How to Draw a Virtual Cubical Perspective Box

António Bandeira Araújo[1] and Lucas Fabian Olivero[2]

[1]Univ. Aberta and CIAC, Portugal; antonio.araujo@uab.pt
[2]Univ. Aberta, Univ. Algarve, and CIAC, Portugal; lucas.olivero@uab.pt

Abstract

In this workshop we will learn how to draw a cubical perspective by hand and how to visualize the resulting drawing as a VR panorama, creating a kind of virtual perspective box. We will do this by viewing cubical perspective as a special case of spherical perspective and considering how spherical geodesics project on the cube.

Introduction

Spherical perspectives are handmade drawings that capture all the visual information around the eye of an observer, and render it onto a compact region of the plane [5]. These drawings are characterized by rendering spatial lines into plane curves, and by having exactly two vanishing points for each line. Spherical perspectives are becoming popular among artists and architects [9] in part because they can be turned into immersive visualizations as VR panoramas. In Bridges 2018, A. B. Araújo delivered a workshop on how to draw handmade equirectangular spherical perspectives using a dynamic grid method that exploits the translational symmetries of that perspective [3]. In Bridges 2019, the same author presented a different method to draw in the azimuthal equidistant ("360-degree fisheye") perspective [2], making use of its rotational symmetries. In the present workshop the authors propose a different approach to obtain an immersive drawing: a *cubical perspective*. We will see that cubical perspective can be seen as a special case of spherical perspective, which although devoid of the nice symmetry group actions of the equirectangular or fisheye cases, compensates for this through its connection with classical perspective, which simplifies the rendering of line images.

Cubical Perspective as a Spherical Perspective

A *cubical perspective* is a plane drawing obtained as follows: place the observer's eye at a point O in space, at the center of a cube; project the points of the 3D environment radially onto O and mark where the projection's rays touch the cube's faces. You get an immersive anamorphosis, that is, a 2D drawing on the cube's surface that looks exactly like the original 3D scene if you see it from O. Now cut and flatten the cube (Figure 1). You obtain a flat drawing that is no longer a *trompe l'oeil* but still codifies the same visual information as the original anamorphosis. This flat drawing is called a *cubical perspective*. It can be seen as a mechanism for storing on a compact subset of the plane the visual information of the immersive anamorphosis. Now consider how you would do this procedure in reverse. Could you, from the coordinates of the 3D environment, make the flat cubical perspective drawing that you might then fold onto a cube to see from within? How would you perform these operations efficiently by hand? That's what we call *solving* a perspective: obtaining all line images and vanishing points on the flat drawing from a given (usually small) sampling of the 3D data. You might of course just solve a set of 6 classical perspectives [8], one for each face of the cube, as you would normally do to construct a classical perspective box [10], [11]. However, as first shown in [6], it is far more elegant to think of this as solving a single *spherical perspective*.

A spherical perspective, as formalized in [4] is a conical projection onto a sphere followed by a flattening onto the plane via a cartographic mapping verifying certain continuity conditions - basically projecting onto a compact connected set and being a homeomorphism almost everywhere. The flattening step can of course vary a lot, since every cartographic projection defines a different perspective, hence the various spherical perspectives of one same scene can be extremely varied in appearance. But since the first step – anamorphosis – is uniquely defined, every spherical perspective verifies the principle of *radial occlusion*, that is, 3D points in the same ray from the eye will project onto the same point of the plane. This principle, all by itself, guarantees that each line will have exactly two vanishing points [4], [5].

