



HAL
open science

Taming Bounded Depth with Nested Sequents

Agata Ciabattoni, Lutz Straßburger, Matteo Tesi

► **To cite this version:**

Agata Ciabattoni, Lutz Straßburger, Matteo Tesi. Taming Bounded Depth with Nested Sequents. Advances in Modal Logic 2022, Aug 2022, Rennes, France. hal-03909534

HAL Id: hal-03909534

<https://hal.inria.fr/hal-03909534>

Submitted on 21 Dec 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Taming Bounded Depth with Nested Sequents

Agata Ciabattoni

Vienna University of Technology

Lutz Straßburger

Inria Saclay & LIX, Ecole Polytechnique

Matteo Tesi

Scuola Normale Superiore di Pisa

Abstract

Bounded depth refers to a property of Kripke frames that serve as semantics for intuitionistic logic. We introduce nested sequent calculi for the intermediate logics of bounded depth. Our calculi are obtained in a modular way by adding suitable structural rules to a variant of Fitting’s calculus for intuitionistic propositional logic, for which we present the first syntactic cut elimination proof. This proof modularly extends to the new nested sequent calculi introduced in this paper.

Keywords: Nested Sequents, Cut Elimination, Intermediate Logics, Bounded Depth Kripke Models.

1 Introduction

Nested sequents are a natural extension of ordinary Gentzen sequents, (re)discovered several times in different contexts [15,27,2]. Whereas Gentzen sequents are lists (or multisets) of formulas, nested sequents are trees of multisets of formulas. The tree structure makes nested sequents well-suited to handle logics having Kripke-style semantics. Indeed, nested sequents have been employed to provide internal analytic calculi for modal logics for which this was not possible before, for example, for the modal logic KB in [2]. Here, *internal* means that there are no “external” semantic constructs (like labels) in the syntax, and *analytic* means that all derivations in these calculi have the property that they only contain formulas which are subformulas of the conclusion (subformula property). This feature of analytic calculi renders them a natural starting point for proving meta-logical properties as decidability, complexity, and interpolation, and for developing automated reasoning methods.

Nested sequents have been effective for classical modal logics in the S5-cube [2], as well as for intuitionistic modal logics in the IS5-cube [28,24,17],

and more generally, for all extensions of \mathbf{IK} with Horn-Scott-Lemmon axioms [22]. Furthermore, they have been used to provide focused proof systems for classical and intuitionistic modal logics [3,4], to construct interpolation proofs [21], to implement proof search [11,26], and to “tame” modal logics with path-axioms [12], where “taming” means to have proof systems suited for proof search. In all these calculi, the tree structure of the nested sequents corresponds to the accessibility relation between the worlds in the Kripke frame.

An analytic nested calculus has also been proposed for intuitionistic propositional logic (\mathbf{IPL}) by Fitting [8]; there the tree structure of the nested sequents corresponds to the future relation between the worlds in the Kripke frame. His calculus, that has been obtained as a notational variant of prefixed tableaux, is not equipped with a direct cut elimination proof, which seemed hard to define.

This leads to the first contribution of this paper. We import insights from the nested sequent calculi for modal logics to introduce \mathbf{NIPL} — a variant of Fitting’s calculus for \mathbf{IPL} that allows for a direct cut elimination proof. In the spirit of Belnap’s conditions for cut elimination in display calculi [1], our proof relies on the abstract conditions (N1)–(N5), which will be presented in Section 3. The rules of \mathbf{NIPL} are all invertible, and therefore \mathbf{NIPL} offers a purely formula-driven approach to proof search.

The next question we address is: Can we extend \mathbf{NIPL} by additional structural rules to capture intermediate logics (i.e., logics between intuitionistic and classical logic), in a similar way as the nested systems for the modal logics \mathbf{K} and \mathbf{IK} are extended with additional rules corresponding to additional axioms [24,13]? Moreover, can this be done in such a way that the cut elimination property is preserved, and can this be proved in a modular way, i.e., can we reuse the existing proof and only add the cases concerning the new rules? ¹

For hypersequents—which are disjunctions of ordinary sequents—we have such results, provided the additional axioms for intermediate logics follow a certain shape [5]. This is not the case for the family of intermediate logics characterized by Kripke models of bounded depth, usually denoted by \mathbf{BD}_n . The logic \mathbf{BD}_1 is just classical logic and the logic \mathbf{BD}_2 is one of the seven interpolable intermediate logics [23]. Analytic hypersequent calculi for \mathbf{BD}_n are indeed provably not obtainable [18] by using the methodology in [5] which extends the base hypersequent calculus for \mathbf{IPL} with suitable structural rules corresponding to the additional axioms.

Modular analytic calculi for \mathbf{BD}_n have been defined using frameworks more powerful than hypersequents (and nested sequents); however, the objects that the resulting calculi manipulate (labelled or display sequents [25,6]) cannot be translated into formulas of the logic, and hence there is no real subformula property even when the calculus is analytic; moreover the rules of these calculi are not invertible.

This brings us to our second contribution: modular nested sequent calculi

¹ Girard has argued in [10] that the lack in modularity is one of the main technical limitations in structural proof theory.

for the logics BD_n (for all $n \geq 1$). The calculi are obtained by adding suitable structural rules to NIPL. We show that the cut elimination proof for NIPL scales to the new systems, as our conditions (N1)–(N5) are preserved. It is interesting to note that the additional rules for BD_n , with $n \geq 2$, have more than one premise. To our knowledge, this is a novelty. So far, there are no nested sequent systems with multi-premise structural rules.

2 Preliminaries

The *formulas* of intuitionistic propositional logic (IPL), denoted by A, B, C, \dots , are generated from a countable set of atoms $\{p, q, \dots\}$ via the grammar

$$A ::= p \mid \perp \mid A \vee A \mid A \wedge A \mid A \supset A$$

We define the *degree* of a formula to be the number of connectives in it. A *nested sequent* is a finite tree of multisets of formulas. In ordinary sequents for intuitionistic logic we distinguish between the left and the right hand side of the turnstile. To make this distinction in nested sequents, we use polarities on formulas. There are two polarities, input (intuitively as if on the left of the turnstile in the conventional sequent calculus), denoted by a \bullet superscript and output (intuitively as if on the right of the turnstile), denoted by a \circ superscript. Now, a nested sequent can be written as:

$$\Gamma = A_1^\bullet, \dots, A_m^\bullet, B_1^\circ, \dots, B_n^\circ, [\Gamma_1], \dots, [\Gamma_k] \quad (1)$$

where $A_1^\bullet, \dots, A_m^\bullet, B_1^\circ, \dots, B_n^\circ$ is the multiset of formulas at the root of the sequent tree of Γ , and where $\Gamma_1, \dots, \Gamma_k$ are its immediate subtrees. We use \emptyset to denote the *empty sequent*, i.e., where $m = n = k = 0$ in (1) above. We use capital Greek letters $\Gamma, \Delta, \Sigma, \dots$, to denote nested sequents, and we assume that the associativity and commutativity of the comma is implicit in our systems, and that \emptyset acts as its unit. We write Γ^\bullet for $A_1^\bullet, \dots, A_m^\bullet$ and Γ° for $B_1^\circ, \dots, B_n^\circ, [\Gamma_1], \dots, [\Gamma_k]$ if Γ is as in (1) above. In other words, for every nested sequent Γ we have that $\Gamma = \Gamma^\bullet, \Gamma^\circ$. More generally, we will write $\Gamma^\bullet, \Delta^\bullet, \Sigma^\bullet, \dots$, for multisets of input formulas (i.e., all formulas have \bullet -polarity, and there are no nestings), and we will write $\Gamma^\circ, \Delta^\circ, \Sigma^\circ, \dots$, for sequents that have only \circ -formulas at their root nodes (i.e., there are no \bullet -formulas at the root, but there can be nestings with \bullet -formulas inside).

