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Abstract

This paper presents a sliding mode observer (SMO) design method to estimate the states and unknown inputs (UIs) in a class of non-infinitely observable (NIO) descriptor systems that contain UIs in both the state and output equations. Existing works on SMO design for NIO systems did not consider UIs in the output equation. In order to overcome the difficulty caused by UIs in output channels and the NIO condition, we reformulated the original system and introduced new UIs to replace the original UIs to obtain an equivalent infinitely observable descriptor system whose output does not contain any UI. Based on the developed equivalent system, a new SMO method is proposed to estimate both the states and the UIs. Subsequently, the necessary and sufficient conditions for the existence of the SMO are derived in terms of the original system matrices, which thus makes the conditions easy to be examined. Finally, an example is used to verify the effectiveness of the proposed method.

Key words: Sliding mode observer; Non-infinitely observable; Linear descriptor system; Existence conditions.

1 Introduction

In the past decades, the sliding mode observer (SMO) has been widely used for estimating states and unknown inputs (UIs) due to its robustness to the UIs. The main feature of the SMO is the nonlinear switching function which drives the output estimation error to zero in finite time. Following that, both the states and the UIs can be estimated asymptotically without requiring the derivatives of outputs to be available [1]; in particular, the switching function estimates the UIs. In recent years, SMOs have gained even more attention and many excellent results have been reported, see for instance [2,3].

The descriptor system, also known as differential-algebraic system, singular system or generalized system, is a special class of systems governed by both dynamic and algebraic equations. Due to these characteristics, the descriptor system can represent a wider range of systems such as electrical circuits, biological systems, constrained mechanics, chemical processes and so on [4]. Although there have been fruitful results in general observer design for descriptor systems (see [5–8] and references therein), only few works have used SMOs [9–22]. Note that in [9–15] the so-called *infinitely observable* condition (IOC) was required to be satisfied, which might be a restrictive condition for many physical systems, such as electrical circuits [23] and the chemical systems [20] which are *non-infinitely observable* (NIO). Indeed, for a descriptor system to satisfy the IOC, the output matrix needs to have a certain number of sensors, which could increase the cost and the complexity of the system [18]. Recently, there are some works in designing SMOs for NIO descriptor systems [18–22], where some states of the original system were reformulated as UIs; this is however conservative because it increases the

⋆ This work is supported in part by National Natural Science Foundation of China under Grants 61803181 and 61973138, and in part by the China Postdoctoral Science Foundation under Grant 2019M651695. E-mail: jcz@jiangnan.edu.cn (J. Zhang), tan.chee.pin@monash.edu (C. P. Tan), gang.zheng@inria.fr (G. Zheng), wangyan88@jiangnan.edu.cn (Y. Wang, Corresponding author)

number of UIs and thus reduces the feasibility of the SMO. In addition, the works of [18–22] only presented the sufficient conditions for the so-called minimum-phase condition without giving the necessary one.

Moreover, it should also be pointed out that [18–22] focused on the case where the UIs only occur in state equations, and the case where the output containing UIs has not been considered. In fact, if the output contains UIs, the traditional SMO design would immediately become difficult. This is because in such situations, the nonlinear switching term does not drive the output estimation error to zero, and the switching term no longer captures the UI asymptotically. However, in practical situations the measurement outputs are usually influenced by UIs such as disturbances or sensor faults.

Motivated by the above observations, this paper is dedicated to developing a systematic SMO design method for a class of NIO descriptor systems where UIs exist in both the state and output equations. The main contributions as well as the novelties of this paper are threefold as follows: (I) Reformulating the original system and introducing new UIs to replace the original UIs yields an equivalent descriptor system which satisfies the IOC and has no UIs in the output, thus enabling the SMO to be designed for it. (II) Based on the equivalent system, both the states and the UIs are estimated by using a SMO together with a new developed UI algebraic reconstruction method. (III) The necessary and sufficient conditions for the existence of the SMO are given in terms of the original system matrices which will make it easier for the designer/user to determine at the outset if the proposed scheme is applicable to their system.

The remainder of the paper is organised as follows. Section 2 gives the problem formulation and preliminaries. In Section 3, as the main results the SMO developments and the discussions of the existence conditions are presented. In Section 4, a practical example is employed to verify the effectiveness of the proposed methods. Finally, conclusions are given in Section 5.

Throughout the paper, I_n is an $n \times n$ identity matrix. $X \iff Y$ means X is equivalent to Y. For matrix Θ, Θ^+ is a general inverse of Θ which satisfies $\Theta\Theta^+\Theta = \Theta$, and $Sym(\Theta)$ represents $\Theta + \Theta^T$. Symbol $diag\{A_1, A_2, \dots, A_s\}$ denotes a diagonal matrix with matrices $A_1, A_2, \dots, A_s$ being the diagonal elements. For any vector or matrix v, $\|v\|$ denotes the 2−norm.

2 Problem formulation and preliminaries

Consider a class of linear descriptor systems as follows:

$$E\dot{x}(t) = Ax(t) + Bu(t) + Df(t) \tag{1a}$$
$$y(t) = Cx(t) + Ff(t) \tag{1b}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$ are the state, measurable output and control input vectors, respectively. $f \in \mathbb{R}^q$ is the unknown input which could represent disturbances, faults or measurement noise and so on. $E \in \mathbb{R}^{n\times n}, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{n\times q}$ and $F \in \mathbb{R}^{p\times q}$ are constant matrices, and $x(0), f(t)$ are unknown but bounded vectors, i.e., $\|x(0)\| \le x_0$ and $\|f(t)\| \le f_0$ ($x_0, f_0 > 0$ are known constants). It is assumed that system (1) is regular and impulse-free [23]. Besides, assume generally that $\text{rank}(E) = n_0 < n$ and $\text{rank}\left[\begin{smallmatrix} E \\ C \end{smallmatrix}\right] < n$ (i.e., the system (1) is NIO [18]). Also assume generally that $\text{rank}\left[\begin{smallmatrix} D \\ F \end{smallmatrix}\right] = q$, because it is always possible to find matrices $\mathscr{D}$, $\mathscr{F}$ and Ω such that $\left[\begin{smallmatrix} D \\ F \end{smallmatrix}\right] f = \left[\begin{smallmatrix} \mathscr{D} \\ \mathscr{F} \end{smallmatrix}\right] \Omega f$ with $\left[\begin{smallmatrix} \mathscr{D} \\ \mathscr{F} \end{smallmatrix}\right]$ being full column rank.

As mentioned in the introduction, SMOs can be easily constructed for descriptor systems that satisfy the IOC, i.e. $\text{rank}\left[\begin{smallmatrix} E \\ C \end{smallmatrix}\right] = n$ [9–15]. Then, Tan et al. relaxed the IOC, and proposed several SMO-based methods [18–22] for NIO descriptor systems where the UI appears only in the state equation, but they are inapplicable for system (1) where the UIs appear in both state and output equations. Moreover, in [18–22], certain states were expressed as UIs, which inevitably increases the number of UIs and reduces the possibility of a successful SMO design. This paper seeks to overcome that limitation by developing a new SMO scheme for system (1) and giving the sufficient and necessary conditions.

3 Main results

In this section, by reformulating system (1) and introducing new UIs to replace the original UIs, we obtain an equivalent descriptor system that is infinitely observable and without any UIs in the output. Following that, a SMO is developed to estimate the state and UIs. Then, the existence conditions of the SMO are discussed in detail.

