URAL MATHEMATICAL JOURNAL, Vol. 8, No. 2, 2022, pp. 94–114

DOI: 10.15826/umj.2022.2.008

ON LOCAL IRREGULARITY OF THE VERTEX COLORING OF THE CORONA PRODUCT OF A TREE GRAPH

Arika Indah Kristiana[1,†], M. Hidayat[1], Robiatul Adawiyah[1], D. Dafik[1], Susi Setiawani[1], Ridho Alfarisi[2]

[1]Department of Mathematics Education, University of Jember,
Jalan Kalimantan 37, 68126, Jember, Jawa Timur, Indonesia

[2]Department of Elementary School Education, University of Jember,
Jalan Kalimantan 37, 68126, Jember, Jawa Timur, Indonesia

[†]arika.fkip@unej.ac.id

Abstract: Let $G = (V, E)$ be a graph with a vertex set V and an edge set E. The graph G is said to be with a local irregular vertex coloring if there is a function f called a local irregularity vertex coloring with the properties: (i) $l : (V(G)) \to \{1, 2, ..., k\}$ as a vertex irregular k-labeling and $w : V(G) \to N$, for every $uv \in E(G)$, $w(u) \neq w(v)$ where $w(u) = \sum_{v\in N(u)} l(i)$ and (ii) $\mathrm{opt}(l) = \min\{\max\{l_i : l_i$ is a vertex irregular labeling$\}\}$. The chromatic number of the local irregularity vertex coloring of G denoted by $\chi_{lis}(G)$, is the minimum cardinality of the largest label over all such local irregularity vertex colorings. In this paper, we study a local irregular vertex coloring of $P_m \bigodot G$ when G is a family of tree graphs, centipede C_n, double star graph $(S_{2,n})$, Weed graph $(S_{3,n})$, and E graph $(E_{3,n})$.

Keywords: Local irregularity, Corona product, Tree graph family.

1. Introduction

Let $G(V, E)$ be a connected and simple graph with a vertex set V and an edge set E. In this paper, we combine two concepts, namely the local antimagic vertex coloring and the distance irregular labelling, with a local irregularity of vertex coloring. This concept firstly was introduced by Kristiana [2, 3], et. al. The latest research was conducted by Azzahra [4], who examined the local irregularity vertex coloring of a grid graph family. In this paper we study the local irregularity of vertex coloring of corona product graph of a tree graph family.

Definition 1. *Suppose* $l : V(G) \to \{1, 2, ..., k\}$ *and* $w : V(G) \to N$*, where*

$$w(u) = \sum_{v\in N(u)} l(v),$$

then $l(v)$ *is called the vertex irregular k-labeling and* $w(u)$ *is called the local irregularity of vertex coloring if*

(i) $\mathrm{opt}(l) = \min\{\max\{l_i\} : \ l_i$ *vertex irregular labeling*$\}$;

(ii) *for every* $uv \in E(G), w(u) \neq w(v)$.

Definition 2. *The chromatic number of local irregular graph* G *denoted by* $\chi_{lis}(G)$*, is the minimum of cardinality of the local irregularity of vertex coloring.*

In this paper, we will use the following lemma which gives a lower bound on the chromatic number of local irregular vertex coloring:

Lemma 1 [2]. *Let G be a simple and connected graph, then $\chi_{lis}(G) \geq \chi(G)$.*

Proposition 1 [2]. *Let G be a graph each two adjacent vertices of which have a different vertex degree then* $\mathrm{opt}(l) = 1$.

Proposition 2 [2]. *Let G be a graph each two adjacent vertices have the same vertex degree then* $\mathrm{opt}(l) \geq 2$.

Definition 3 [1]. *Let G and H be two connected graphs. Let o be a vertex of H. The corona product of the combination of two graphs G and H is defined as the graph obtained by taking a duplicate of graph G and $|V(G)|$ a duplicate of graph H, namely H_i; $i = 1, 2, 3, ...|V(G)|$ then connects each vertex i in G to each vertex in H_i. The corona product of the graphs G and H is denoted by $G \bigodot H$.*

2. Result and discussion

In this paper, we analyze the new result of the chromatic number of local irregular vertex coloring of corona product by family of tree graph ($P_m \bigodot G$) where G is centipede graph (C_n), double star graph ($S_{2,n}$), and Weed graph ($S_{3,n}$).

Theorem 1. *Let $G = P_m \bigodot Cp_n$, be a corona product of a path graph of order m and a centipede graph of order n for $n, m \geq 2$, then*

$$\chi_{lis}(P_m \bigodot Cp_n) = \begin{cases} 5, & \text{for} \quad m = 3 \quad \text{and} \quad n = 2, 3, \\ 6, & \text{for} \quad m = 2 \quad \text{and} \quad n = 2, 3 \quad \text{or} \quad \text{for} \quad m = 3 \quad \text{and} \quad n \geq 4, \\ 7, & \text{for} \quad m = 2 \quad \text{and} \quad n \geq 4 \quad \text{or} \quad \text{for} \quad m \geq 4 \quad \text{and} \quad n = 2, 3, \\ 8, & \text{for} \quad m \geq 4 \quad \text{and} \quad n \geq 4, \end{cases}$$

with $\mathrm{opt}(l)$ *defined as*

$$\mathrm{opt}(l)(P_m \bigodot Cp_n) = \begin{cases} 1, & \text{for} \quad m = 3 \quad \text{and} \quad n = 3, \\ 1, 2, & \text{for} \quad m = 2 \quad \text{and} \quad n = 2 \quad \text{or} \\ & \text{for} \quad m = 3 \quad \text{and} \quad n = 2 \quad \text{or} \\ & \text{for} \quad m \geq 3 \quad \text{and} \quad n \geq 4. \end{cases}$$

P r o o f. Vertex set is

$$V(P_m \bigodot CP_n) = \{x_i; 1 \leq i \leq m\} \cup \{x_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{y_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$$

and the edge set is

$$E(P_m \bigodot CP_n) = \{x_i x_{i+1}; 1 \leq i \leq m-1\} \cup \{x_{ij} x_{ij+1}; 1 \leq i \leq m, 1 \leq j \leq n-1\}$$
$$\cup \{x_{ij} y_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_i x_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_i y_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\},$$

the *order* and *size* respectively are $2mn + m$ and $4mn - 1$.

Case 1: $m \neq p,\ m \geq 2,\ p \geq 2,\ n \geq 3$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 4$, let $\chi_{lis}(P_m \bigodot Cp_n) = 4$, if $l(x_1) = l(x_3) = 1$, $l(x_2) = 2$, $l(x_{ij}) = l(y_{ij}) = 1$ then $w(x_1) = w(x_2)$, then there are 2 adjacent vertices that have the same color, it contradicts the definition of vertex coloring. If

$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \leq i \leq 3, \quad j = 1, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq 3, \quad j = 2,$$
$$l(x_i) = 1 \rightarrow w(x_i) \neq w(x_{i+1}), \quad w(x_{i1}) \neq (x_{i2}),$$

then $\chi_{lis}(P_m \bigodot Cp_n) \geq 5$. Based on this, we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 5$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$. Furthermore, the upper bound for the chromatic number of local irregular $(P_m \bigodot Cp_n)$, we define $l : V(P_m \bigodot Cp_n) \rightarrow \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(x_{ij}) = 1,$$
$$l(y_{ij}) = \begin{cases} 1, & \text{for} \quad 1 \leq i \leq 3 \text{ and } j = 1, \\ 2, & \text{for} \quad 1 \leq i \leq 3 \text{ and } j = 2. \end{cases}$$

Hence, $\mathrm{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 1, 3, \\ 8, & \text{for } i = 2, \end{cases}$$
$$w(x_{ij}) = \begin{cases} 3, & \text{for } 1 \leq i \leq 3 \text{ and } j = 1, \\ 4, & \text{for } 1 \leq i \leq 3 \text{ and } j = 2, \end{cases}$$
$$w(y_{ij}) = 2, \quad \text{for } 1 \leq i \leq 3 \text{ and } j = 1, 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \leq 5$, and we have $5 \leq \chi_{lis}(P_m \bigodot Cp_n) \leq 5$, so $\chi_{lis}(P_m \bigodot Cp_n) = 5$ for $m = 3$ and $n = 2$.