The *corresponding formula* of the sequent in (1) above is defined as

$$fm(\Gamma) = \bigwedge_{i=1}^m A_i \supset \left(\bigvee_{j=1}^n B_j \vee \bigvee_{l=1}^k fm(\Gamma_l) \right) \quad (2)$$

A (*sequent*) *context* is a nested sequent with a hole $\{ \}$, taking the place of a formula. Contexts are denoted by $\Gamma\{ \}$, and $\Gamma\{\Delta\}$ is the sequent obtained from $\Gamma\{ \}$ by replacing the occurrence of $\{ \}$ with Δ . We write $\Gamma\{\emptyset\}$ for the sequent obtained from $\Gamma\{ \}$ by removing the $\{ \}$ (i.e., the hole is filled with nothing). The depth of a context $\Gamma\{ \}$, denoted by $depth(\Gamma\{ \})$, is the length

Initial Sequents

$$\frac{}{\Gamma\{p^\bullet, \Delta\{p^\circ\}\}} \text{ax}$$

$$\frac{}{\Gamma\{\perp^\bullet\}} \perp^\bullet$$

Logical Rules

$$\frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \wedge^\bullet$$

$$\frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}} \wedge^\circ$$

$$\frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \vee^\bullet$$

$$\frac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{A \vee B^\circ\}} \vee^\circ$$

$$\frac{\Gamma\{A \supset B^\bullet, \Delta\{\Sigma, A^\circ\}\} \quad \Gamma\{A \supset B^\bullet, \Delta\{\Sigma, B^\bullet\}\}}{\Gamma\{A \supset B^\bullet, \Delta\{\Sigma\}\}} \supset^\bullet$$

$$\frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}} \supset^\circ$$

Figure 1. The calculus NIPL

$$\frac{\Gamma\{A^\circ\} \quad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}} \text{cut}$$

Figure 2. The cut rule

$$\frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \text{w} \quad \frac{\Gamma\{\Delta, \Delta\}}{\Gamma\{\Delta\}} \text{c} \quad \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}} \text{t} \quad \frac{\Gamma\{\Sigma^\circ\}}{\Gamma\{[\Sigma^\circ]\}} \text{4} \quad \frac{\Gamma\{[\Sigma^\bullet, \Delta]\}}{\Gamma\{\Sigma^\bullet, [\Delta]\}} \text{lift} \quad \frac{\Gamma\{\Sigma^\circ, [\Delta]\}}{\Gamma\{[\Sigma^\circ, \Delta]\}} \text{lower}$$

Figure 3. Admissible structural rules

of the path in the sequent tree from the root to the hole $\{ \}$. It is defined inductively as follows: $\text{depth}(\{ \}) = 0$ and $\text{depth}(\Gamma', \Gamma\{ \}) = \text{depth}(\Gamma\{ \})$ and $\text{depth}([\Gamma\{ \}]) = \text{depth}(\Gamma\{ \}) + 1$.

We will also use the notation $\Gamma\{\Delta\}$ as abbreviation for $\Gamma\{[\Delta]\}$.

Example 2.1 Let $\Gamma\{ \} = A^\bullet, B^\circ, [\{ \}, [D^\bullet, C^\circ]]$. We have that

$$\Gamma\{B^\circ\} = A^\bullet, B^\circ, [B^\circ, [D^\bullet, C^\circ]]$$

and

$$\Gamma\{\emptyset\} = A^\bullet, B^\circ, [[D^\bullet, C^\circ]].$$

Let $\Delta = F^\bullet, [G^\circ]$, then

$$\Gamma\{\Delta\} = A^\bullet, B^\circ, [F^\bullet, [G^\circ], [D^\bullet, C^\circ]]$$

and

$$\Gamma\{\Delta\} = A^\bullet, B^\circ, [[F^\bullet, [G^\circ]], [D^\bullet, C^\circ]].$$

3 Nested Sequent Calculus for IPL

An elegant nested sequent calculus for Intuitionistic Propositional Logic IPL was introduced by Fitting [8], as a notational variant of prefixed tableaux. The lack of a direct cut elimination proof in his calculus has prevented its extension to cover intermediate logics. Indeed, to the best of our knowledge there are

no analytic nested calculi for any intermediate logic (other than classical and intuitionistic logic). Even methods to extract such calculi from more powerful frameworks (like the structural refinement method in [22,20] that used labelled calculi as starting point), do not seem to work for intermediate logics.² In order to define analytic nested calculi for intermediate logics, the nested sequent formalism has been extended in various ways, giving rise to, e.g., linear nested calculi [19], and injective nested calculi [16].

In general, proving syntactic cut elimination for nested calculi is harder than for other proof theoretic formalisms, e.g., (hyper)sequent or display calculus. Often this result is obtained by translating the nested calculus at hand to other formalisms, as e.g. in [13,8]. The few existing cut elimination proofs for nested calculi are indeed tailored to specific systems [27,2,24,28], and their proofs do not seem to be generalizable (in particular, to deal with multi-premise structural rules).

In this section we present NIPL, a variant of Fitting’s calculus for IPL designed to have all invertible rules, and to admit a direct cut elimination proof. The system NIPL, whose rules are shown in Figure 1, is obtained from Fitting’s calculus by using multisets instead of sets and by absorbing the rule *lift* into the initial sequents and the rule \supset^\bullet . Observe indeed that *ax* and \supset^\bullet can be simulated in Fitting’s calculus by repeated applications of *lift*. As an immediate consequence we obtain the soundness of NIPL with respect to IPL.

Terminology: As in standard sequent calculi, we call *context* the part left unchanged from premises to conclusions, we call *principal* the introduced formula in a logical rule, and the rest *active* part/formulas (active formulas in the initial sequents are p^\bullet , p° , and \perp^\bullet).

As we will show in the next section, NIPL satisfies the following properties that guarantee a relatively simple proof of the elimination of the cut rule depicted in Figure 2.

- (N1) *All rules are height-preserving invertible.*
- (N2) *Dedicated structural rules are height-preserving admissible.* These rules, displayed in Figure 3, are the usual weakening (*w*) and contraction (*c*), the *lift*-rule from [8], variations of the rules for the modal axioms *t* and *4*, from [24], and the new *lower*-rule which can be seen as the inverse of *lift*.
- (N3) *A cut over formulas that are not principal can be shifted upwards over its premises.* This condition is implied by Belnap’s sufficient conditions (C2)–(C7) for cut elimination in display calculi [1].
- (N4) *All logical rules are reductive.* This means that they allow the replacement of a cut whose cut formula is principal in the left and right premise of the cut rule by cuts on smaller formulas (possibly using the dedicated structural rules from (N2)). This property is the nested sequent formulation of Belnap’s (C8) condition [1].

² See also [14] for the correspondence between labeled systems and nested sequents.

(N5) *Cuts having an initial sequent as one of their premises can be removed.*

Let us mention two useful features of NIPL. The first is standard in well-designed sequent-style calculi: the general form of the **ax**-rule is derivable.

Lemma 3.1 (Axiom expansion) *The sequent $\Gamma\{A^\bullet, \Pi\{A^\circ, \Delta\}\}$ is derivable in NIPL for every context Γ, Π, Δ and every formula A .*

Proof By induction on the degree of the formula A . We detail the case in which A is of the shape $B \supset C$, the other cases being similar.

$$\frac{\frac{\Gamma\{B \supset C^\bullet, \Pi\{B^\bullet, B^\circ, C^\circ, \Delta\}\} \quad \Gamma\{B \supset C^\bullet, \Pi\{B^\bullet, C^\bullet, C^\circ, \Delta\}\}}{\Gamma\{B \supset C^\bullet, \Pi\{B^\bullet, C^\circ, \Delta\}\}} \supset^\bullet}{\Gamma\{B \supset C^\bullet, \Pi\{B \supset C^\circ, \Delta\}\}} \supset^\circ$$

The premises are derivable by induction hypothesis. \square

The second feature concerns the admissibility of the necessitation rule

$$\frac{\Gamma}{[\Gamma]} \text{ nec}$$

which will be used in the heuristic for generating the structural rules for BD_n in Section 5. Note that unlike all other rules, **nec** is shallow, as it cannot be applied inside a context.

Proposition 3.2 *If a sequent Γ is derivable, then so is $[\Gamma]$.*

Proof The proof is by induction on the height n of the derivation of Γ . If Γ is an initial sequent, so is $[\Gamma]$. If $n > 0$, then apply the induction hypothesis to the premise(s) of the rule and then rule again. \square

4 Cut elimination for NIPL

We are going to show that NIPL satisfies conditions (N1)–(N5) and how these conditions entail the cut elimination theorem.