3.1 System reformulation

Pre-multiplying (1a) with matrices EE^+ and $I - EE^+$, yields the following equations respectively

$$E\dot{x}(t) = EE^+Ax(t) + EE^+Bu(t) + EE^+Df(t) \tag{2a}$$
$$0 = (I - EE^+)(Ax(t) + Bu(t) + Df(t)) \tag{2b}$$

Combining (1)-(2) yields

$$\overline{E}\dot{x}(t) = \overline{A}x(t) + \overline{B}u(t) + \overline{D}f(t) \tag{3a}$$
$$\overline{y}(t) = \overline{C}x(t) + \overline{F}f(t) \tag{3b}$$

where $\overline{E} = E$, $\overline{A} = EE^+A$, $\overline{B} = EE^+B$, $\overline{D} = EE^+D$, $\overline{C} = \left[\begin{smallmatrix} (I-EE^+)A \\ C \end{smallmatrix}\right]$, $\overline{F} = \left[\begin{smallmatrix} (I-EE^+)D \\ F \end{smallmatrix}\right]$ and $\overline{y} = \left[\begin{smallmatrix} -(I-EE^+)Bu \\ y \end{smallmatrix}\right]$. It can be easily seen that the assumption rank $\begin{bmatrix} D \\ F \end{bmatrix} = q$ results in rank $\begin{bmatrix} \overline{D} \\ \overline{F} \end{bmatrix} = q$. According to the general matrix inverse theory [5], the general solution for f in (3b) is

$$f(t) = \overline{F}^{+}(\overline{y}(t) - \overline{C}x(t)) + (I - \overline{F}^{+}\overline{F})\eta(t) \tag{4}$$

where η is a definite but unknown vector which will be considered as a new UI to replace f. Then, substituting (4) into (3a) and pre-multiplying (3b) with $I - \overline{F}\,\overline{F}^{+}$ yields

$$\widetilde{E}\dot{x} = \widetilde{A}x + \widetilde{B}u + \overline{D}\,\overline{F}^{+}\overline{y} + \overline{D}(I - \overline{F}^{+}\overline{F})\eta \tag{5a}$$

$$\widetilde{y} = \widetilde{C}x \tag{5b}$$

where $\widetilde{E} = \overline{E}$, $\widetilde{A} = \overline{A} - \overline{D}\,\overline{F}^{+}\overline{C}$, $\widetilde{B} = \overline{B}$, and $\widetilde{C} \in \mathbb{R}^{\overline{p}\times n}$ is a matrix of full row rank, constructed by selecting all independent rows of matrix $(I - \overline{F}\,\overline{F}^{+})\overline{C}$. Then, the vector $\widetilde{y}$ in (5b), selected from the corresponding rows of $(I - \overline{F}\,\overline{F}^{+})\overline{y}$, is also measurable. Denote $\bar{q} = \text{rank}(\overline{D}(I - \overline{F}^{+}\overline{F}))$, and thus there exists a full column rank matrix $\widetilde{D} \in \mathbb{R}^{n\times\bar{q}}$ and a matrix $\Lambda \in \mathbb{R}^{\bar{q}\times q}$ such that $\overline{D}(I - \overline{F}^{+}\overline{F}) = \widetilde{D}\Lambda$, where $\bar{q} \leq q$. Therefore, the term $\overline{D}(I - \overline{F}^{+}\overline{F})\eta(t)$ in (5a) can be rewritten as $\widetilde{D}\Lambda\eta(t)$. Then, let $\omega = \Lambda\eta$, and system (5) becomes

$$\widetilde{E}\dot{x}(t) = \widetilde{A}x(t) + \widetilde{B}u(t) + \overline{D}\,\overline{F}^{+}\overline{y} + \widetilde{D}\omega(t) \tag{6a}$$

$$\widetilde{y}(t) = \widetilde{C}x(t) \tag{6b}$$

To perform the SMO design, assume that the following conditions hold

$$\text{rank}\begin{bmatrix} \widetilde{E} & \widetilde{D} \\ \widetilde{C} & 0 \end{bmatrix} = n + \text{rank}(\widetilde{D}) \tag{7a}$$

$$\text{rank}\begin{bmatrix} s\widetilde{E}-\widetilde{A} & \widetilde{D} \\ \widetilde{C} & 0 \end{bmatrix} = n + \text{rank}(\widetilde{D}), \forall\, s \in \mathbb{C}^{+}. \tag{7b}$$

Remark 1. In system (1), both the NIO condition and UIs in the output prevent a SMO from estimating the states and UIs. However, after the transformations in (2)-(5), system (6) is obtained, which removes the UIs from the output and, also, makes it possible for system (6) to satisfy the IOC, and thus enabling the SMO design. Therefore, if conditions (7a)-(7b) are satisfied, it is then possible to design a SMO for system (6) [9] to estimate x and ω. The implications and equivalent conditions of (7a)-(7b) will be given in Section 3.3.

For matrix $\widetilde{C}$, by using the Smith orthogonal procedure we can find an orthogonal matrix W such that $\widehat{C} = \widetilde{C}W = [0\ \widehat{R}]$ with $\widehat{R}$ being nonsingular. On the other hand, (7a) implies that there exists a nonsingular matrix $\widetilde{T}$ and a matrix $\widetilde{H}$ such that $\widetilde{T}\widetilde{E} + \widetilde{H}\widetilde{C} = I_n$ [4].

Thus, pre-multiplying system (6) with $\widetilde{T}$ and performing a state transformation $x = W\overline{x}$ leads to

$$\widehat{E}\dot{\overline{x}}(t) = \widehat{A}\overline{x}(t) + \widehat{B}u(t) + \widehat{\Delta}\overline{y}(t) + \widehat{D}\omega(t) \tag{8}$$

where

$$\widehat{E} = W^{-1}\widetilde{T}\widetilde{E}W \triangleq \left[\begin{array}{c|c} \widehat{E}_{11} & \widehat{E}_{12} \\ \hline \widehat{E}_{21} & \widehat{E}_{22} \end{array}\right] = \left[\begin{array}{c|c} I_{n-\overline{p}} & -\widehat{H}_1\widehat{R} \\ \hline 0 & I_{\overline{p}} - \widehat{H}_2\widehat{R} \end{array}\right]$$
$$\widehat{A} = W^{-1}\widetilde{T}\widetilde{A}W \triangleq \left[\begin{array}{c|c} \widehat{A}_{11} & \widehat{A}_{12} \\ \hline \widehat{A}_{21} & \widehat{A}_{22} \end{array}\right], \widehat{B} = W^{-1}\widetilde{T}\widetilde{B} \triangleq \left[\begin{array}{c} \widehat{B}_1 \\ \hline \widehat{B}_2 \end{array}\right]$$
$$\widehat{\Delta} = W^{-1}\widetilde{T}\overline{D}\,\overline{F}^{+} \triangleq \left[\begin{array}{c} \widehat{\Delta}_1 \\ \hline \widehat{\Delta}_2 \end{array}\right], \widehat{D} = W^{-1}\widetilde{T}\widetilde{D} \triangleq \left[\begin{array}{c} \widehat{D}_1 \\ \hline \widehat{D}_2 \end{array}\right]. \tag{9}$$

and $\widehat{H}_1, \widehat{H}_2$ are $\begin{bmatrix} \widehat{H}_1 \\ \widehat{H}_2 \end{bmatrix} = W^{-1}\tilde{H}$ where $\widehat{H}_1 \in \mathbb{R}^{(n-\bar{p})\times\bar{p}}$. Decomposing vector $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$ with $\bar{x}_1 \in \mathbb{R}^{n-\bar{p}}$, and based on the partitions in (9), system (8) can be rewritten as

$$\dot{\overline{x}}_1 + \widehat{E}_{12}\dot{\overline{x}}_2 = \widehat{A}_{11}\overline{x}_1 + \widehat{A}_{12}\overline{x}_2 + \widehat{B}_1u + \widehat{\Delta}_1\overline{y} + \widehat{D}_1\omega \tag{10a}$$

$$\widehat{E}_{22}\dot{\overline{x}}_2 = \widehat{A}_{21}\overline{x}_1 + \widehat{A}_{22}\overline{x}_2 + \widehat{B}_2u + \widehat{\Delta}_2\overline{y} + \widehat{D}_2\omega \tag{10b}$$

where $\overline{x}_2 = \widehat{R}^{-1}\widetilde{y}$. Now, we give Lemma 1 which will facilitate the SMO design.

Lemma 1. The conditions (7a) and (7b) hold, if and only if (iff) there exist appropriately dimensioned symmetric positive definite (SPD) matrices $\widetilde{P}, \widetilde{Q} > 0$ and matrices $\widetilde{J}, \widetilde{K}$ such that

$$\widetilde{T}\widetilde{D} = \widetilde{J}\widetilde{C}\widetilde{T}\widetilde{D} \tag{11a}$$

$$Sym\left(\widetilde{P}[(I - \widetilde{J}\widetilde{C})\widetilde{T}\widetilde{A} - \widetilde{K}\widetilde{C}]\right) = -\widetilde{Q} \tag{11b}$$

Proof. To this end, we will show $(7a) \iff (11a)$ and $(7b) \iff (11b)$.

To prove that $(7a) \iff (11a)$: Since $\underbrace{\begin{bmatrix} I & 0 \\ -\widetilde{C} & I \end{bmatrix}}_{T_2}\underbrace{\begin{bmatrix} \widetilde{T} & \widetilde{H} \\ 0 & I \end{bmatrix}}_{T_1}\begin{bmatrix} \widetilde{E} & \widetilde{D} \\ \widetilde{C} & 0 \end{bmatrix} = \begin{bmatrix} I_n & \widetilde{T}\widetilde{D} \\ 0 & -\widetilde{C}\widetilde{T}\widetilde{D} \end{bmatrix}$ and T_1, T_2 are invertible, it follows that condition (7a) holds iff rank $\begin{bmatrix} I_n & \widetilde{T}\widetilde{D} \\ 0 & -\widetilde{C}\widetilde{T}\widetilde{D} \end{bmatrix} = n + \text{rank}(\widetilde{T}\widetilde{D})$, i.e. $\text{rank}(\widetilde{C}\widetilde{T}\widetilde{D}) = \text{rank}(\widetilde{T}\widetilde{D})$ which holds iff there exists a matrix $\widetilde{J}$ that satisfies (11a).