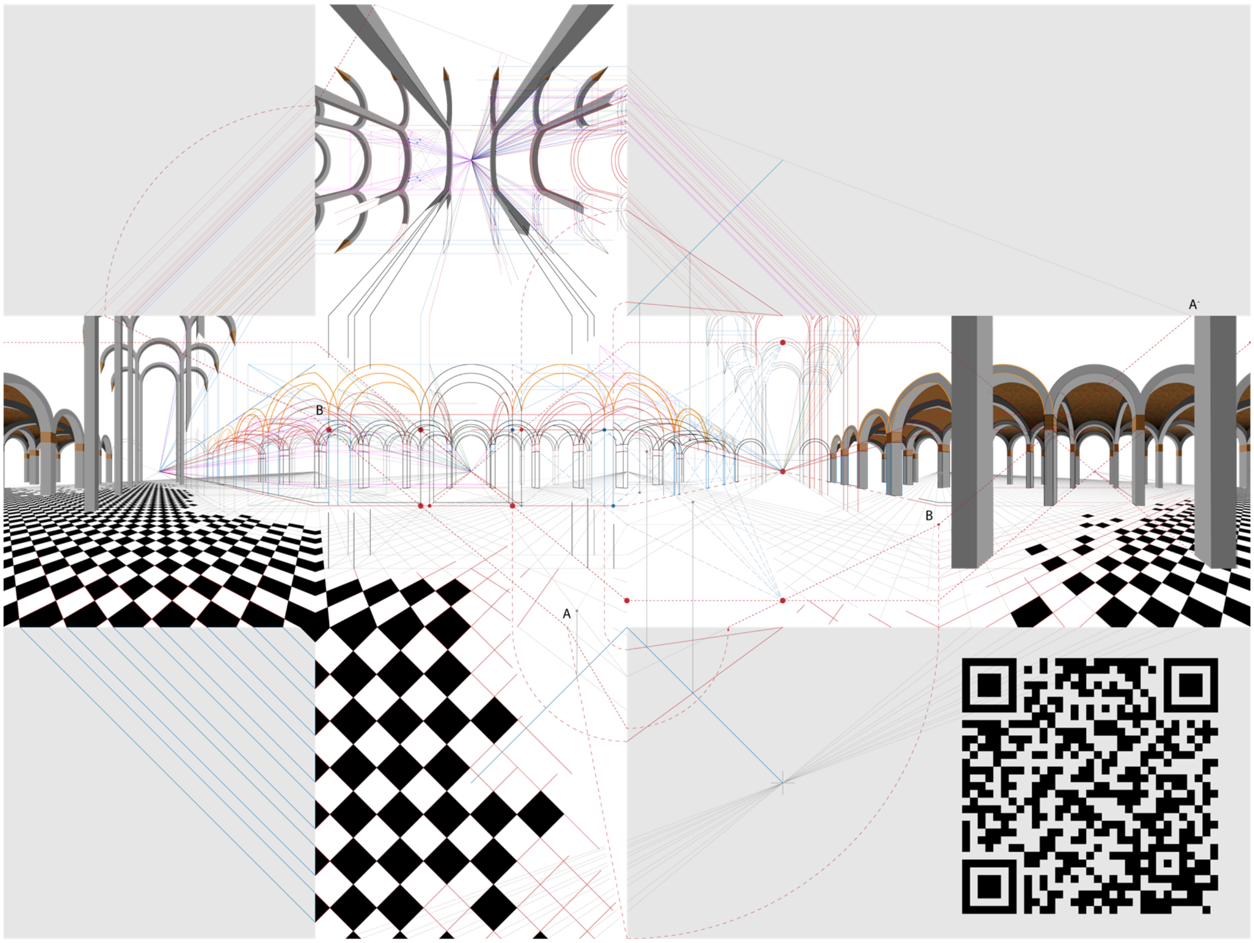

Figure 1: *An elaborate example of a cubical perspective (handmade vectorial drawing by L. F. Olivero). A 3D scene is projected onto a cube which then is cut and flattened. Scan the QR-code to see the VR panorama online.*