The preservation of the height of a derivation is crucial for all our arguments. Formally, the *height* of a derivation is the length of the longest path in the tree from its root to one of its leaves. An inference rule with premises $\Gamma_1, \dots, \Gamma_n$ and conclusion Γ is *height-preserving invertible*, if for every derivation of Γ , there are derivations of each of $\Gamma_1, \dots, \Gamma_n$ with at most the same height. The rule is *height-preserving admissible* if, whenever the premises are derivable, the conclusion has a derivation whose height is not bigger than any derivation of a premise.

Lemma 4.1 *The weakening rule **w** is height-preserving admissible in NIPL.*

Proof By induction on the height n of the derivation of $\Gamma\{\emptyset\}$. If $n = 0$, then $\Gamma\{\emptyset\}$ is an initial sequent and so is $\Gamma\{\Delta\}$. If $n > 0$, we apply the induction hypothesis to the premise(s) of the last rule applied and then the rule again. \square

Lemma 4.2 *Every rule in NIPL is height-preserving invertible.*

Proof By induction on the height n of the derivation of the conclusion of each rule. The proofs for conjunction and disjunction are standard. The rule \supset^\bullet is

height-preserving invertible by the height-preserving admissibility of the rule of weakening. We discuss the rule \supset° . If $\Gamma\{A \supset B^\circ\}$ is an initial sequent, then $\Gamma\{[A^\bullet, B^\circ]\}$ is an initial sequent too. If $n > 0$, then we apply the induction hypothesis to each of the premise(s) and then we apply the rule again. \square

Lemma 4.3 *The contraction rule c is height-preserving admissible in NIPL.*

Proof By induction on the height n of the derivation. If $\Gamma\{\Delta, \Delta\}$ is an initial sequent, the conclusion easily follows. If $n > 0$ and the principal formula is not in Δ , we apply the induction hypothesis to each of the premises and then the rule again. If $n > 0$ and the principal formula is in Δ we exploit the height-preserving invertibility of the logical rules as shown below, where ρ stands for an arbitrary rule instance and $\bar{\rho}$ stands for its inversion (which does not count for the overall height, as ρ is height-preserving invertible):

$$\frac{\frac{\Gamma\{\Delta', \Delta\}}{\Gamma\{\Delta, \Delta\}} \rho}{\Gamma\{\Delta\}} c \quad \rightsquigarrow \quad \frac{\frac{\Gamma\{\Delta', \Delta\}}{\Gamma\{\Delta', \Delta'\}} \bar{\rho}}{\Gamma\{\Delta'\}} c}{\Gamma\{\Delta\}} \rho$$

The application of c is removed invoking the induction hypothesis. The case where ρ is a binary rule is analogous and we omit the details. \square

The way we formulated the rules in NIPL allows us to establish the admissibility of the lift-rule. A variant of this rule was instead explicitly present in Fitting's system. Its absence (in combination with w and c) permits the use of the additive version of cut, which simplifies the cut elimination argument.

Lemma 4.4 *The lift-rule is height-preserving admissible in NIPL.*

Proof Proceed by induction on the height n of the derivation of the premise $\Gamma\{[\Sigma^\bullet, \Delta]\}$ of the rule. If $n = 0$ and no formula in Σ^\bullet is active, then we can remove it. Otherwise, $\Gamma\{\Sigma^\bullet, [\Delta]\}$ is again an instance of ax . If $n > 0$ and no formula in Σ is principal, we apply the induction hypothesis to the premise(s) of the rule and then the rule again.

If a formula A^\bullet in Σ^\bullet is principal in \wedge^\bullet or \vee^\bullet , we apply the induction hypothesis (possibly twice). E.g.,

$$\frac{\Gamma\{[\Sigma'^\bullet, A^\bullet, B^\bullet, \Delta]\}}{\Gamma\{[\Sigma'^\bullet, A \wedge B^\bullet, \Delta]\}} \wedge^\bullet \quad \rightsquigarrow \quad \frac{\frac{\Gamma\{[\Sigma'^\bullet, A^\bullet, B^\bullet, \Delta]\}}{\Gamma\{[\Sigma'^\bullet, A^\bullet, B^\bullet, [\Delta]]\}} \text{lift}}{\Gamma\{[\Sigma'^\bullet, A \wedge B^\bullet, [\Delta]]\}} \wedge^\bullet$$

If a formula A^\bullet in Σ^\bullet is principal in \supset^\bullet as in

$$\frac{\Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, \Delta\{\Pi, A^\circ\}]\} \quad \Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, \Delta\{\Pi, B^\bullet\}]\}}{\Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, \Delta\{\Pi\}]\}} \supset^\bullet$$

we apply the induction hypothesis and the rule \supset^\bullet , as in

$$\frac{\frac{\Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, \Delta\{\Pi, A^\circ\}]\}}{\Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, [\Delta\{\Pi, A^\circ\}]\}} \text{lift} \quad \frac{\Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, \Delta\{\Pi, B^\bullet\}]\}}{\Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, [\Delta\{\Pi, B^\bullet\}]\}} \text{lift}}{\Gamma\{[\Sigma'^\bullet, A \supset B^\bullet, [\Delta\{\Pi\}]\}} \supset^\bullet$$

□

Note that with the admissibility of the lift-rule we immediately obtain completeness of NIPL with respect to IPL via Fitting's system [8].

Lemma 4.5 *The 4-rule is height-preserving admissible in NIPL.*

Proof By induction on the height n of the derivation of the rule premise. If $\Gamma\{\Sigma^\circ\}$ is an initial sequent, then so is $\Gamma\{\Sigma^\circ\}$. If $n > 0$ we assume that a formula in Σ° is principal, otherwise the proof is trivial. We apply the induction hypothesis to the premise(s) of the rule and then the rule again. For example, if the last rule applied is \supset° , we have:

$$\frac{\Gamma\{\Delta, [A^\bullet, B^\circ]\}}{\Gamma\{\Delta, A \supset B^\circ\}} \supset^\circ \quad \rightsquigarrow \quad \frac{\Gamma\{\Delta, [A^\bullet, B^\circ]\}}{\Gamma\{\Delta, [A^\bullet, B^\circ]\}} \supset^\circ \quad \frac{\Gamma\{\Delta, [A^\bullet, B^\circ]\}}{\Gamma\{\Delta, A \supset B^\circ\}} \supset^\circ$$

□

Lemma 4.6 *The lower-rule is height-preserving admissible in NIPL.*

Proof The lower-rule is derivable with the following height-preserving steps:

$$\frac{\frac{\Gamma\{\Sigma^\circ, [\Delta]\}}{\Gamma\{\Sigma^\circ, [\Delta]\}} \supset^\circ \quad \frac{\Gamma\{\Sigma^\circ, [\Delta]\}}{\Gamma\{\Sigma^\circ, [\Delta]\}} \supset^\circ}{\frac{\Gamma\{\Sigma^\circ, [\Delta]\}}{\Gamma\{\Sigma^\circ, [\Delta]\}} \supset^\circ} \supset^\circ$$

□

Lemma 4.7 *The t-rule is height-preserving admissible in NIPL.*

Proof By induction on the height n of the premise $\Gamma\{\Delta\}$. If $n = 0$, then $\Gamma\{\Delta\}$ is an initial sequent and so is $\Gamma\{\Delta\}$. If $n > 0$, we apply the induction hypothesis to the premise(s) and then the rule again. As an example, consider the case in which the last rule applied is \supset^\bullet and formulas are introduced (bottom-up) in $[\Delta]$. We have:

$$\frac{\Gamma\{A \supset B^\bullet, [\Delta, A^\circ]\} \quad \Gamma\{A \supset B^\bullet, [\Delta, B^\bullet]\}}{\Gamma\{A \supset B^\bullet, [\Delta]\}} \supset^\bullet$$

We construct the following derivation:

$$\frac{\frac{\Gamma\{A \supset B^\bullet, [\Delta, A^\circ]\}}{\Gamma\{A \supset B^\bullet, \Delta, A^\circ\}} t \quad \frac{\Gamma\{A \supset B^\bullet, [\Delta, B^\bullet]\}}{\Gamma\{A \supset B^\bullet, \Delta, B^\bullet\}} t}{\Gamma\{A \supset B^\bullet, \Delta\}} \supset^\bullet$$

where the applications of t are removed by induction hypothesis. □

This completes the proof of the properties (N1) and (N2). To eliminate cut, we also need (N3)–(N5), which will be shown below.