To prove that $(7b) \iff (11b)$: We only need to show $(7b)$ holds iff there exist matrices $\widetilde{J}, \widetilde{K}$ such that matrix

$\widetilde{\Upsilon} \triangleq (I - \widetilde{J}\widetilde{C})\widetilde{T}\widetilde{A} - \widetilde{K}\widetilde{C}$ is Hurwitz. Condition (11a) gives the solution of $\widetilde{J}$ as

$$\widetilde{J} = \widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+} + Z(I - \widetilde{C}\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+}). \tag{12}$$

Substituting (12) into matrix $\widetilde{\Upsilon}$ yields

$$\begin{aligned}\widetilde{\Upsilon} &= \widetilde{T}\widetilde{A} - \widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+}\widetilde{C}\widetilde{T}\widetilde{A} \\ &\quad - Z(I - \widetilde{C}\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+})\widetilde{C}\widetilde{T}\widetilde{A} - K\widetilde{C} \\ &= \Pi_1 - \Gamma\Pi_2\end{aligned} \tag{13}$$

where $\Gamma = [Z\ \ K]$, $\Pi_1 = \widetilde{T}\widetilde{A} - \widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+}\widetilde{C}\widetilde{T}\widetilde{A}$ and $\Pi_2 = \left[\begin{smallmatrix} (I-\widetilde{C}\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+})\widetilde{C}\widetilde{T}\widetilde{A} \\ \widetilde{C} \end{smallmatrix}\right]$. Therefore, proving $(7b) \iff (11b)$ can be reduced to showing that $(7b)$ holds iff the pair (Π_1, Π_2) is detectable. Next, we will show the equivalence.

Define the following matrices with full column rank, $S_1 = \left[\begin{smallmatrix} \widetilde{T} & \widetilde{H} \\ 0 & I \end{smallmatrix}\right]$, $S_2 = \left[\begin{smallmatrix} I & 0 \\ (\widetilde{C}\widetilde{T}\widetilde{D})^{+}\widetilde{C} & 0 \\ 0 & I \end{smallmatrix}\right]$, $S_3 = \left[\begin{smallmatrix} I & 0 \\ (\widetilde{C}\widetilde{T}\widetilde{D})^{+}\widetilde{C}\widetilde{T}\widetilde{A} & I \end{smallmatrix}\right]$, $S_4 = \left[\begin{smallmatrix} I & 0 \\ -\widetilde{C} & 0 \\ 0 & I \end{smallmatrix}\right]$ and $S_5 = \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & sI \\ 0 & 0 & I \end{smallmatrix}\right]$. Then, since $\widetilde{C}\widetilde{T}\widetilde{D}$ has full column rank, for any $s \in \mathbb{C}^{+}$ we have

$$\begin{aligned}
&\text{rank}\left[\begin{smallmatrix} s\widetilde{E}-\widetilde{A} & \widetilde{D} \\ \widetilde{C} & 0 \end{smallmatrix}\right] = \text{rank}\left(S_2 S_1 \left[\begin{smallmatrix} s\widetilde{E}-\widetilde{A} & \widetilde{D} \\ \widetilde{C} & 0 \end{smallmatrix}\right] S_3\right) \\
&= \text{rank}\left[\begin{smallmatrix} sI_n-(\widetilde{T}\widetilde{A}-\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+}\widetilde{C}\widetilde{T}\widetilde{A}) & \widetilde{T}\widetilde{D} \\ s(\widetilde{C}\widetilde{T}\widetilde{D})^{+}\widetilde{C} & I_{\bar{q}} \\ \widetilde{C} & 0 \end{smallmatrix}\right] \\
&= \bar{q} + \text{rank}\left[\begin{smallmatrix} sI_n-(\widetilde{T}\widetilde{A}-\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+}\widetilde{C}\widetilde{T}\widetilde{A}) \\ \widetilde{C} \end{smallmatrix}\right] \\
&= \bar{q} + \text{rank}\left(S_4\left[\begin{smallmatrix} sI_n-(\widetilde{T}\widetilde{A}-\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+}\widetilde{C}\widetilde{T}\widetilde{A}) \\ \widetilde{C} \end{smallmatrix}\right]\right) \\
&= \bar{q} + \text{rank}\left[\begin{smallmatrix} sI_n-(\widetilde{T}\widetilde{A}-\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+}\widetilde{C}\widetilde{T}\widetilde{A}) \\ -s\widetilde{C}+(I-\widetilde{C}\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+})\widetilde{C}\widetilde{T}\widetilde{A} \\ \widetilde{C} \end{smallmatrix}\right] \\
&= \bar{q} + \text{rank}\left(S_5\left[\begin{smallmatrix} sI_n-(\widetilde{T}\widetilde{A}-\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+}\widetilde{C}\widetilde{T}\widetilde{A}) \\ -s\widetilde{C}+(I-\widetilde{C}\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+})\widetilde{C}\widetilde{T}\widetilde{A} \\ \widetilde{C} \end{smallmatrix}\right]\right) \\
&= \bar{q} + \text{rank}\left[\begin{smallmatrix} sI_n-(\widetilde{T}\widetilde{A}-\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+}\widetilde{C}\widetilde{T}\widetilde{A}) \\ (I-\widetilde{C}\widetilde{T}\widetilde{D}(\widetilde{C}\widetilde{T}\widetilde{D})^{+})\widetilde{C}\widetilde{T}\widetilde{A} \\ \widetilde{C} \end{smallmatrix}\right] \\
&= \bar{q} + \text{rank}\left[\begin{smallmatrix} sI_n-\Pi_1 \\ \Pi_2 \end{smallmatrix}\right].
\end{aligned} \tag{14}$$

Recalling that $\text{rank}(\widetilde{T}\widetilde{D}) = \text{rank}(\widetilde{D}) = \bar{q}$, it follows from (14) that (7b) holds iff $\text{rank}\left[\begin{smallmatrix} sI_n-\Pi_1 \\ \Pi_2 \end{smallmatrix}\right] = n$, i.e., the pair (Π_1, Π_2) is detectable, which completes the proof. □

Remark 2. Matrices $\widetilde{P}$ and $\widetilde{Q}$ associated with (11b) can be obtained as follows: Based on the definition $\widetilde{\Upsilon} \triangleq (I - \widetilde{J}\widetilde{C})\widetilde{T}\widetilde{A} - \widetilde{K}\widetilde{C}$, substituting (12) and (13) into (11b) yields $Sym(\widetilde{P}\Pi_1 - \widetilde{P}\Gamma\Pi_2) < 0$. Treating $\widetilde{P}\Gamma$ as a new variable X, and solving LMI $Sym(\widetilde{P}\Pi_1 - X\Pi_2) < 0$ yields $\widetilde{P}$ and X, and then Γ can be solved as $\Gamma = \widetilde{P}^{-1}X$. This together with (13) and (11b) gives matrix $\widetilde{Q}$.

