Generalized Adaptors with Memory for Nonlinear Wave Digital Structures

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ABSTRACT

The problem of modeling a nonlinear resistor in the Wave Digital domain can be seen as that of applying to its nonlinear characteristic the affine transformation that maps Khirchhoff variables into wave variables. When dealing with nonlinear elements with memory, such as nonlinear capacitors and inductors, the above approach cannot be applied, as affine transformations are memoryless.

In this paper, a new approach is proposed for modeling nonlinear elements with memory in the wave domain. The method we propose defines a more general class of wave variables and adaptors with memory that, under some conditions, can incorporate the "memory" of a nonlinear circuit and allow us to treat some nonlinear elements with memory as if they were instantaneous.

1 Introduction

Wave Digital Filters (WDF's) [1] are known to possess several desirable properties not found in other digital filter implementations. In fact, not only are WDF structures based on analog circuits, but they tend to preserve most of the characteristics of their analog counterpart. For example, *passivity* and *losslessness* of analog filters are preserved by their wave digital implementation. In addition, the behavior of WDF is little sensitive to coefficient quantization, therefore we may have good dynamical range performance with modest accuracy requirements. The inner resemblance of WDF's to their physical counterpart can also be useful when needing to reproduce the qualitative behavior of some physical system without giving up the flexibility of computing systems.

Modeling nonlinearities in the wave domain is possible in many cases where the nonlinear element is memoryless [2], in which case its nonlinear characteristic can be mapped directly into the wave domain through the affine transformation that defines the wave variables as a function of the Khirchhoff variables. In order to model reactive nonlinearities or, more generally, nonlinear elements with memory, classical WDF principles are no longer adequate. In this paper we propose an extension of the classical WDF theory, based on a new class of wave variables and generalized adaptors that, besides allowing us to model a variety of nonlinear elements with memory in the wave domain, gives us a new perspective on the classical WDF theory as well.

2 Preliminaries

Basic elements of WDF's are adaptors, which are memoryless devices whose task is to perform transformations between pairs of wave variables that are referred to different levels of port resistance. In order to avoid noncomputability problems in the interconnection of the wave models of all circuit elements, wave adaptors can be made reflection-free at one port by adding a constraint to the levels of reference port resistance [1].

Under mild conditions we can simulate the behavior of a circuit containing a nonlinear resistor by connecting an appropriate instantaneous map with a reflection-free port of an adaptor [2]. The characteristic $F(v,i)=0$ (in the Khirchhoff domain) of a nonlinear resistor, in fact, can be transformed into the *Wave* domain through the change of variables $v^+ = v + Ri$, $v^- = v - Ri$. The corresponding *wave* characteristic $f(v^+, v^-) = 0$ is thus given by

$$f(v^+, v^-) = F\left(\frac{v^+ + v^-}{2}, \frac{v^+ - v^-}{2R}\right) .$$

The conditions under which the reflected wave $v^-(t)$ can be written as a function $v^- = g(v^+)$ of the incident wave $v^+(t)$ can be derived from the *implicit function theorem*. For example, in the case of the piece-wise continuous characteristic [2] of a voltage-controlled resistor $i = i(v)$, the explicitability of v^- is guaranteed if either one of the following conditions is satisfied

$$\inf_{v_2 \neq v_1} \frac{i(v_2) - i(v_1)}{v_2 - v_1} > -\frac{1}{R} , \quad \sup_{v_2 \neq v_1} \frac{i(v_2) - i(v_1)}{v_2 - v_1} < -\frac{1}{R} .$$

3 Adaptors with Memory

Instantaneous nonlinearities (nonlinear resistors) can be mapped onto the wave domain as they are described by an algebraic equation, therefore it is sufficient to use the definition of wave variables as a transformation of variables in order to obtain the wave counterpart of such nonlinear elements. This coordinate transformation warps the nonlinear characteristic through a combination of rotation and shear.

When dealing with nonlinear elements with memory, i.e. circuit elements that are described by a differential equation rather than an algebraic one, traditional tools provided by classical WDF theory are no longer sufficient. As a matter of fact, even the simplest case of purely reactive element, such as a nonlinear capacitor or a nonlinear inductor, cannot be directly mapped onto the *wave* domain by using classical WDF principles, as we would have to cope with non-computable connections by solving an implicit equation per output sample [3]. In order to be able to deal with nonlinear elements with memory, we define a new class of wave *adaptors with memory*, that can be used for incorporating the memory of any linear (and, in some case, nonlinear) circuit that they are connected to.

