delay system theory.

5. FIAGBEDZI-PEARSON'S TRANSFORMATION

The purpose of this section is to study the existence of $f \in \hom_{\mathcal{H}}(\mathcal{M}, \mathcal{M}')$ defined by $f(\pi(\lambda)) = \pi'(\lambda\, P)$, where

$$
P = \begin{pmatrix} a_0\,\tau\,I\,a_1 + a_2\,I\,a_3 + a_4 & b_0\,\delta\,I\,b_1 + b_2\,I\,b_3 + b_4 \\ 0 & c_0 \end{pmatrix},
$$

and $a_i \in \mathcal{A}^{n\times n}$, $b_j \in \mathcal{A}^{n\times m}$, and $c_0 \in \mathcal{A}^{m\times m}$ are matrices to be determined. According to Section 2, P defines a left $\mathcal{H}$-homomorphism from $\mathcal{M}$ to $\mathcal{M}'$ if and only if there exists $Q \in \mathcal{H}^{n\times n}$ satisfying $R\,P = Q\,R'$, i.e., such that:

$$
\begin{aligned}
(\partial\,\mathbb{1}_n - A)(a_0\,\tau\,I\,a_1 + a_2\,I\,a_3 + a_4) & \\
= Q\,(\partial\,\mathbb{1}_n - A_0 - A_1\,\tau), & \qquad (14) \\
(\partial\,\mathbb{1}_n - A)\,(b_0\,\delta\,I\,b_1 + b_2\,I\,b_3 + b_4) - B\,c_0 & \\
= -Q\,(B_0 + B_1\,\delta). & \qquad (15)
\end{aligned}
$$

The normal form of the left-hand side of (14) is then:

$$
\begin{aligned}
&(\dot{a_0} - A\,a_0)\,\tau\,I\,a_1 + (\dot{a_2} - A\,a_2)\,I\,a_3 + a_4\,\partial + a_0\,\tau(a_1)\,\tau \\
&+\dot{a_4} - A\,a_4 + a_2\,a_3.
\end{aligned}
$$

Comparing this normal form with the right-hand side of (14) yields $\deg_\partial Q = 0$, $\deg_\delta Q = 0$, $\deg_\tau Q = 0$, $\deg_I Q = 0$, i.e., $Q \in \mathcal{A}$. Thus, (14) yields $Q = a_4$ and:

$$
\begin{aligned}
&(\dot{a_0} - A\,a_0)\,\tau\,I\,a_1 + (\dot{a_2} - A\,a_2)\,I\,a_3 \\
&+(a_0\,a_1(\,\cdot - r) + a_4\,A_1)\,\tau + a_2\,a_3 + \dot{a}_4 - A\,a_4 + a_4\,A_0 = 0.
\end{aligned}
$$

Let us suppose that $a_1 \neq 0$ and $a_3 \neq 0$. Thus, we get:

$$
\begin{cases}
\dot{a}_0 - A\,a_0 = 0, \\
\dot{a}_2 - A\,a_2 = 0, \\
a_0\,a_1(\,\cdot - r) + a_4\,A_1 = 0, \\
a_2\,a_3 + \dot{a}_4 - A\,a_4 + a_4\,A_0 = 0.
\end{cases} \qquad (16)
$$

Let Φ be the transition matrix of $\dot{a}_0 - A\,a_0 = 0$. Integrating (16), we then obtain:

$$
\begin{cases}
a_0(t) = \Phi(t,t_0)\,c_0, \quad c_0 \in \mathbb{R}^{n\times n}, \\
a_2(t) = \Phi(t,t_0)\,c_2, \quad c_2 \in \mathbb{R}^{n\times n}, \\
c_0\,a_1(t) = -\Phi(t_0,t+r)\,a_4(t+r)\,A_1(t+r), \\
c_2\,a_3(t) = -\Phi(t_0,t)\,(\dot{a}_4(t) - A(t)\,a_4(t) + a_4(t)\,A_0(t)).
\end{cases}
$$

Using that c_0 and c_2 are two constant matrices, we get:

$$
\begin{aligned}
P_{11} &= a_0\,\tau\,I\,a_1 + a_2\,I\,a_3 + a_4 \\
&= \Phi(\,\cdot\,,t_0)\,c_0\,\tau\,I\,a_1 + \Phi(\,\cdot\,,t_0)\,c_2\,I\,a_3 + a_4 \\
&= \Phi(\,\cdot\,,t_0)\,(\tau\,I\,c_0\,a_1 + I\,c_2\,a_3) + a_4, \\
P_{11}(t) &= a_4(t) - \Phi(t,t_0)\,\left(\tau\,I\,\Phi(t_0,t+r)\,a_4(t+r)\,A_1(t+r)\right. \\
&\left.+I\,\Phi(t_0,t)\,(\dot{a}_4(t) - A(t)\,a_4(t) + a_4(t)\,A_0(t))\right).
\end{aligned}
$$

