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Generalized Second-Order G-Wolfe Type Fractional Symmetric Program and their Duality Relations under Generalized Assumptions

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Abstract
In this article, we formulate the concept of generalize bonvexity/pseudobonvexity functions. We formulate duality results for second-order fractional symmetric dual programs of G-Wolfe-type model. In the next section, we explain the duality theorems under generalize bonvexity/pseudobonvexity assumptions. We identify a function lying exclusively in the class of generalize pseudobonvex and not in class of generalize bonvex functions. Our results are more generalized several known results in the literature.

Keywords- Second-order, Symmetric duality, Fractional programming problem, Generalize bonvexity, G-Wolfe model.

1. Introduction

In the recent years, there has been a great deal of research on the fractional optimization problem with multiple objective functions, leading to a number of optimality and duality results for these problems because there are more factors involved. Second order duality has the practical benefit of offering tighter boundaries for the value of the objective function of the primary problem. Numerical programming's dualities have meritorious applications in a variety of computational and speculative advancements, like in financial problems, control hypotheses, business issues, and other diverse sectors. In-depth research on the fractional optimization problem with multiple objective functions has been going on for a while now (Dubey et al., 2020) and duality conclusions have been produced for these situations.

Numerous writers (Mond, 1978; Zhang and Mond, 1998) have examined fractional programming problems containing square roots of positive semidefinite quadratic forms. The commonality of this kind of issue stems from the fact that the dual can be defined simply even though the target and limitation capacities cannot be differentiated. Using psuedo-invex functions and cone constraints, Kim et al. (1998) investigated pair of multiobjective symmetric dual programs. Devi (1998) developed pair of second-order symmetric dual problems and involving second-order invex discovered dualities. The second order symmetric dual nonlinear programming problem was described by Mishra (2000) as follows:

Primal (RP)

Minimize $K(x_{11}, x_{12}) - [x_{12}{}^T \nabla_{x_{12}} \mu^T K(x_{11}, x_{12})]e - \frac{1}{2}\xi_0{}^T \nabla_{x_{12}x_{12}} \mu^T K(x_{11}, x_{12})\xi_0$

Subject to $(x_{11}, x_{12}) \in C_1 \times C_2$,
$\nabla_{x_{12}} \mu^T K(x_{11}, x_{12}), \nabla_{x_{12}x_{12}} \mu^T K(x_{11}, x_{12})\xi_0 \in C_2^*$,
$\mu \geq 0, \quad \mu^T e = 1$,

Dual (RD)

Maximize $K(u_{11}, u_{12}) - [u_{11}{}^T \nabla_{x_{11}} \mu^T K(u_{11}, u_{12})]e - \frac{1}{2}\xi_1{}^T \nabla_{x_{11}x_{11}} \mu^T K(u_{11}, u_{12})\xi_1$

Subject to $(u_{11}, u_{12}) \in C_1 \times C_2$,
$-\nabla_{x_{11}} \mu^T K(u_{11}, u_{12}), \nabla_{x_{11}x_{11}} \mu^T K(u_{11}, u_{12})\xi_1 \in C_1^*$,
$\mu \geq 0,\ \mu^T e = 1$,

where, K is a ξ −vector and $e = (1,1,\dots,1) \in \mathcal{R}^{\xi}$.
Yang et al. (2005) discussed following Wolfe type second-order multi-objective symmetric duality problem:

Primal (RP)

$$\min_{u_{11},u_{12},\mu,x_{12}} F_{\xi}(x_{11}, x_{12}, \mu, \xi) = \Phi(x_{11}, x_{12}) - [x_{12}{}^T \nabla_{x_{12}} \mu^T \Phi(x_{11}, x_{12})]e_k - [x_{12}{}^T \nabla_{x_{12}x_{12}} \mu^T \Phi(x_{11}, x_{12})\xi]e_k - \frac{1}{2}\xi^T [\nabla_{x_{12}x_{12}} \mu^T \Phi(x_{11}, x_{12})\xi]e_k,$$

Subject to $\nabla_{x_{12}} \mu^T \Phi(x_{11}, x_{12}) + \nabla_{x_{12}x_{12}} \mu^T \Phi(x_{11}, x_{12})\xi \leqq 0$,
$\mu > 0,\ \mu^T e_k = 1$,

Dual (RD)

$$\max_{u_{11},u_{12},\mu,x_{12}} F_D(u_{11}, u_{12}, \mu, y) = \Phi(u_{11}, u_{12}) - [u_{12}{}^T \nabla_{u_{12}} \mu^T \Phi(u_{11}, u_{12})]e_k - [u_{12}{}^T \nabla_{u_{12}u_{12}} \mu^T \Phi(u_{11}, u_{12})x_{12}]e_k - \frac{1}{2}x_{12}{}^T [\nabla_{u_{12}u_{12}} \mu^T \Phi(u_{11}, u_{12})x_{12}]e_k,$$

Subject to $\nabla_{u_{12}} \mu^T \Phi(u_{11}, u_{12}) + \nabla_{u_{12}u_{12}} \mu^T \Phi(u_{11}, u_{12})ţ \geqq 0$,
$\mu > 0,\ \mu^T e_k = 1$,
where, $\Phi: \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^k, \xi \in \mathcal{R}^m, u_{12} \in \mathcal{R}^n, \mu \in \mathcal{R}^k, and\ e_k = (1,1,\dots,1)^T \in \mathcal{R}^k$.

Due to a few global qualities that it possesses, convexity is one of the most frequently used assumptions in simplifying hypotheses. Since convexity presumptions are routinely violated in real-world problems, it was necessary to weaken them. The quasi/pseudo-convexity is one of the ways to generalization of convexity problems. The higher-order cone-convex functions listed below were defined by Suneja et al. (2018):

Primal (RP)

K-Minimize $\Phi(x_{11})$

Subject to $-\Psi(x_{11}) \in Q,$

where, $\Phi: S \to \mathcal{R}^m$, $\Psi: S \to \mathcal{R}^\xi$ vector-valued differentiable functions, S is non-empty open subset of pointed cones $\mathcal{R}^n, K$ & Q are closed non-empty interiors convex in $\mathcal{R}^m$ and $\mathcal{R}^\xi$ respectively. $S_0 = \{x_{11} \in S; -\Psi(x_{11}) \in Q\}$ is the set of all feasible solutions.

Dual (RD)

K-Maximize $\Phi(u_{11}) + \mathrm{h}(u_{11}, \xi) - \nabla_\xi h(u_{11}, \xi)\xi + \left[\mu^T \Psi(u_{11}) + \mu^T \mathrm{k}(u_{11}, \xi) - \xi^T \nabla_\xi\left(\mu^T \mathrm{k}(u_{11}, \xi)\right)\right] e$

Subject to $[\nabla_\xi \mu^T h(, u_{11}, \xi) + \nabla_\xi \mu^T k(, u_{11}, \xi)] = 0$ and $\mu^T e = 1.$

$e \in \operatorname{int} k, \mu = K^+ \setminus \{0\}, u_{11} \in S.$

Ying (2012) discussed the following multiobjective fractional symmetric dual problem:

Primal (RP)

Minimize $L(x_{11}, x_{12}, \xi) = \{L_1(x_{11}, x_{12}, \xi_1), \dots, L_k(x_{11}, x_{12}, \xi_k)\}^T$

Subject to

$$\sum_{i=1}^{k} \mu_i \left[\left(\nabla_{x_{12}} f_i(x_{11}, x_{12}) - x_{ii} + \nabla_{\xi_i} H_i(x_{11}, x_{12}, \xi_i)\right) - L_i(x_{11}, x_{12}, \xi_i)\left(\nabla_{x_{12}} \Psi_i(x_{11}, x_{12}) + x_{ij} + \nabla_{\xi_i} G_i(x_{11}, x_{12}, \xi_i)\right)\right] \leqq 0,$$

$$x_{12}{}^T \sum_{i=1}^{k} \mu_i \left[\left(\nabla_{x_{12}} f_i(x_{11}, x_{12}) - x_{ii} + \nabla_{\xi_i} H_i(x_{11}, x_{12}, \xi_i)\right) - L_i(x_{11}, x_{12}, \xi_i)\left(\nabla_{x_{12}} \Psi_i(x_{11}, x_{12}) + x_{ij} + \nabla_{\xi_i} G_i(x_{11}, x_{12}, \xi_i)\right)\right] \geqq 0,$$

$$\mu > 0, \mu^T e = 1, x_{ii} \in D_i, x_{ij} \in F_i, i = 1,2,3 \dots, k; i \le j.$$

Dual (RD)