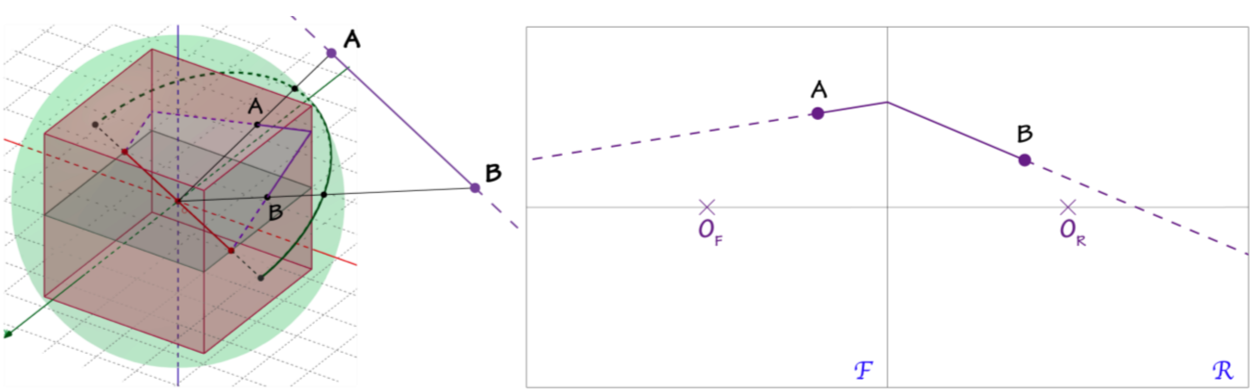

Figure 2: *A spatial line AB projects on the sphere as a meridian (half a geodesic). Flattening the cube turns the geodesic onto a connected set of line segments that may change direction at the cube's edges.*

It is easy to see that cubical perspective is a special case of a spherical perspective: just consider a sphere concentric with the cube; project radially from the sphere to the cube and then cut open the cube; since the first projection is a homeomorphism, then the entailment results in a flattening of the sphere in the conditions of [4], hence cubical perspective is just a special type of spherical perspective.

Now, in order to solve a general spherical perspective, A. B. Araújo [4], [5] proposes a general strategy that consists in classifying all the geodesics of a specific perspective and then finding a method for rendering them by hand, with special attention to the symmetries coming from the duality of vanishing points/antipodes, and the natural symmetries of each flattening. This strategy was the basis of the methods proposed in both spherical perspectives mentioned above.

The focus on geodesics comes from this: solving a perspective drawing means finding the images of all lines and their vanishing points. Yet, the easiest way to go about it is to consider not each line directly, but the plane though O that contains the line. This plane will intersect the sphere at a geodesic (great circle) and the line image will be half of this: a meridian between two diametrically opposite (antipodal) vanishing points. These vanishing points in turn are obtained thus: translate the line to O and intersect the translated line with the sphere, to obtain two points. These are the vanishing points of the line. The strategy for rendering lines is to draw their geodesics first and then to crop them at these vanishing point pairs.

The advantage of cubical perspective is that, although its symmetries are less useful than in other spherical perspectives, the rendering step is easier since the geodesics project as line segments in each face of the cube (since each face is just a classical perspective). A geodesic in cubic anamorphosis is just a closed cycle of lines around the cube, which can be comprised of either 4 or 6 segments. The problem that remains is to find how those segments change slope (in the flat perspective) as we cross from one face of the cube to another (see Figure 2). The way this is determined depends on the classification of the geodesics, which is done according to the number of sides in the cycle and the relative position of the given points. The systematic characterization of this has a relatively large number of sub-cases and we will not repeat it here, nor during the workshop, as it can be found in [6]. It is simpler and more useful in the context of a workshop to present the idea thus: when you have a spatial line whose perspective image you want to render, you will project some points of that line onto the faces of the cube. You will need no more than two points, chosen according to convenience. The problem is how to render a complete geodesic from any two given points on the cube.

We need some notation here. Note that we cut open the cube according to Figure 3 and we mark reference points **F**ront (O_F), **L**eft (O_L), **R**ight (O_R), **B**ack (O_B), **U**p (O_U) and **D**own (O_D), at the center of each face. These are where orthogonal axes centered on O would intersect the cube. We call horizon to both the plane $O_F O O_L$, and the horizontal line $O_L O_F$ onto which it projects.

When you have two points of a geodesic on the same face, you simply join them with a straightedge. To complete a cycle around the cube you need points on other faces, which you find by the following methods:

1-Antipodes: For each point P of a geodesic g the antipode P^- (the point diametrically opposite to P on the cube/sphere) also belongs to g. When P is on one of the faces F, B, L, R (resp. U, D) the image of P^- is obtained from that of P by a vertical reflection across the horizon followed by a horizontal translation of two cube side lengths (resp. horizontal reflection across axis $O_U O_D$ followed by vertical translation of two cube side lengths). In Figure 3 we use antipodes to extend the segment $\overline{AB}$ to a full geodesic (see caption).

2-Identification (double points): because the cube must be cut to be flattened, a point on the edge of a cut will have a corresponding point on the edge that identifies with it on a connecting face. See for instance the double point N in the example of Figure 5, imaged in both the top and front faces. Point N is first obtained in face R and then the geodesic is extended to face U by identification.