From system (10) and equation (11), we will describe the SMO design in the following. Let $\widehat{L}$ be a matrix which will be determined later. Then, pre-multiplying (10b) with $\widehat{L}$ and subtracting the result from (10a) yields

$$\begin{aligned}&(\widehat{E}_{12} - \widehat{L}\widehat{E}_{22})\dot{\overline{x}}_2 + \dot{\overline{x}}_1 = (\widehat{A}_{11} - \widehat{L}\widehat{A}_{21})\overline{x}_1 + (\widehat{A}_{12} - \widehat{L}\widehat{A}_{22})\overline{x}_2 \\ &+ (\widehat{B}_1 - \widehat{L}\widehat{B}_2)u + (\widehat{\Delta}_1 - \widehat{L}\widehat{\Delta}_2)\overline{y} + (\widehat{D}_1 - \widehat{L}\widehat{D}_2)\omega\end{aligned} \tag{15}$$

On the other hand, let

$$\begin{aligned}\widehat{P} &= W^{-1}\widetilde{P}W = \left[\begin{smallmatrix} \widehat{P}_1 & \widehat{P}_2 \\ \widehat{P}_2^T & \widehat{P}_3 \end{smallmatrix}\right], \widehat{K} = W^{-1}\widetilde{K} = \left[\begin{smallmatrix} \widehat{K}_1 \\ \widehat{K}_2 \end{smallmatrix}\right], \\ \widehat{Q} &= W^{-1}\widetilde{Q}W = \left[\begin{smallmatrix} \widehat{Q}_1 & \widehat{Q}_2 \\ \widehat{Q}_2^T & \widehat{Q}_3 \end{smallmatrix}\right], \widehat{J} = W^{-1}\widetilde{J} = \left[\begin{smallmatrix} \widehat{J}_1 \\ \widehat{J}_2 \end{smallmatrix}\right].\end{aligned} \tag{16}$$

It follows from (11a) that $W^{-1}\widetilde{T}\widetilde{D} = W^{-1}\widetilde{J}\widetilde{C}\,W\,W^{-1}\widetilde{T}\widetilde{D}$, i.e. $\left[\begin{smallmatrix} \widehat{D}_1 \\ \widehat{D}_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widehat{J}_1 \\ \widehat{J}_2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & \widehat{R} \end{smallmatrix}\right]\left[\begin{smallmatrix} \widehat{D}_1 \\ \widehat{D}_2 \end{smallmatrix}\right]$. Now, choose matrix

$$\widehat{L} = \widehat{J}_1\widehat{R} - \widehat{P}_1^{-1}\widehat{P}_2(I - \widehat{J}_2\widehat{R}) \tag{17}$$

and it is easy to verify that $\widehat{D}_1 - \widehat{L}\widehat{D}_2 = 0$. Thus, system (15) reduces to

$$\begin{aligned}&(\widehat{E}_{12} - \widehat{L}\widehat{E}_{22})\dot{\overline{x}}_2 + \dot{\overline{x}}_1 = (\widehat{A}_{11} - \widehat{L}\widehat{A}_{21})\overline{x}_1 + \\ &(\widehat{A}_{12} - \widehat{L}\widehat{A}_{22})\overline{x}_2 + (\widehat{B}_1 - \widehat{L}\widehat{B}_2)u + (\widehat{\Delta}_1 - \widehat{L}\widehat{\Delta}_2)\overline{y}.\end{aligned} \tag{18}$$

In the following subsection, a SMO will be constructed.

3.2 SMO design

Based on the reformulation given in Section 3.1 and the conditions (7a) and (7b), a new SMO is developed for systems (18) and (10b) as follows

$$\begin{aligned}\dot{\zeta}_1 = (\widehat{A}_{11} - \widehat{L}\widehat{A}_{21})\widehat{\overline{x}}_1 + (\widehat{A}_{12} - \widehat{L}\widehat{A}_{22})\widehat{R}^{-1}\widetilde{y} \\ +(\widehat{B}_1 - \widehat{L}\widehat{B}_2)u + (\widehat{\Delta}_1 - \widehat{L}\widehat{\Delta}_2)\overline{y}\end{aligned} \tag{19}$$

$$\widehat{\overline{x}}_1 = \zeta_1 - (\widehat{E}_{12} - \widehat{L}\widehat{E}_{22})\widehat{R}^{-1}\widetilde{y} \tag{20}$$

$$\dot{\zeta}_2 = \widehat{A}_{21}\widehat{\overline{x}}_1 + \widehat{A}_{22}\widehat{R}^{-1}\widetilde{y} + \widehat{B}_2 u + \widehat{\Delta}_2\overline{y} + v \tag{21}$$

$$\widehat{\overline{x}}_2 = \zeta_2 + (I - \widehat{E}_{22})\widehat{R}^{-1}\widetilde{y} \tag{22}$$

where

$$v(t) = \rho\frac{e_{\widetilde{y}}}{\|e_{\widetilde{y}}\|},\ \rho > (\rho_0 + \|\widehat{A}_{21}\| \cdot \overline{e}_1 + \|\widehat{D}_2\| \cdot \bar{\omega}), \tag{23}$$

ζ_1 and ζ_2 are the SMO states, $\widehat{\overline{x}}_1$ and $\widehat{\overline{x}}_2$ are the estimates of $\overline{x}_1$ and $\overline{x}_2$, $e_1 = \overline{x}_1 - \widehat{\overline{x}}_1$ and $e_{\widetilde{y}} = \overline{x}_2 - \widehat{\overline{x}}_2 = \widehat{R}^{-1}\widetilde{y} - \widehat{\overline{x}}_2$

are estimation errors. $\rho_0 > 0$, $\overline{e}_1$ and $\bar{\omega}$ are the upper bounds of $\|e_1\|_2$ and $\|\omega\|_2$, respectively. Next, in Theorem 1, we analyze the performance of SMO (19)-(22).

Theorem 1. Based on (7a) and (7b) and the above developments, the SMO (19)-(22) can achieve $e_1 \to 0$, $t \to \infty$ and sliding motion $e_{\tilde{y}} = 0$ in finite time. Furthermore, the state x and unknown input f can be asymptotically estimated by SMO (19)-(22).

Proof. Firstly, it will be proven that $e_1 \to 0$, $t \to \infty$. Considering systems (18) and (19)-(22) we have $\dot{e}_1 = (\widehat{A}_{11} - \widehat{L}\widehat{A}_{21})e_1$. In the following, it will be shown that $\widehat{A}_{11} - \widehat{L}\widehat{A}_{21}$ is Hurwitz. For equation (11b), pre-multiply with W^{-1} (from (9) and (16)) and post-multiply with W to get

$$Sym\left(\widehat{P}\left((I - \widehat{J}\widehat{C})\widehat{A} - \widehat{K}\widehat{C}\right)\right) = -\widehat{Q}. \tag{24}$$

Based on the decompositions of matrices $\widehat{P}$, $\widehat{J}$, $\widehat{A}$, $\widehat{K}$ and $\widehat{C}$ in (16), the sub-block on the first row and the first column of $\widehat{P}\left((I - \widehat{J}\widehat{C})\widehat{A} - \widehat{K}\widehat{C}\right)$ is

$$\begin{aligned}
&\begin{bmatrix} I & \mathbf{0} \end{bmatrix} \widehat{P}\left((I - \widehat{J}\widehat{C})\widehat{A} - \widehat{K}\widehat{C}\right)\begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} \\
&= \begin{bmatrix} \widehat{P}_1 & \widehat{P}_2 \end{bmatrix}\left((I - \widehat{J}\widehat{C})\widehat{A} - \widehat{K}\widehat{C}\right)\begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} \\
&= \begin{bmatrix} \widehat{P}_1 & \widehat{P}_2 \end{bmatrix}(I - \widehat{J}\widehat{C})\widehat{A}\begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} \\
&= \widehat{P}_1\left(\widehat{A}_{11} - \left(\widehat{J}_1\widehat{R} - \widehat{P}_1^{-1}\widehat{P}_2\left(I - \widehat{J}_2\widehat{R}\right)\right)\widehat{A}_{21}\right) \\
&= \widehat{P}_1\left(\widehat{A}_{11} - \widehat{L}\widehat{A}_{21}\right).
\end{aligned} \tag{25}$$

Then, equations (24) and (25) imply

$$\widehat{P}_1(\widehat{A}_{11} - \widehat{L}\widehat{A}_{21}) + (\widehat{A}_{11} - \widehat{L}\widehat{A}_{21})^T\widehat{P}_1 = -\widehat{Q}_1 < 0 \tag{26}$$

which indicates that $\widehat{A}_{11} - \widehat{L}\widehat{A}_{21}$ is Hurwitz. Thus, we have $e_1 \to 0$, $t \to \infty$. Also, since the initial state $x(0)$ is bounded by x_0, from (26) we have $e_1^T(t)\widehat{P}_1 e_1(t) \leq e_1^T(0)\widehat{P}_1 e_1(0)$, which implies that we can find a scalar $\overline{e}_1 \triangleq \sqrt{\frac{\lambda_{max}(\widehat{P}_1)}{\lambda_{min}(\widehat{P}_1)}}\|\begin{bmatrix} I & \mathbf{0} \end{bmatrix} W^T\|x_0$ such that $\|e_1\|_2 \leq \overline{e}_1$. Besides, from the definition of $\omega(t)$ in (6) we have $\omega(t) = \widetilde{D}^+\overline{D}(I - \overline{F}^+\overline{F})f(t)$. Thus, there exists an upper bound of $\omega(t)$ as $\overline{\omega} = \|\widetilde{D}^+\overline{D}(I - \overline{F}^+\overline{F})\|f_0$ satisfying $\|\omega(t)\| \leq \overline{\omega}$. Next, by using these results we will show the achievement of sliding motion $e_{\tilde{y}} = 0$ in a finite time.