Theorem 4.8 (Cut elimination) *The cut-rule is admissible for NIPL.*

Proof We consider an uppermost cut and proceed by induction on the lexicographically ordered pair (c, n) where c is the degree of its cut formula and n is

the height of the derivation of $\Gamma\{A^\bullet\}$.³

(N5) If $n = 0$, then $\Gamma\{A^\bullet\}$ is an initial sequent. If A^\bullet is not active, $\Gamma\{\emptyset\}$ is an initial sequent too. If A^\bullet is active in ax, we have $A^\bullet = p^\bullet$ for some p , and

$$\frac{\Gamma\{p^\circ, \Delta\{p^\circ\}\} \quad \overline{\Gamma\{p^\bullet, \Delta\{p^\circ\}\}}^{\text{ax}}}{\Gamma\{\Delta\{p^\circ\}\}}_{\text{cut}}$$

The cut is eliminated as follows:

$$\frac{\frac{\Gamma\{p^\circ, \Delta\{p^\circ\}\}}{\Gamma\{\Delta\{p^\circ, p^\circ\}\}}^{\text{lower}}}{\Gamma\{\Delta\{p^\circ\}\}}_c$$

The case of the axiom \perp^\bullet is handled similarly, noticing that from the derivability in NIPL of $\Gamma\{\perp^\circ\}$ follows the derivability of $\Gamma\{\emptyset\}$.

(N3) If $n > 0$ and A^\bullet is not principal, we apply the invertibility of the corresponding rule to $\Gamma\{A^\circ\}$, permute the cut upwards, and remove it by secondary induction hypothesis. For example, consider the following derivation, where ρ is some binary rule from NIPL:

$$\frac{\Gamma\{A^\circ\} \quad \frac{\Gamma\{A^\bullet\} \quad \Gamma''\{A^\bullet\}}{\Gamma\{A^\bullet\}}_\rho}{\Gamma\{\emptyset\}}_{\text{cut}}$$

We construct the following derivation, where we again use the height-preserving invertibility of ρ :

$$\frac{\frac{\frac{\Gamma\{A^\circ\}}{\Gamma\{A^\circ\}}^{\bar{\rho}} \quad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}}_{\text{cut}} \quad \frac{\frac{\Gamma\{A^\circ\}}{\Gamma''\{A^\circ\}}^{\bar{\rho}} \quad \Gamma''\{A^\bullet\}}{\Gamma''\{\emptyset\}}_\rho}{\Gamma\{\emptyset\}}_{\text{cut}}}$$

(N4) If A^\bullet is principal in \wedge or \vee , the case is handled in the usual way using the invertibility of the rules. For example

$$\frac{\Gamma\{B \vee C^\circ\} \quad \frac{\Gamma\{B^\bullet\} \quad \Gamma\{C^\bullet\}}{\Gamma\{B \vee C^\bullet\}}^{\vee^\bullet}}{\Gamma\{\emptyset\}}_{\text{cut}}$$

is eliminated as follows (where $\bar{\vee}^\circ$ is the inversion of \vee° and each cut is on a formula of lesser degree):

$$\frac{\frac{\frac{\Gamma\{B \vee C^\circ\}}{\Gamma\{B^\circ, C^\circ\}}^{\bar{\vee}^\circ} \quad \frac{\Gamma\{B^\bullet\}}{\Gamma\{B^\bullet, C^\circ\}}^w}{\Gamma\{C^\circ\}}_{\text{cut}} \quad \Gamma\{C^\bullet\}}{\Gamma\{\emptyset\}}_{\text{cut}}$$

³ It is enough to consider only the height of the left premise as every right rule is invertible.

The case below in which A^\bullet is principal in \supset^\bullet :

$$\frac{\Gamma\{B \supset C^\bullet, \Pi\{\Sigma\}\} \quad \frac{\Gamma\{B \supset C^\bullet, \Pi\{B^\circ, \Sigma\}\} \quad \Gamma\{B \supset C^\bullet, \Pi\{C^\bullet, \Sigma\}\}}{\Gamma\{B \supset C^\bullet, \Pi\{\Sigma\}\}} \supset^\bullet}{\Gamma\{\Pi\{\Sigma\}\}} \text{cut}$$

is handled using some of the structural rules from (N2). We first construct a derivation of $\Gamma\{\Pi\{B^\circ, \Sigma\}\}$:

$$\frac{\frac{\Gamma\{B \supset C^\circ, \Pi\{\Sigma\}\}}{\Gamma\{B \supset C^\circ, \Pi\{B^\circ, \Sigma\}\}} \text{w} \quad \Gamma\{B \supset C^\bullet, \Pi\{B^\circ, \Sigma\}\}}{\Gamma\{\Pi\{B^\circ, \Sigma\}\}} \text{cut}$$

The cut is removed by secondary induction hypothesis. A symmetrical derivation yields $\Gamma\{\Pi\{C^\bullet, \Sigma\}\}$, and the reduction is completed as follows:

$$\frac{\frac{\Gamma\{\Pi\{B^\circ, \Sigma\}\}}{\Gamma\{\Pi\{B^\circ, C^\circ, \Sigma\}\}} \text{w} \quad \frac{\frac{\Gamma\{B \supset C^\circ, \Pi\{\Sigma\}\}}{\Gamma\{\Pi\{B \supset C^\circ, \Sigma\}\}} \text{lower} \quad \frac{\Gamma\{\Pi\{B^\bullet, C^\circ, \Sigma\}\}}{\Gamma\{\Pi\{B^\bullet, C^\circ, \Sigma\}\}} \text{t}}{\Gamma\{\Pi\{B^\bullet, C^\circ, \Sigma\}\}} \text{t}}{\Gamma\{\Pi\{C^\circ, \Sigma\}\}} \text{cut} \quad \frac{\Gamma\{\Pi\{C^\bullet, \Sigma\}\}}{\Gamma\{\Pi\{\Sigma\}\}} \text{cut} \quad \square$$

We can now show completeness independently from Fitting's calculus:

Corollary 4.9 *NIPL is complete with respect to IPL.*

Proof It is easy to check that every axiom of IPL can be proved in NIPL and modus ponens can be simulated by cut. The claim follows by Theorem 4.8. \square

5 Intermediate Logics of Bounded Depth

We introduce nested calculi for the propositional intermediate logics BD_n (Bounded Depth n) which are semantically characterized by intuitionistic Kripke frames in which every chain is of length less or equal to n . Our calculi are defined in a modular way by extending NIPL with suitable structural rules, which preserve conditions (N1)–(N5) and therefore cut elimination.

Definition 5.1 A *bounded depth n Kripke frame* is a pair $\langle P, \leq \rangle$, where P is a non empty set of *worlds*, denoted by x, y, z, \dots , and \leq is a partial order on P with

$$\forall x_0 \dots x_n. \left(\bigwedge_{0 \leq i \leq n-1} x_i \leq x_{i+1} \supset \bigvee_{0 \leq i \leq n-1} x_{i+1} \leq x_i \right). \quad (3)$$

A *bounded depth Kripke model* is a triple $\langle P, \leq, v \rangle$, where $\langle P, \leq \rangle$ is a bounded depth Kripke frame and v a function which maps propositional atoms to subsets of P , such that for all worlds x and y , if $x \leq y$ and $x \in v(p)$ then

$y \in v(p)$ (monotonicity condition). Truth conditions for a formula in a world are defined as usual, see, e.g. [9]. A formula is *true in a model* if it is true in every world of the model. A formula is *valid* if it is true in every bounded depth Kripke model. A sequent is true in a world (resp. true in a model, resp. valid) if the corresponding formula is.