Maximize $M(u_{11}, u_{12}, \upsilon) = \{M_1(u_{11}, u_{12}, \upsilon_1), \dots, M_k(u_{11}, u_{12}, \upsilon_k)\}^T$

Subject to

$$\sum_{i=1}^{k} \mu_i \left[\left(\nabla_{x_{11}} f_i(u_{11}, u_{12}) + r_i + \nabla_{\upsilon_i} \phi_i(u_{11}, u_{12}, \upsilon_i)\right) - M_i(u_{11}, u_{12}, \upsilon_i)\left(\nabla_{x_{11}} g_i(u_{11}, u_{12}) - z_i + \nabla_{\upsilon_i} \psi_i(u_{11}, u_{12}, \upsilon_i)\right)\right] \geqq 0,$$

$$\upsilon^T \sum_{i=1}^{k} \mu_i \left[\left(\nabla_{x_{11}} f_i(u_{11}, u_{12}) + r_i + \nabla_{\upsilon_i} \phi_i(u_{11}, u_{12}, \upsilon_i)\right) - M_i(u_{11}, u_{12}, \upsilon_i)\left(\nabla_{x_{11}} g_i(u_{11}, u_{12}) - z_i + \nabla_{\upsilon_i} \psi_i(u_{11}, u_{12}, \upsilon_i)\right)\right] \leqq 0,$$

$$\mu > 0,\ \mu^T e = 1, r_i \in C_i, z_i \in E_i, i = 1,2,3 \dots, k,$$

where, $i = 1,2,3 \dots, k$,

$$L_i(x_{11}, x_{12}, \xi_i) = \frac{f_i(x_{11}, x_{12}) + S(x_{11}|C_i) - {x_{12}}^T z_i + H_i(x_{11}, x_{12}, \xi_i) - \xi_i^T \nabla_{\xi_i} H_i(x_{11}, x_{12}, \xi_i)}{g_i(x_{11}, x_{12}) - S(x_{11}|E_i) + {x_{12}}^T r_i + G_i(x_{11}, x_{12}, \xi_i) - \xi_i^T \nabla_{\xi_i} G_i(x_{11}, x_{12}, \xi_i)}$$

and

$$M_i(u_{11}, u_{12}, \upsilon_i) = \frac{f_i(u_{11}, u_{12}) - S(u_{12}|D_i) + {u_{11}}^T r_i + \phi_i(u_{11}, u_{12}, \upsilon_i) - \upsilon_i^T \nabla_{\upsilon_i} \phi_i(u_{11}, u_{12}, \upsilon_i)}{g_i(u_{11}, u_{12}) + S(u_{12}|F_i) - {u_{11}}^T z_i + \psi_i(u_{11}, u_{12}, \upsilon_i) - \upsilon_i^T \nabla_{\upsilon_i} \psi_i(u_{11}, u_{12}, \upsilon_i)},$$

where,
$\Phi_i: \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}$; $\Psi_i: \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}$; $H_i, G_i: \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}$ and $\phi_i, \psi_i: \mathcal{R}^n \times \mathcal{R}^m \times \mathcal{R}^n \to \mathcal{R}$ are twice differential functions for all $i = 1,2,3 \dots, k.$ C_i, E_i are compact convex sets in $\mathcal{R}^n$, and D_i, F_i are compact convex sets in $\mathcal{R}^m$, $i = 1,2,3 \dots, k$, $e = (1,1, \dots, 1)^T \in \mathcal{R}^k$, $\xi_i \in \mathcal{R}^m, \upsilon_i \in \mathcal{R}^n$, $\xi = (\xi_1, \xi_2, \dots, \xi_k), \upsilon = (\upsilon_1, \upsilon_2 \dots, \upsilon_k)$.

In order to create a vector optimization program under cones, they were able to develop higher-order sufficient optimality assumptions and duality theorems (Kumar et al., 2022). The extended Jacobian has recently been used by Gutierrez et al. (2015) to create various conceptions of (K_1, K_2)-pseudoinvexity-I and II using $K_1, K_2 \in \{C_0^c, (Int. C)^c\}$ for a locally Lipschitz function, where $C \subseteq \mathcal{R}^n$ has a non-empty interior, a closed convex pointed cone and $C_0 = C \setminus \{0\}$. They used them to take variational-like discrepancies with Lipschitz functions into account while calculating productivity. In a recent formulation, Kumar et al. (2021) created a nondifferentiable multiobjective Schiable type dual and established duality results for second-order $(K \times Q) - C -$ type $-$ I functions. Readers are urged to consult (Khurana, 2005; Ojha, 2012) and for additional information on fractional programming (Stancu-Minasian, 2013; Dubey, 2019; Dubey and Mishra, 2019; Dubey and Mishra, 2020).

The structure of this article is as follows: we provide some definitions, introductions, and the fundamental concept of (G, λ, θ) $-$bonvex/pseudobonvex functions. in Section 2. We also illustrate a non-trivial numerical example, which is function of (G, λ, θ) $-$pseudobonvex, but not (G, λ, θ) $-$bonvex functions. In section 3, we formulate a new type fractional $G -$ Wolfe type problem and derive duality outcomes under aforesaid assumptions. In the final section, we add conclusion section.

2. Abbreviation and Definitions

Assume that $S_1 \subseteq \mathcal{R}^n$ & $S_2 \subseteq \mathcal{R}^m$ are open set and function $\Phi(x_{11}, x_{12})$ is real-valued differential and defined on $S_1 \times S_2$. Assume $G: \mathcal{R} \to \mathcal{R}$ be a function of strictly increasing in the rank $G: I_\Phi(S_1 \times S_2) \to \mathcal{R}$ and $I_\Phi(S_1 \times S_2)$ is the range of $\Phi, \eta_1, \eta_2: S_1 \times S_2 \to \mathcal{R}^n, \lambda \in \mathcal{R}$ & $\theta: (S_1 \times S_2) \to \mathcal{R}$.

Definition 2.1 $\Phi(x_{11}, x_{12})$ is (G, λ, θ) $-$bonvex at $u_{11} \in S_1$ for fixed $u_{12} \in S_2$ w.r.t. η_1, $\exists$ λ and θ, such that for $x_{11} \in S_1$ & $\upsilon \in \mathcal{R}^n$, we have

$$G\big(f(x_{11}, u_{12})\big) - G\big(\Phi(u_{11}, u_{12})\big) \geq {\eta_1}^T(x_{11}, u_{11}) \Big[G'\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12}) + \{G''\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\left(\nabla_{x_{11}}\Phi(u_{11}, u_{12})\right)^T + G'\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\}\upsilon\Big] - \frac{1}{2}\upsilon^T \Big[\{G''\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\left(\nabla_{x_{11}}\Phi(u_{11}, u_{12})\right)^T + G'\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\}\Big]\upsilon + \lambda\|\theta(x_{11}, u_{11})\|^2.$$

Remarks 2.1 When sign changes in above inequality to $\leq$, then Φ is referred to as (G, λ, θ) $-$boncave *at* $u_{11} \in S_1$ w.r.t. η_1.

Definition 2.2 $\Phi(u_{11},u_{12})$ is (G,λ,θ) −bonvex at $u_{12}\in S_2$ for fixed $u_{11}\in S_1$ w.r.t. η_2, if ∃ λ and θ, such that for $x_{12}\in S_2$ & $\xi\in \mathcal{R}^m$, we have

$$G\big(f(u_{11},x_{12})\big)-G\big(\Phi(u_{11},u_{12})\big)\geq {\eta_2}^T(x_{12},u_{12})\Big[G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})+\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\}\xi\Big]-\frac{1}{2}\xi^T\Big[\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\}\Big]\xi+\lambda\|\theta(x_{12},u_{12})\|^2.$$

Remarks 2.2 When sign changes in above inequality to ≤, then Φ is referred to as $(\mathrm{G},\lambda,\theta)$ −boncave at $u_{12}\in S_2$ *w.r.t.* η_2.

Definition 2.3 $\Phi(x_{11},x_{12})$ is (G,λ,θ) −pseudobonvex at $u_{11}\in S_1$ for fixed $u_{12}\in S_2$ w.r.t. η_1, if ∃ λ and θ, such that for $x_{11}\in S_1$ & $\upsilon\in \mathcal{R}^n$, we have

$$\begin{aligned}&{\eta_1}^T(x_{11},u_{11})\Big[G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\\&\qquad+\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T\\&\qquad+G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\}\upsilon\Big]\geq 0\\&\Rightarrow G\big(f(x_{11},u_{12})\big)-G\big(\Phi(u_{11},u_{12})\big)\\&\qquad+\frac{1}{2}\upsilon^T\Big[\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T\\&\qquad+G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\}\Big]\upsilon-\lambda\|\theta(x_{11},u_{11})\|^2\geq 0.\end{aligned}$$

Remarks 2.3 When sign changes in above inequality to ≤, then Φ is referred to as $(\mathrm{G},\lambda,\theta)$ −pseudobonvex at $u_{11}\in S_1$ w.r.t. η_1.