Using the identities $\partial\,\delta = \delta\,\partial$, $\partial\,I = \mathbb{1}$, $\delta\,b_1 = b_1(\cdot - h)\,\delta$, we then get the following normal form of (15):

$$
\begin{aligned}
&(\dot{b}_0 - A\,b_0)\,\delta\,I\,b_1 + (b_0\,b_1(\,\cdot - h) + a_4\,B_1)\,\delta + (\dot{b}_2 - A\,b_2)\,I\,b_3 \\
&+b_4\,\partial + b_2\,b_3 + \dot{b}_4 - A\,b_4 - B\,c_0 + a_4\,B_0 = 0,
\end{aligned}
$$

Suppose that $b_1 \neq 0$ and $b_3 \neq 0$. The last equation yields:

$$
\Longleftrightarrow \begin{cases}
\dot{b}_0 - A\,b_0 = 0, \\
b_0\,b_1(\,\cdot - h) + a_4\,B_1 = 0, \\
\dot{b}_2 - A\,b_2 = 0, \\
b_4 = 0, \\
b_2\,b_3 + \dot{b}_4 - A\,b_4 - B\,c_0 + a_4\,B_0 = 0,
\end{cases}
$$

$$
\Longleftrightarrow \begin{cases}
\dot{b}_0 - A\,b_0 = 0, \\
\dot{b}_2 - A\,b_2 = 0, \\
b_4 = 0, \\
b_0\,b_1(\,\cdot - h) + a_4\,B_1 = 0, \\
b_2\,b_3 - B\,c_0 + a_4\,B_0 = 0.
\end{cases}
$$

Integrating the last system, we then obtain:

$$
\begin{cases}
b_0(t) = \Phi(t,t_0)\,d_0, \qquad d_0 \in \mathbb{R}^{n\times n}, \\
b_2(t) = \Phi(t,t_0)\,d_2, \qquad d_2 \in \mathbb{R}^{n\times n}, \\
b_4 = 0, \\
d_0\,b_1(t) = -\Phi(t_0,t+h)\,a_4(t+h)\,B_1(t+h), \\
d_2\,b_3(t) = \Phi(t_0,t)\,(B(t)\,c_0(t) - a_4(t)\,B_0(t)).
\end{cases}
$$

Using that c_0 and c_2 are two constant matrices, we obtain:

$$
\begin{aligned}
P_{12} &= b_0\,\delta\,I\,b_1 + b_2\,I\,b_3 + b_4 \\
&= \Phi(\,\cdot\,,t_0)\,(d_0\,\delta\,I\,b_1 + d_2\,I\,b_3) \\
&= \Phi(\,\cdot\,,t_0)\,(\delta\,I\,d_0\,b_1 + I\,d_2\,b_3), \\
P_{12}(t) &= \Phi(t,t_0)\,(-\delta\,I\,\Phi(t_0,t+h)\,a_4(t+h)\,B_1(t+h) \\
&\quad + I\,\Phi(t_0,t)\,(B(t)\,c_0(t) - a_4(t)\,B_0(t))).
\end{aligned}
$$

Theorem 1. A left $\mathcal{H}$-homomorphism from $\mathcal{M}$ to $\mathcal{M}'$ is defined by $f(\pi(\lambda)) = \pi'(\lambda\,P)$ for all $\lambda \in \mathcal{H}^{1\times(n+m)}$, where P is a four block matrix defined by $P_{21} = 0$, $P_{22}(t) = c_0(t)$,

$$
\begin{aligned}
P_{11}(t) &= a_4(t) \\
&-\Phi(t,t_0)\,\left(\tau\,I\,\Phi(t_0,t+r)\,a_4(t+r)\,A_1(t+r)\right. \\
&\left.+I\,\Phi(t_0,t)\,(\dot{a}_4(t) - A(t)\,a_4(t) + a_4(t)\,A_0(t))\right), \\
P_{12}(t) &= \Phi(t,t_0)\,\left(-\delta\,I\,\Phi(t_0,t+h)\,a_4(t+h)\,B_1(t+h)\right. \\
&\left.+I\,\Phi(t_0,t)\,(B(t)\,c_0(t) - a_4(t)\,B_0(t))\right),
\end{aligned}
$$