Let us consider the Laplace-transform $(V(s), I(s))$ of a pair of Khirchhoff variables $(v(t), i(t))$. Instead of defining a wave pair with reference to a generic resistance R (see [1]) we define the new pair of wave variables $V^+(s) = V(s)+Z(s)I(s)$, and $V^+(s) = V(s)-Z(s)I(s)$, $Z(s)$ being a *reference impedance* which, in fact, could be the transfer function of any linear circuit. By doing so, we incorporate part of the past history of the Khirchhoff variables into the wave variables $v^+(t)$ and $v^-(t)$.

An immediate consequence of the above definition is in the structure of the adaptors between wave pairs that are referred to different reference impedances. Let us consider, for example, the case of a scattering junction (see Figure 1) between two wave pairs $(V_1^+(s), V_1^-(s))$ and $(V_2^+(s), V_2^-(s))$, which are referred to $Z_1(s)$ and $Z_2(s)$, respectively. From the definition of the wave pairs

$$V_1^+ = V_1 + Z_1 I_1 \qquad V_1^- = V_1 - Z_1 I_1$$
$$V_2^+ = V_2 + Z_2 I_2 \qquad V_2^- = V_2 - Z_2 I_2$$

and the continuity constraints, $V_1 = V_2$ and $I_1 = I_2$, at the scattering junction, we may quite easily express the Laplace-transforms of the waves that are entering the junction as a function of those that are exiting it

$$\begin{aligned} V_1^- &= K V_1^+ + (1-K) V_2^- \\ V_2^+ &= (1+K) V_1^+ - K V_2^- \ , \end{aligned}$$

where

$$K(s) = \frac{Z_2(s) - Z_1(s)}{Z_2(s) + Z_1(s)}$$

is the transfer function of the “reflection” filter that characterizes the scattering junction with memory.

The digital version of the above junction can be obtained through any suitable mapping from the Laplace-transform plane to the Zeta-transform plane, such as the

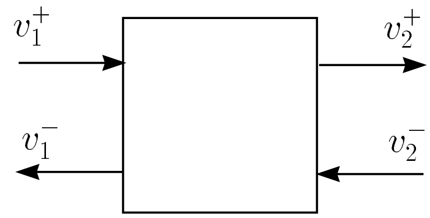

Figure 1: Scattering junction.

bilinear transformation

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \ .$$

It is important to notice that, when the *instantaneous* portions of Z_1 e Z_2 are equal, both ports of the scattering junction result as having no instantaneous reflection.

Particularly interesting is the case in which Z_1 is purely resistive ($Z_1 = R$), while Z_2 is purely reactive. For example, in the case of ideal inductance $Z_2(s) = sL$, the scattering equations in the Zeta domain are

$$\begin{aligned} V_1^-(z) &= K(z) V_1^+(z) + (1 - K(z)) V_2^-(z) \\ V_2^+(z) &= (1 + K(z)) V_1^+(z) - K(z) V_2^-(z) \ , \end{aligned}$$

where

$$K(z) = \frac{Z_2(z) - R}{Z_2(z) + R} \ , \quad Z_2(z) = \frac{2L}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \ .$$

Notice that we may eliminate the instantaneous reflections at both ports by letting $R = 2L/T$, in which case we have $r = -z^{-1}$.

This last result is quite interesting as we have just found that, in order to “adapt” a linear inductor to a resistor, we may use a scattering cell where the reflection coefficient is replaced by a pure delay with sign change, and whose second port is left open. In this particular case, the whole scattering junction can be replaced with a pure delay with sign change, which is how the classical WDF theory [1] deals with linear inductors.

It is also important to notice that the adaptation between a purely resistive impedance and a purely inductive one can be used for extending the results of Meerkotter [2] on nonlinear resistors (see Section 2) to the case of nonlinear inductors. In fact, the output of an scattering junction that adapts R to sL, $L > 0$ is given by the wave pair

$$\begin{aligned} V_2^+(s) &= V(s) + LsI(s) = V(s) + LJ(s) \\ V_2^-(s) &= V(s) - LsI(s) = V(s) - LJ(s) \end{aligned}$$

where $J(s)$ is the Laplace-transform of $j(t) = di(t)/dt$. As a consequence, since a nonlinear inductor can be described by an algebraic relationship of the form

$M(v,j) = 0$ between the voltage v and the derivative of the current j, we may use the results of Section 2 on nonlinear resistors [2] by letting L play the role of a "reference inductance" in the affine transformation that maps the Khirchhoff characteristic of the nonlinear inductor onto the wave domain.