Definition 2.4 $\Phi(x_{11},x_{12})$ is (G,λ,θ) −pseudobonvex at $u_{12}\in S_2$ for fixed $u_{11}\in S_1$ w.r.t. $\eta_{2,}$, if ∃ λ and θ, such that for $x_{12}\in S_2$ & $\xi\in \mathcal{R}^m$, we have

$$\begin{aligned}&{\eta_2}^T(x_{12},u_{12})\Big[G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\\&\qquad+\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T\\&\qquad+G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\}\xi\Big]\geq 0,\\&\Rightarrow G\big(f(u_{11},x_{12})\big)-G\big(\Phi(u_{11},u_{12})\big)\\&\qquad+\frac{1}{2}\xi^T\Big[\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T\\&\qquad+G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\}\Big]\xi-\lambda\|\theta(x_{12},u_{12})\|^2\geq 0.\end{aligned}$$

Remarks 2.4 When sign changes in above inequality to ≤, then Φ is referred to as $(\mathrm{G},\lambda,\theta)$ −pseudoboncave at $u_{12}\in S_2$ w.r.t. η_2.

In the following numerical example, we will try to show that a function which is pseudobonvex, but it is not bonvex at the same point.

Example 2.1 Let $X = \left[0, \frac{\pi}{2}\right]$, consider the function $\Phi: X \times X \to \mathcal{R}$ defineD by

$$\Phi(x_{11}, x_{12}) = x_{11} + sinx_{12}.$$

Next, consider $G: X \to \mathcal{R}$ given by

$$G(r) = r^2.$$

Let $\eta: X \times X \to \mathcal{R}$ given by

$$\eta(x_{11}, u_{11}) = x_{11} - u_{11},$$

and

$$\theta\,(x_{11}, u_{11}) = x_{11}\, u_{11}.$$

Also, let $\upsilon = 1 \in \left[0, \frac{\pi}{2}\right]$.

Next, we have to claim that Φ is pseudobonvex at $u_{11} = 0 \in \left[0, \frac{\pi}{2}\right]$, i.e., it is sufficient to show that

$$\eta_1^T(x_{11}, u_{11}) \Big[G'\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12}) + \{G''\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\left(\nabla_{x_{11}}\Phi(u_{11}, u_{12})\right)^T + G'\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\}\upsilon\Big] \geq 0.$$

$$\Rightarrow G\big(f(x_{11}, u_{12})\big) - G\big(\Phi(u_{11}, u_{12})\big) + \frac{1}{2}\upsilon^T \Big[\{G''\big(\Phi(u_{11}, u_{12})\big)\,\nabla_{x_{11}}\,\Phi(u_{11}, u_{12})\left(\nabla_{x_{11}}\Phi(u_{11}, u_{12})\right)^T + G'\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\}\Big]\upsilon - \lambda\|\theta(x_{11}, u_{12})\|^2 \geq 0.$$

Let

$$\tau_1 = \eta_1^T(x_{11}, u_{11}) \Big[G'\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12}) + \{G''\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\left(\nabla_{x_{11}}\Phi(u_{11}, u_{12})\right)^T + G'\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\}\upsilon\Big].$$

Substituting the values of $\Phi, G_\Phi, \eta_1^T, \lambda$ & υ, we get

$$\tau_1 = (x_{11} - u_{11})\left[2u_{11} + 2\,sin\,u_{12} + \big(2 + 2(u_{11} + sin\,u_{12})\big)\upsilon\right].$$

The value of above expression at the point $u_{11} = 0 \in \left[0, \frac{\pi}{2}\right]$, we have

$$\tau_1 = x_{11}[2\sin u_{12} + 2\upsilon + 2\sin u_{12}].$$

At $\upsilon = 1 \in \left[0, \frac{\pi}{2}\right]$, the above expression reduced to

$$\tau_1 = x_{11}[4\sin u_{12} + 2].$$

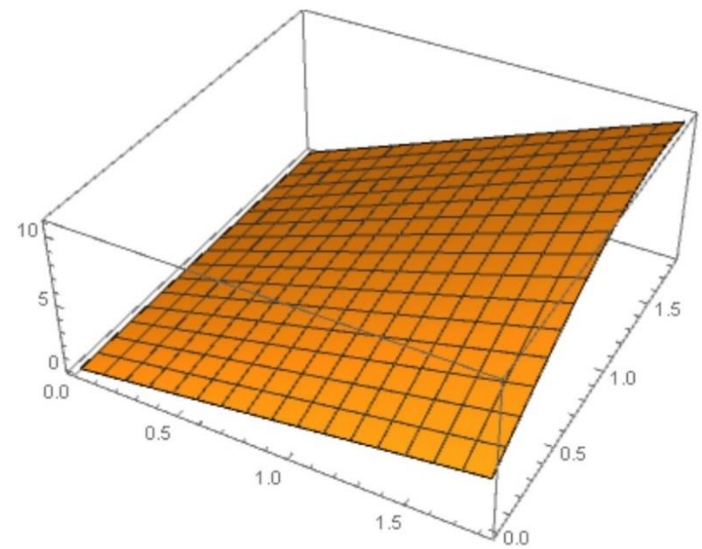

Figure 1. $x_{11}[4\sin u_{12}+2]$.

It is clear from the Figure 1, that $\tau_1 \geq 0$.

Next, Let

$$\tau_2 = G\big(\Phi(x_{11},u_{12})\big) - G\big(\Phi(u_{11},u_{12})\big) + \frac{1}{2}\upsilon^T\Big[\{G^{''}\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T + G^{'}\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\}\upsilon\Big] + \lambda\|\theta(x_{12},u_{12})\|^2.$$

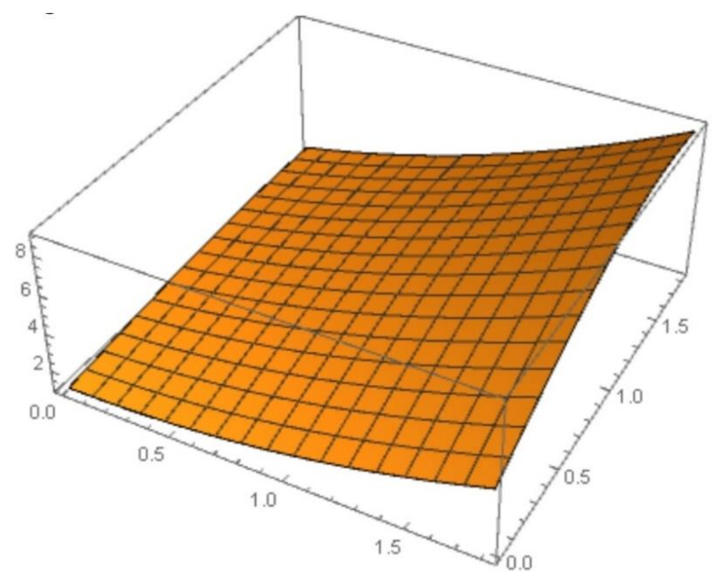

Figure 2. ${x_{11}}^2 + 2x_{11}\sin u_{12} + 1 + \sin u_{12}$.

Putting the values of Φ, G, λ & θ, we get

$$\tau_2 = (x_{11} + \sin u_{12})^2 - (u_{11} + \sin u_{12})^2 + \frac{1}{2}\upsilon^2[2 + 2u_{11} + 2\sin u_{12}] - \lambda\|x_{11}u_{11}\|^2.$$

On simplifying and the value of above expression at the point $u_{11} = 0 \in \left[0, \frac{\pi}{2}\right]$, we have

$$\tau_2 = {x_{11}}^2 + 2x_{11}\, sin\, u_{12} + 1 + \sin u_{12}.$$

It is clear from Figure 2, that $\tau_2 \geq 0, \forall\, x_{11}, u_{12} \in \left[0, \frac{\pi}{2}\right]$.

Hence, Φ is pseudobonvex at $u_{11} = 0 \in \left[0, \frac{\pi}{2}\right]$.

Then, we assert that at point $u_{11} = 0 \in \left[0, \frac{\pi}{2}\right]$, Φ is not (G, λ, θ) −bonvex at the same point.