Case 2: $m = n = 3$.

Based on Proposition 1, $\mathrm{opt}(l) = 1$. So the lower bound of $(P_m \bigodot Cp_n)$ is

$$\chi_{lis}(P_m \bigodot Cp_n) \geq 5.$$

Hence $\mathrm{opt}(l) = 1$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 1, 3, \\ 8, & \text{for } i = 2, \end{cases}$$
$$w(y_{ij}) = \begin{cases} 3, & \text{for } 1 \leq i \leq 3 \text{ and } j \equiv 1, 3 \pmod 4, \\ 4, & \text{for } 1 \leq i \leq 3 \text{ and } j = 2, \end{cases}$$
$$w(x_{ij}) = 2, \quad \text{for } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 3.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \leq 5$. We have $5 \leq \chi_{lis}(P_m \bigodot Cp_n) \leq 5$, so $\chi_{lis}(P_m \bigodot Cp_n) = 5$ for $m = 3$ and $n = 3$.

Case 3: $m = n = 2$.

First step here is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 5$, if $l(x_1) = 1$, $l(x_2) = 2$, $l(x_{ij}) = l(y_{ij}) = 1$, then $w(x_{11}) = w(x_{12})$ and there are 2 adjacent vertices, that have the same color, it contradicts the definition of vertex coloring. If

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = 1, \quad l(y_{i1}) = 1, \quad i = 1, 2, \quad l(y_{i2}) = 2, \quad i = 1, 2,$$

then $w(x_1) \neq w(x_2)$, $w(x_{i1}) \neq w(x_{i2})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 6$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2, \end{cases} \quad l(x_{ij}) = 1, \quad l(y_{ij}) = \begin{cases} 1, & \text{for } i = 1, 2 \text{ and } j = 1, \\ 2, & \text{for } i = 1, 2 \text{ and } j = 2. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 2, \\ 7, & \text{for } i = 1, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i = 1 \text{ and } j = 1 \\ 4, & \text{for } i = 1 \text{ and } j = 1, \text{ or for } i = 2 \text{ and } j = 1, \\ 5, & \text{for } i = 2 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i = 1 \text{ and } j = 1, 2, \\ 3, & \text{for } i = 2 \text{ and } j = 1, 2. \end{cases}$$

We have the following upper bound $\chi_{lis}(P_m \bigodot Cp_n) \leq 6$. We have $6 \leq \chi_{lis}(P_m \bigodot Cp_n) \leq 6$, so $\chi_{lis}(P_m \bigodot Cp_n) = 6$ for $m = 2$ and $n = 2$.

Case 4: $m = 2$ and $n = 3$.

First step here is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 5$, if

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{1j}) = 1, \quad l(y_{2j}) = 1, \quad j = 1, 2, \quad l(y_{i3}) = 2,$$

then $w(x_{22}) = w(x_{23})$, so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = 1, \quad l(y_{ij}) = 1,$$

then $w(x_1) \neq w(x_2)$, $w(x_{i,1}) \neq w(x_{i,2})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 6$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2, \end{cases} \quad l(x_{ij}) = 1, \quad w(y_{ij}) = 1.$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 2, \\ 8, & \text{for } i = 1, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i = 1 \text{ and } j = 1, 3, \\ 4, & \text{for } i = 1 \text{ and } j = 2, \text{ or for } i = 2 \text{ and } j = 1, 3, \\ 5, & \text{for } i = 2 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i = 1 \text{ and } 1 \le j \le 3, \\ 3, & \text{for } i = 2 \text{ and } 1 \le j \le 3. \end{cases}$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \le 6$. So we have $\chi_{lis}(P_m \bigodot Cp_n) = 6$ for $m = 2$ and $n = 3$.

Case 5: $m = 3$ and $n \ge 4$.

First step to prove this theorem in this case is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \ge \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 5$, if $l(x_1) = l(x_3) = 1$, $l(x_2) = 2$, $l(x_{ij}) = l(y_{ij}) = 1$, then $w(x_1) = w(x_2)$ so there are 2 adjacent vertices with the have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \le i \le 3, \quad j = 1, n, \quad j \equiv 0 \pmod 2,$$
$$l(y_{ij}) = 2, \quad 1 \le i \le 3, \quad j \equiv 1, 3 \pmod 4, \quad j \ne 1, n,$$

with the $w(x_i) \ne w(x_{i+1})$, $w(x_{ij}) = w(x_{ij+1})$. Therefore we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \ge 6$.

After that, we will find the upper bound for $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \le i \le 3 \text{ and } j = 1, n \quad \text{or} \quad \text{for } 1 \le i \le 3 \text{ and } j \equiv 0 \pmod 2, \\ 2, & \text{for } 1 \le i \le 3 \text{ and } j \equiv 1, 3 \pmod 4, \ j \ne 1, n. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n + n/2, & \text{for } i = 1, 3 \text{ and } n \equiv 0 \pmod 2, \\ 2n + \lfloor n/2 \rfloor, & \text{for } i = 1, 3 \text{ and } n \equiv 1, 3 \pmod 4, \\ 3n + 1 - n/2, & \text{for } i = 2 \text{ and } n \equiv 0 \pmod 2, \\ 3n + 1 - \lceil n/2 \rceil, & \text{for } i = 2 \text{ and } n \equiv 1, 3 \pmod 4, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } 1 \le i \le 3 \text{ and } j = 1, n, \\ 4, & \text{for } 1 \le i \le 3 \text{ and } j \equiv 0 \pmod 2, \\ 5, & \text{for } 1 \le i \le 3 \text{ and } j \equiv 1, 3 \pmod 4, \quad j \ne 1, n, \end{cases}$$

$$w(y_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \le 6$. So $\chi_{lis}(P_m \bigodot Cp_n) = 6$ for $m = 3$ and $n \ge 4$.

Case 6: $m \equiv 0 \pmod 2$, $m \ge 4$ and $n = 2$.