It follows from (10b) and (19)-(22) that the estimation error dynamics of $e_{\tilde{y}}$ are

$$\dot{e}_{\tilde{y}} = \widehat{A}_{21}e_1 + \widehat{D}_2\omega - v. \tag{27}$$

Consider a Lyapunov function $V_2 = \frac{1}{2}e_{\tilde{y}}^T e_{\tilde{y}}$, and then from (27) it follows that

$$\begin{aligned}
\dot{V}_2 &= e_{\tilde{y}}^T\left(\widehat{A}_{21}e_1 + \widehat{D}_2\omega - v\right) \\
&\leq \|e_{\tilde{y}}\|\left(\|\widehat{A}_{21}\| \cdot \overline{e}_1 + \|\widehat{D}_2\| \cdot \overline{\omega}\right) - \rho\|e_{\tilde{y}}\| \\
&\leq -\rho_0\|e_{\tilde{y}}\|
\end{aligned} \tag{28}$$

which yields $\dot{V}_2 \leq -\sqrt{2}\rho_0\sqrt{V_2}$. Thus, the *reachability condition* has been satisfied, and according to the sliding mode theory, sliding motion $\dot{e}_{\tilde{y}} = e_{\tilde{y}} = 0$ is attained in finite time $t_f = t_0 + \frac{\sqrt{2V_2(0)}}{\rho_0}$ [1]. Then, we have $\widehat{\overline{x}}_1 \to \overline{x}_1, t \to \infty$ and $\widehat{\overline{x}}_2 \equiv \overline{x}_2, t \geq t_f$, which also implies $\begin{bmatrix} \widehat{\overline{x}}_1 \\ \widehat{\overline{x}}_2 \end{bmatrix} \to \overline{x}, t \to \infty$. Therefore, for the state estimate $\widehat{x} = W\begin{bmatrix} \widehat{\overline{x}}_1 \\ \widehat{\overline{x}}_2 \end{bmatrix}$, since matrix W is orthogonal and thus invertible, we have $\widehat{x} \to x, t \to \infty$.

Then, since $e_1 \to 0$, it follows from (27) that after the sliding motion is achieved we have $v_{eq} \to \widehat{D}_2\omega$, where v_{eq}, the so-called *equivalent output error injection*, is the low-frequency component of v required to maintain the sliding motion, and can be approximated online to any degree of accuracy as $v_{eq} \approx \rho\frac{e_{\tilde{y}}}{\|e_{\tilde{y}}\| + \delta}$ with $\delta > 0$ being a small constant [18–22]. Besides, since $\widehat{D}_1 - \widehat{L}\widehat{D}_2 = 0$ we have $\begin{bmatrix} \widehat{L} \\ I \end{bmatrix} v_{eq} \to \widehat{D}\omega$ which implies $\widetilde{T}^{-1}W\begin{bmatrix} \widehat{L} \\ I \end{bmatrix} v_{eq} \to \widetilde{D}\omega$.

On the other hand, pre-multiplying (4) with $\overline{D}$ yields

$$\overline{D}f(t) = \overline{D}\,\overline{F}^+(\overline{y}(t) - \overline{C}x(t)) + \overline{D}(I - \overline{F}^+\overline{F})\eta(t) \tag{29}$$

or

$$\overline{D}f(t) = \overline{D}\,\overline{F}^+(\overline{y}(t) - \overline{C}x(t)) + \widetilde{D}\omega(t). \tag{30}$$

Since matrix $\begin{bmatrix} \overline{D} \\ \overline{F} \end{bmatrix}$ has full column rank, by solving the algebraic equations (3b) and (30) the original UI f can be re-expressed as

$$f = \left(\begin{bmatrix} \overline{D} \\ \overline{F} \end{bmatrix}^T\begin{bmatrix} \overline{D} \\ \overline{F} \end{bmatrix}\right)^{-1}\begin{bmatrix} \overline{D} \\ \overline{F} \end{bmatrix}^T\begin{bmatrix} \overline{D}\,\overline{F}^+(\overline{y}(t) - \overline{C}x(t)) + \widetilde{D}\omega(t) \\ \overline{y}(t) - \overline{C}x(t) \end{bmatrix}. \tag{31}$$

Define the following measurable signal

$$\widehat{f} \triangleq \left(\begin{bmatrix} \overline{D} \\ \overline{F} \end{bmatrix}^T\begin{bmatrix} \overline{D} \\ \overline{F} \end{bmatrix}\right)^{-1}\begin{bmatrix} \overline{D} \\ \overline{F} \end{bmatrix}^T\begin{bmatrix} \overline{D}\,\overline{F}^+(\overline{y} - \overline{C}\widehat{x}) + \widetilde{T}^{-1}W\begin{bmatrix} \widehat{L}v_{eq} \\ v_{eq} \end{bmatrix} \\ \overline{y} - \overline{C}\widehat{x} \end{bmatrix} \tag{32}$$

where $\widehat{f}$ is the estimate of f. Since $\widehat{x} \to x$ and $\widetilde{T}^{-1}W\begin{bmatrix} \widehat{L} \\ I \end{bmatrix} v_{eq} \to \widetilde{D}\omega$, $t \to \infty$, we have $\widehat{f} \to f$, $t \to \infty$. This completes the proof. □

Remark 3. It should be emphasized that the sliding motion $e_{\tilde{y}} = 0$ is crucial in state and UI estimation. On the one hand, once $e_{\tilde{y}} = 0$ is achieved, we have $\widehat{\overline{x}}_2(t) \equiv \overline{x}_2(t)$, which together with the estimate $\widehat{\overline{x}}_1$ defined in (20) gives the state estimation $\widehat{x}(t)$. On the other hand, the achievement of $e_{\tilde{y}} = 0$ also allows us to estimate $\widehat{D}_2\omega$ by using the equivalent output error injection v_{eq}, and further gives the estimation of f in (32).

Remark 4. Note that conditions (7a) and (7b) contain intermediate matrices $\widetilde{E}, \widetilde{A}, \widetilde{D}, \widetilde{C}$ that make the conditions difficult to be examined. In the next subsection, the equivalent conditions of (7a) and (7b) will be given in terms of original system matrices, which will make it easier for the designer/user to determine at the outset if the proposed scheme is applicable to their system.

Remark 5. The requirement of the prior knowledge of x_0 and f_0 (which respectively are the bounds of $x(0)$ and $f(t)$) used to design SMO parameter ρ in (23), is a standard assumption in SMO research [18–22]. The values of x_0 and f_0 can normally be determined by knowing the physical properties of the system or by simulating its operation. In the case that x_0 and f_0 cannot be determined, a time-varying $\rho(t)$ given by the following adaptive law

$$\dot{\rho}(t) = \alpha_0 \|e_{\tilde{y}}\|, \rho(0) \geq 0, \alpha_0 > 0 \tag{33}$$

can also achieve sliding motion. The mechanism is this: with the adaptive gain $\rho(t)$, equation (28) can be rewritten as $\dot{V}_2 \leq -\left(\rho(t) - \|\widehat{A}_{21}\| \cdot \overline{e}_1 - \|\widehat{D}_2\| \cdot \overline{\omega}\right) \|e_{\tilde{y}}\|$. Then, with initial value $\rho(0) \geq 0$, $\rho(t)$ keeps increasing until sliding motion $e_{\tilde{y}} = 0$ is achieved, when $\rho(t)$ is large enough such that $\rho(t) > \|\widehat{A}_{21}\| \cdot \overline{e}_1 + \|\widehat{D}_2\| \cdot \overline{\omega}$ (no matter how large the (unknown) expression $\|\widehat{A}_{21}\| \cdot \overline{e}_1 + \|\widehat{D}_2\| \cdot \overline{\omega}$ is). Then, $\rho(t)$ stops increasing and maintains its value. The estimation performance of the SMO (19)-(22) under the adaptive $\rho(t)$ given by (33) will be illustrated in the simulation.

Remark 6. In addition to the smoothing function $v_{eq} \approx \rho\frac{e_{\tilde{y}}}{\|e_{\tilde{y}}\|+\delta}$ above, another way to approximate the equivalent signal v_{eq} is the low-pass filter method given by Utkin in [24], where $v_{eq,i}$, the i-th component of v_{eq}, can be obtained by passing the component $v_i(t)$ through a first-order low-pass filter of time constant τ satisfying $\tau\dot{v}_{eq,i} + v_{eq,i} = v_i$. Both the methods above have been proven to be able to get the low-frequency components of $v(t)$ [18–22,24]. However, the low-pass filter method may cause singularity, but the smooth function does not cause any singularity, and is a trade-off between getting ideal performance($e_{\tilde{y}} = 0$ exactly) and a smooth low-frequency component of $v(t)$ [25].