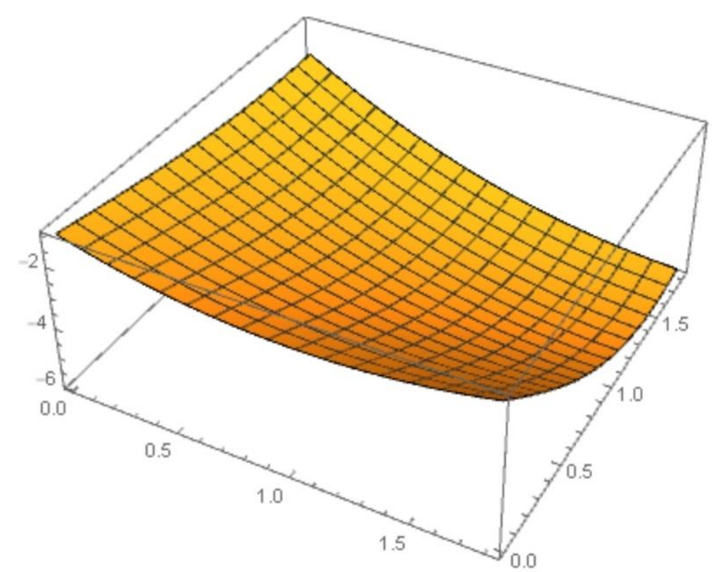

Figure 3. ${x_{11}}^2 + 2x_{11}\, sin\, u_{12} - 2x_{11}[2\, sin\, u_{12} + 1] + [1 + sin\, u_{12}]$.

For this, we have to show that

$$G\big(\Phi(x_{11}, u_{12})\big) - G\big(\Phi(u_{11}, u_{12})\big) - \eta_1^T(x_{11}, u_{11})\Big[G'\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12}) + \{G''\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\left(\nabla_{x_{11}}\Phi(u_{11}, u_{12})\right)^T + G'\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\}\upsilon\Big] + \frac{1}{2}\upsilon^T\Big[\{G''\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\left(\nabla_{x_{11}}\Phi(u_{11}, u_{12})\right)^T + G'\big(\Phi(u_{11}, u_{12})\big)\nabla_{x_{11}}\Phi(u_{11}, u_{12})\}\upsilon\Big] - \lambda\|\theta(x_{12}, u_{12})\|^2 < 0.$$

Let

$\tau_3 = G\big(\Phi(x_{11},u_{12})\big) - G\big(\Phi(u_{11},u_{12})\big) - \eta_1^T(x_{11},u_{11})\Big[G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12}) + \{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T + G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\}\upsilon\Big] + \frac{1}{2}\upsilon^T\Big[\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T + G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\}\upsilon\Big] - \lambda\|\theta(x_{12},u_{12})\|^2 < 0.$

Substituting the values and simplifying, at the point $\upsilon = 0$, we get

$$\tau_3 = {x_{11}}^2 + 2x_{11}\sin u_{12} - 2x_{11}[2\sin u_{12} + 1] + [1 + \sin u_{12}].$$

It is clear from Figure 3, that $\tau_3 < 0$. Hence, function Φ is not (G,λ,θ) −bonvex at $u_{11} = 0 \in \left[0,\frac{\pi}{2}\right]$.

3. Second-Order Fractional Symmetric Duality Model

In this section, we discuss the following model:

Primal (SWP):

Minimize

$$\frac{G(\Phi(x_{11},x_{12})) - {x_{12}}^T\left[\begin{array}{c} G'(\Phi(x_{11},x_{12}))\nabla_{x_{12}}\Phi(x_{11},x_{12}) \\ +\left\{G''(\Phi(x_{11},x_{12}))\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T + G'(\Phi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\right\}\xi\end{array}\right]}{G(\Psi(x_{11},x_{12})) - {x_{12}}^T\left[\begin{array}{c} G'(\Psi(x_{11},x_{12}))\nabla_{x_{12}}\Psi(x_{11},x_{12}) \\ +\left\{G''(\Psi(x_{11},x_{12}))\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T + G'(\Psi(x_{11},x_{12}t))\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\right\}\xi\end{array}\right]}$$

$$\frac{-\frac{1}{2}\xi^T\left\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T + G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\right\}\xi}{-\frac{1}{2}\xi^T\left\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T + G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\right\}\xi}$$

Subject to

$\Big[G\big(\Psi(x_{11},x_{12})\big) - {x_{12}}^T\Big[G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12}) + \Big\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T + G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\xi\Big]\Big] - \frac{1}{2}\xi^T\Big[\Big\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T + G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\xi\Big] \Big[G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12}) + \Big\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T + G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi\Big] - \Big[G\big(\Phi(x_{11},x_{12})\big) - {x_{12}}^T\Big[G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12}) + \Big\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T + G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi\Big] - \frac{1}{2}\xi^T\Big\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T + G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi\Big]$

$$\left[G'\left(\Psi(x_{11},x_{12})\right)\nabla_{x_{12}}\Psi(x_{11},x_{12})+\left\{G''\left(\Psi(x_{11},x_{12})\right)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+ G'\left(\Psi(x_{11},x_{12})\right)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\right\}\xi\right]\leq 0,$$

$$x_{11}\geq 0.$$

Dual (SWD):
Maximize

$$\frac{G(\Phi(u_{11},u_{12}))-u_{11}{}^T\left[\begin{array}{c}G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12})\\ +\left\{G''(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\right\}\upsilon\end{array}\right]}{G(\Psi(u_{11},u_{12}))-u_{11}{}^T\left[\begin{array}{c}G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12})\\ +\left\{G''(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T+G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\right\}\upsilon\end{array}\right]}$$

$$\frac{-\frac{1}{2}\upsilon^T\left\{G''\left(\Phi(u_{11},u_{12})\right)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'\left(\Phi(u_{11},u_{12})\right)\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\right\}\upsilon}{-\frac{1}{2}\upsilon^T\left\{G''\left(\Psi(u_{11},u_{12})\right)\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T+G'\left(\Psi(u_{11},u_{12})\right)\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\right\}\upsilon}$$

Subject to

$$G\left(\Psi(u_{11},u_{12})\right)-u_{11}{}^T\left[G'\left(\Psi(u_{11},u_{12})\right)\nabla_{x_{11}}\Psi(u_{11},u_{12})+\left\{G''\left(\Psi(u_{11},u_{12})\right)\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T+G'\left(\Psi(u_{11},u_{12})\right)\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\right\}\upsilon\right]-\frac{1}{2}\upsilon^T\left[\left\{G''\left(\Psi(u_{11},u_{12})\right)\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T+G'\left(\Psi(u_{11},u_{12})\right)\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\right\}\upsilon\right]\left[G'\left(\Phi(u_{11},u_{12})\right)\nabla_{x_{11}}\Phi(u_{11},u_{12})+\left\{G''\left(\Phi(u_{11},u_{12})\right)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'\left(\Phi(u_{11},u_{12})\right)\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\right\}\upsilon\right]-\left[G\left(\Phi(u_{11},u_{12})\right)-u_{11}{}^T\left[G'\left(\Phi(u_{11},u_{12})\right)\nabla_{x_{11}}\Phi(u_{11},u_{12})+\left\{G''\left(\Phi(u_{11},u_{12})\right)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'\left(\Phi(u_{11},u_{12})\right)\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\right\}\upsilon\right]\right]-\frac{1}{2}\upsilon^T\left[\left\{G''\left(\Phi(u_{11},u_{12})\right)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'\left(\Phi(u_{11},u_{12})\right)\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\right\}\upsilon\right]\left[G'\left(\Psi(u_{11},u_{12})\right)\nabla_{ṣ}\Psi(u_{11},u_{12})+\left\{G''\left(\Psi(u_{11},u_{12})\right)\nabla_{ṣ}\Psi(u_{11},u_{12})\left(\nabla_{ṣ}\Psi(u_{11},u_{12})\right)^T+G'\left(\Psi(u_{11},u_{12})\right)\nabla_{ṣṣ}\Psi(u_{11},u_{12})\right\}\upsilon\right]\leq 0,$$

$$u_{12}\geq 0,$$

where, $\Phi : S_1\times S_2\to\mathcal{R}$ *and* $\Psi : S_1\times S_2\to\mathcal{R}_+\setminus\{0\}$ are differentiable functions. Vectors ξ & υ are in $\mathcal{R}^m$ & $\mathcal{R}^m$ respectively. The aforementioned primal-dual problem can be expressed as:

(ESWP) Minimize r:
Subject to

$$G\big(\Phi(x_{11},x_{12})\big)-x_{12}{}^{T}\Big[G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})+$$
$$\Big\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^{T}+G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi\Big]-$$
$$rG\big(\Psi(x_{11},x_{12})\big)-x_{12}{}^{T}\Big[G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})+$$
$$\Big\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^{T}+G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\xi\Big]=0, \quad (1)$$
$$G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})+\Big\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^{T}+$$
$$G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi-rG'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})+$$
$$\Big\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^{T}+G'\big(\Psi(ș,ț)\big)\nabla_{x_{12}x_{12}}\Psi(ș,ț)\Big\}\xi\leq 0, \quad (2)$$
$$x_{11}\geq 0. \quad (3)$$

(ESWD) Minimize z:
Subject to

$$G\big(\Phi(u_{11},u_{12})\big)-u_{11}{}^{T}\Big[G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})+$$
$$\Big\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^{T}+G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\Big\}\upsilon\Big]-$$
$$zG\big(\Psi(u_{11},u_{12})\big)-x_{12}{}^{T}\Big[G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})+$$
$$\Big\{G''\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^{T}+G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\Big\}\upsilon\Big]=0, \quad (4)$$
$$G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})+\Big\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^{T}+$$
$$G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\Big\}\upsilon-zG'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})+$$
$$\Big\{G''\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^{T}+G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\Big\}\upsilon\geq 0, \quad (5)$$
$$u_{12}\geq 0. \quad (6)$$

Let set P^0 and Q^0 be the set of feasible solution of (ESWP) & (ESWD), respectively.

Theorem 3.1 (Weak Duality Theorem). Let $(x_{11},x_{12},\mathrm{r},\xi)\in P^0$ & $(u_{11},u_{12},\mathrm{z},\upsilon)\in Q^0$. Let

(i) $\Phi(.,u_{12})$ be (G,λ_1,θ_1)-bonvex and $\Psi(.,u_{12})$ be (G,λ_2,θ_2)-boncave at u_{11} for fixed u_{12} w.r.t. η_1,
(ii) $\Phi(x_{11},.)$ be (G,λ_3,θ_3)-boncave and $\Psi(x_{11},.)$ be (G,λ_4,θ_4)-bonvex at x_{12} for fixed x_{11} w.r.t. η_2,
(iii) $\eta_1(x_{11},u_{11})+u_{11}\in C_1$ and $\eta_2(u_{12},x_{12})+x_{12}\in C_2$,
(iv) $G\big(\Psi(x_{11},u_{12})\big)>0$,
(v) $\lambda_1\|\theta_1(x_{11},u_{11})\|^2-z\lambda_2\|\theta_2(x_{11},u_{11})\|^2\geq 0$,
(vi) $\lambda_3\|\theta_3(x_{11},u_{11})\|^2-r\lambda_4\|\theta_4(x_{11},u_{11})\|^2\leq 0$.

Then, $r\geq z$.

Proof: Using hypothesis (i), we get

$$G(\Phi(x_{11},u_{12})) - G(\Phi(u_{11},u_{12})) + \frac{1}{2}\upsilon^T\left[\left\{G''(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T + G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\right\}\upsilon\right] \geq \eta_1{}^T(x_{11},u_{11})\left[G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12}) + \left\{G''(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T + G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\right\}\upsilon\right] + \lambda_1\|\theta_1(x_{11},u_{11})\|^2, \quad (7)$$

and

$$-G(\Psi(x_{11},u_{12})) + G(\Psi(u_{11},u_{12})) - \frac{1}{2}\upsilon^T\left[\left\{G''(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T + G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\right\}\upsilon\right] \geq -\eta_1{}^T(x_{11},u_{11})\left[G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12}) + \left\{G''(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T + G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\right\}\upsilon\right] - \lambda_2\|\theta_2(x_{11},u_{11})\|^2. \quad (8)$$

Inequality (8) Multiplying by z and combining with (7), we have

$$G(\Phi(x_{11},u_{12})) - zG(\Psi(x_{11},u_{12})) - G(\Phi(u_{11},u_{12})) + zG(\Psi(u_{11},u_{12})) + \frac{1}{2}\upsilon^T\left[\left\{G''(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T + G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\right\} - z\left\{G''(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T + G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\right\}\right]\upsilon \geq \eta_1{}^T(x_{11},u_{11})\left[G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12}) + \left\{G''(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T + G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\right\} - z\left\{G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12}) + \left\{G''(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T + G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\right\}\right\}\right]\upsilon + \lambda_1\|\theta_1(x_{11},u_{11})\|^2 - z\lambda_2\|\theta_2(x_{11},u_{11})\|^2,$$

Use hypothesis (v), the aforementioned inequality follows that

$$G(\Phi(x_{11},u_{12})) - zG(\Psi(x_{11},u_{12})) - G(\Phi(u_{11},u_{12})) + zG(\Psi(u_{11},u_{12})) + \frac{1}{2}\upsilon^T\left[\left\{G''(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T + G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\right\} - z\left\{G''(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T + G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\right\}\right]\upsilon \geq \eta_1{}^T(x_{11},u_{11})\left[G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12}) - z\left\{G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12}) + \left\{G''(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T + G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\right\} + \left\{G''(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T + G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\right\}\upsilon\right\}\right]. \quad (9)$$

Next by hypothesis (ii), gives

$$-G(\Phi(x_{11},u_{12}))-G(\Phi(x_{11},x_{12}))-\frac{1}{2}\xi^T\Big[\Big\{G''(\Phi(x_{11},x_{12}))\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+G'(\Phi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\Big]\xi\geq-\eta_2{}^T(u_{12},x_{12})\Big[G'(\Phi(x_{11},x_{12}))\nabla_{x_{12}}\Phi(x_{11},x_{12})+\Big\{G''(\Phi(x_{11},x_{12}))\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+G'(\Phi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi\Big]-\lambda_3\|\theta_3(u_{12},x_{12})\|^2. \tag{10}$$

and

$$G(\Psi(x_{11},u_{12}))+G(\Psi(x_{11},x_{12}))+\frac{1}{2}\xi^T\Big[\Big\{G''(\Psi(x_{11},x_{12}))\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+G'(\Psi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\Big]\xi\geq\eta_2{}^T(u_{12},x_{12})\Big[G'(\Psi(x_{11},x_{12}))\nabla_{x_{12}}\Psi(x_{11},x_{12})+\Big\{G''(\Psi(x_{11},x_{12}))\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+G'(\Psi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\xi\Big]+\lambda_4\|\theta_4(u_{12},x_{12})\|^2. \tag{11}$$

Multiplying r in inequality (11) and combining with inequality (10) then, we get

$$-G(\Phi(x_{11},u_{12}))+rG(\Psi(x_{11},u_{12}))-G(\Phi(x_{11},x_{12}))+rG(\Psi(x_{11},x_{12}))-\frac{1}{2}\xi^T\Big[\Big\{G''(\Phi(x_{11},x_{12}))\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+G'(\Phi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}-r\Big\{G''(\Psi(x_{11},x_{12}))\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+G'(\Psi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\Big]\xi\geq-\eta_2{}^T(u_{12},x_{12})\Big[G'(\Phi(x_{11},x_{12}))\nabla_{x_{12}}\Phi(x_{11},x_{12})-r\{G'(\Psi(x_{11},x_{12}))\nabla_{x_{12}}\Psi(x_{11},x_{12})\}+\Big\{G''(\Phi(x_{11},x_{12}))\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+G'(\Phi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi-r\Big\{G''(\Psi(x_{11},x_{12}))\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+G'(\Psi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\xi\Big]-\lambda_3\|\theta_3(u_{12},x_{12})\|^2+\lambda_4\|\theta_4(u_{12},x_{12})\|^2. \tag{12}$$

The inequality above indicates that under hypothesis (vi),

$$-G(\Phi(x_{11},u_{12}))+rG(\Psi(x_{11},u_{12}))-G(\Phi(x_{11},x_{12}))+rG(\Psi(x_{11},x_{12}))-\frac{1}{2}\xi^T\Big[\Big\{G''(\Phi(x_{11},x_{12}))\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+G'(\Phi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}-r\Big\{G''(\Psi(x_{11},x_{12}))\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+G'(\Psi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\Big]\xi\geq-\eta_2{}^T(u_{12},x_{12})\Big[G'(\Phi(x_{11},x_{12}))\nabla_{x_{12}}\Phi(x_{11},x_{12})-r\{G'(\Psi(x_{11},x_{12}))\nabla_{x_{12}}\Psi(x_{11},x_{12})\}+\Big\{G''(\Phi(x_{11},x_{12}))\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+G'(\Phi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi-r\Big\{G''(\Psi(x_{11},x_{12}))\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+G'(\Psi(x_{11},x_{12}))\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\xi\Big]. \tag{13}$$

