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Convergence rate for the hybrid iterative technique to explore the real root of nonlinear problems

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ABSTRACT

This study explored the convergence rate of the hybrid numerical iterative technique (HNIT) for the solution of nonlinear problems (NLPs) of one variable $\left(f(x)=0\right)$. It is sightseen that convergence rate is two for the HNIT. By the HNIT, several algebraic and transcendental NLPs of one variable have been illustrated as an approximate real root for efficient performance. In many instances, HNIT is more vigorous and attractive than well-known conventional iterative techniques (CITs). The computational tool MATLAB has been used for the solution of iterative techniques.

1. Introduction

It is acknowledged that the arts and sciences serve as the foundational components of scientific ways since the twenty-first century. The age of computational mathematics now includes cutting-edge research in both mathematics and computer science, and these studies have generated a number of cutting-edge methods for the estimation of numerical problems. The challenge to determine the solutions of NLPs has become an important research field in most scientific real-world applications. Such solutions to real-world challenges and problems can be logically approximated with applied mathematics including algebra, geometry, engineering, physical, social, biological, natural sciences and so on. Various studies have discovered the different ways of order iterative methods to solve nonlinear problems [1]. Initially, Traub [2] presented the study of iterative strategies for solving nonlinear equations and proposed a fundamental quadratic convergent Newton iterative method for solving nonlinear equations, often frequently taking advantage of the literature. The approximate estimation of NLPs $\left(f(x)=0\right)$ has the most significant iterative procedure in terms of numerical demonstrations [3]. A research procedure requires their convergence order for numerical iterative techniques (NITs) to estimate the approximate solution for $f(x)=0$. The NITs are a mathematical approach for generating a set of successful approximate solutions to the problems [4-5]. A standard convergence scheme is adopted with a class of iterative procedural manners and applications for the methods [6]. Analysis for the NITs could converge or diverge at the initial guesses; if the technique is converged, then it should be repeated until the requisite or less than requisite accuracy (accuracy estimated by error methodologies) is obtained; if the technique has diverged, it should be terminated because there could not have any solution [7-8]. Contemplate NITs to find an approximate root of $f(x)=0$, where $f:D\subset\mathbb{R}\rightarrow\mathbb{R}$ for the bracketing interval D is a scalar function. The solution to $f(x)=0$ by the HNIT has the most potent systematically desired response than well-known CITs. The iterative structure of HNIT (1) has been developed using crossbreed thoughts between Newton Raphson's iterative technique (NRIT) and Taylor series expansions (TSE).

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_{n-1})+(x_n-x_{n-1})f''(x_{n-1})} \tag{1}$$

This study searches the convergence rate of (1). For the strong and methodical estimations of the models have approximated the real root solution of the various NLPs by MATLAB and Excel computational tools, which play a fundamental role in the numerical models. The numerical solution of the various NLPs is depicted with the comparison of the conventional models.

2. Analysis of criteria for convergence rate

The numerical iterative research process is considered with the derivation, analysis and application of methods for estimating the convergence rate of numerical accuracy. The convergence rate of the numerical iterative models is analysed for closeness to the exact analytical solution of the NLPs. An NIT is convergent when a sequence of NIT approximations with progressively distinguished outcomes domain reaches a stable approximation. Also, An NIT is consistent when it converges for the approximate root of the $f(x)=0$, controlling its corporeal phenomenon.

2.1 Theorem

Let the sequence of approximations $\{x_n\}$ converges to $r\in D$ be the simple zero of satisfactorily differentiable function $f:D\subseteq R\rightarrow R$ for bracketing interval D. If $k\in Z^+$ and p is the arbitrary constant and ε_n and ε_{n+1} are the errors [9], such that

$$\varepsilon_{n+1}=p\varepsilon_n^k \tag{2}$$

Where k is the convergence rate for an NIT. Whenever x_{n-1} and x_n are adequately close to r, then the HNIT well-defined by the algorithm for the convergence rate.

2.1.1 Proof

Let r be the simple zero of f is satisfactorily differentiable. Between ε_n and ε_{n+1} has developed a bond of r for the convergence rate of HNIT (1).

Suppose

$$\varepsilon_{n+1}=x_{n+1}-y',\ \varepsilon_n=x_n-y' \text{ and } \varepsilon_{n-1}=x_{n-1}-y \tag{3}$$

Substituting (3) into (1), we have

$$\varepsilon_{n+1}+y=\varepsilon_n+y-\frac{f(\varepsilon_n+y)}{f'(\varepsilon_{n-1}+y)+(\varepsilon_n+y-\varepsilon_{n-1}-y)f''(\varepsilon_{n-1}+y)} \tag{4}$$

Using cancellation law, assimilated TSE [10-11] and obtained Eq. (5).