The case of the nonlinear capacitors is very similar to that of the nonlinear inductors. In fact, the scattering junction that adapts a purely capacitive impedance to a purely resistive one will be the same as the above, with the only difference in that $R(z) = z^{-1}$, as expected from the classic WDF theory. Similarly to what we have seen above, we can use this type of adaptors in the case of nonlinear capacitors. In fact, the output of an scattering junction that adapts R to $1/(sC)$, $C > 0$, is given by the wave pair $V_2^+ = V + I/(sC)$, $V_2^- = V - I/(sC)$, i.e. $\Phi_2^+ = \Phi + I/C$, $\Phi_2^- = \Phi - I/C$, where $\Phi(s)$ is the Laplace-transform of $\phi(t)$, $\dot{\phi}(t) = v(t)$. As a nonlinear capacitor can be described by an algebraic relationship of the form $P(\phi, i) = 0$ between the integral of the voltage ϕ and the current j, we can use the results of Section 2 by letting $1/C$ play the role of "reference admittance" in the affine transformation that maps the Khirchhoff characteristic of the nonlinear inductor onto the wave domain.

A more general case is that in which $Z_1 = R$ and $Z_2 = Z(z)$ is a generic impedance of the form

$$Z_2(z) = \frac{A(z)}{B(z)} = \frac{a_0 + \sum_{i=0} a_i z^{-i}}{1 + \sum_{i=1} b_i z^{-i}} \ .$$

The instantaneous portion of $Z_2(z)$ is represented by the coefficient a_0, therefore the absence of instantaneous reflections at the two ports can be achieved by letting $R = a_0$. This choice yields

$$R(z) = \frac{Z_2(z) - a_0}{Z_2(z) + a_0} = \frac{\sum_{i=1}(a_i - a_0 b_i) z^{-i}}{2a_0 + \sum_{i=1}(a_i + a_0 b_i) z^{-i}} \ .$$

Once again, when $Z_2(s) = 1/(sC)$, and $R = T/(2C)$, the "reflection filter" assumes the form $K(z) = 2a_0 z^{-1}/2a_0 = z^{-1}$.

When, as it often happens, $Z_2(z)$ is a causal FIR filter, its transfer function can be written as $Z_2(z) = Z_0[1 + z^{-1}H(z)]$. In this case, the adaptation condition $Z_1 = R = Z_0$ results in a reflection filter of the form

$$K(z) = \frac{z^{-1}H(z)}{2 + z^{-1}H(z)} \ ,$$

which can be implemented by properly inserting the FIR filter $z^{-1}H(z)/2$ in a feedback configuration.

The approach proposed above for deriving scattering junctions with memory can be readily extended to multiport parallel or series junctions by combining the new definitions of wave variables at the various ports, with the Khirchhoff equations that characterize a parallel or a series connection. The multiport adaptors turn out to be structured in same way as those derived by Fettweis [1] provided that the reflection coefficients are replaced by reflection filters. For example, a series connection of n ports with port impedances $Z_1(z)$ to $Z_n(z)$ is characterized by the Khirchhoff equations $V_1(z) + \ldots + V_n(z) = 0$ and $I_1 = \ldots = I_n(z)$. The Zeta-transforms of the m-th output wave, $m = 1, \ldots, n$, can thus be written as a function of all input waves as

$$V_m^-(z) = V_m^+(z) - \Gamma_m(z)(V_1^+(z) + \ldots + V_n^+(z)) \ ,$$

where

$$\Gamma_m(z) = \frac{2Z_m(z)}{Z_1(z) + \ldots + Z_n(z)}$$

are the reflection filters. In order to make one port of the junction reflection-free, for example the n-th one, it is sufficient for the "instantaneous" coefficient of $\Gamma(z)$ to be equal to one.

4 Examples of Applications

Chaotic behavior in electrical circuits is due in most cases to a nonlinear resistance. There are, however, several examples of circuits that contain a nonlinear reactance and exhibit, in certain conditions, particularly interesting phenomena such as period doubling (generation of subharmonic oscillation) and chaotic dynamics. The accuracy of the computer simulation of such circuits is usually quite sensitive to the errors caused by discretization. An example of this type, whose simulation in the wave digital domain has been studied in depth by Felderhoff [3], is represented by the anharmonic oscillator [4] of Fig. 2. This simple RLC circuit is characterized by a nonlinear voltage-controlled capacitance, whose $q - v$ characteristic

$$q = C_0 \frac{v}{\sqrt{1 + v/v_0}} \ , v > -v_0$$

is shown in Fig. 3. The parameters used for the simulation of such a circuit are $v_0 = 0.6V$, $R = 180\Omega$, $L = 100\mu H$, $C_0 = 80pF$, and the voltage supplied by the ideal generator is $v(t) = e_0 \sin(2\pi f_0 t)$, $f_0 = 1/(2\pi\sqrt{LC_0})$. For $v \leq v_0$ the nonlinear element behaves like an active resistor, but since we are studying the chaotic behavior of the circuit, we can assume $v > v_0$ holds throughout the simulation, provided the initial conditions are properly chosen. The behavior of the varactor oscillator is studied for different values e_0 of the voltage generator amplitude.