On adding inequalities (9) and (13), we have

$$G(\Phi(x_{11},u_{12}))-zG(\Psi(x_{11},u_{12}))-G(\Phi(x_{11},x_{12}))+zG(\Psi(x_{11},x_{12}))+\frac{1}{2}\upsilon^T\Big[\Big\{G''(\Phi(u_{11},u_{12}))\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'(\Phi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\Big\}-z\Big\{G''(\Psi(u_{11},u_{12}))\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T+G'(\Psi(u_{11},u_{12}))\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\Big\}\Big]\upsilon-$$

$$G\big(\Phi(x_{11},u_{12})\big)+rG\big(\Psi(x_{11},u_{12})\big)-G\big(\Phi(x_{11},x_{12})\big)+rG\big(\Psi(x_{11},x_{12})\big)-$$
$$\frac{1}{2}\xi^T\bigg[\Big\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}-$$
$$r\Big\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\bigg]\xi\geq$$
$$\eta_1{}^T(x_{11},u_{11})\Big[G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})-z\Big\{G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})+$$
$$\Big\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'(\Phi u_{11},u_{12})\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\Big\}+$$
$$\Big\{G''\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T+$$
$$G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\Big\}\upsilon\Big\}\Big]-\eta_2{}^T(u_{12},x_{12})\Big[G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})-$$
$$r\{G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\}+\Big\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+$$
$$G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi-r\Big\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+$$
$$G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\xi\Big].\tag{14}$$

Using hypothesis (iii) and constraint (5), we get

$$(\eta_1(x_{11},u_{11})+u_{11})^T\Big[G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})+$$
$$\Big\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\Big\}\upsilon-$$
$$\mathrm{z}G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})+\Big\{G''\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T+$$
$$G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\Big\}\upsilon\Big]\geq 0,$$

or

$$(\eta_1(x_{11},u_{11}))^T\Big[G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})+$$
$$\Big\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\Big\}\upsilon-$$
$$\mathrm{z}G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})+\Big\{G''\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T+$$
$$G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\Big\}\upsilon\Big]\geq-\upsilon^T\Big[G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})+$$
$$\Big\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\Big\}\upsilon-$$
$$\mathrm{z}G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})+\Big\{G''\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}}\Psi(u_{11},u_{12})\left(\nabla_{x_{11}}\Psi(u_{11},u_{12})\right)^T+$$
$$G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Psi(u_{11},u_{12})\Big\}\upsilon\Big].\tag{15}$$

Similarly using inequality (2) and hypothesis (iii), we get

$$-(\eta_2(u_{12},x_{12})+x_{12})^T\Big[G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})+$$
$$\Big\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi-$$

$$rG'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})+\Big\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+ G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\xi\Big]\geq 0,$$

or

$$-(\eta_2(u_{12},x_{12}))^T\Big[G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})+ \Big\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi- rG'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})+\Big\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+ G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\xi\Big]\geq {x_{12}}^T\Big[G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})+ \Big\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}\xi- rG'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})+\Big\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+ G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\xi\Big]\geq 0. \tag{16}$$

From inequalities (14), (15) and (16), we get

$$G\big(\Phi(x_{12},u_{12})\big)-zG\big(\Psi(x_{12},u_{12})\big)-G\big(\Phi(x_{11},x_{12})\big)+zG\big(\Psi(x_{11},x_{12})\big)+ \frac{1}{2}\upsilon^T\Big[\Big\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}}\Phi(u_{11},u_{12})\left(\nabla_{x_{11}}\Phi(u_{11},u_{12})\right)^T+ G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{11}x_{11}}\Phi(u_{11},u_{12})\Big\}\Big]\upsilon-G\big(\Phi(x_{11},u_{12})\big)+rG\big(\Psi(x_{11},u_{12})\big)-G\big(\Phi(x_{11},x_{12})\big)+ rG\big(\Psi(x_{11},x_{12})\big)-\frac{1}{2}\xi^T\Big[\Big\{G''\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}}\Phi(x_{11},x_{12})\left(\nabla_{x_{12}}\Phi(x_{11},x_{12})\right)^T+ G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})\Big\}-r\Big\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+ G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\Big]\xi\geq -{u_{11}}^T\Big[G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{12}}\Phi(u_{11},u_{12})+ \Big\{G''\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{12}}\Phi(u_{11},u_{12})\left(\nabla_{x_{12}}\Phi(u_{11},u_{12})\right)^T+G'\big(\Phi(u_{11},u_{12})\big)\nabla_{x_{12}x_{12}}\Phi(u_{11},u_{12})\Big\}.\upsilon- zG'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{12}}\Psi(u_{11},u_{12})+\Big\{G''\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{12}}\Psi(u_{11},u_{12})\left(\nabla_{x_{12}}\Psi(u_{11},u_{12})\right)^T+ G'\big(\Psi(u_{11},u_{12})\big)\nabla_{x_{12}x_{12}}\Psi(u_{11},u_{12})\Big\}\upsilon\Big]+G'\big(\Phi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Phi(x_{11},x_{12})- r\{G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\}+\Big\{G''\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}}\Psi(x_{11},x_{12})\left(\nabla_{x_{12}}\Psi(x_{11},x_{12})\right)^T+ G'\big(\Psi(x_{11},x_{12})\big)\nabla_{x_{12}x_{12}}\Psi(x_{11},x_{12})\Big\}\xi.$$

Using equations (1) and (4), it follows that

$$(r-z)G\big(\Psi(x_{11},u_{12})\big)\geq 0.$$

By using hypothesis (iv), this gives

$$r\geq z.$$

Hence, the result.

Remark 3.1 Due to the fact that all bonvex functions are pseudobonvex. Similarly, can say every $(\mathrm{G},\lambda,\theta)$ −bonvex function is $(\mathrm{G},\lambda,\theta)$ −pseudobonvex, consequently, the aforementioned weak duality theorem follows in a similar path.

Theorem 3.2 (Weak Duality Theorem). Let $(x_{11},x_{12},r,\xi)\in P^0$ *and* $(x_{11},x_{12},\mathrm{z},\upsilon)\in Q^0$. Let

(i) $\Phi(.,u_{12})$ be (G,λ_1,θ_1)- pseudobonvex and $\Psi(.,u_{12})$ be (G,λ_2,θ_2)-pseudoboncave at u_{11} for fixed u_{12} with respect to η_1,

(ii) $\Phi(x_{11},.)$ be (G,λ_3,θ_3)- pseudoboncave and $\Psi(x_{11},.)$ be (G,λ_4,θ_4)- pseudobonvex at x_{12} for fixed x_{11} with respect to η_2,

(iii) $\eta_1(x_{11},u_{11})+\ u_{11}\in C_1$ and $\eta_2(u_{12},x_{12})+x_{12}\in C_2$,

(iv) $G(\Psi(x_{11},u_{12})>0$,

(v) $(\lambda_1\|\theta_1(x_{11},u_{11})\|^2-z\lambda_2\|\theta_2(x_{11},u_{11})\|^2\geq 0$,

(vi) $(\lambda_3\|\theta_3(x_{11},u_{11})\|^2-r\lambda_4\|\theta_4(x_{11},u_{11})\|^2\leq 0$.

Then, $r\geq z$.

Proof*:* This proof adheres to the similar framework as theorem 3.1.

Theorem 3.3 (Strong Duality Theorem). Assume the function Φ & Ψ be differentiable and $(\bar{x}_{11},\bar{x}_{12},\bar{r},\bar{\xi})$ be an optimal solution of (ESWP). Consider

(i) $\Big[\Big\{G''\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\Big\}-$ $\bar{r}\Big\{G''\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\Big]$ is non-singular,

(ii) $\bar{x}_{11}{}^TG'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{11}}\Psi(\bar{x}_{11},\bar{x}_{12})-\bar{x}_{12}{}^TG'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\left(G\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\right)+$ $\bar{x}_{12}{}^TG'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)-\bar{x}_{11}{}^TG'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{11}}\Phi(\bar{x}_{11},\bar{x}_{12})\left(G\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\right)=$ 0,

(iii) The equality

$$\bar{\xi}^T\Big[\nabla_{x_{12}}\Big\{G''\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)^T$$
$$+G'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}$$
$$-\bar{r}\Big\{G''\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T$$
$$+G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}\Big]=0,$$

$\Rightarrow\qquad \bar{\xi}=0.$

Then, $(\bar{x}_{11},\bar{x}_{12},\bar{r},\bar{\upsilon})\in Q^0$ and objective values of (ESWP) & (ESWD) are equal. Additionally, the solution $(\bar{x}_{11},\bar{x}_{12},\bar{r},\bar{\upsilon})=0)$ is the optimal for the weak duality theorem if all of its hypotheses are met (ESWP).