The logics BD_n are axiomatized by the schemata

$$\mathbf{bd}_1 := p_1 \vee (p_1 \rightarrow \perp) \quad \text{and} \quad \mathbf{bd}_{n+1} := p_{n+1} \vee (p_{n+1} \rightarrow \mathbf{bd}_n)$$

where with an abuse of notation here and below we use $p_{(i)}, q$ as metavariables, to be substituted by arbitrary formulas. For each n , the logic BD_n is sound and complete with respect to the class of bounded depth n Kripke frames [9]. BD_1 turns out to be classical logic, and BD_2 one of the seven interpolable intermediate logics [23].

Before presenting the peculiar structural rule capturing BD_n , we sketch its genesis, using BD_2 as case study. The starting point for the rule's definition is (an adaptation to nested sequents of) the algorithm for transforming Hilbert axioms into structural hypersequent rules given in [5], and into display rules in [6,7]. Indeed, we start with the axiom schema $\mathbf{bd}_2 := p \vee (p \supset (q \vee \neg q))$ characterizing BD_2 . By using the invertible rules of NIPL, the axiom is decomposed into the equivalent nested sequent in which all connectives are removed:

$$p^\circ, [p^\bullet, q^\circ, [q^\bullet]] \quad (4)$$

This sequent can be transformed into the equivalent (interderivable) rule

$$\frac{\Delta_0, [\Delta_0, \Delta_1, \Sigma, [\Delta_2]] \quad \Delta_0, [\Delta_1, \Sigma, [\Delta_2, \Sigma]]}{\Delta_0, [\Delta_1, \Sigma, [\Delta_2]]} \text{bd}'_2$$

following the steps below:

- First, move p° and q° to the premise, as shown below left, and then p^\bullet and q^\bullet , as in the rule below right (Δ_i, Σ are *fresh* metavariables for sequents):

$$\frac{\Sigma, q^\bullet \quad \Delta_0, p^\bullet}{\Delta_0, [p^\bullet, \Sigma, [q^\bullet]]} \quad \frac{\Sigma, q^\bullet \quad \Delta_0, p^\bullet \quad \Delta_0, [\Delta_1, p^\circ, \Sigma, [\Delta_2]] \quad \Delta_0, [\Delta_1, \Sigma, [\Delta_2, q^\circ]]}{\Delta_0, [\Delta_1, \Sigma, [\Delta_2]]} r_2$$

It is easy to see that each of these rules derives the sequent (4) in NIPL (instantiate $\Delta_0 := p^\circ$ and $\Delta_1 := p^\bullet$ and $\Sigma := q^\circ$ and $\Delta_2 := q^\bullet$). The converse direction follows by *nec* (Proposition 3.2), *w*, *c* and *cut* with the sequent in (4).

- The rule (bd'_2) is obtained from the (r_2) rule above by cutting the premises of the latter in all possible ways (using first *nec* and *w*). This ensures that (bd'_2) derives (r_2) in NIPL. For the converse direction we show that from any instance of the premisses of (bd'_2) we can derive an instance of the premisses of (r_2) for suitable p and q . Take $p := \text{fm}(\Delta_0^\circ)$, and $q := \text{fm}(\Sigma^\circ)$.

As shown below, the (bd'_2) rule allows the derivation of the axiom \mathbf{bd}_2 , and therefore its addition to NIPL result in a complete calculus for the BD_2 logic.

$$\frac{\frac{\frac{\frac{A^\circ, [A^\circ, A^\bullet, B^\circ, [B^\bullet]]}{A^\circ, [A^\bullet, B^\circ, [B^\bullet, B^\circ]]} \text{bd}'_2}{\frac{A^\circ, [A^\bullet, B^\circ, [B^\bullet]]}{A^\circ, [A^\bullet, B^\circ, \neg B^\circ]} \supset^\circ}{\frac{A^\circ, [A^\bullet, B \vee \neg B^\circ]}{A^\circ, A \supset (B \vee \neg B)}^\circ} \vee^\circ}{A \vee (A \supset (B \vee \neg B))^\circ} \vee^\circ$$

However, (bd'_2) does not preserve cut elimination. Prior identifying Δ_1 and Σ , we modify (bd'_2) from shallow to *deep* (using the terminology in [13]), to permit its application on structures in any arbitrary node in the tree. This leads to:

$$\frac{\Gamma\{\Delta_0\{\Delta_0\{\emptyset\}, \Delta_1\{\Delta_2\}\}\} \quad \Gamma\{\Delta_0\{\Delta_1\{\Delta_1\{\emptyset\}, \Delta_2\}\}\}}{\Gamma\{\Delta_0\{\Delta_1\{\Delta_2\}\}\}} \text{bd}_2$$

The argument applies also to BD_n ($n \geq 1$) resulting in a rule having n premises.

Definition 5.2 NIPLBD_n is the calculus NIPL extended with the rule bd_n :

$$\frac{\Gamma\{\Delta_0\{\dots\{\Delta_{i-1}\{\emptyset\}, \Delta_i\{\dots\{\Delta_n\}\dots\}\}\} \mid 1 \leq i \leq n}{\Gamma\{\Delta_0\{\Delta_1\{\dots\Delta_{n-1}\{\Delta_n\}\dots\}\}} \text{bd}_n$$

Proposition 5.3 *For every n , the rule bd_n is sound with respect to intuitionistic Kripke frames with depth bounded by n .*

Proof By contradiction. We assume that $\Gamma\{\Delta_0\{\dots\{\Delta_n\}\dots\}\}$ is not valid. Hence there are worlds x_0, \dots, x_n such that:

- $x_i \leq x_{i+1}$, with $0 \leq i \leq n-1$.
- $x_i \Vdash \bigwedge \Delta_i^\bullet$ and $x_i \not\Vdash \bigvee \Delta_i^\circ$.

where $\Delta_i^\bullet, (\Delta_i^\circ)$ are the input (output) formulas (formulas and boxed sequents) in Δ_i . By (3) we have $x_{i+1} \leq x_i$ for some $i \in \{0, \dots, n-1\}$. In each case, by monotonicity (w.r.t. compound formulas) we get $x_{i+1} \Vdash \bigwedge \Delta_i^\bullet$ and thus $x_{i+1} \Vdash \bigvee \Delta_i^\circ$, which (again by monotonicity) yields $x_i \Vdash \bigvee \Delta_i^\circ$, which is a contradiction. \square

6 Cut Elimination for NIPLBD_n

We show that cut elimination holds for NIPLBD_n . The proof extends that for NIPL in a modular way: only the cases concerning the new rules need to be considered. We start by showing properties (N1) and (N2): invertibility of all rules, and height-preserving admissibility of the dedicated rules in Fig. 3.

Lemma 6.1 *The weakening rule is height-preserving admissible in NIPLBD_n .*

Proof Proceeds as the proof of Lemma 4.1. \square

Lemma 6.2 *Every rule is height-preserving invertible in NIPLBD_n .*

Proof By induction on the height of the derivation for every rule of the system. The structural rule bd_n is height-preserving invertible by using Lemma 6.1. The

addition of the rule bd_n preserves the invertibility of the other rules. The strategy consists in applying (possibly twice, due to the repetition of the contexts) the induction hypothesis to each premise of the rule. \square

Lemma 6.3 *The contraction rule is height-preserving admissible in NIPLBD $_n$.*

Proof By induction on the height of the derivation. The only additional case concerns the rule bd_n . Since the principal formulas are repeated in each premise of bd_n we just apply the induction hypothesis and then the rule again. We give a concrete example of this qualitative analysis; to improve the readability we consider the particular case of the bd_2 rule.

$$\frac{\Gamma\{\Delta_0\{\Delta_0\{\emptyset\}, \Delta_1\{\Delta_2\}\}, [\Delta_1\{\Delta_2\}]\} \quad \Gamma\{\Delta_0\{[\Delta_1\{\Delta_1\{\emptyset\}, \Delta_2\}], [\Delta_1\{\Delta_2\}]\}}}{\Gamma\{\Delta_0\{[\Delta_1\{\Delta_2\}], [\Delta_1\{\Delta_2\}]\}}}_{\text{bd}_2}$$