First step here is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 5$, let $\chi_{lis}(P_m \bigodot Cp_n) = 5$, if

$$l(x_i) = 1, \quad i \equiv 1,3 \pmod 4, \quad i \equiv 2 \pmod 4, \quad l(x_i) = 2, \quad i \equiv 0 \pmod 4, \quad l(x_{ij}) = l(y_{ij}) = 1$$

then $w(x_{ij}) = w(x_{ij+1})$, then there are 2 adjacent vertices that have same color, this contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1,3 \pmod 4, \quad i \equiv 2 \pmod 4, \quad l(x_i) = 2, \quad i \equiv 0 \pmod 4,$$
$$l(x_{ij}) = 1 l(y_{ij}) = 1, \quad j = 2, \quad l(y_{ij}) = 2, \quad j = 1,$$

then $w(x_{ij}) \neq w(x_{ij+1}); w(x_{i+1}) \neq w(x_{i+2})$. So we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4 \text{ or for } i \equiv 2 \pmod 4, \\ 2, & \text{for } i \equiv 0 \pmod 4, \end{cases}$$
$$l(x_{ij}) = 1, \quad l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j = 2. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 1, m, \\ 7, & \text{for } i \equiv 0 \pmod 2, \quad i \neq m, \\ 8, & \text{for } i \equiv 1,3 \pmod 4, \quad i \neq 1, \end{cases}$$
$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 1 \quad \text{or} \quad \text{for } i \equiv 2 \pmod 4 \text{ and } j = 1, \\ 4, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 2 \quad \text{or} \quad \text{for } i \equiv 2 \pmod 4 \text{ and } j = 2 \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 4 \text{ and } j = 1, \\ 5, & \text{for } i \equiv 0 \pmod 4 \text{ and } j = 2, \end{cases}$$
$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 1,2 \quad \text{or} \quad \text{for } i \equiv 2 \pmod 4 \text{ and } j = 1,2, \\ 3, & \text{for } i = 0 \pmod 4 \text{ and } j = 1,2. \end{cases}$$

We have the upper bound $\chi_{lis}(P_m \bigodot Cp_n) \leq 7$. So $\chi_{lis}(P_m \bigodot Cp_n) = 7$ for $m \geq 4$ and $n = 2$.

Case 7: $m \equiv 0 \pmod 2$, $m \geq 4$ and $n = 3$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) = 5$, in this case if

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \leq i \leq m, \quad j = 3, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq m, \quad j = 1,2,$$

then $w(x_i) = w(x_{i+1})$, then there are 2 adjacent vertices that have the same color, this contradicts the definition of vertex coloring. If

$$l(x_i) = 1 \quad i \equiv 1,3 \pmod 4, \quad i \equiv 2 \pmod 4, \quad l(x_i) = 2, \quad i \equiv 0 \pmod 2, \quad l(y_{ij}) = l(x_{ij}) = 1,$$

then $w(x_{i+1}) \neq w(x_{i+2})$, $w(x_{i1}) \neq w(x_{i2})$, $w(x_{i1}) \neq w(y_{i2})$. Therefore we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4 \text{ or for } i \equiv 2 \pmod 4, \\ 2, & \text{for } i \equiv 0 \pmod 4, \end{cases}$$

$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1.$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 1, m, \\ 8, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \\ 9, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 1,3 \text{ or for } i \equiv 2 \pmod 4 \text{ and } j = 1,3, \\ 4, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 2 \text{ or for } i \equiv 2 \pmod 4 \text{ and } j = 2 \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 4 \text{ and } j = 1,3, \\ 5, & \text{for } i \equiv 0 \pmod 4 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } 1 \leq j \leq 3 \text{ or for } i \equiv 2 \pmod 4 \text{ and } 1 \leq j \leq 3, \\ 3, & \text{for } i = 0 \pmod 4 \text{ and } 1 \leq j \leq 3. \end{cases}$$

We have the upper bound $\chi_{lis}(P_m \bigodot Cp_n) \leq 7$. So $\chi_{lis}(P_m \bigodot Cp_n) = 7$ for $m \geq 4$ and $n = 3$.

Case 8: $m = 2$ and $n \geq 4$.

First step here is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) < 7$, let $\chi_{lis}(P_m \bigodot Cp_n) = 6$, if

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then $w(x_{ij+1}) = w(x_{ij+2})$, then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_1) = 1, \quad l(x_2) = 2, \quad l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad j \equiv 0 \pmod 2, \quad j = 1, n,$$
$$l(y_{ij}) = 2, \quad j \equiv 1,3 \pmod 4, \quad j \neq 1, \quad n \to w(x_1) \neq w(x_2), \quad w(x_{ij+1}) \neq w(x_{ij+2}),$$

then we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i = 1, \\ 2, & \text{for } i = 2, \end{cases}$$

$$l(x_{ij}) = 1,$$

$$l(y_{ij}) = \begin{cases} 1, & \text{for } i = 1,2 \text{ and } j = 1, n \text{ or for } i = 1,2 \text{ and } j \equiv 0 \pmod 2, \\ 2, & \text{for } i = 1,2 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n+1-n/2, & \text{for } i=1 \text{ and } n \equiv 0 \pmod 2, \\ 3n+1-\lceil n/2 \rceil, & \text{for } i=1 \text{ and } n \equiv 1,3 \pmod 4, \\ 2n+n/2, & \text{for } i=2 \text{ and } n \equiv 0 \pmod 2, \\ 2n+\lfloor n/2 \rfloor, & \text{for } i=2 \text{ and } n \equiv 1,3 \pmod 4, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i=1 \text{ and } j=1,n, \\ 4, & \text{for } i=1 \text{ and } j \equiv 0 \pmod 2,\ j \neq n \text{ or for } i=2 \text{ and } j=1,n, \\ 5, & \text{for } i=1 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1,n \text{ or for } i=2 \text{ and } j \equiv 0 \pmod 2,\ j \neq n, \\ 6, & \text{for } i=2 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1,n, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i=1 \text{ and } 1 \leq j \leq n, \\ 3, & \text{for } i=2 \text{ and } 1 \leq j \leq n. \end{cases}$$

The upper bound $\chi_{lis}(P_m \bigodot Cp_n) \leq 7$ is true. So $\chi_{lis}(P_m \bigodot Cp_n) = 7$ for $m = 2$ and $n \geq 4$.

Case 9: $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n = 2$.

First step to prove this theorem in this case is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) < 7$, and let $\chi_{lis}(P_m \bigodot Cp_n) = 6$, if

$$l(x_i) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2, \quad l(x_i) = 2, \quad i \equiv 3 \pmod 4, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then $w(x_{i1}) = w(x_{i2})$, $w(x_{i+1}) = w(x_{i+2})$, then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2, \quad l(x_i) = 2, \quad i \equiv 3 \pmod 4,$$
$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \quad 1 \leq i \leq m, \quad j = 1, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq m, \quad j = 2,$$

then $w(x_{i+1}) \neq w(x_{i+2})$, $w(x_{ij+1}) \neq w(x_{ij+2})$. Therefore we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod 4 \text{ or for } i \equiv 0 \pmod 2, \\ 2, & \text{for } i \equiv 3 \pmod 4, \end{cases}$$

$$w(x_{ij}) = 1; \quad l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j = 2. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 6, & \text{for } i = 1, m, \\ 7, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1, \\ 8, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 1 \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 1, \\ 4, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 2 \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 2 \text{ or} \\ & \quad \text{for } i \equiv 3 \pmod 4 \text{ and } j = 1, \\ 5, & \text{for } i \equiv 3 \pmod 4 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 1,2 \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 1,2, \\ 3, & \text{for } i = 3 \pmod 4 \text{ and } j = 1,2. \end{cases}$$

The upper bound $\chi_{lis}(P_m \bigodot Cp_n) \leq 7$ is true. So $\chi_{lis}(P_m \bigodot Cp_n) = 7$ for $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n = 2$.

Case 10: $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n = 3$.