$$\varepsilon_{n+1}=\varepsilon_n-\frac{f(y)+\varepsilon_n f'(y)+\frac{\varepsilon_n^2}{2!}f''(y)+\frac{\varepsilon_n^3}{3!}f'''(y)+\cdots}{\left[f'(y)+\varepsilon_{n-1}f''(y)+\frac{\varepsilon_{n-1}^2}{2!}f'''(y)+\cdots\right]+(\varepsilon_n-\varepsilon_{n-1})\left[f''(y)+\varepsilon_{n-1}f'''(y)+\cdots\right]} \tag{5}$$

Neglecting third and higher-order prime terms, then obtained (6)

$$\varepsilon_{n+1}=\varepsilon_n-\frac{f(y)+\varepsilon_n f'(y)+\frac{\varepsilon_n^2}{2!}f''(y)}{f'(y)+\varepsilon_n f''(y)}=\frac{\frac{\varepsilon_n^2}{2}f''(y)-f(y)}{f'(y)+\varepsilon_n f''(y)} \tag{6}$$

By the definition of the NLPs $(f(y)=0)$, substitute in Eq. (6) and becomes the following.

$$\varepsilon_{n+1}=\varepsilon_n^2\frac{f''(y)}{2f'(y)}\left[1+\varepsilon_n\frac{f''(y)}{f'(y)}\right]^{-1} \tag{7}$$

Expand the term of Eq. (7) by the Binomial theorem, then Eq. (8) developed.

$$\varepsilon_{n+1}=\varepsilon_n^2\frac{f''(y)}{2f'(y)}\left[1-\varepsilon_n\frac{f''(y)}{f'(y)}+\cdots\right] \tag{8}$$

After multiplication, neglecting the least valued terms of Eq. (8), then acquired Eq. (9).

$$\varepsilon_{n+1}=\frac{f''(y)}{2f'(y)}\varepsilon_n^2 \tag{9}$$

Assumed $p=\dfrac{f''(y)}{2f'(y)}$ and assimilated in Eq. (9), then Eq. (10) becomes

$$\varepsilon_{n+1}=p\varepsilon_n^2 \tag{10}$$

Hence, after comparing Eq. (10) and Eq. (2) found $k=2$, therefore, the convergence rate of the HNIT (1) is two. Furthermore, it is observed that the error at

$(n+1)^{st}$ step is proportional to the square of the n^{th} step. Also, it has been examined that the number of truthful significances is approximately two with each iteration.

3. Hybrid Numerical Iterative Technique

Efforts were made with an efficient approximate solution of the various NLP types of $f(x)=0$ for exploring the modified and hybrid NIT scheme, i.e., HNIT (1). The HNIT model was developed as the family of bracketing iterative techniques (BITs) with NRIT and TSE crossbreed contemplations and found an excellent work performance. Capability to the HNIT model is mapped as where the NRIT is not performed due to the slope being zero and it works efficiently. It is also observed that the convergence performance of HNIT is close to the NRIT and better than conventional BITs in term of the iterations. Therefore, the solution to the various algebraic/transcendental problems $f(x)=0$ describes the robust efficiency of the HNIT model and is more consistent than eminent CITs.

4. Comparisons of Numerical Iterative Techniques

Numerical solutions of the various real-world, algebraic and transcendental NLPs of one variable are illustrated in Tabs. 1-5 and then compete with the number of iterations until zero tolerance between the NHIT and CITs including the Bisection (BIT), the False Position (FPIT) and Newton Raphson Raphson's (NRIT) iterative techniques. It has been analysed from the tables that NHIT is a strong loftier iterative technique than CITs [12].

The NITs for determining the approximate real root of NLPs has executed in the below three steps.

Step-1: For NITs, if the initial guesses value(s) are not given, search these, where lies the numerical solution (real root approximation).

Step-2: Employ the iterative systematic procedure with norms of an NIT.

Step-3: Whenever required accuracy is reached procedure could be terminated.

4.1 Real-World Numerical Problems

Estimating the approximate real root of NLPs $\left(f(x)=0\right)$ is a classic approach in numerical interpretation. Therefore, here discussing some real-world numerical problems in Tabs. 1-4 using various NITs approaches.

1. A plastic ball is floating in a bucket filled with water and its immersed height (h) has measured from the bottom of the partially submerged ball. The $\mathrm{r}^3=\dfrac{0.25\rho_l \mathrm{h}^2(3\mathrm{r}-\mathrm{h})}{\rho_s}$ equation has been determined by the rule of Archimedes with the assumption of the radius r and densities ρ_1 and ρ_s, apparently, $\rho_s<\rho_l$. Estimate the immersed height with zero tolerance using the appropriate radius by NITs.

Table 1

Solution of (i) by NITs and comparison of the models.