We construct the following derivation:

$$\frac{\frac{\Gamma\{\Delta_0\{\Delta_0\{\emptyset\}, \Delta_1\{\Delta_2\}\}, [\Delta_1\{\Delta_2\}]\}}{\Gamma\{\Delta_0\{\Delta_0\{\emptyset\}, \Delta_1\{\Delta_2\}\}, [\Delta_0\{\emptyset\}, \Delta_1\{\Delta_2\}]\}}_w \quad \frac{\Gamma\{\Delta_0\{[\Delta_1\{\Delta_1\{\emptyset\}, \Delta_2\}], [\Delta_1\{\Delta_2\}]\}}}{\Gamma\{\Delta_0\{[\Delta_1\{\Delta_1\{\emptyset\}, \Delta_2\}], [\Delta_1\{\Delta_1\{\emptyset\}, \Delta_2\}]\}}_w}{\Gamma\{\Delta_0\{\Delta_0\{\emptyset\}, \Delta_1\{\Delta_2\}\}}}_c \quad \frac{\Gamma\{\Delta_0\{[\Delta_1\{\Delta_1\{\emptyset\}, \Delta_2\}], [\Delta_1\{\Delta_1\{\emptyset\}, \Delta_2\}]\}}}{\Gamma\{\Delta_0\{[\Delta_1\{\Delta_1\{\emptyset\}, \Delta_2\}]\}}}_{\text{bd}_2}}{\Gamma\{\Delta_0\{[\Delta_1\{\Delta_2\}]\}}}_c$$

\square

Lemma 6.4 *The rule lift is height-preserving admissible in NIPLBD $_n$.*

Proof By induction on the height of the derivation. To simplify the notation we consider only bd_2 ; the generalization to bd_n is immediate.

Let $\Gamma\{[\Sigma^\bullet, \Delta]\}$ be the conclusion of bd_2 , we need to consider two subcases. Either $[\Sigma^\bullet, \Delta]$ is moved by the rule or not. In the latter case we simply apply the induction hypothesis to the premises of the rule and then the rule again. In the former case we need to distinguish two further subcases. Either $[\Sigma^\bullet, \Delta]$ is Δ_i for $i \in \{0, 1, 2\}$ or not. In the latter case we apply the induction hypothesis to the premises and then the rule again. In the former case assume $[\Delta_1\{\ \}] = [\Sigma^\bullet, \Delta\{\ \}]$. We have:

$$\frac{\Gamma\{\Delta_0\{\Delta_0\{\emptyset\}, \Sigma^\bullet, \Delta\{\Delta_2\}\}\} \quad \Gamma\{\Delta_0\{\Sigma^\bullet, \Delta\{\Sigma^\bullet, \Delta\{\emptyset\}, \Delta_2\}\}\}}{\Gamma\{\Delta_0\{\Sigma^\bullet, \Delta\{\Delta_2\}\}\}}_{\text{bd}_2}$$

We construct the following derivation:

$$\frac{\frac{\Gamma\{\Delta_0\{\Delta_0\{\emptyset\}, \Sigma^\bullet, \Delta\{\Delta_2\}\}\}}{\Gamma\{\Delta_0\{\Sigma^\bullet, [\Delta_0\{\emptyset\}, \Delta\{\Delta_2\}]\}\}}_{\text{lift}} \quad \frac{\Gamma\{\Delta_0\{\Sigma^\bullet, \Delta\{\Sigma^\bullet, \Delta\{\emptyset\}, \Delta_2\}\}\}}{\Gamma\{\Delta_0\{\Sigma^\bullet, \Sigma^\bullet, [\Delta\{\Delta\{\emptyset\}, \Delta_2\}]\}\}}_{\text{several lift}}}{\frac{\Gamma\{\Delta_0\{\Sigma^\bullet, [\Delta_0\{\Sigma^\bullet\}, \Delta\{\Delta_2\}]\}\}}{\Gamma\{\Delta_0\{\Sigma^\bullet, [\Delta\{\Delta\{\emptyset\}, \Delta_2\}]\}\}}_w \quad \frac{\Gamma\{\Delta_0\{\Sigma^\bullet, \Sigma^\bullet, [\Delta\{\Delta\{\emptyset\}, \Delta_2\}]\}\}}{\Gamma\{\Delta_0\{\Sigma^\bullet, [\Delta\{\Delta\{\emptyset\}, \Delta_2\}]\}\}}_c}_{\text{bd}_2}}{\Gamma\{\Delta_0\{\Sigma^\bullet, [\Delta\{\Delta_2\}]\}\}}_{\text{bd}_2}}$$

\square

Lemma 6.5 *The rule 4 is height-preserving admissible in NIPLBD $_n$.*

Proof By induction on the height of the derivation. Assume the last applied rule is bd_n . We observe that the general form of Σ° is $A_1^\circ, \dots, A_n^\circ, [\Sigma_1], \dots, [\Sigma_m]$. We distinguish three cases: either Σ° does not move, or it entirely moves

to another boxed sequent, or different components of Σ° move to different sequents. In the first and second case we simply apply the induction hypothesis to the premises and then the rule again. In the latter case we have that Σ° is of shape $\Sigma^\circ = \Sigma' \{ \Sigma'' \{ \Sigma''' \} \}$ and we have

$$\frac{\dots \frac{\Gamma\{\Delta_0 \{ \dots \{ \Delta'_i, \Sigma' \{ \Delta'_i, \Sigma' \{ \emptyset \}, \Sigma'' \{ \dots \{ \Delta_n \} \dots \} \} \dots \} \} \}}{\Gamma\{\Delta_0 \{ \dots \{ \Delta'_i, \Sigma' \{ \Sigma'' \{ \dots \{ \Delta_n \} \dots \} \} \dots \} \} \}} \quad \dots}{\Gamma\{\Delta_0 \{ \dots \{ \Delta'_i, [\Sigma' \{ \Sigma'' \{ \dots \{ \Delta_n \} \dots \} \} \dots \} \} \}} \quad \dots}{\Gamma\{\Delta_0 \{ \dots \{ \Delta'_i, [\Sigma' \{ \Sigma'' \{ \dots \{ \Delta_n \} \dots \} \} \dots \} \} \}} \quad \dots} \text{bd}_n$$

and we can proceed as follows:

$$\frac{\dots \frac{\frac{\Gamma\{\Delta_0 \{ \dots \{ \Delta'_i, \Sigma' \{ \Delta'_i, \Sigma' \{ \emptyset \}, \Sigma'' \{ \dots \{ \Delta_n \} \dots \} \} \dots \} \} \}}{\Gamma\{\Delta_0 \{ \dots \{ \Delta'_i, [\Sigma' \{ \Delta'_i, \Sigma' \{ \emptyset \}, \Sigma'' \{ \dots \{ \Delta_n \} \dots \} \} \dots \} \} \}} \quad \dots}{\Gamma\{\Delta_0 \{ \dots \{ \Delta'_i, [\Sigma' \{ \Delta'_i, [\Sigma' \{ \emptyset \}], \Sigma'' \{ \dots \{ \Delta_n \} \dots \} \} \dots \} \} \}} \quad \dots}{\Gamma\{\Delta_0 \{ \dots \{ \Delta'_i, [\Sigma' \{ \Sigma'' \{ \dots \{ \Delta_n \} \dots \} \} \dots \} \} \}} \quad \dots} \text{bd}_n \quad \square$$

Lemma 6.6 *The rule lower is height-preserving admissible in NIPLBD_n.*

Proof This proof is literally the same as for NIPL, applying 4, w and c. \square

We now need to prove the admissibility of the **t** rule, which removes boxes. To simplify the proof in the presence of **bd_n**, we consider the auxiliary rule **lift*** below.