First step here is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) < 7$, let $\chi_{lis}(P_m \bigodot Cp_n) = 6$, if

$$l(x_i) = l(x_{ij}) = 1, \quad l(y_{i1}) = 1, \quad l(y_{ij}) = 2, \quad j = 2,3,$$

then $w(x_{i+1}) = w(x_{i+2})$, then we have that there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2, \quad l(x_i) = 2, \quad i \equiv 3 \pmod 4, \quad l(x_{ij}) = 1; l(y_{ij}) = 1,$$

then $w(x_{i+1}) \neq w(x_{i+2})$, $w(x_{ij+1}) \neq w(x_{ij+2})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \rightarrow \{1,2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod 4 \text{ or for } i \equiv 0 \pmod 2, \\ 2, & \text{for } i \equiv 3 \pmod 4, \end{cases}$$

$$l(x_{ij}) = 1, \quad l(y_{ij}) = 1.$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 7, & \text{for } i = 1, m, \\ 8, & \text{for } i \equiv 1,3 \pmod 4, \ i \neq 1, m, \\ 9, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 1,3 \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 1,3, \\ 4, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 2 \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 2 \text{ or} \\ & \quad \text{for } i \equiv 3 \pmod 4 \text{ and } j = 1,3, \\ 5, & \text{for } i \equiv 3 \pmod 4 \text{ and } j = 2, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1 \pmod 4 \text{ and } 1 \leq j \leq 3 \text{ or for } i \equiv 0 \pmod 2 \text{ and } 1 \leq j \leq 3, \\ 3, & \text{for } i = 3 \pmod 4 \text{ and } 1 \leq j \leq 3. \end{cases}$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \leq 7$. So $\chi_{lis}(P_m \bigodot Cp_n) = 7$ for $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n = 3$.

Case 11: $m \equiv 0 \pmod 2$ $m \geq 4$ and $n \geq 4$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) < 8$, let $\chi_{lis}(P_m \bigodot Cp_n) = 7$, if

$$l(x_i) = 1, \quad i \equiv 1,3 \pmod 4, \quad i \equiv 2 \pmod 4, \quad l(x_i) = 2, \quad i \equiv 0 \pmod 4, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then $w(x_{ij+1}) = w(x_{ij+2})$, so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad i \equiv 1,3 \pmod 4, \quad i \equiv 2 \pmod 4, \quad l(x_i) = 2, \quad i \equiv 0 \pmod 4, \quad l(x_{ij}) = 1,$$
$$l(y_{ij}) = 1, \quad 1 \leq i \leq m, \quad j = 1, n, \quad j \equiv 0 \pmod 2, \quad l(y_{ij}) = 2,$$
$$1 \leq i \leq m, \quad j \equiv 1,3 \pmod 4, \quad j \neq 1, n,$$

then $w(x_{i+1}) \neq w(x_{i+2})$, $w(x_{ij+1}) \neq w(x_{ij+2})$, $w(x_{ij}) \neq w(y_{ij})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 8$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } 1 \leq j \leq n \text{ or for } i \equiv 2 \pmod 4 \text{ and } 1 \leq j \leq n, \\ 2, & \text{for } i \equiv 0 \pmod 4 \text{ and } 1 \leq j \leq n, \end{cases}$$

$$l(x_{ij}) = 1,$$

$$l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, n \text{ or for } 1 \leq i \leq m \text{ and } j \equiv 0 \pmod 2, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, m. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n + n/2, & \text{for } i = 1, m \text{ and } n \equiv 0 \pmod 2, \\ 2n + \lfloor n/2 \rfloor, & \text{for } i = 1, m \text{ and } n \equiv 1,3 \pmod 4, \\ 3n + 1 - n/2, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1 \text{ and } n \equiv 0 \pmod 2, \\ 3n + 1 - \lceil n/2 \rceil, & \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } n \equiv 1,3 \pmod 4, \\ 3n + 2 - n/2, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1 \text{ and } n \equiv 0 \pmod 2, \\ 3n + 1 - \lfloor n/2 \rfloor, & \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } n \equiv 1,3 \pmod 4, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 1, n \text{ or for } i \equiv 2 \pmod 4 \text{ and } j = 1, n, \\ 4, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j \equiv 0 \pmod 2,\ j \neq n \text{ or} \\ & \quad \text{for } i \equiv 2 \pmod 4 \text{ and } j \equiv 0 \pmod 2,\ j \neq n \text{ or for } i \equiv 0 \pmod 4 \text{ and } j = 1, n, \\ 5, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n \text{ or} \\ & \quad \text{for } i \equiv 2 \pmod 4 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 4 \text{ and } j \equiv 0 \pmod 2,\ j \neq n, \\ 6, & \text{for } i \equiv 0 \pmod 4 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } 1 \leq j \leq n \text{ or for } i \equiv 2 \pmod 4 \text{ and } 1 \leq j \leq n, \\ 3, & \text{for } i \equiv 0 \pmod 4 \text{ and } 1 \leq j \leq n. \end{cases}$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \leq 8$. So $\chi_{lis}(P_m \bigodot Cp_n) = 8$ for $m \equiv 0 \pmod 4$, $m \geq 4$ and $n \geq 4$.

Case 12: $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n \geq 4$.

First step to prove this theorem in this case is to find the lower bound of $V(P_m \bigodot CP_n)$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot Cp_n) \geq \chi(P_m \bigodot Cp_n) = 3$.

Assume $\chi_{lis}(P_m \bigodot Cp_n) < 8$, let $\chi_{lis}(P_m \bigodot Cp_n) = 7$, if

$$l(x_i) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2, \quad l(x_i) = 2, \quad i \equiv 3 \pmod 4, \quad l(x_{ij}) = l(y_{ij}) = 1,$$

then $w(x_{ij+1}) = w(x_{ij+2})$, so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$\begin{gathered} l(x_i) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2, \quad l(x_i) = 2, \quad i \equiv 3 \pmod 4, \quad l(x_{ij}) = 1, \quad l(y_{ij}) = 1, \\ 1 \leq i \leq m, \quad j = 1, n, \quad j \equiv 0 \pmod 2, \quad l(y_{ij}) = 2, \quad 1 \leq i \leq m, \quad j \equiv 1,3 \pmod 4, \\ j \neq 1, n \to w(x_{i+1}) \neq w(x_{i+2}), \quad w(x_{ij+1}) \neq w(x_{ij+2}), \quad w(x_{ij}) \neq w(y_{ij}), \end{gathered}$$

therefore we have the lower bound $\chi_{lis}(P_m \bigodot Cp_n) \geq 8$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot Cp_n)$.

Furthermore, we define $l : V(P_m \bigodot Cp_n) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = \begin{cases} 1, & \text{for } i \equiv 1 \pmod 4 \text{ or for } i \equiv 0 \pmod 2, \\ 2, & \text{for } i \equiv 3 \pmod 4, \end{cases}$$

$$l(x_{ij}) = 1,$$

$$l(y_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq m \text{ and } j = 1, n \text{ or for } 1 \leq i \leq m \text{ and } j \equiv 0 \pmod 2, \\ 2, & \text{for } 1 \leq i \leq m \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n + n/2, & \text{for } i = 1, m \text{ and } n \equiv 0 \pmod 2, \\ 3n + 2 - n/2, & \text{for } i = 0 \pmod 2 \text{ and } n \equiv 0 \pmod 2, \\ 3n + 1 - n/2, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } n \equiv 0 \pmod 2, \\ 2n + \lfloor n/2 \rfloor, & \text{for } i = 1, m \text{ and } n \equiv 1,3 \pmod 4, \\ 3n - \lfloor n/2 \rfloor, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1 \text{ and } n \equiv 1,3 \pmod 4, \\ 3n + 1 - \lfloor n/2 \rfloor, & \text{for } i = 0 \pmod 4 \text{ and } n \equiv 1,3 \pmod 4, \end{cases}$$