for all $a_4 \in \mathcal{A}^{n\times n}$ and $c_0 \in \mathcal{A}^{m\times m}$, where Φ is transition matrix of $\dot{a} - A\,a = 0$. Thus, the solutions of (1) are sent to solutions of (2) by the following integral transformation:

$$
\begin{cases}
z(t) = a_4(t)\,x(t) \\
\quad - \displaystyle\int_{t_0}^{t-r} \Phi(t,\tau+r)\,a_4(\tau+r)\,A_1(\tau+r)\,x(\tau)\,d\tau \\
\quad + \displaystyle\int_{t_0}^{t} \Phi(t,\tau)\,(A(\tau)\,a_4(\tau) - a_4(\tau)\,A_0(\tau) - \dot{a}_4(\tau))\,x(\tau)\,d\tau \\
\quad - \displaystyle\int_{t_0}^{t-h} \Phi(t,\tau+h)\,a_4(\tau+h)\,B_1(\tau+h)\,u(\tau)\,d\tau \\
\quad + \displaystyle\int_{t_0}^{t} \Phi(t,\tau)\,(B(\tau)\,c_0(\tau) - a_4(\tau)\,B_0(\tau))\,u(\tau)\,d\tau, \\
v(t) = c_0(t)\,u(t).
\end{cases} \qquad (17)
$$

If the following relations hold

$$\begin{cases} \dot{a}_4 - A\, a_4 + a_4\, A_0 = -\Phi(\,\cdot\,, \cdot + r)\, a_4(\cdot + r)\, A_1(\cdot + r), \\ B\, c_0 - a_4\, B_0 = \Phi(\,\cdot\,, \cdot + h)\, a_4(\cdot + h)\, B_1(\cdot + h), \end{cases} \tag{18}$$

then the matrix P simplifies into:

$$\begin{cases} P_{11} = a_4 + \Phi(\,\cdot\,, t_0)\, (\mathbb{1} - \tau)\, I\, \Phi(t_0, \cdot + r)\, a_4(\cdot + r)\, A_1(\cdot + r), \\ P_{12} = \Phi(\,\cdot\,, t_0)\, (\mathbb{1} - \delta)\, I\, \Phi(t_0, \cdot + h)\, a_4(\cdot + h)\, B_1(\cdot + h). \end{cases}$$

If we suppose that a_4 and c_0 are invertible, then (18) yields:

$$\begin{cases} A = \dot{a}_4\, a_4^{-1} + a_4\, A_0\, a_4^{-1} \\ \qquad + \Phi(\,\cdot\,, \cdot + r)\, a_4(\cdot + r)\, A_1(\cdot + r)\, a_4^{-1}, \\ B = a_4\, B_0\, c_0^{-1} + \Phi(\,\cdot\,, \cdot + h)\, a_4(\cdot + r)\, B_1(\cdot + r)\, c_0^{-1}. \end{cases} \tag{19}$$

Finally, if we set $a_4 = \mathbb{1}_n$ and $c_0 = \mathbb{1}_m$ in (19), we then obtain the following corollary of Theorem 1.

Corollary 2. With the notations of Theorem 1, if the matrice A and B are chosen so that

$$\begin{cases} A(t) = A_0(t) + \Phi(t, t + r)\, A_1(t + r), \\ B(t) = B_0(t) + \Phi(t, t + h)\, B_1(t + h), \end{cases} \tag{20}$$

then the matrices P_{11} and P_{12} of Theorem 1 become:

$$\begin{cases} P_{11}(t) = \mathbb{1}_n + \Phi(t, t_0)\, (\mathbb{1} - \tau)\, I\, \Phi(t_0, t + r)\, A_1(t + r) \\ P_{12}(t) = \Phi(t, t_0)\, (\mathbb{1} - \delta)\, I\, \Phi(t_0, t + h)\, B_1(t + h). \end{cases}$$

Thus, the solutions of (1) are sent to solutions of (2) by:

$$\begin{cases} z(t) = x(t) + \displaystyle\int_{t-r}^{t} \Phi(t, \tau + r)\, A_1(\tau + r)\, x(\tau)\, d\tau \\ \qquad + \displaystyle\int_{t-h}^{t} \Phi(t, \tau + h)\, B_1(\tau + h)\, u(\tau)\, d\tau, \\ v(t) = u(t). \end{cases} \tag{21}$$