Lemma 6.7 *The following rule is derivable with {t, c, 4, lift, w}.*

$$\frac{\Gamma\{\Sigma\{\Delta, \Sigma\{\Delta\}\}\}}{\Gamma\{\Sigma\{\Delta, \Sigma\{\emptyset\}\}\}} \text{lift}^*$$

Proof For better readability we only show the case where the depth of $\Sigma\{ \}$ is 1, i.e., $\Sigma\{ \} = \Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, \{ \}]$:

$$\frac{\frac{\frac{\frac{\frac{\Gamma\{\Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, [\Delta, \Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, \Delta]]\}}{\Gamma\{\Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, [\Delta, \Sigma_1, \Sigma_2^\bullet, \Sigma_2^\circ, \Delta]]\}} \quad \text{t}}{\Gamma\{\Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, [\Delta, \Sigma_1, \Sigma_2^\bullet, \Sigma_2^\circ]]\}} \quad \text{c}}{\Gamma\{\Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, \Sigma_2^\circ, [\Delta, \Sigma_1, \Sigma_2^\circ]]\}} \quad \text{lift}}{\Gamma\{\Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, [\Delta, \Sigma_1, \Sigma_2^\circ]]\}} \quad \text{c}}{\frac{\Gamma\{\Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, [\Delta, \Sigma_1, \Sigma_2^\circ]]\}}{\Gamma\{\Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, [\Delta, \Sigma_1, [\Sigma_2^\circ]]\}} \quad \text{4}}{\Gamma\{\Sigma_1, [\Sigma_2^\bullet, \Sigma_2^\circ, [\Delta, \Sigma_1, [\Sigma_2^\circ]]\}} \quad \text{w}} \quad \text{4}} \quad \text{w}$$

If the depth of $\Sigma\{ \}$ is n , all steps in this derivation have to be repeated n times. \square

Lemma 6.8 *The rule **t** is height-preserving admissible in NIPLBD_n.*

Proof The only new case to consider is the **bd_n**-rule. Again, for the sake of clarity, we discuss **bd₂**. We distinguish the following subcases: either we apply **t** to $[\Delta_i]$ for some $i \in \{1, 2\}$, or to some other boxed sequent. In the latter case, we apply the induction hypothesis (possibly twice) to the premises and then the rule again. In the former case we assume w.l.o.g. that we apply **t** to Δ_1 :

$$\frac{\frac{\Gamma\{\Delta_0\{\Delta_0\{\emptyset\}, \Delta_1\{\Delta_2\}\}\}}{\Gamma\{\Delta_0\{\Delta_1\{\Delta_2\}\}\}} \quad \frac{\Gamma\{\Delta_0\{\Delta_1\{\Delta_1\{\emptyset\}, \Delta_2\}\}\}}{\Gamma\{\Delta_0\{\Delta_1\{\Delta_2\}\}\}}}{\Gamma\{\Delta_0\{\Delta_1\{\Delta_2\}\}\}} \text{bd}_2$$

We have two subcases to distinguish here:

- $[\Delta_1]$ is an immediate child of Δ_0 in the nested sequent tree. We have:

$$\frac{\frac{\Gamma\{\Delta_0, [\Delta_0, \Delta_1\{\Delta_2\}]\}}{\Gamma\{\Delta_0, \Delta_0, \Delta_1\{\Delta_2\}\}} \text{t}}{\Gamma\{\Delta_0, \Delta_1\{\Delta_2\}\}} \text{c}$$

by applying the induction hypothesis to the left subproof, discarding the right one.

- $[\Delta_1]$ is not an immediate child of Δ_0 in the nested sequent tree, so we can assume that Δ_0 contains at least one nesting, thus we have:

$$\Delta_0\{ \} = \Delta'_0\{\Delta''_0\{ \} \}$$

for some contexts $\Delta'_0\{ \}$ and $\Delta''_0\{ \}$. We construct the derivation

$$\frac{\frac{\frac{\Gamma\{\Delta'_0\{\Delta''_0, [\Delta'_0\{\Delta''_0\{\emptyset\}], \Delta_1\{\Delta_2\}\}\}}{\Gamma\{\Delta'_0\{\Delta''_0, \Delta'_0\{\Delta''_0\{\emptyset\}\}, \Delta_1\{\Delta_2\}\}\}} \text{t}}{\Gamma\{\Delta'_0\{\Delta''_0, \Delta'_0\{\Delta''_0\{\emptyset\}\}, \Delta_1\{\Delta_2\}\}\}} \text{t}}{\Gamma\{\Delta'_0\{\Delta''_0, \Delta'_0\{\emptyset\}, \Delta_1\{\Delta_2\}\}\}} \text{lift}^* \quad \frac{\frac{\Gamma\{\Delta'_0\{\Delta''_0, [\Delta_1\{\Delta_1\{\emptyset\}, \Delta_2\}\}\}}{\Gamma\{\Delta'_0\{\Delta''_0, \Delta_1\{\Delta_1\{\emptyset\}, \Delta_2\}\}\}} \text{t}}{\Gamma\{\Delta'_0\{\Delta''_0, \Delta_1\{\Delta''_0, \Delta_1\{\emptyset\}, \Delta_2\}\}\}} \text{w}}{\Gamma\{\Delta'_0\{\Delta''_0\{\Delta_1\{\Delta_2\}\}\}} \text{bd}_2}$$

which yields the desired conclusion. \square

This completes the proofs of conditions (N1) and (N2).

Theorem 6.9 (Cut elimination) *The cut rule is eliminable in NIPLBD_n.*

Proof The proof proceeds as for NIPL. We only need to check that the cut rule can be shifted up over bd_n . That means we only have to verify (N3), as (N4) and (N5) are not affected by adding new structural rules to the system (and (N1) and (N2) still hold).

If the cut formula A^\bullet is not active in bd_n , we can permute the cut upwards and remove it by induction hypothesis. If A^\bullet is active, assume w.l.o.g. that the cut formula is in $[\Delta_i]$. We distinguish two cases according to its position.

- If A^\bullet is in the branch from Δ_0 to Δ_n , we have:

$$\frac{\frac{\Gamma\{\Delta_0\{A^\bullet, \Delta_1\{\Delta_2 \dots \{\Delta_n\}\}\}} \quad \dots \quad \frac{\Gamma\{\Delta_0\{A^\bullet, \Delta_1\{A^\bullet, \Delta_1\{\emptyset\}, \Delta_2 \dots \{\Delta_n\}\}\}}{\Gamma\{\Delta_0\{A^\bullet, \Delta_1\{\Delta_2 \dots \{\Delta_n\}\}\}} \text{bd}_n}{\Gamma\{\Delta_0\{A^\bullet, \Delta_1\{\Delta_2 \dots \{\Delta_n\}\}\}} \text{cut}}{\Gamma\{\Delta_0\{\Delta_1\{\Delta_2 \dots \{\Delta_n\}\}\}}}$$

(we display the premise in which the position of A^\bullet changes). For each premise different from $\Gamma\{\Delta_0\{A^\bullet, \Delta_1\{A^\bullet, \Delta_1\{\emptyset\}, \Delta_2 \dots \{\Delta_n\}\}\}$, we proceed by height-preserving admissibility of w and cross-cuts which are removed by induction hypothesis. Then we construct the following derivation:

$$\frac{\frac{\Gamma\{\Delta_0\{A^\circ, \Delta_1\{\Delta_2 \dots \{\Delta_n\}\}\}\}}{\Gamma\{\Delta_0\{A^\circ, \Delta_1\{\Delta_1\{\emptyset\}, \Delta_2 \dots \{\Delta_n\}\}\}} \text{ w} \quad \frac{\Gamma\{\Delta_0\{A^\bullet, \Delta_1\{A^\bullet, \Delta_1\{\emptyset\}, \Delta_2 \dots \{\Delta_n\}\}\}}{\Gamma\{\Delta_0\{A^\bullet, \Delta_1\{\Delta_1\{\emptyset\}, \Delta_2 \dots \{\Delta_n\}\}\}} \text{ lift}}{\Gamma\{\Delta_0\{\Delta_1\{\Delta_1\{\emptyset\}, \Delta_2 \dots \{\Delta_n\}\}\}} \text{ cut}}$$

to which we can now apply the rule bd_n and obtain the desired conclusion.