$$w(x_{ij}) = \begin{cases} 3, & \text{for } i \equiv 1 \pmod 4 \text{ and } j = 1, n \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 1, n, \\ 4, & \text{for } i \equiv 1 \pmod 4 \text{ and } j \equiv 0 \pmod 2,\ j \neq n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2 \text{ and } j \equiv 0 \pmod 2,\ j \neq n \text{ or} \\ & \quad \text{for } i \equiv 3 \pmod 4 \text{ and } j = 1, n, \\ 5, & \text{for } i \equiv 1 \pmod 4 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n \text{ or} \\ & \quad \text{for } i \equiv 3 \pmod 4 \text{ and } j \equiv 0 \pmod 2,\ j \neq n, \\ 6, & \text{for } i \equiv 3 \pmod 4 \text{ and } j \equiv 1,3 \pmod 4,\ j \neq 1, n, \end{cases}$$

$$w(y_{ij}) = \begin{cases} 2, & \text{for } i \equiv 1 \pmod 4 \text{ and } 1 \leq j \leq n \text{ or for } i \equiv 0 \pmod 2 \text{ and } 1 \leq j \leq n, \\ 3, & \text{for } i \equiv 3 \pmod 4 \text{ and } 1 \leq j \leq n. \end{cases}$$

The upper bound is true: $\chi_{lis}(P_m \bigodot Cp_n) \leq 8$. So $\chi_{lis}(P_m \bigodot Cp_n) = 8$ for $m \geq 5$ and $n \geq 4$. □

Theorem 2. *Let $G = P_m \bigodot S_{2,n}$ for $n, m \geq 2$, then the chromatic number of local irregular G is*

$$\chi_{lis}(P_m \bigodot S_{2,n}) = \begin{cases} 5, & \text{for } m = 3 \text{ and } n \geq 2, \\ 6, & \text{for } m = 2 \text{ and } n \geq 2, \\ 7, & \text{for } m \geq 4 \text{ and } n \geq 2, \end{cases}$$

with $\text{opt}(l)(P_m \bigodot S_{2,n}) = 1, 2,$ *for* $m \geq 2$ *and* $n \geq 2$.

P r o o f. Vertex set is

$$V(P_n \bigodot S_{2,n}) = \{x_i; 1 \leq i \leq m\} \cup \{a_i; 1 \leq i \leq m\} \cup \{b_i; 1 \leq i \leq m\}$$
$$\cup \{a_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{b_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$$

and the edge set is

$$E(P_n \bigodot S_{2,n}) = \{x_i x_{i+1},\ 1 \leq i \leq m-1\} \cup \{a_i b_i; 1 \leq i \leq m\} \cup \{x_i a_i; 1 \leq i \leq m\}$$
$$\cup \{x_i b_i; 1 \leq i \leq m\} \cup \{x_i a_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{x_i b_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}$$
$$\cup \{a_i a_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{b_i b_{ij}; 1 \leq i \leq m, 1 \leq j \leq n\}.$$

The order and the size respectively are $2mn + 3m$ and $4mn + 4m - 1$. This proof is divided into 4 cases as follows.

Case 1: $m = 3$ and $n \geq 2$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot S_{2,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{2,n}) \geq \chi(P_m \bigodot S_{2,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{2,n}) = 4$, if $l(a_i) = l(b_i) = 1$, $l(x_i) = l(a_{ij}) = l(b_{ij}) = 1$ then $w(a_i) = w(b_i)$, then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = l(b_{ij}) = 1, \quad 1 \leq j \leq n-1, \quad l(b_{in}) = 2,$$

then

$$w(a_i) \neq w(b_i), \quad w(x_1) = w(x_3) \neq w(x_2),$$

therefore we have the lower bound $\chi_{lis}(P_m \bigodot S_{2,n}) \geq 5$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{2,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{2,n}) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1,$$
$$l(b_{ij}) = \begin{cases} 1, & \text{for } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq n-1, \\ 2, & \text{for } 1 \leq i \leq 3 \text{ and } j = n. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n+4, & \text{for } i = 1, 3, \\ 2n+5, & \text{for } i = 2, \end{cases}$$
$$w(a_i) = n + 2, \quad \text{for } 1 \leq i \leq 3,$$
$$w(b_i) = n + 3, \quad \text{for } 1 \leq i \leq 3,$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound $\chi_{lis}(P_m \bigodot S_{2,n}) \leq 5$ is true. So $\chi_{lis}(P_m \bigodot S_{2,n}) = 5$ for $m = 3$ and $n \geq 2$.

Case 2: $m = 2$ and $n \geq 2$.

First step here is to find the lower bound of $V(P_m \bigodot S_{2,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{2,n}) \geq \chi(P_m \bigodot S_{2,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{2,n}) = 5$, if

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = l(b_{2j}) = 1, \quad l(b_{1j}) = 1, \quad 1 \leq j \leq n-1, \quad l(b_{1n}) = 2,$$

and then $w(a_2) = w(b_2)$, and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1,$$
$$l(b_{1,j}) = 1, \quad l(b_{1,n}) = 2, \quad l(b_{2,j}) = 2, \quad j = 1, n, \quad l(b_{2j}) = 1, \quad 2 \leq j \leq n-1,$$

then $w(a_i) \neq w(b_i)$, $w(x_1) \neq w(x_2)$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot S_{2,n}) \geq 6$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{2,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{2,n}) \to \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1,$$
$$l(b_{ij}) = \begin{cases} 1, & \text{for } i = 1 \text{ and } 1 \leq j \leq n-1 \text{ or for } i = 2 \text{ and } 2 \leq j \leq n-1, \\ 2, & \text{for } i = 1 \text{ and } j = n \text{ or for } i = 2 \text{ and } j = 1, n. \end{cases}$$

Hence, $\mathrm{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n+4, & \text{for } i = 1, \\ 2n+5, & \text{for } i = 2, \end{cases}$$
$$w(a_i) = n+2, \quad \text{for } i = 1, 2,$$
$$w(b_i) = \begin{cases} n+3, & \text{for } i = 1, \\ n+4, & \text{for } i = 2, \end{cases}$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{2,n}) \leq 6$. So $\chi_{lis}(P_m \bigodot S_{2,n}) = 6$ for $m = 2$ and $n \geq 2$.

Case 3: $m \equiv 0 \pmod 4$, $m \geq 4$ and $n \geq 2$.

First step to prove this theorem in this case is to find the lower bound of $V(P_m \bigodot S_{2,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{2,n}) \geq \chi(P_m \bigodot S_{2,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{2,n}) = 6$, if

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1, 3 \pmod 4, \quad i \equiv 2 \pmod 4,$$
$$l(b_{ij}) = 1, \quad i \equiv 0 \pmod 4, \quad j \neq 1, n, \quad l(b_{ij}) = 2, \quad i \equiv 0 \pmod 4, \quad j = 1, n,$$

then $w(a_i) = w(b_i)$, so there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1, 3 \pmod 4, \quad i = m, \quad 1 \leq j \leq n-1,$$
$$i \equiv 0 \pmod 2, \quad i \neq m, \quad 2 \leq j \leq n-1, \quad l(b_{ij}) = 2, \quad i \equiv 1, 3 \pmod 4,$$
$$i = m, \quad j = n, \quad i \equiv 0 \pmod 2, \quad i \neq m, \quad j = 1, n,$$

then $w(a_i) \neq w(b_i)$, $w(x_i) \neq w(x_{i+1})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot S_{2,n}) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{2,n})$

Furthermore, we define $l : V(P_m \bigodot S_{2,n}) \rightarrow \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1,$$

$$l(b_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod 4 \text{ and } 1 \leq j \leq n-1 \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } 2 \leq j \leq n-1 \text{ or} \\ & \quad \text{for } i = m, \text{ and } 1 \leq j \leq n-1, \\ 2, & \text{for } i \equiv 1, 3 \pmod 4 \text{ and } j = n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = 1, n \text{ or} \\ & \quad \text{for } i = m, \text{ and } j = n. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n+4, & \text{for } i = 1, m, \\ 2n+5, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \\ 2n+6, & \text{for } i \equiv 1, 3 \pmod 4,\ i \neq 1, \end{cases}$$

$$w(a_i) = n+2, \quad \text{for } 1 \leq i \leq m,$$

$$w(b_i) = \begin{cases} n+3, & \text{for } i \equiv 1, 3 \pmod 4,\ i = m, \\ n+4, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{2,n}) \leq 7$. So $\chi_{lis}(P_m \bigodot S_{2,n}) = 7$ for $m \equiv 0 \pmod 2$, $m \geq 4$ and $n \geq 2$.