If $A_1 = 0$, then (20) and (21) yield $A = A_0$ and:

$$P = \begin{pmatrix} \mathbb{1}_n & \Phi(\,\cdot\,, t_0)\, (\mathbb{1} - \delta)\, I\, \Phi(t_0, \cdot + h)\, B_1(\cdot + h) \\ 0 & \mathbb{1}_m \end{pmatrix}.$$

Example 1. Let us consider the time-invariant case, i.e., A, A_0, $A_1 \in \mathbb{R}^{n\times n}$ and B, $B_0 \in \mathbb{R}^{n\times m}$. Then, we first get $\Phi(t, t_0) = e^{A\,(t - t_0)}$. Setting $t_0 = 0$, $a_4 \in \mathbb{R}^{n\times n}$ and $c_0 \in \mathbb{R}^{n\times m}$ to simply the expressions, Theorem 1 yields:

$$\begin{cases} z(t) = a_4\, x(t) \\ \qquad - \displaystyle\int_0^{t-r} e^{A\,(t-\tau-r)}\, a_4\, A_1\, x(\tau)\, d\tau \\ \qquad + \displaystyle\int_0^{t} e^{A\,(t-\tau)}\, (A\, a_4 - a_4\, A_0)\, x(\tau)\, d\tau \\ \qquad - \displaystyle\int_0^{t-h} e^{A\,(t-\tau-h)}\, a_4\, B_1\, u(\tau)\, d\tau \\ \qquad + \displaystyle\int_0^{t} e^{A\,(t-\tau)}\, (B\, c_0 - a_4\, B_0)\, u(\tau)\, d\tau, \\ v(t) = c_0\, u(t). \end{cases}$$

If a_4 and c_0 are invertible, then (19) yields

$$\begin{cases} A = a_4\, A_0\, a_4^{-1} + e^{-A\,r}\, a_4\, A_1\, a_4^{-1}, \\ B = a_4\, B_0\, c_0^{-1} + e^{-A\,h}\, a_4\, B_1\, c_0^{-1}, \end{cases}$$

and the solutions of (1) are sent to solutions of (2) by:

$$\begin{cases} z(t) = a_4\, x(t) + \displaystyle\int_{t-r}^{t} e^{A\,(t-\tau)}\, a_4\, A_1\, x(\tau)\, d\tau \\ \qquad + \displaystyle\int_{t-h}^{t} e^{A\,(t-\tau)}\, a_4\, B_1\, u(\tau)\, d\tau, \\ v(t) = c_0\, u(t). \end{cases}$$

Now, if we set $a_4 = \mathbb{1}_n$ and $c_0 = \mathbb{1}_m$, then we obtain:

$$P = \begin{pmatrix} \mathbb{1}_n + e^{A\,t}\, (\mathbb{1} - \tau)\, I\, e^{-A\,(t+r)}\, A_1 & e^{A\,t}\, (\mathbb{1} - \delta)\, I\, e^{-A\,(t+h)}\, B_1 \\ 0 & \mathbb{1}_m \end{pmatrix}.$$

$$\begin{cases} A = A_0 + e^{-A\,r}\, A_1, \\ B = B_0 + e^{-A\,h}\, B_1, \end{cases}$$

Thus, the solutions of (1) are sent to solutions of (2) by:

$$\begin{cases} z(t) = x(t) + \displaystyle\int_{t-r}^{t} e^{A\,(t-\tau-r)}\, A_1\, x(\tau)\, d\tau \\ \qquad + \displaystyle\int_{t-h}^{t} e^{A\,(t-\tau-h)}\, B_1\, u(\tau)\, d\tau, \\ v(t) = u(t). \end{cases}$$

We have just found again *Fiagbedzi-Pearson's transformation* introduced in Fiagbedzi et al. (1986).

Finally, if $A_1 = 0$, i.e., if (1) has no state delay, then $A = A_0$, $B = B_0 + e^{-A_0\,h}\, B_1$, and the matrix P becomes:

$$P = \begin{pmatrix} \mathbb{1}_n & e^{A\,t}\, (\mathbb{1} - \delta)\, I\, e^{-A\,(t+h)}\, B_1 \\ 0 & \mathbb{1}_m \end{pmatrix}.$$

We then find again *Artstein's reduction* (Artstein (1982))

$$\begin{cases} z(t) = x(t) + \displaystyle\int_{t-h}^{t} e^{A\,(t-\tau-h)}\, B_1\, u(\tau)\, d\tau, \\ v(t) = u(t), \end{cases}$$

which bijectively maps the solutions of (1) with $A_1 = 0$ to the solutions of (2). See Artstein (1982); Quadrat (2015).

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