3-Edge leap: Given two points A and B on two adjacent faces F_A and F_B sharing an edge e, we wish to find the point S where the geodesic AB crosses e. This may be found by the descriptive geometry diagram of Figure 4. The operation is described in detail in [6]. In short, the diagram represents F_A, F_B, and a plane ε through O, perpendicular to both faces. On plane ε we obtain the orthographic projection C_ε of the intersection of AB with the plane that bisects the dihedral angle $F_A F_B$. We lift C_ε to F_A to obtain its image C on the orthographic projection of AB onto F_A. Finally, a line from O_A through C finds S at edge e. The geodesic AB projects on F_A and F_B as the union of AS with SB. See [6] for further details.

4. Half-leap: Given the image $\overline{AB}$ of an arc of a geodesic g on a cube face H, the following construction (from [6]) obtains two further points of the image of g: Let $l = AB$. Let l_O be the translation of l to O. Then l_O is in the plane of g and intersects the cube at a pair of antipodal points M, M^- in the image of g. These points lie in the midlines of two antipodal faces adjacent to H. In Figure 5 we use this construction to obtain point M and close the 6-cycle generated by the segment $\overline{AB}$. Line AB is translated to the center O_F to find the height of the projection of M onto the vertical midline of the face R of the cube. This is applied again in Figure 6 to obtain M and thus extend the projection of a line l on face F to its image on face R.

The operation to use depends on the relative position of the points you already have. Though a full classification has been presented in [6], we will dodge the litany of cases through inspection: at each step we just check what operations can be used with the points available. This is more intuitive and enjoyable in a workshop context, even if it does not guarantee a construction with the least possible number of steps.

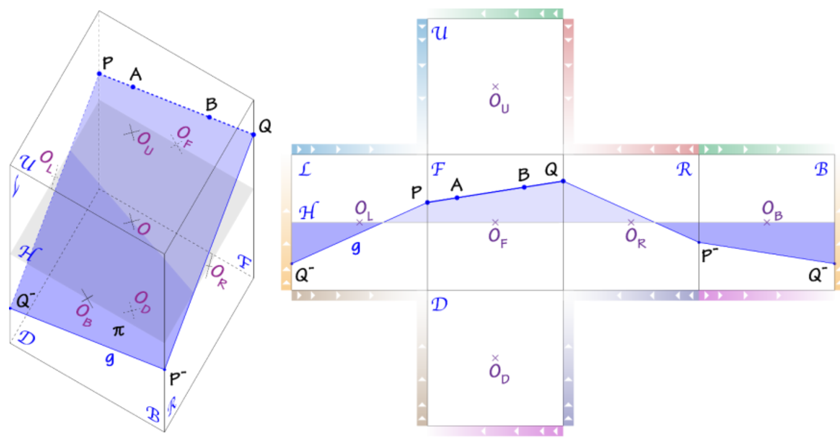

Figure 3: *The flattening of the cube. A geodesic defined by a plane AOB becomes a 4-cycle (a closed cycle of 4-line segments) upon flattening of the cube. Starting from points A and B, points P and Q are obtained at the edges of the frontal face. Then points P^- and Q^- are obtained by taking antipodes. These points are enough to determine the full 4-cycle.*

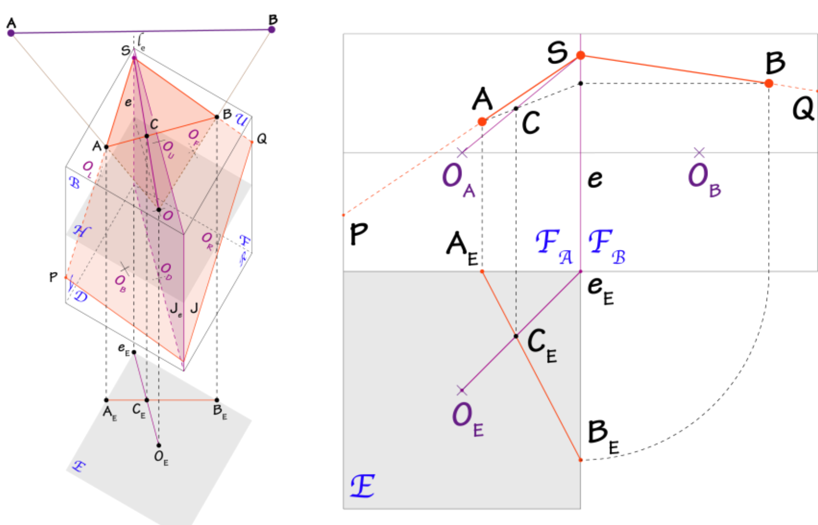

Figure 4: *Descriptive geometry construction for the transition of a geodesic across cube faces.*