Case 4: $m \equiv 1, 3 \pmod 4$, $m \geq 5$ and $n \geq 2$.

First step here is to find the lower bound of $V(P_m \bigodot S_{2,n})$. Based on Lemma 1, we have

$$\chi_{lis}(P_m \bigodot S_{2,n}) \geq \chi(P_m \bigodot S_{2,n}) = 3.$$

Assume $\chi_{lis}(P_m \bigodot S_{2,n}) = 6$, if

$$l(x_i) = l(a_i) = l(b_i) = l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1 \pmod 4, \quad i \equiv 0 \pmod 2,$$
$$l(b_{ij}) = 1, \quad i \equiv 3 \pmod 4, \quad j \neq n, \quad l(b_{ij}) = 2, \quad i \equiv 3 \pmod 4, \quad j = n,$$

then $w(a_i) = w(b_i)$, and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad i \equiv 1, 3 \pmod 4, \quad 1 \leq j \leq n-1,$$
$$i \equiv 0 \pmod 2, \quad 2 \leq j \leq n-1, \quad l(b_{ij}) = 2, \quad i \equiv 1, 3 \pmod 4, \quad j = n, \quad i \equiv 0 \pmod 2, \quad j = 1, n,$$

then $w(a_i) \neq w(b_i), w(x_i) \neq w(x_{i+1})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot S_{2,n}) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{2,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{2,n}) \rightarrow \{1, 2\}$ with the vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(a_{ij}) = 1,$$

$$l(b_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod 4 \text{ and } 1 \leq j \leq n-1 \text{ or for } i \equiv 0 \pmod 2, \text{ and } 2 \leq j \leq n-1, \\ 2, & \text{for } i \equiv 1, 3 \pmod 4 \text{ and } j = n \text{ or for } i \equiv 0 \pmod 2 \text{ and } j = 1, n. \end{cases}$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 2n+4, & \text{for } i = 1, m, \\ 2n+5, & \text{for } i \equiv 1,3 \pmod 4, \\ 2n+6, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$
$$w(a_i) = n+2, \quad \text{for } 1 \le i \le m,$$
$$w(b_i) = \begin{cases} n+3, & \text{for } i \equiv 1,3 \pmod 4, \\ n+4, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{2,n}) \le 7$. So $\chi_{lis}(P_m \bigodot S_{2,n}) = 7$ for $m \equiv 1,3 \pmod 4$, $m \ge 5$ and $n \ge 2$. □

Theorem 3. *Let $G = P_m \bigodot S_{3,n}$ for $n, m \ge 2$, then the chromatic number of local irregular G is*

$$\chi_{lis}(P_m \bigodot S_{3,n}) = \begin{cases} 5, & \text{for } m = 3 \text{ and } n \ge 2, \\ 6, & \text{for } m = 2 \text{ and } n \ge 2, \\ 7, & \text{for } m \ge 4 \text{ and } n \ge 3, \end{cases}$$

with

$$\text{opt}(l)(P_m \bigodot S_{3,n}) = \begin{cases} 1, & \text{for } m = 3 \text{ and } n = 3, \\ 1,2, & \text{for } m = 2 \text{ and } n = 2 \text{ or for } m = 3 \text{ and } n = 2 \text{ or} \\ & \quad \text{for } m \ge 4 \text{ and } n \ge 2. \end{cases}$$

P r o o f. The vertex set is

$$V(P_m \bigodot S_{3,n}) = \{x_i; 1 \le i \le m\} \cup \{a_i; 1 \le i \le m\} \cup \{b_i; 1 \le i \le m\} \cup \{c_i; 1 \le i \le m\}$$
$$\cup \{a_{ij}; 1 \le i \le m, 1 \le j \le n\} \cup \{b_{ij}; 1 \le i \le m, 1 \le j \le n\} \cup \{c_{ij}; 1 \le i \le m, 1 \le j \le n\}$$

and the edge set is

$$V(P_m \bigodot S_{3,n}) = \{x_i x_{i+1}; 1 \le i \le m-1\} \cup \{x_i y_i; 1 \le i \le m\} \cup \{x_i a_i; 1 \le i \le m\}$$
$$\cup \{x_i b_i; 1 \le i \le m\} \cup \{x_i c_i; 1 \le i \le m\} \cup \{y_i a_i; 1 \le i \le m\} \cup \{y_i b_i; 1 \le i \le m\}$$
$$\cup \{y_i c_i; 1 \le i \le m\} \cup \{x_i a_{ij}; 1 \le i \le m; 1 \le j \le n\} \cup \{x_i b_{ij}; 1 \le i \le m; 1 \le j \le n\}$$
$$\cup \{x_i c_{ij}; 1 \le i \le m; 1 \le j \le n\} \cup \{a_i a_{ij}; 1 \le i \le m; 1 \le j \le n\}$$
$$\cup \{b_i b_{ij}; 1 \le i \le m; 1 \le j \le n\} \cup \{c_i c_{ij}; 1 \le i \le m; 1 \le j \le n\}.$$

The order and size respectively are $3mn + 5m$ and $6mn + 8n - 1$. This proof can be divided into 8 following cases.

Case 1: $m = 3$ and $n = 2$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{3,n}) \ge \chi(P_m \bigodot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{3,n}) = 4$, if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1,$$

then $w(a_i) = w(b_i) = w(c_i) = w(y_i)$, and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(a_i) = l(b_i) = l(c_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1, \quad l(y_i) = 2,$$

then $(w(a_i) = w(b_i) = w(c_i)) \neq w(y_i)$, $w(x_1) \neq w(x_2)$. Therefore we have the lower bound $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 5$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \rightarrow \{1, 2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1, \quad l(c_{ij}) = 1.$$

Hence, opt$(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & \text{for } i = 1, 3, \\ 13, & \text{for } i = 2, \end{cases}$$

$$w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5, \quad w(c_i) = 5, \quad w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{3,n}) \leq 5$. So $\chi_{lis}(P_m \bigodot S_{3,n}) = 5$ for $m = 3$ and $n = 2$.

Case 2: $m = 3$ and $n = 3$.

Based on Proposition 1, we have opt$(l) = 1$. So the lower bound $(P_m \bigodot S_{3,n})$ is $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 5$

Since opt$(l) = 1$, the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n + 5, & \text{for } i = 1, 3, \\ 3n + 6, & \text{for } i = 2, \end{cases}$$

$$w(y_i) = 4,$$
$$w(a_i) = n + 1 \quad \text{for } 1 \leq i \leq 3,$$
$$w(b_i) = n + 1 \quad \text{for } 1 \leq i \leq 3,$$
$$w(c_i) = n + 1 \quad \text{for } 1 \leq i \leq 3,$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{3,n}) \leq 5$. So $\chi_{lis}(P_m \bigodot S_{3,n}) = 5$ for $m = 3$ and $n \geq 2$.