By using an adapted scattering junction that transforms a purely resistive port resistance into a purely capacitive one, as explained in Section 3, we may use the results of Section 2 for mapping the nonlinear characteristic of the capacitor (Fig. 3) onto the wave domain (Fig. 4).

Unlike Felderhoff's implementation [3], the varactor oscillator can now be implemented in the wave domain without "computability" problems, therefore there is no

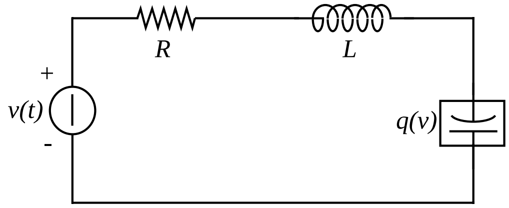

Figure 2: Electrical circuit of the anharmonic oscillator.

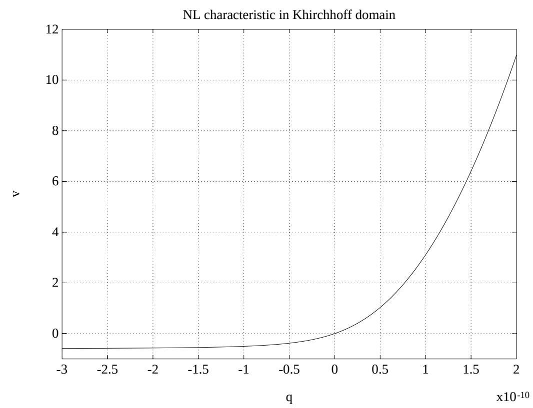

Figure 3: Nonlinear characteristic of the capacitor of the anharmonic oscillator in the Khirchhoff domain.

need of solving implicit equations for overcoming noncomputability problems. A phase portrait of the varactor's trajectories in the state space is shown in Fig. 5 for $T = \frac{1}{32 f_0}$. Such a simulation is very little sensitive to discretization errors, and the complexity associated to it is now rather modest.

5 Conclusions

In this paper, a generalization of the Wave Digital Filter theory has been proposed in order to be able to implement nonlinear elements with memory, such as nonlinear reactances, in the wave domain. The proposed extension of WDF theory gives us a new perspective on classical Wave Digital Filters. In fact, the well-known WDF structures associated to linear circuits, can be now be re-obtained in a different way, together with alternative structures for implementing them.

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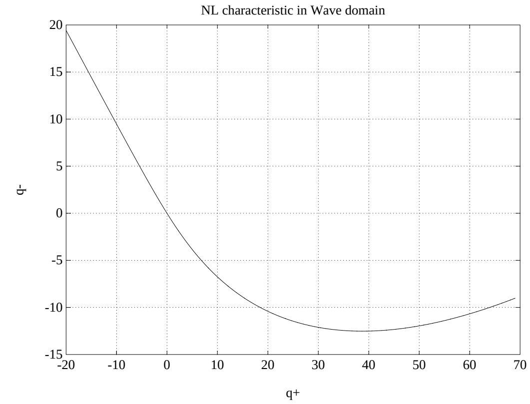

Figure 4: Nonlinear characteristic of the capacitor of the anharmonic oscillator in the wave domain.

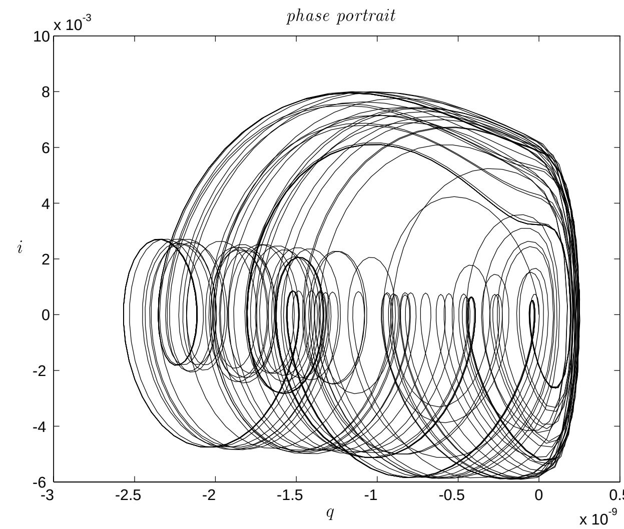

Figure 5: Phase portrait for $e_0 = 3.57V$ and $T = \frac{1}{32 f_0}$.

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