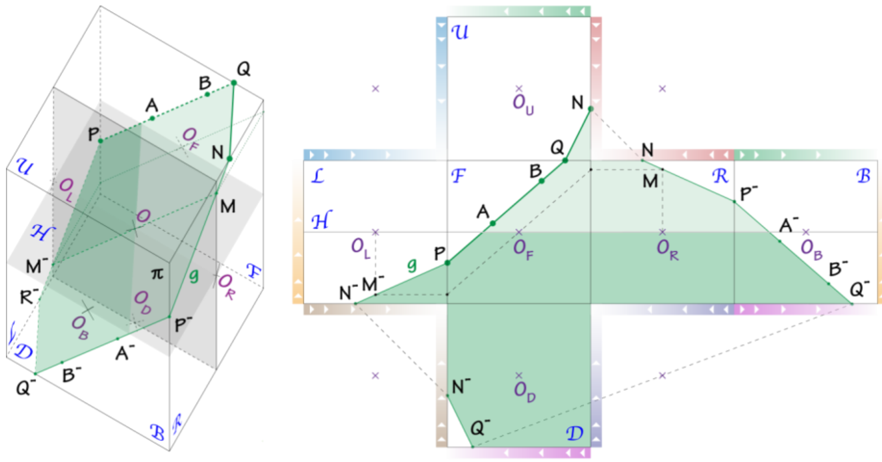

Figure 5: *The flattening of a 6-cycle geodesic. Point M is obtained by the "half-leap" construction: line AB is translated to O and it hits the vertical through O_R at point M which must belong to the geodesic.*

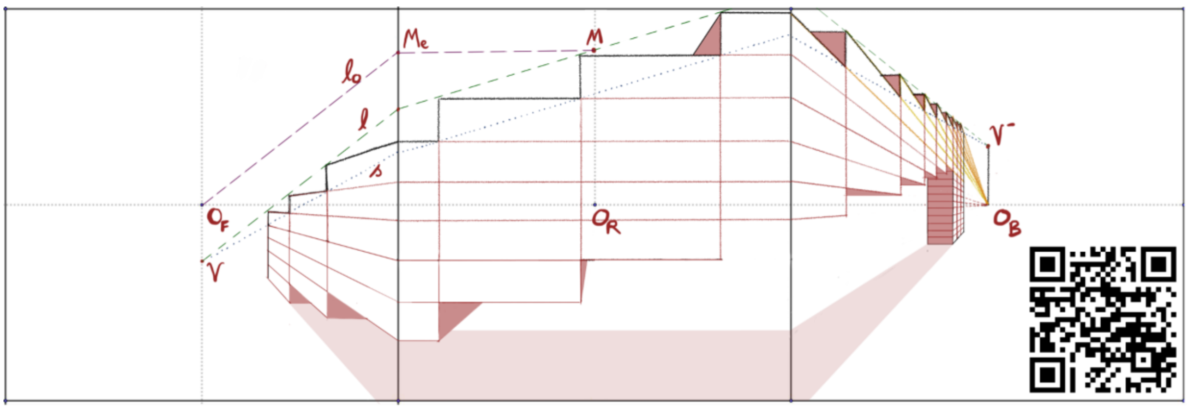

Figure 6: *A stairway climbing from a vanishing point V under the frontal reference point O_F to its antipode V^- above O_B (drawn by A. B. Araújo). We draw the steps by bouncing verticals and horizontals between two sloping lines l and s going to V. Drawing cropped to show only faces F, R, B. Scan the QR-code to see the VR panorama online.*

Practical Work

The workshop should have 20 participants or less, and the recommended age is 16 or older. Participants will be given a template with a flattened cube (as in Figure 3), pencils, erasers, and rulers. A compass will not be needed as we will use compositions of reflections to avoid rotations. This is also practical for outdoor sketching, where compasses are awkward to use. The workshop will proceed as follows:

1. Brief introduction to spherical perspectives, geodesics and vanishing points. Description of the equirectangular and fisheye cases. Contrast with cubical case. Examples of applications and artworks.