3.3 Existence conditions

Here, the necessary and sufficient conditions of (7) are given by Lemma 2 and Lemma 3 which are in terms of the original system matrices E, A, D, C, F.

Lemma 2. Matrices $\widetilde{E}, \widetilde{C}, \widetilde{D}$ satisfy (7a) iff the following equation holds

$$\operatorname{rank}\begin{bmatrix} E & A & D & 0 \\ 0 & E & 0 & D \\ 0 & C & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix} = n + \operatorname{rank}\begin{bmatrix} E & D \\ 0 & F \end{bmatrix} + \operatorname{rank}\begin{bmatrix} D \\ F \end{bmatrix}. \tag{34}$$

Proof. Let $S_6 = \begin{bmatrix} EE^+ \\ I-EE^+ \end{bmatrix}$, $S_7 = \operatorname{diag}\{S_6, S_6, I, I\}$, $S_8 = \begin{bmatrix} I & -E^+A & -E^+D & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$, $S_9 = \operatorname{diag}\{I, I, [\overline{F}^+\overline{F} \quad I - \overline{F}^+\overline{F}]\}$, $S_{10} = \begin{bmatrix} I & 0 & -\overline{D}\,\overline{F}^+ \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$, $S_{11} = \begin{bmatrix} \overline{F}\,\overline{F}^+ \\ I-\overline{F}\,\overline{F}^+ \end{bmatrix}$, $S_{12} = \operatorname{diag}\{I, S_{11}\}$, $S_{13} = \begin{bmatrix} I & 0 & 0 \\ -\overline{F}^+\overline{C} & I & 0 \\ 0 & 0 & I \end{bmatrix}$ and $S_{14} = \operatorname{diag}\{S_{11}, I\}$, we have

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix} E & A & D & 0 \\ 0 & E & 0 & D \\ 0 & C & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix} &= \operatorname{rank}\left(S_7 \begin{bmatrix} E & A & D & 0 \\ 0 & E & 0 & D \\ 0 & C & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix} S_8\right) \\
&= \operatorname{rank}\begin{bmatrix} E & 0 & 0 & 0 \\ 0 & (I-EE^+)A & (I-EE^+)D & 0 \\ 0 & E & 0 & EE^+D \\ 0 & 0 & 0 & (I-EE^+)D \\ 0 & C & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix} \\
&= \operatorname{rank}E + \operatorname{rank}\begin{bmatrix} E & 0 & EE^+D \\ (I-EE^+)A & (I-EE^+)D & 0 \\ C & F & 0 \\ 0 & 0 & (I-EE^+)D \\ 0 & 0 & F \end{bmatrix} \\
&= \operatorname{rank}E + \operatorname{rank}\begin{bmatrix} \overline{E} & 0 & \overline{D} \\ \overline{C} & \overline{F} & 0 \\ 0 & 0 & \overline{F} \end{bmatrix} \\
&= \operatorname{rank}E + \operatorname{rank}\left(S_{10}\begin{bmatrix} \overline{E} & 0 & \overline{D} \\ \overline{C} & \overline{F} & 0 \\ 0 & 0 & \overline{F} \end{bmatrix} S_9\right) \\
&= \operatorname{rank}E + \operatorname{rank}\begin{bmatrix} \widetilde{E} & 0 & 0 & \widetilde{D} \\ \overline{C} & \overline{F} & 0 & 0 \\ 0 & 0 & \overline{F} & 0 \end{bmatrix} \\
&= \operatorname{rank}E + \operatorname{rank}\overline{F} + \operatorname{rank}\begin{bmatrix} \widetilde{E} & 0 & \widetilde{D} \\ \overline{C} & \overline{F} & 0 \end{bmatrix} \\
&= \operatorname{rank}E + \operatorname{rank}\overline{F} + \operatorname{rank}\left(S_{12}\begin{bmatrix} \widetilde{E} & 0 & \widetilde{D} \\ \overline{C} & \overline{F} & 0 \end{bmatrix} S_{13}\right) \\
&= \operatorname{rank}E + \operatorname{rank}\overline{F} + \operatorname{rank}\begin{bmatrix} \widetilde{E} & 0 & \widetilde{D} \\ 0 & \overline{F} & 0 \\ \widetilde{C} & 0 & 0 \end{bmatrix} \\
&= \operatorname{rank}E + 2\operatorname{rank}\overline{F} + \operatorname{rank}\begin{bmatrix} \widetilde{E} & \widetilde{D} \\ \widetilde{C} & 0 \end{bmatrix}.
\end{aligned} \tag{35}$$

On the other hand,

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix} E & D \\ 0 & F \end{bmatrix} &= \operatorname{rank}\left(S_{14}\begin{bmatrix} E & D \\ 0 & F \end{bmatrix}\right) = \operatorname{rank}\begin{bmatrix} E & EE^+D \\ 0 & (I-EE^+)D \\ 0 & F \end{bmatrix} \\
&= \operatorname{rank}E + \operatorname{rank}\begin{bmatrix} (I-EE^+)D \\ F \end{bmatrix} = \operatorname{rank}E + \operatorname{rank}\overline{F}.
\end{aligned} \tag{36}$$

Furthermore,

$$\begin{aligned}\operatorname{rank}\left[\begin{smallmatrix} D \\ F \end{smallmatrix}\right] &= \operatorname{rank}\left(\left[\begin{smallmatrix} D \\ F \end{smallmatrix}\right]\left[\overline{F}^{+}\overline{F} \quad I-\overline{F}^{+}\overline{F}\right]\right) = \\ \operatorname{rank}&\left[\begin{smallmatrix} D\overline{F}^{+}\overline{F} & \widetilde{D} \\ \overline{F} & 0 \end{smallmatrix}\right] = \operatorname{rank}\overline{F} + \operatorname{rank}\widetilde{D}.\end{aligned} \tag{37}$$

Then, it follows from (35)-(37) that $(7a) \iff (34)$. □

Lemma 3. Matrices $\widetilde{E}, \widetilde{A}, \widetilde{C}, \widetilde{D}$ satisfy (7b) iff for matrices E, A, C, D, F the following equation holds

$$\operatorname{rank}\left[\begin{smallmatrix} sE-A & -D \\ C & F \end{smallmatrix}\right] = n + \operatorname{rank}\left[\begin{smallmatrix} D \\ F \end{smallmatrix}\right], \ \forall s \in \mathbb{C}^{+}. \tag{38}$$

Proof. Let $S_{15} = \operatorname{diag}\{S_6, I\}$, $S_{16} = \operatorname{diag}\{I, [\overline{F}^{+}\overline{F} \ \ I - \overline{F}^{+}\overline{F}]\}$ and $S_{17} = \left[\begin{smallmatrix} I & D\overline{F}^{+} \\ 0 & I \end{smallmatrix}\right]$, and for any $s \in \mathbb{C}^{+}$ we have

$$\begin{aligned}\operatorname{rank}\left[\begin{smallmatrix} sE-A & -D \\ C & F \end{smallmatrix}\right] &= \operatorname{rank}\left(S_{17}S_{15}\left[\begin{smallmatrix} sE-A & -D \\ C & F \end{smallmatrix}\right]S_{16}\right) \\ = \operatorname{rank}\left[\begin{smallmatrix} s\widetilde{E}-\widetilde{A} & 0 & -\widetilde{D} \\ \widetilde{C} & \overline{F} & 0 \end{smallmatrix}\right] &= \operatorname{rank}\left(S_{12}\left[\begin{smallmatrix} s\widetilde{E}-\widetilde{A} & 0 & -\widetilde{D} \\ \widetilde{C} & \overline{F} & 0 \end{smallmatrix}\right]S_{13}\right) \\ = \operatorname{rank}\left[\begin{smallmatrix} s\widetilde{E}-\widetilde{A} & 0 & -\widetilde{D} \\ 0 & \overline{F} & 0 \\ \widetilde{C} & 0 & 0 \end{smallmatrix}\right] &= \operatorname{rank}\left[\begin{smallmatrix} s\widetilde{E}-\widetilde{A} & -\widetilde{D} \\ \widetilde{C} & 0 \end{smallmatrix}\right] + \operatorname{rank}\overline{F}\end{aligned} \tag{39}$$

which together with equation (36) implies that $(7b) \iff (38)$. □

Following the discussions on the existence of the SMO (19)-(22) in Lemmas 2 and 3, it can be concluded that for the NIO descriptor system (1), there exists a SMO in the form of (19)-(22) that is able to estimate the state x and the UI f iff the conditions (34) and (38) are satisfied.

Now, we summarize the SMO design procedure in Algorithm 1.