Proof: Since primal problem (ESWP) has the optimal solution $(\bar{x}_{11},\bar{x}_{12},\bar{r},\bar{\xi})$, $\alpha\in\mathcal{R},\beta\in\mathcal{R},\gamma\in\mathcal{R}^m,\mu\in\mathcal{R}^n$ so the following necessary conditions of Fritz-John (John, 1948) are met at $(\bar{x}_{11},\bar{x}_{12},\bar{r},\bar{\xi})$:

$\Big[\beta\left(G'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{11}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)-\bar{r}G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})+(\gamma-$ $\beta\bar{x}_{12})^T\left(G''\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{11}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)+G'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}x_{11}}\Phi(\bar{x}_{11},\bar{x}_{12})-$

$$\bar{r}\left\{G''\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{11}}\Psi(\bar{x}_{11},\bar{x}_{12})+G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{11}}\Psi(\bar{x}_{11},\bar{x}_{12})\right\}\Big]+\Big(\gamma-\beta\bar{x}_{12}-\frac{\beta\bar{\xi}}{2}\Big)\Big[\nabla_{x_{11}}\Big\{G''\left(\Phi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Phi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}-\bar{r}\Big\{G''\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}\Big]=\mu. \tag{17}$$

$$\left(\gamma-\beta\bar{x}_{12}-\beta\bar{\xi}\right)^T\Big[\Big\{G''\left(\Phi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Phi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\Big\}-\bar{r}\Big\{G''\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}\Big]+\Big(\gamma-\beta\bar{x}_{12}-\frac{\beta\bar{\xi}}{2}\Big)\Big[\nabla_{x_{12}}\Big\{G''\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}-\bar{r}\Big\{G''\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}\Big]=0, \tag{18}$$

$$\gamma^T\Big[G'\left(\Phi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})-\bar{r}G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})+\Big\{G''\left(\Phi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Phi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}-\bar{r}\Big\{G''\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}\Big]=0, \tag{19}$$

$$\alpha-\beta\Big[G\left(\Phi(\bar{x}_{11},\bar{x}_{12})\right)-\bar{x}_{12}{}^TG'\left(\Phi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})+\Big\{G''\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}-\frac{\bar{\xi}}{2}\Big\{G''\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}\Big]-\gamma\Big[G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})+\Big\{G''\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\bar{\xi}\Big]=0, \tag{20}$$

$$\left(\gamma-\beta\bar{x}_{12}-\beta\bar{\xi}\right)^T\Big[\Big\{G''\left(\Phi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Phi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\Big\}-\bar{r}\Big\{G''\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T+G'\left(\Psi(\bar{x}_{11},\bar{x}_{12})\right)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\Big]=0, \tag{21}$$

$$\mu^T\bar{x}_{11}=0, \tag{22}$$

$$(\alpha,\beta,\gamma,\mu)\neq 0,(\alpha,\beta,\gamma,\mu)\geq 0. \tag{23}$$

Since

$$\Big[\Big\{G''\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)^T + G'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\Big\} - \bar{r}\Big\{G''\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)^T + G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\Big\}\Big]$$

is non-singular, then from (21), we get

$$\gamma = \beta\bar{x}_{12} + \beta\bar{\xi}. \tag{24}$$

Next, we want to demonstrate that $\beta \neq 0$. If it's possible, assume that $\beta = 0$, in that case from (24), we get $\gamma = 0$. From (20), we have $\alpha = 0$, which contradicts equation (23). This combined with (17), then we get $\mu = 0$. Hence, $\beta \neq 0 \;\Rightarrow\; \beta > 0$.

Now, it gives that (18) and (24) and condition (iii) that $\bar{\xi} = 0$. By (24), $\beta > 0$ and since $\gamma \geq 0$. From inequality (17), we get

$$G'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{11}}\Phi(\bar{x}_{11},\bar{x}_{12})\right) - \bar{r}\{G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{11}}\Psi(\bar{x}_{11},\bar{x}_{12})\} = \frac{\mu}{\beta} \geq 0. \tag{25}$$

Therefore, $(\bar{x}_{11},\bar{x}_{12},\bar{r},\bar{\upsilon}) \in Q^0$.

The objective values of the problem must then be asserted to be equal. It is sufficient to establish

$$\frac{G\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big) - \bar{x}_{11}{}^T G'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{11}}\Phi(\bar{x}_{11},\bar{x}_{12})}{G\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big) - \bar{x}_{11}{}^T G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{11}}\Psi(\bar{x}_{11},\bar{x}_{12})} = \frac{G\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big) - \bar{x}_{12}{}^T G'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})}{G\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big) - \bar{x}_{12}{}^T G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})}.$$

Now, multiplying (25) by $\bar{\varsigma}^T$ and using (22), we have

$$\frac{\bar{x}_{11}{}^T G'(\Phi(\bar{x}_{11},\bar{x}_{12}))\left(\nabla_{x_{11}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)}{\bar{x}_{11}{}^T G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{11}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)} = \bar{r}. \tag{26}$$

Again, since $\bar{\xi} = 0$, then from (24) we get

$$\gamma = \beta\bar{x}_{12}. \tag{27}$$

Further, using $\bar{\xi} = 0$ with (27) and (19), we get

$$\frac{\bar{x}_{12}{}^T G'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})}{\bar{x}_{12}{}^T G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})} = \bar{r}. \tag{28}$$

From (26) and (28), we have

$$\frac{\bar{x}_{11}{}^T G'(\Phi(\bar{x}_{11},\bar{x}_{12}))\left(\nabla_{x_{11}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)}{\bar{x}_{11}{}^T G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{11}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)} = \frac{\bar{x}_{12}{}^T G'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)}{\bar{x}_{12}{}^T G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)}$$

i.e.

$$\left\{\bar{x}_{11}{}^T G'(\Phi(\bar{x}_{11},\bar{x}_{12}))\left(\nabla_{x_{11}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)\right\}\left\{\bar{x}_{12}{}^T G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{12}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)\right\} = \left\{\bar{x}_{12}{}^T G'\big(\Phi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{12}}\Phi(\bar{x}_{11},\bar{x}_{12})\right)\right\}\left\{\bar{x}_{11}{}^T G'\big(\Psi(\bar{x}_{11},\bar{x}_{12})\big)\left(\nabla_{x_{11}}\Psi(\bar{x}_{11},\bar{x}_{12})\right)\right\}. \tag{29}$$

By hypothesis (ii), we get

$$\bar{x}_{11}{}^{T} G'(\Psi(\bar{x}_{11}, \bar{x}_{12}))\nabla_{x_{11}}\Psi(\bar{x}_{11}, \bar{x}_{12}) G(\Phi(\bar{x}_{11}, \bar{x}_{12})) + \bar{x}_{12}{}^{T} G'(\Phi(\bar{x}_{11}, \bar{x}_{12}))\left(\nabla_{x_{12}}\Phi(\bar{x}_{11}, \bar{x}_{12})\right) G(\Psi(\bar{x}_{11}, \bar{x}_{12}))$$
$$= \bar{x}_{11}{}^{T} G'(\Phi(\bar{x}_{11}, \bar{x}_{12}))\left(\nabla_{x_{11}}\Phi(\bar{x}_{11}, \bar{x}_{12})\right) G(\Psi(\bar{x}_{11}, \bar{x}_{12})) + \bar{x}_{12}{}^{T} G'(\Psi(\bar{x}_{11}, \bar{x}_{12}))\left(\nabla_{x_{12}}\Psi(\bar{x}_{11}, \bar{x}_{12})\right) G(\Phi(\bar{x}_{11}, \bar{x}_{12})). \tag{30}$$