Case 3: $m = 2$ and $n = 2$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{3,n}) \geq \chi(P_m \bigodot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{3,n}) = 5$, if

$$l(a_i) = l(b_i) = l(c_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1, \quad l(y_1) = 1, \quad l(y_2) = 2,$$

then $w(a_2) = w(b_2) = w(c_2) = w(y_2)$ and there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(c_i) = l(b_{ij}) = 1, \quad l(y_i) = 2, \quad l(c_{1j}) = 1, \quad l(c_{2,1}) = 1, \quad l(c_{2,2}) = 2,$$

then $w(x_1) \neq w(x_2)$, $w(y_i) \neq ((w(a_i) = w(b_i) = w(c_i))$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 6$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \to \{1,2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i = 1 \text{ and } j = 1,2 \text{ or for } i = 2 \text{ and } j = 1, \\ 2, & \text{for } i = 2 \text{ and } j = 2. \end{cases}$$

Hence, $\mathrm{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & \text{for } i = 1, \\ 13, & \text{for } i = 2, \end{cases}$$

$$w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5,$$

$$w(c_i) = \begin{cases} 5, & \text{for } i = 1, \\ 6, & \text{for } i = 2, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{2,n}) \leq 6$. So $\chi_{lis}(P_m \bigodot S_{2,n}) = 6$ for $m = 2$ and $n = 2$.

Case 4: $m = 2$ and $n \geq 3$.

First step here is to find the lower bound of $V(P_m \bigodot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{3,n}) \geq \chi(P_m \bigodot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{3,n}) = 5$, if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i = 1,2, \quad 1 \leq j \leq n-1$$
$$l(c_{ij}) = 2, \quad i = 1,2, \quad j = n,$$

then $w(x_1) = w(x_2)$, then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i = 1, \quad 1 \leq j \leq n, \quad i = 2, \quad 1 \leq j \leq n-1,$$
$$l(c_{ij}) = 2, \quad i = 2, \quad j = n,$$

then $w(x_1) \neq w(x_2), w(y_i) \neq ((w(a_i) = w(b_i) = w(c_i))$. Therefore we have the lower bound $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 6$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \to \{1,2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i = 1 \text{ and } 1 \leq j \leq n \text{ or for } i = 2 \text{ and } 1 \leq j \leq n-1, \\ 2, & \text{for } i = 2 \text{ and } j = n. \end{cases}$$

Hence, $\mathrm{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n+5, & \text{for } i = 1, \\ 3n+6, & \text{for } i = 2, \end{cases}$$

$$w(y_i) = 4,$$

$$w(a_i) = n+1, \quad \text{for } i = 1,2,$$

$$w(b_i) = n+1, \quad \text{for } i = 1,2,$$

$$w(c_i) = \begin{cases} n+1, & \text{for } i = 1, \\ n+2, & \text{for } i = 2, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \odot S_{3,n}) \leq 6$. So $\chi_{lis}(P_m \odot S_{3,n}) = 6$ for $m = 2$ and $n \geq 3$.

Case 5: $m \equiv 0 \pmod 2$ $m \geq 4$ and $n = 2$.

First step to prove this theorem in this case is to find the lower bound of $V(P_m \odot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \odot S_{3,n}) \geq \chi(P_m \odot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \odot S_{3,n}) = 6$, if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1,3 \pmod 4,$$
$$j = 1,2, \quad i \equiv 0 \pmod 4, \quad j = 1,2, \quad l(c_{ij}) = 1, \quad i \equiv 2 \pmod 4,$$
$$j = 1, \quad l(c_{ij}) = 2, \quad i \equiv 2 \pmod 4, \quad j = 2,$$

then $w(y_i) = w(a_i)$. Then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad l(a_i) = l(b_i) = l(c_i) = 1, \quad l(y_i) = 2, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1,$$
$$i \equiv 0 \pmod 2, \quad j = 1, \quad i \neq m, \quad i \equiv 1,3 \pmod 4, \quad j = 1,2, \quad l(c_{ij}) = 2,$$
$$i \equiv 0 \pmod 2, \quad i \neq m, \quad j = 2,$$

then $w(x_{i+1}) \neq w(x_{i+2}; w(y_i) \neq w(a_i)$. Therefore we have the lower bound $\chi_{lis}(P_m \odot S_{3,n}) \geq 7$. After that, we will find the upper bound of $\chi_{lis}(P_m \odot S_{3,n})$.

Furthermore, we define $l : V(P_m \odot S_{3,n}) \to \{1,2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$
$$l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } j = 1,2 \text{ or for } i = m \text{ and } j = 1,2 \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = 1, \\ 2, & \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = 2. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & \text{for } i = 1, m, \\ 13, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1, \\ 14, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$
$$w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5,$$
$$w(c_i) = \begin{cases} 5, & \text{for } i \equiv 1,3 \pmod 4, \\ 6, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2,$$

The upper bound is true: $\chi_{lis}(P_m \odot S_{3,n}) \leq 7$. So $\chi_{lis}(P_m \odot S_{3,n}) = 7$ for $m \equiv 0 \pmod 2$; $m \geq 4$ and $n = 2$.

Case 6: $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n = 2$.

First step here is to find the lower bound of $V(P_m \odot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \odot S_{3,n}) \geq \chi(P_m \odot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \odot S_{3,n}) = 6$, if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1,$$
$$i \equiv 1 \pmod 4, \quad j = 1,2, \quad i \equiv 0 \pmod 2, \quad j = 1,2, \quad l(c_{ij}) = 1,$$
$$i \equiv 3 \pmod 4, \quad j = 1, \quad l(c_{ij}) = 2, \quad i \equiv 3 \pmod 4, \quad j = 2,$$

then $w(y_i) = w(a_i)$, then there are 2 adjacent vertices that have same color, it contradicts to definition of vertex coloring. If

$$l(x_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1, 3 \pmod 4,$$
$$j = 1, 2, \quad i \equiv 0 \pmod 2, \quad j = 1, \quad l(c_{ij}) = 2, \quad i \equiv 0 \pmod 2, \quad j = 2, \quad l(y_i) = 2,$$

then

$$w(x_{i+1}) \neq w(x_{i+2}, \quad w(y_i) \neq w(a_i), \quad w(y_i) \neq w(b_i), \quad w(y_i) \neq w(c_i).$$

We have the lower bound $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \to \{1, 2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 2, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$
$$l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1, 3 \pmod 4 \text{ and } j = 1, 2 \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = 1, \\ 2, & \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = 2. \end{cases}$$

Hence, $\text{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 12, & \text{for } i = 1, m \\ 13, & \text{for } i \equiv 1, 3 \pmod 4,\ i \neq 1, m, \\ 14, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$
$$w(y_i) = 4, \quad w(a_i) = 5, \quad w(b_i) = 5,$$
$$w(c_i) = \begin{cases} 5, & \text{for } i \equiv 1, 3 \pmod 4,\ i = 1, m, \\ 6, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{3,n}) \leq 7$. So $\chi_{lis}(P_m \bigodot S_{3,n}) = 7$ for $m \equiv 1, 3 \pmod 4$; $m \geq 5$ and $n = 2$.

Case 7: $m \equiv 0 \pmod 2$ $m \geq 4$ and $n \geq 3$.