2. Projection of points from orthographic diagrams. Explanation of the surveying process for making a plan and elevation diagram of a room (the room where the workshop takes place may be used if adequate, but we will bring a template of an imaginary room that lends itself to the exercises).

3. Construction of the cubical perspective of the room from its plan and elevation drawing. We will assume the walls to be parallel to the cubes faces and see how lines go to the principal vanishing points at the centers of the faces, and how slope changes discontinuously at the edges of each face. Discussion of vanishing point duality in spherical perspective.

4. Construction of an imaginary wall at an arbitrary angle to the cube's faces (or an arbitrary rotation of the observer's referential). Finding the transition point of a line (for instance the top edge of the wall)

joining two known points in adjacent faces. This uses the method from Figure 4. Experiments with the use of antipodal points and identification points for continuation of a 4-cycle geodesic.

5. Construction of repeating patterns using vanishing points. Tilings of the floor and walls. Making measurements and placing objects in the scene. Free sketching of small objects directly onto the perspective drawing once the overall large-scale schema is obtained.

6. Construction of slopes and stairs using vanishing points (see Figure 6). Use of the half-leap points in 6-cycle geodesics.

7. Brief instruction on how to render the obtained drawing as a VR panorama. This is a matter of scanning and cropping the picture and sending it to a suitable website (such as [1]) or desktop application (such as [7]). As these resources keep evolving, and updated list will be provided in the workshop.

Summary and Conclusions

Cubical perspective is a nice special case of spherical perspective, that compensates its lack of symmetries with the familiar rendering of linear perspective, allowing for easy insertion of freehand drawn objects into a scene. It is therefore a good tool for artists and an interesting exercise for geometers.

Acknowledgements

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References

[1] O. Alshaker. 360 Tool Kit. https://360toolkit.co
[2] A. B. Araújo. “A Fisheye Gyrograph: Taking Spherical Perspective for a Spin.” *Bridges Conference Proceedings*, Linz, Austria, Jul. 16-20, 2019, pp. 659–664. https://archive.bridgesmathart.org/2019/bridges2019-659.html
[3] A. B. Araújo. “Let’s Sketch in 360º: Spherical Perspectives for Virtual Reality Panoramas.” *Bridges Conference Proceedings*, Stockholm, Sweden, Jul. 25-29, 2018, pp. 637–644. https://archive.bridgesmathart.org/2018/bridges2018-637.html
[4] A. B. Araújo. “Ruler, compass, and nail: constructing a total spherical perspective.” *Journal of Mathematics and the Arts*, vol. 12, no. 2–3, pp. 144–169, 2018. DOI: 10.1080/17513472.2018.1469378
[5] A. B. Araújo. “Spherical Perspective.” *Handbook of the Mathematics of the Arts and Sciences*, B. Sriraman, Ed. Cham: Springer International Publishing, 2021, pp. 1–61. DOI: 10.1007/978-3-319-70658-0_100-1
[6] A. B. Araújo, L. F. Olivero, and A. Rossi. “A Descriptive Geometry Construction of VR panoramas in Cubical Spherical Perspective.” *Diségno*, no. 6, pp. 35–46, Jun. 2020. DOI: https://doi.org/10.26375/disegno.6.2020.06
[7] P. Harvey. *ExifTool*. 2022. https://exiftool.org
[8] L. F. Olivero, A. Rossi, and S. Barba. “A Codification of the Cubic Projection to Generate Immersive Models.” *Diségno*, no. 4, pp. 53–63, 2019. DOI: 10.26375/disegno.4.2019.07
[9] A. Rossi, L. F. Olivero, and A. Araújo. “For Representation, a New Reality: Hybrid Immersive Models.” *Faces of Geometry*, 2nd ed., vol. 172, Springer International Publishing, 2021, pp. 263–275. DOI: 10.1007/978-3-030-63702-6_20
[10] J. Spencer. “Illusion as ingenuity: Dutch perspective boxes in the Royal Danish Kunstkammer’s ‘Perspective Chamber.’” *Journal of the History of Collections*, vol. 30, no. 2, pp. 187–201, 2018.
[11] A. Verweij. “Perspective in a box.” *Architecture, Mathematics and Perspective*, Springer, 2010, pp. 47–62.