Nonlinear formation	Initial guess	f(h)	Immersed Height (f(h)=0)	Number of iterations till to zero tolerance by NITs			
				BIT	FPIT	NRIT	HNIT
$\mathrm{f}(\mathrm{h})=0.25\mathrm{h}^3-0.75\mathrm{h}^2+50$	-4.5 -5.5	12 -31	|-5|	>20	15	5	4

2. As it is a natural process that the volume of the gas increases corresponding with the temperature rises. Two volumes of the gases $\mathrm{V}_1=\mathrm{e}^{\mathrm{T}}$ and $\mathrm{V}_2=4\mathrm{T}$ have assumed under the ideal conditions. Using NITs models, analysis the temperature (T) where volumes of the gases become identical.

Table 2

Solution of (ii) by NITs and comparison of the models.

Nonlinear formation	Initial guess	f(T)	Temperature (f(T)=0)	Number of iterations till to zero tolerance by NITs			
				BIT	FPIT	NRIT	HNIT
$f(T) = e^{T} - 4T$	0 0.5	1 -0.351278729	0.357402956	>20	15	5	5

3. The resistance $R = A + B \cdot I^{3/2}$ in the electrical network varies with the current (I) by Ohm's law $(V = R \cdot I)$. Determine current (I) with zero tolerance using $V = 5, A = 100$ and $B = 10$.

Table 3

Solution of (iii) by NITs and comparison of the models.

Nonlinear Problem	Initial guess	f(I)	Current (f(I)=0)	Number of iterations till to zero tolerance by NITs			
				BIT	FPIT	NRIT	HNIT
$f(I) = 2I^{5/2} + 20I - I$	0 0.2	-1 3.178885438	0.048918058	>15	>12	12	12

4. In the analysis of the anti-symmetric buckling of beams, a factor ϕ satisfies $0 < \phi \le \pi/2$ and $6\cos\phi = \gamma\ \phi\sin\phi$, where γ depends on the geometry and the critical stress on the beams. Determine ϕ using NITs along zero tolerance.

Table 4

Solution of (iv) by NITs and comparison of the models.

Nonlinear Problem	Initial guess	f(φ)	Exact Real Root (f(φ)=0)	Number of iterations till to zero tolerance by NITs			
				BIT	FPIT	NRIT	HNIT
$f(\phi) = \phi - 60\cot(\phi)$	1.4 1.6	-8.94860355 3.352718692	1.545051164	>10	>8	6	6

4.2. Algebraic and transcendental Numerical Problems

Numerous algebraic and transcendental NLPs $(f(x) = 0)$ are interpreted in Tab.5 with numerical solutions by NITs approaches.

Table 5

Solution of the numerous NLPs by NITs.

Sr. No.	Nonlinear Problems	Initial guess	f(x)	Exact Real Root (f(x)=0)	Number of iterations till to zero tolerance by NITs			
					BIT	FPIT	NRIT	HNIT
1	$x^3 + 4x - 15$	1.5 2	-2.625 9	1.631980806	>25	8	6	6
2	$x^2 Sin(x) - Cos(x)$	0.5 1	-0.757726177 0.301168679	0.895206045	51	15	7	7
3	$Cos(x) - Sinh(x)$	0 1	1 -0.634898888	0.703290659	53	17	6	6
4	$2Cosh(x)Sin(x) - 1$	0.4 0.5	-0.158021178 0.081225371	0.466833756	57	18	5	5
5	$Cos(x) - Sqrt(x)$	0.5 1	0.170475781 -0.459697694	0.641714371	>30	6	6	6
6	$e^x - 4Sin(x)$	0 1	1 -0.647602111	0.370558096	25	21	Fail	20
7	$e^x - Sin(x) - 2Cos(x) + 1/2$	0 1	-0.5 1.296206232	0.547127317	47	33	Fail	12
8	$e^x + 2^{-x} + 2Cos(x) - 6$	1.5 2	-1.023283136 0.806762426	1.829383602	>35	18	13	12
9	$Sin^{-1}(x) - x^2 + 1$	-0.75 -0.5	-0.410562079 0.226401224	-0.598569653	>30	16	6	5
10	$Cos^{-1}(x) - Sin(x^2)$	0.7 0.9	0.324772942 -0.273260363	0.815537502	>30	14	5	6

5. Conclusions

This article established the convergence rate for newly developed hybrid NIT and found that it is binomially convergent. Therefore, the HNIT and CITs have analyzed and compared significant performance, convergence, and accuracy on an approximate root of numerous NLPs of one variable $\left(f\left(x\right)=0\right)$, described in Tabs. 1-5. Therefore, the NHIT is working robustly and the exactness of the root with zero tolerance precision also observed that the number of iterations of the model is reduced compared to the BITs and close to or equal to NRIT. Similarly, it is observed that where BITs work slow and NRIT has numerical difficulties or fails, there, NHIT has fast and successful converge, respectively. NITs have a vital role in various physical eras to estimate an approximate solution, but the NHIT is a well-founded iterative procedure for every type of NLPs of one variable $\left(f\left(x\right)=0\right)$.

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Conflict of Interests

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