- If A^\bullet is not in the branch from Δ_0 to Δ_n , then A^\bullet must be inside a bracket in this branch, as in:

$$\frac{\Gamma\{\Delta_0\{[\Delta\{A^\bullet\}], \Delta_1\{\Delta_2 \dots \{\Delta_n\}\}\}}{\Gamma\{\Delta_0\{[\Delta\{\emptyset\}], \Delta_1\{\Delta_2 \dots \{\Delta_n\}\}\}} \quad \frac{\dots \quad \Gamma\{\Delta_0\{[\Delta\{A^\bullet\}], \Delta_1\{[\Delta\{A^\bullet\}], \Delta_1\{\emptyset\}, \Delta_2 \dots \{\Delta_n\}\}\} \quad \dots}{\Gamma\{\Delta_0\{[\Delta\{A^\bullet\}], \Delta_1\{\Delta_2 \dots \{\Delta_n\}\}\}} \text{ bd}_n}{\Gamma\{\Delta_0\{[\Delta\{\emptyset\}], \Delta_1\{\Delta_2 \dots \{\Delta_n\}\}\}} \text{ cut}}$$

We construct the following derivation:

$$\frac{\frac{\Gamma\{\Delta_0\{[\Delta\{A^\circ\}], \Delta_1\{\Delta_2 \dots \{\Delta_n\}\}\}}{\Gamma\{\Delta_0\{\Delta_1\{[\Delta\{A^\circ\}], \Delta_2 \dots \{\Delta_n\}\}\}} \text{ lower} \quad \frac{\Gamma\{\Delta_0\{[\Delta\{A^\bullet\}], \Delta_1\{[\Delta\{A^\bullet\}], \Delta_1\{\emptyset\}, \Delta_2 \dots \{\Delta_n\}\}\}}{\Gamma\{\Delta_0\{\Delta_1\{[\Delta\{A^\bullet\}], \Delta_1\{\emptyset\}, \Delta_2 \dots \{\Delta_n\}\}\}} \text{ lower, c}}{\Gamma\{\Delta_0\{\Delta_1\{[\Delta\{\emptyset\}], \Delta_1\{\emptyset\}, \Delta_2 \dots \{\Delta_n\}\}\}} \text{ cut}}$$

The cut is removed invoking the secondary induction hypothesis. With respect to the other premises we permute the cut upwards and we remove it applying the secondary induction hypothesis. The desired sequent then follows by an application of bd_n . \square

Corollary 6.10 *NIPLBD_n is complete with respect to BD_n.*

Proof Follows from Corollary 4.9, the derivability of the axiom bd_n , and the cut elimination theorem. \square

Concluding Remark

The cut elimination proof contained in this paper makes use of the auxiliary rules t and 4 , originally introduced in the modal logics context [24]. Our results can be seen as “transfer of knowledge” from (the proof theory of) modal logics to intermediate logics. We consider it an important aspect of our future research to also initiate a transfer back to modal logics. In fact, notice that our cut elimination proof holds for NIPL extended by any structural rule that preserves properties (N1)–(N5). This paves the way for the definition of an algorithm, along the line of that in [5,6], to introduce analytic nested calculi for a large class of intermediate logics starting from their axiomatizations. Many such intermediate logics have indeed modal counterparts that have not yet been investigated from a proof theoretical point of view. We plan to do so using our work on nested sequents.

References

- [1] N. D. Belnap, Jr. Display logic. *J. Philos. Logic*, 11(4):375–417, 1982.
- [2] K. Brännler. Deep sequent systems for modal logic. *Archive for Mathematical Logic*, 48(6):551–577, 2009.
- [3] K. Chaudhuri, S. Marin, and L. Straßburger. Focused and synthetic nested sequents. In B. Jacobs and C. Löding, editors, *Proceedings of FoSSaCS*, 2016.

- [4] K. Chaudhuri, S. Marin, and L. Straßburger. Modular focused proof systems for intuitionistic modal logics. In D. Kesner and B. Pientka, editors, *Proceedings of FSCD*, volume 52 of *LIPICs*, pages 16:1–16:18. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.
- [5] A. Ciabattoni, N. Galatos, and K. Terui. From axioms to analytic rules in nonclassical logics. In *Proceedings of LICS'08*, pages 229–20. IEEE, 2008.
- [6] A. Ciabattoni and R. Ramanayake. Power and limits of structural display rules. *ACM Trans. Comput. Log.*, 17(3):17:1–17:39, 2016.
- [7] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for distributive modal logic. *Annals of Pure and Applied Logic*, 163(3):338–376, 2012.
- [8] M. Fitting. Nested sequents for intuitionistic logics. *Notre Dame Journal of Formal Logic*, 55(1):41–61, 2014.
- [9] D. Gabbay. *Semantical Investigations in Heyting's Intuitionistic Logic*. Reidel, 1983.
- [10] J.-Y. Girard. *Proof Theory and Logical Complexity, Volume I*, volume 1 of *Studies in Proof Theory*. Bibliopolis, edizioni di filosofia e scienze, 1987.
- [11] M. Girlando and L. Straßburger. MOIN: A nested sequent theorem prover for intuitionistic modal logics (system description). In N. Peltier and V. Sofronie-Stokkermans, editors, *Proceedings of IJCAR 2020*, volume 12167 of *LNCS*, pages 398–407. Springer, 2020.
- [12] R. Goré, L. Postniece, and A. Tiu. Taming displayed tense logics using nested sequents with deep inference. In M. Giese and A. Waaler, editors, *Automated Reasoning with Analytic Tableaux and Related Methods*, volume 5607 of *LNCS*, pages 189–204. Springer, 2009.
- [13] R. Goré, L. Postniece, and A. Tiu. On the correspondence between display postulates and deep inference in nested sequent calculi for tense logics. *Log. Methods Comput. Sci.*, 7(2):2:8, 38, 2011.
- [14] R. Goré and R. Ramanayake. Labelled tree sequents, tree hypersequents and nested (deep) sequents. In T. Bolander, T. Braüner, S. Ghilardi, and L. S. Moss, editors, *Advances in Modal Logic 9, papers from the ninth conference on "Advances in Modal Logic," held in Copenhagen, Denmark, 22-25 August 2012*, pages 279–299. College Publications, 2012.
- [15] R. Kashima. Cut-free sequent calculi for some tense logics. *Studia Logica*, 53(1):119–136, 1994.
- [16] R. Kuznets and B. Lellmann. Interpolation for intermediate logics via injective nested sequents. *J. Log. Comput.*, 31(3):797–831, 2021.
- [17] R. Kuznets and L. Straßburger. Maehara-style modal nested calculi. *Arch. Math. Log.*, 58(3-4):359–385, 2019.
- [18] F. Lauridsen. Intermediate logics admitting a structural hypersequent calculus. *Studia Logica*, 107:247–282, 2019.
- [19] B. Lellmann. Linear nested sequents, 2-sequents and hypersequents. In H. de Nivelle, editor, *Automated Reasoning with Analytic Tableaux and Related Methods TABLEAUX*, volume 9323 of *LNCS*, pages 135–150. Springer, 2015.
- [20] T. Lyon. On the correspondence between nested calculi and semantic systems for intuitionistic logics. *J. Log. Comput.*, 31(1):213–265, 2021.
- [21] T. Lyon, A. Tiu, R. Goré, and R. Clouston. Syntactic interpolation for tense logics and bi-intuitionistic logic via nested sequents. In M. Fernández and A. Muscholl, editors, *Computer Science Logic, CSL 2020*, volume 152 of *LIPICs*, pages 28:1–28:16. Leibniz-Zentrum für Informatik, 2020.
- [22] T. S. Lyon. Nested sequents for intuitionistic modal logics via structural refinement. In A. Das and S. Negri, editors, *Automated Reasoning with Analytic Tableaux and Related Methods*, volume 12842 of *LNCS*, pages 409–427. Springer, 2021.
- [23] L. Maksimova. Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-boolean algebras. *Algebra and Logic*, 16:427–455, 1977.
- [24] S. Marin and L. Straßburger. Label-free modular systems for classical and intuitionistic modal logics. In *Advances in Modal Logic 10*, pages 387–406. College Publications, 2014.
- [25] S. Negri. Proof analysis in modal logics. *Journal of Philosophical Logic*, 34(5-6):507–544, 2005.

- [26] N. Olivetti and G. L. Pozzato. Nescond: An implementation of nested sequent calculi for conditional logics. In S. Demri, D. Kapur, and C. Weidenbach, editors, *Proceedings of IJCAR 2014*, pages 511–518. Springer, 2014.
- [27] F. Poggiolesi. The method of tree-hypersequents for modal propositional logic. In *Towards mathematical philosophy*, volume 28 of *Trends Log. Stud. Log. Libr.*, pages 31–51. Springer, Dordrecht, 2009.
- [28] L. Straßburger. Cut elimination in nested sequents for intuitionistic modal logics. In F. Pfenning, editor, *FoSSaCS'13*, volume 7794 of *LNCS*, pages 209–224. Springer, 2013.