Algorithm 1:
Step 1. Check if (34) and (38) are satisfied. If so, go to Step 2; otherwise, the observer design fails.
Step 2. According to the transformations defined in (2)-(6), compute matrices $\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$, and then choose matrices $W, \widetilde{T}, \widetilde{H}$ according to [4].
Step 3. Compute matrices $\widehat{E}, \widehat{A}, \widehat{B}, \widehat{D}, \widehat{\Delta}$ and obtain their decompositions according to (9).
Step 4. Based on Remark 2, considering (12) and solving the LMI $\widetilde{P}\widetilde{\Upsilon} + \widetilde{\Upsilon}^T\widetilde{P} < 0$ with the definition of $\widetilde{\Upsilon}$ in (13) gives $\widetilde{P}, \widetilde{J}$. Then, compute $\widehat{P}, \widehat{J}$ and their decompositions $\widehat{P}_1, \widehat{P}_2, \widehat{J}_1, \widehat{J}_2$ given by (16), and finally compute $\widehat{L}$ according to (17).
Step 5. Construct SMO (19)-(22) and get the state and UI estimates as $\widehat{x} = W\left[\begin{smallmatrix} \widehat{\overline{x}}_1 \\ \widehat{\overline{x}}_2 \end{smallmatrix}\right]$ and $\widehat{f}$ given by (32).

Remark 7. This paper has presented a new SMO scheme to circumvent the IOC required in [9–15]. Existing works that overcame the IOC restriction [18–22] considered only UIs in the state equation, and could only develop sufficient conditions (and not necessary conditions) for the stability condition (which is also known as the minimum phase condition in SMO research) to be satisfied, whereas this paper considers a more general case of faults in the output, and has developed the necessary and sufficient conditions for the stability condition. Moreover, the proposed reformulation of the NIO descriptor system did not treat certain states as UIs (as in [18–22]) and thus is a more feasible approach to yield a successful SMO design.

Remark 8. In [26], for descriptor systems containing UI f in both the state and output equations, the authors proposed a high-order sliding mode observer (HOSMO) technique which can also estimate simultaneously the state and the UI. The main idea of [26] is to express the estimate as a function of $\dot{y}, \ddot{y}, \ldots, y^{(\bar{s})}$, which are the high-order derivatives of y (where $\bar{s}$ is a positive integer). However, if output y contains f, and if $f^{(\bar{s})}$ does not exist, then $y^{(\bar{s})}$ does not exist either, and hence the HOSMO technique [26] will fail to estimate x and f. By comparison, the proposed SMO technique in this paper does not need the assumption that $f^{(\bar{s})}$ exists, and thus can estimate more general types of UIs such as the square wave signal. This point will be further elaborated in the simulation as a comparison.

4 Simulation results

In this section, a modified version of the chemical mixing tank system [9,20] is employed for simulation to show the effectiveness of the proposed methods and their superiority over the existing methods.

Table 1
Variables of the chemical mixing tank system

Variable	Variable type	Unit
c_3	Connection	mol/l
q_3	Flow rate	l/s
c_5	Connection	mol/l
q_5	Flow rate	l/s
q_1	Flow rate	l/s
$q_{1,e}$	Flow rate reference	l/s
q_4	Flow rate	l/s
f_1	Fault signal	l/s
f_2	Fault signal	l/s
f_3	Fault signal	mol/l

From [20], for the chemical mixing tank system the actuator signal q_1 is affected by a fault f_1, and the leakage volume from the pipe connecting the two tanks is f_2. Besides, it is assumed that the sensor fault f_3 occurs in the measurement of the concentration c_5. Thus, with the variables described in Table 1, the dynamics of the chemical mixing tank system can be written in the notation of (1) with $x = \left[\begin{smallmatrix} c_3^T & q_3^T & c_5^T & q_5^T & q_1^T \end{smallmatrix}\right]^T$, $u = \left[\begin{smallmatrix} q_{1,e}^T & q_4^T \end{smallmatrix}\right]^T$,

$y = \left[\, q_3^T \; c_5^T \; q_1^T \,\right]^T$, $f = \left[\, f_1^T \; f_2^T \; f_3^T \,\right]^T$ and matrices

$$E = \begin{bmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1 \end{bmatrix}, B = \begin{bmatrix} 0&0\\ 0&0\\ 0&0.02\\ 0&1\\ 2&0 \end{bmatrix}, D = \begin{bmatrix} 0&0&0\\ 0&-1&0\\ 0&0&0\\ 0&0&0\\ 1&0&0 \end{bmatrix}, F = \begin{bmatrix} 0&0&0\\ 0&0&1\\ 0&0&0 \end{bmatrix},$$

$$A = \begin{bmatrix} -0.3750&-0.0667&0&0&0.1\\ 0&-1&0&0&1\\ 0.3&0.0533&-0.5&-0.04&0\\ 0&1&0&-1&0\\ 0&0&0&0&-2 \end{bmatrix}, C = \begin{bmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1 \end{bmatrix}.$$

Obviously, here $\operatorname{rank}\left[\begin{smallmatrix} E\\ C \end{smallmatrix}\right] = 4 < 5$, and thus it is an NIO system indicating that the traditional SMO methods [9–15] cannot be used. On the other hand, the works in [18–22] also cannot be applied because they do not consider the UIs in the output. However, it is shown next that the proposed method in this paper can still estimate the states and UIs. On the other hand, we will also show the superiorities of the proposed SMO method over the HOSMO scheme [26] in estimating more general types of UIs f when $f^{(\bar{s})}$ does not exist. By following the steps in Algorithm 1, the construction of the SMO (19)-(22) will be given in the following.

Step 1. It is examined that the conditions (34) and (38) are satisfied.

Step 2. After performing the specified operations presented in Section 3.1, we obtain

$$\widetilde{E} = \begin{bmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1 \end{bmatrix}, \widetilde{A} = \begin{bmatrix} -0.3750&-0.0667&0&0&0.1\\ 0&0&0&0&0\\ 0.3&0.0533&-0.5&-0.04&0\\ 0&0&0&0&0\\ 0&0&0&0&-2 \end{bmatrix},$$

$$\widetilde{B} = \begin{bmatrix} 0&0\\ 0&0\\ 0&0.02\\ 0&0\\ 2&0 \end{bmatrix}, \widetilde{D} = \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 1 \end{bmatrix}, \widetilde{C} = \begin{bmatrix} 0&1&0&-1&0\\ 0&1&0&0&0\\ 0&0&0&0&1 \end{bmatrix}.$$ Choose

$$\widetilde{T} = I_5,\; W = \begin{bmatrix} 1&0&0&0&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \text{ and } \widetilde{H} = \begin{bmatrix} 0&0&0\\ 0&1&0\\ 0&0&0\\ -1&1&0\\ 0&0&0 \end{bmatrix}.$$

Step 3. Compute matrices

$$\widehat{E} = \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1 \end{bmatrix}, \widehat{A} = \begin{bmatrix} -0.375&0&-0.0667&0&0.1\\ 0.3&-0.5&0.0533&-0.04&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&-2 \end{bmatrix},$$

$$\widehat{B} = \begin{bmatrix} 0&0\\ 0&0.02\\ 0&0\\ 0&0\\ 2&0 \end{bmatrix}, \widehat{D} = \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 1 \end{bmatrix} \text{ and } \widehat{\Delta} = \mathbf{0}_{5\times 8}.$$

Step 4. According to Remark 2, substituting (12) into (11b) and solving it gives $\widetilde{P} = \begin{bmatrix} 2.7232&0&0.163&0&0\\ 0&2.5&0&0&0\\ 0.1630&0&2.3979&0&0\\ 0&0&0&2.5&0\\ 0&0&0&0&2.5 \end{bmatrix}$,

$\widetilde{J} = \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&1 \end{bmatrix}$. Then, compute $\widehat{P}_1 = \left[\begin{smallmatrix} 2.7232&0.1630\\ 0.1630&2.3979 \end{smallmatrix}\right]$,

$\widehat{P}_2 = \mathbf{0}_{2\times 3}$, $\widehat{J}_1 = \mathbf{0}_{2\times 3}$ and $\widehat{J}_2 = \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&1 \end{bmatrix}$, and then according to (17) compute $\widehat{L} = \mathbf{0}_{2\times 3}$.

Step 5. Constructing SMO (19)-(22) gives the estimations of the states and UIs according to Algorithm 1.

Since the control input u does not change the observability for linear systems, for the purpose of simulation, we set $u = 0$. Moreover, in order to compare the performance of the proposed SMO with the HOSMO in [26], for $f = [f_1 \; f_2 \; f_3]^T$, the case when $f^{(\bar{s})}$ exists and the case when $f^{(\bar{s})}$ does not exist will be tested, respectively (according to [26], here $\bar{s} = 1$).