First step to prove this theorem is to find the lower bound of $V(P_m \bigodot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{3,n}) \geq \chi(P_m \bigodot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{3,n}) = 6$, it is true if

$$l(a_i) = l(b_i) = l(c_i) = l(y_i) = l(a_{ij}) = l(b_{ij}) = 1, \quad l(c_{ij}) = 1, \quad 1 \leq i \leq m,$$
$$1 \leq j \leq n - 1, \quad l(c_{ij}) = 2, \quad 1 \leq i \leq m, \quad j = n,$$

then $w(x_{i+1}) = w(x_{i+2})$, then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = l(y_i) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1, 3 \pmod 4, \quad 1 \leq j \leq n,$$
$$i \equiv 0 \pmod 2, \quad i \neq m, \quad 1 \leq j \leq n - 1, \quad i = m, \quad 1 \leq j \leq n, \quad l(c_{ij}) = 2, \quad i \equiv 0 \pmod 2,$$
$$i \neq m, \quad i \neq m, \quad j = n, \quad w(x_{i+1}) \neq w(x_{i+2}),$$

we have the lower bound of $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 7$. After that, we will find the upper bound $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \to \{1,2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4 \text{ and } 1 \leq j \leq n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2,\ i \neq m, \text{ and } 1 \leq j \leq n-1 \text{ or} \\ & \quad \text{for } i = m, \text{ and } 1 \leq j \leq n, \\ 2, & \text{for } i \equiv 0 \pmod 2,\ i \neq m \text{ and } j = n. \end{cases}$$

Hence, $\mathrm{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n+5, & \text{for } i = 1, m, \\ 3n+6, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1, \\ 3n+7, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$

$$w(y_i) = 4,$$

$$w(a_i) = n+2, \quad \text{for } 1 \leq i \leq m,$$

$$w(b_i) = n+2, \quad \text{for } 1 \leq i \leq m,$$

$$w(c_i) = \begin{cases} n+2, & \text{for } i = m, \text{ or for } i \equiv 1,3 \pmod 4, \\ n+3, & \text{for } i \equiv 0 \pmod 2,\ i \neq m, \end{cases}$$

$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2,$$

The upper bound $\chi_{lis}(P_m \bigodot S_{3,n}) \leq 7$. So $\chi_{lis}(P_m \bigodot S_{3,n}) = 7$ for $m \equiv 0 \pmod 2$; $m \geq 4$ and $n \geq 3$.

Case 8: $m \equiv 1,3 \pmod 4$, $m \geq 5$ and $n \geq 3$.

First step to prove the theorem in this case is to find the lower bound of $V(P_m \bigodot S_{3,n})$. Based on Lemma 1, we have $\chi_{lis}(P_m \bigodot S_{3,n}) \geq \chi(P_m \bigodot S_{3,n}) = 3$.

Assume $\chi_{lis}(P_m \bigodot S_{3,n}) = 6$, if

$$l(x_i) = l(y_i) = l(a_{ij}) = l(b_{ij}) = l(c_{ij}) = 1, \quad l(a_i) = l(b_i) = 1, \quad l(c_i) = 2,$$

then $w(x_{i+1}) = w(x_{i+2})$. Then there are 2 adjacent vertices that have same color, it contradicts the definition of vertex coloring. If

$$l(x_i) = l(a_i) = l(b_i) = l(c_i) = 1, \quad l(a_{ij}) = l(b_{ij}) = l(y_i) = 1, \quad l(c_{ij}) = 1, \quad i \equiv 1,3 \pmod 4,$$
$$1 \leq j \leq n, \quad i \equiv 0 \pmod 2, \quad 1 \leq j \leq n-1, \quad l(c_{ij}) = 2, \quad i \equiv 0 \pmod 2, \quad j = n,$$

then $w(x_{i+1}) \neq w(x_{i+2})$. Based on that we have the lower bound $\chi_{lis}(P_m \bigodot S_{3,n}) \geq 7$.

After that, we will find the upper bound of $\chi_{lis}(P_m \bigodot S_{3,n})$.

Furthermore, we define $l : V(P_m \bigodot S_{3,n}) \to \{1,2\}$ with vertex irregular 2-labelling as follows:

$$l(x_i) = 1, \quad l(y_i) = 1, \quad l(a_i) = 1, \quad l(b_i) = 1, \quad l(c_i) = 1, \quad l(a_{ij}) = 1, \quad l(b_{ij}) = 1,$$

$$l(c_{ij}) = \begin{cases} 1, & \text{for } i \equiv 1,3 \pmod 4, \text{ and } 1 \leq j \leq n \text{ or} \\ & \quad \text{for } i \equiv 0 \pmod 2, \text{ and } 1 \leq j \leq n-1, \\ 2, & \text{for } i \equiv 0 \pmod 2, \text{ and } j = n. \end{cases}$$

Hence, $\mathrm{opt}(l) = 2$ and the labelling provides the vertex-weight as follows:

$$w(x_i) = \begin{cases} 3n+5, & \text{for } i = 1, m, \\ 3n+6, & \text{for } i \equiv 1,3 \pmod 4,\ i \neq 1, m, \\ 3n+7, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$

$$w(y_i) = 4,$$
$$w(a_i) = n + 2, \text{ for } 1 \le i \le m,$$
$$w(b_i) = n + 2, \text{ for } 1 \le i \le m,$$
$$w(c_i) = \begin{cases} n+2, & \text{for } i \equiv 1,3 \pmod 4, \\ n+3, & \text{for } i \equiv 0 \pmod 2, \end{cases}$$
$$w(a_{ij}) = 2, \quad w(b_{ij}) = 2, \quad w(c_{ij}) = 2.$$

The upper bound is true: $\chi_{lis}(P_m \bigodot S_{3,n}) \le 7$. So $\chi_{lis}(P_m \bigodot S_{3,n}) = 7$ for $m \equiv 1,3 \pmod 4$, $m \ge 5$ and $n \ge 3$. □

3. Conclusion

In this paper, we have studied the coloring of the vertices of the local irregular corona product by the graph of the family tree. We determined the exact value of the local irregular chromatic number of the corona product from the graph of the family tree, namely $\chi_{lis}(P_m \bigodot Cp_n)$, $\chi_{lis}(P_m \bigodot S_{2,n})$ and $\chi_{lis}(P_m \bigodot S_{3,n})$.

Acknowledgements

We gratefully acknowledge the support from University of Jember of year 2023.

REFERENCES

1. Frucht R., Harary F. On the corona of two graphs. *Aequationes Math.*, 1970. Vol. 4. P. 322–325. DOI: 10.1007/BF01844162
2. Kristiana A. I., Dafik, Utoyo M. I., Slamin, Alfarisi R., Agustin I. H., Venkatachalam M. Local irregularity vertex coloring of graphs. *Int. J. Civil Eng. Technol.*, 2019. Vol. 10, No. 3. P. 1606–1616.
3. Kristiana A. I., Utoyo M. I., Dafik, Agustin I. H., Alfarisi R., Waluyo E. On the chromatic number local irregularity of related wheel graph. *J. Phys.: Conf. Ser.*, 2019. Vol. 1211. Art. no. 0120003. P. 1–10. DOI: 10.1088/1742-6596/1211/1/012003
4. Kristiana A. I., Alfarisi R., Dafik, Azahra N. Local irregular vertex coloring of some families of graph. *J. Discrete Math. Sci. Cryptogr.*, 2020. P. 15–30. DOI: 10.1080/09720529.2020.1754541