On subtracting (30) from (29) and after this we adding $G(\Phi\ (\bar{x}_{11}, \bar{x}_{12}))G(\Psi\ (\bar{x}_{11}, \bar{x}_{12}))$ of both sides, we have

$$G(\Phi(\bar{x}_{11}, \bar{x}_{12}))G(\Psi(\bar{x}_{11}, \bar{x}_{12})) - \bar{x}_{11}{}^{T} G'(\Psi(\bar{x}_{11}, \bar{x}_{12}))\left(\nabla_{x_{11}}\Psi(\bar{x}_{11}, \bar{x}_{12})\right) G(\Phi(\bar{x}_{11}, \bar{x}_{12}))$$
$$- \bar{x}_{12}{}^{T} G'(\Phi(\bar{x}_{11}, \bar{x}_{12}))\left(\nabla_{x_{12}}\Phi(\bar{x}_{11}, \bar{x}_{12})\right) G(\Psi(\bar{x}_{11}, \bar{x}_{12}))$$
$$+ \bar{x}_{11}{}^{T} G'(\Phi(\bar{x}_{11}, \bar{x}_{12}))\nabla_{x_{11}}\Phi(\bar{x}_{11}, \bar{x}_{12})\, \bar{x}_{12}{}^{T} G'(\Psi(\bar{x}_{11}, \bar{x}_{12}))\nabla_{x_{12}}\Psi(\bar{x}_{11}, \bar{x}_{12})$$
$$= G(\Phi(\bar{x}_{11}, \bar{x}_{12}))G(\Psi(\bar{x}_{11}, \bar{x}_{12})) - \bar{x}_{11}{}^{T} G'(\Phi(\bar{x}_{11}, \bar{x}_{12}))\left(\nabla_{x_{11}}\Phi(\bar{x}_{11}, \bar{x}_{12})\right) G(\Psi(\bar{x}_{11}, \bar{x}_{12}))$$
$$- \bar{x}_{12}{}^{T} G'(\Psi(\bar{x}_{11}, \bar{x}_{12}))\nabla_{x_{12}}\Psi(\bar{x}_{11}, \bar{x}_{12}) G(\Phi(\bar{x}_{11}, \bar{x}_{12}))$$
$$+ \bar{x}_{11}{}^{T} G'(\Psi(\bar{x}_{11}, \bar{x}_{12}))\nabla_{x_{11}}\Psi(\bar{x}_{11}, \bar{x}_{12})\, \bar{x}_{12}{}^{T} G'(\Phi(\bar{x}_{11}, \bar{x}_{12}))\nabla_{x_{12}}\Phi(\bar{x}_{11}, \bar{x}_{12}).$$

This can be rewritten as:

$$\frac{G(\Phi(\bar{x}_{11}, \bar{x}_{12})) - \bar{x}_{11}{}^{T} G'(\Phi(\bar{x}_{11}, \bar{x}_{12}))\left(\nabla_{x_{11}}\Phi(\bar{x}_{11}, \bar{x}_{12})\right)}{G(\Psi(\bar{x}_{11}, \bar{x}_{12})) - \bar{x}_{11}{}^{T} G'(\Psi(\bar{x}_{11}, \bar{x}_{12}))\left(\nabla_{x_{11}}\Psi(\bar{x}_{11}, \bar{x}_{12})\right)} = \frac{G(\Phi(\bar{x}_{11}, \bar{x}_{12})) - \bar{x}_{12}{}^{T} G'(\Phi(\bar{x}_{11}, \bar{x}_{12}))\left(\nabla_{x_{12}}\Phi(\bar{x}_{11}, \bar{x}_{12})\right)}{G(\Psi(\bar{x}_{11}, \bar{x}_{12})) - \bar{x}_{12}{}^{T} G'(\Psi(\bar{x}_{11}, \bar{x}_{12}))\left(\nabla_{x_{12}}\Psi(\bar{x}_{11}, \bar{x}_{12})\right)}.$$

If $(\bar{x}_{11}, \bar{x}_{12}, \bar{r})$ is not an optimal solution of (ESWD), then under the weak duality theorem, $\exists$ other $(\bar{u}_{11}, \bar{u}_{12}, W) \in Q^0$ such that $\bar{r} \geq W$. We arrive at the conclusion that that $\bar{r} \geq W$, which is in contradiction because $(\bar{x}_{11}, \bar{x}_{12}, \bar{r}) \in P^{0}$. Thus (ESWD) has the optimal solution $(\bar{x}_{11}, \bar{x}_{12}, \bar{r})$. This completes the proof.

Theorem 3.4 (Strict converse duality). Assume function Φ & Ψ be differentiable and (ESWD) has the optimal solution $(\bar{u}_{11}, \bar{u}_{12}, \bar{z}, \bar{v})$. Consider

(i) $\left[\left\{G''(\Phi(\bar{u}_{11}, \bar{u}_{12}))\nabla_{x_{11}}\Phi(\bar{u}_{11}, \bar{u}_{12})\left(\nabla_{x_{11}}\Phi(\bar{u}_{11}, \bar{u}_{12})\right)^{T} + G'(\Phi(\bar{u}_{11}, \bar{u}_{12}))\nabla_{x_{11}x_{11}}\Phi(\bar{u}_{11}, \bar{u}_{12})\right\} - \bar{z}\left\{G''(\Psi(\bar{u}_{11}, \bar{u}_{12}))\nabla_{x_{11}}\Psi(\bar{u}_{11}, \bar{u}_{12})\left(\nabla_{x_{11}}\Psi(\bar{u}_{11}, \bar{u}_{12})\right)^{T} + G'(\Psi(\bar{u}_{11}, \bar{u}_{12}))\nabla_{x_{11}x_{11}}\Psi(\bar{u}_{11}, \bar{u}_{12})\right\}\right]$ is non-singular,

(ii) $\bar{u}_{11}{}^{T} G'(\Psi(\bar{u}_{11}, \bar{u}_{12}))\nabla_{x_{12}}\Psi(\bar{u}_{11}, \bar{u}_{12}) - \bar{u}_{11}{}^{T} G'(\Psi(\bar{u}_{11}, \bar{u}_{12}))\nabla_{x_{11}}\Psi(\bar{u}_{11}, \bar{u}_{12})\left(G(\Phi(\bar{u}_{11}, \bar{u}_{12}))\right) + \bar{u}_{11}{}^{T} G'(\Phi(\bar{u}_{11}, \bar{u}_{12}))\nabla_{x_{11}}\Phi(\bar{u}_{11}, \bar{u}_{12}) - \bar{u}_{11}{}^{T} G'(\Phi(\bar{u}_{11}, \bar{u}_{12}))\nabla_{x_{12}}\Phi(\bar{u}_{11}, \bar{u}_{12})\left(G(\Psi(\bar{u}_{11}, \bar{u}_{12}))\right) = 0$,

(iii) The equality

$$\bar{v}^T \left[\nabla_{x_{11}} \left\{G''\left(\Phi(\bar{u}_{11}, \bar{u}_{12})\right)\nabla_{x_{11}}\Phi(\bar{u}_{11}, \bar{u}_{12}) \left(\nabla_{x_{11}}\Phi(\bar{u}_{11}, \bar{u}_{12})\right)^T + G'\left(\Phi(\bar{u}_{11}, \bar{u}_{12})\right)\nabla_{x_{11}x_{11}}\Phi(\bar{u}_{11}, \bar{u}_{12})\right\}\bar{v} - \bar{z}\left\{G''\left(\Psi(\bar{u}_{11}, \bar{u}_{12})\right)\nabla_{x_{11}}\Psi(\bar{u}_{11}, \bar{u}_{12}) \left(\nabla_{x_{11}}\Psi(\bar{u}_{11}, \bar{u}_{12})\right)^T + G'\left(\Psi(\bar{u}_{11}, \bar{u}_{12})\right)\nabla_{x_{11}x_{11}}\Psi(\bar{u}_{11}, \bar{u}_{12})\right\}\bar{v}\right] = 0$$

$$\Rightarrow \ \bar{v} = 0.$$

Then, $(\bar{u}_{11}, \bar{u}_{12}, \bar{z}, \bar{v}) \in P^0$ and objective values of (ESWP) & (ESWD) are equal. Additionally, the solution $(\bar{u}_{11}, \bar{u}_{12}, \bar{z}, \bar{v} = 0)$ is the optimal for the weak duality theorem if all of its hypotheses are met (ESWP).

Proof: Due to a symmetricity along the lines of Theorem 3.3 the proof proceeds.

4. Conclusion and Future Work

In the present article, we have considered a pair of G-Wolfe type second-order fractional symmetric dual programming problem and derived weak, strong and converse duality theorems under (G, λ, θ) −bonvexity/(G, λ, θ) −pseudobonvexity conditions. This work can be implemented in multi-objective fractional symmetric dual programs with arbitrary constraints. Extension of this work can also be applied in nondiffrentiable programming of multi-objective symmetric problems, in which cone objective function as well as cone constraints. This might be considered the researchers' upcoming task.

Conflicts of Interest

The authors declare no conflict of interest.

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