Case I: When $f^{(\bar{s})}$ exists ($\bar{s} = 1$)

For the case when $f^{(\bar{s})}$ exists, suppose that $f_1 = -0.6t + 6 + 3\sin(t)$, $f_2 = 4\cos(t)$ and $f_3 = 0.6t - 5 + 3\sin(2t)$. With the initial conditions $x_0 = \left[\, 5 \; 2 \; -5 \; 5 \; 5 \,\right]^T$, $\zeta_1 = \left[\, -1 \; 1 \,\right]^T$ and $\zeta_2 = \left[\, -3 \; 4 \; -5 \,\right]^T$, the simulation results are shown in Figs. 1-2, where the solid lines in red are the actual states or UIs, the dotted lines in green are the estimated ones by HOSMO given in [26], and the dotted lines in blue are the estimated ones by the proposed S-

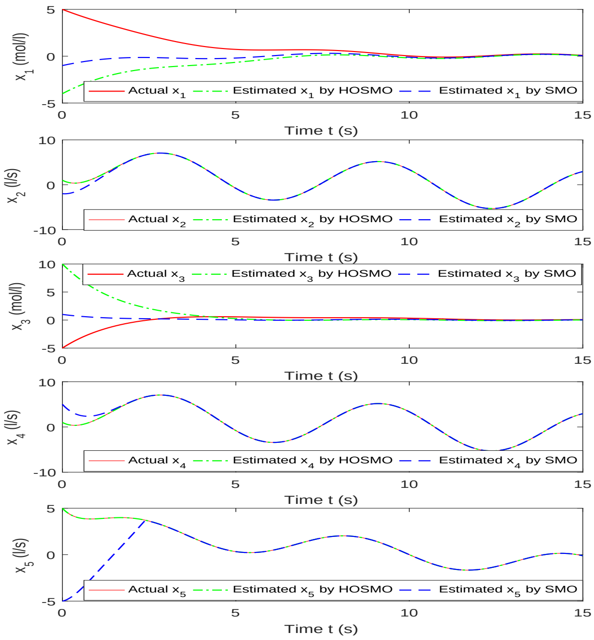

Fig.1 Estimates of x_i by HOSMO and SMO (Case I).

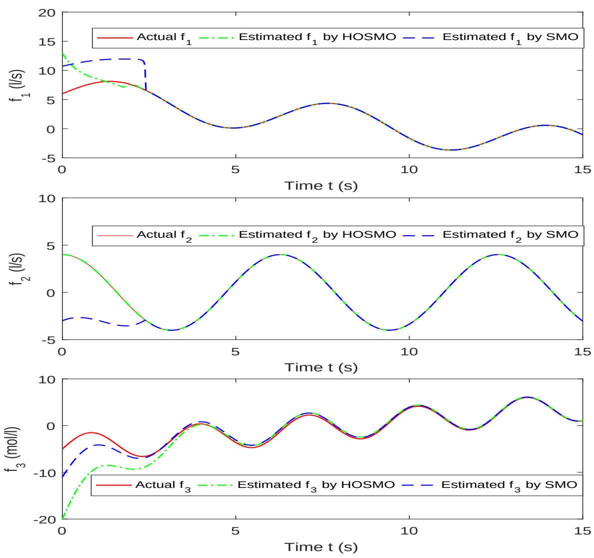

Fig.2 Estimates of f_i by HOSMO and SMO (Case I).

MO in the present paper. The results show that although the SMO methods in [18–22] cannot work for this system, the HOSMO and the proposed SMO scheme can still estimate the states and the UIs successfully.

Case II: When $f^{(\bar{s})}$ does not exist ($\bar{s} = 1$)
For the case when $f^{(\bar{s})}$ does not exist, suppose that

$$f_1 = \begin{cases} 5\sin(2.7t) + 3.5, & 4(\mathrm{N}-1)/3 \le t \le (4\mathrm{N}-2)/3 \\ 5\sin(2.7t) - 3.5, & (4\mathrm{N}-2)/3 \le t < 4\mathrm{N}/3 \end{cases}$$

$$f_2 = \begin{cases} 3, & 5(\mathrm{N}-1)/3 \le t \le (10\mathrm{N}-5)/6 \\ -3, & (10\mathrm{N}-5)/6 \le t < 5\mathrm{N}/3, \end{cases}$$

$\mathrm{N} = 1, 2, \ldots$ and $f_3 = 5.2\sin(t)$. It is clear that f_1 and f_2 experience discontinuities and their first derivatives do not exist at certain time instants. In this scenario, for the SMO (19)-(22), we use the adaptive gain $\rho(t)$ given in (33) to replace the fixed gain ρ with the adaptive parameters being selected as $\rho(0) = 0$ and $\alpha_0 = 2$. For the identical initial conditions with the case when $f^{(\bar{s})}$ exists, we exhibit the simulation results in Figs. 3-4, from which we can see that under the adaptive gain the proposed SMO method can still estimate both the state and UI, but the HOSMO method by [26] fails to reconstruct the UI f_1 in this case. Besides, in order to analyse the evolution of the adaptive gain, we also plot the curve of $\rho(t)$ in Fig. 5 which indicates that before $t = 1s$ or so, $\rho(t)$ keeps increasing, and after the moment when sliding motion $e_{\tilde{y}} = 0$ is achieved, $\rho(t)$ maintains the value, which also validates the correctness of the proposed methods.

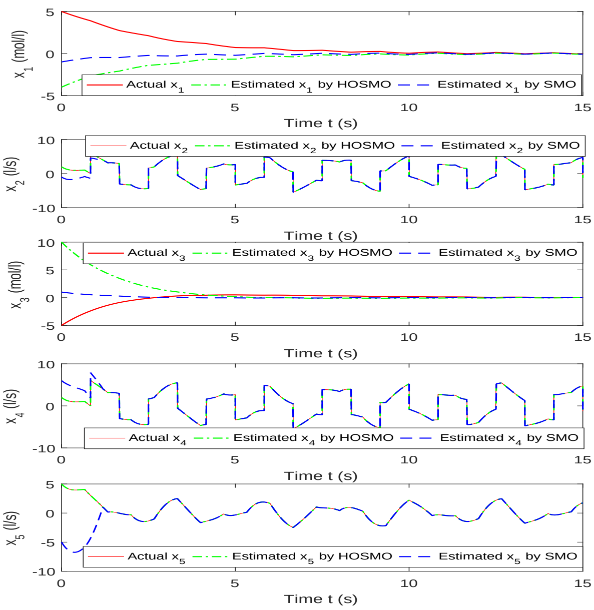

Fig.3 Estimates of x_i by HOSMO and SMO (Case II).

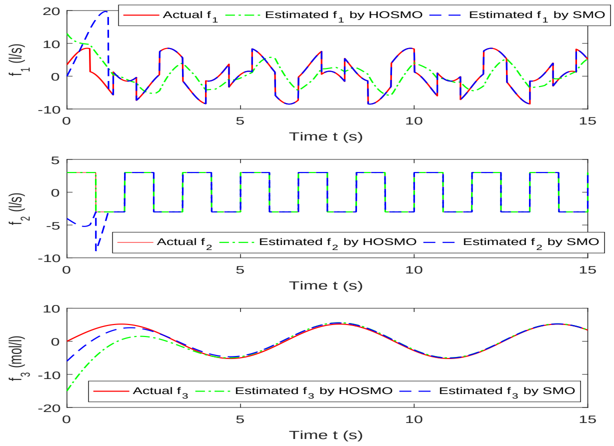

Fig.4 Estimates of f_i by HOSMO and SMO (Case II).

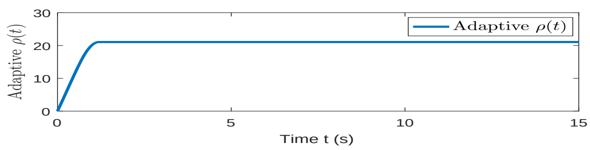

Fig. 5 Evolution of the adaptive gain $\rho(t)$.

5 Conclusions

In this paper, we have proposed a SMO scheme to estimate states and UIs for NIO descriptor systems where the UIs exist in both state and output equations. By reformulating the original system and introducing new UIs an equivalent descriptor system is obtained where both the NIO constraint and UIs in the outputs have been removed. Based on the equivalent descriptor system, a SMO is developed to estimate the states and UIs. Also, the necessary and sufficient conditions for the existence of the SMO are discussed in detail and given in terms of the original system matrices. Future work could consider the sliding mode functional observer design for descriptor systems.

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