Selection of Sampling Grid and Prefilter for Image Decimation Based on Spectral Extension Analysis

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ABSTRACT

Signal decimation aimed at optimal spectral packing has a variety of applications in areas ranging from array processing to image processing. In this article we propose and discuss a new method for determining decimation grid and prefilter that best fit the spectral extension of any 2D signal defined on an arbitrary sampling lattice. The method has been implemented and tested on digital images in order to evaluate quality degradation due to optimal spectral truncation.

1 Introduction

Sampling multidimensional analog signals causes their spectrum to replicate over a regular point structure whose density is inversely proportional to the sampling density. Minimizing the gap among spectral replicas is well-known to reduce information redundancy [1]. This fact could be useful in a variety of applications that range from from array processing [2] to Image Processing [1, 3, 4].

Spectral packing through decimation is not an easy task as it consists not just of a rational selection of data samples, but it also needs a careful spectral truncation for avoiding aliasing. In order to perform anti-aliasing prefiltering, in fact, knowing the area of the spectral extension (spectral occupancy) is not sufficient: we also need to consider its shape. The spectral energy of non-synthetic images, however, usually occupies regions with quite irregular and complex shape [4], which makes the optimal design of a prefilter very difficult. On the other hand, the spectral extension may be defined and estimated according to the class of prefilter we adopt for decimation purposes. More specifically, if we restrict the class of prefilters we are interested in to those having an arbitrary *compact* and *convex* passband, all we need about spectral extension shape is the direction around which the spectral energy is maximally concentrated (principal axis) and a measure of the energy dispersion about this axis. This way of quantifying the anisotropy of the spectral distribution corresponds to approximating the spectral extension with an ellipse whose shape is decided by the ratio between the inertia moments of the power spectrum, while its size is chosen according to the severity of the spectral truncation we are willing to apply.

In this paper we propose and test a method, based on the above convexity assumption, for jointly and automatically determining decimation grid and prefilter for two-dimensional discrete signals defined on arbitrary lattices. The method has been implemented in a fully automatic computer procedure and tested on several images.

2 Mathematical preliminaries

Given a non-singular matrix $\mathbf{A} \in R^M$, the M-dimensional lattice Λ generated by $\mathbf{A}$ is defined as:

$$\Lambda = LAT(\mathbf{A}) = \left\{ \mathbf{x} \in R^M \middle|\ \mathbf{x} = \mathbf{A}\mathbf{n},\ \mathbf{n} \in Z^M \right\} ,$$

which is the set of all possible linear combinations with integer coefficients, of the M linearly independent vectors (*basis* of the lattice) that are given by the columns of $\mathbf{A}$.

Given a basis $\mathbf{A}$, it is possible to derive any other basis $\mathbf{A}'$ of $\Lambda = LAT(\mathbf{A})$, as $\mathbf{A}' = \mathbf{A}\mathbf{U}$, $\mathbf{U}$ being a *unimodular* matrix ($|\det(\mathbf{A})| = 1$).

A fundamental cell $\mathcal{S}$ of an M-dimensional lattice Λ is a closed region of R^M such that the collection $S_\Lambda = \{\mathcal{S} + \mathbf{a}, \mathbf{a} \in \Lambda\}$ of all shifted version of $\mathcal{S}$ on the points of Λ *tiles* R^M without overlapping. Notice that there exist infinite fundamental cells for a single lattice, but their hypervolume is always the same and is given by $d(\Lambda) = |\det(\mathbf{A})|$ (a measure of the lattice *density* is thus $1/d(\Lambda)$). Notice also that there exist no general geometric classifications of all possible fundamental cells of a given lattice. The only results that are available in the literature concern convex cells [5] and are particularly simple in the 2D case. In fact, the only convex regions that tile R^2 are parallelograms and hexagons with central symmetry.

$\Gamma = LAT(\mathbf{B})$ is a *sublattice* of $\Lambda = LAT(\mathbf{A})$ (i.e. a subset of Λ with lattice structure) if and only if there exists a non-singular integer matrix $\mathbf{H}$ such that $\mathbf{B} = \mathbf{A}\mathbf{H}$. The integer number $|\det(\mathbf{H})| = |\det(\mathbf{B})|/|\det(\mathbf{A})|$ corresponds to the *decimation ratio*. All k-th order sublattices of $\Lambda = LAT(\mathbf{A})$ can be obtained as $\Gamma_i = LAT(\mathbf{W}_i)$, where $\mathbf{W}_i = \mathbf{A}\mathbf{H}_i$, $\mathbf{H}_i \in \mathcal{H}_{M,k}$, where $\mathcal{H}_{M,k}$ is the set of integer matrices in Hermite normal form whose determinant is equal to k [4].

The Fourier transform $U(\mathbf{f})$ of a discrete signal $u(\mathbf{x})$, defined on the lattice Λ, is periodic and its periodicity centers are specified by $\Lambda^\star = LAT\left(\mathbf{A}^{-T}\right)$, which is called "reciprocal lattice" of Λ [3]. The Fourier transform $U(\mathbf{f})$ is thus completely specified by its values in any fundamental cell $\mathcal{P}$ of $\Lambda^\star$. Decimating u from $\Lambda = LAT(\mathbf{A})$ to $\Gamma = LAT(\mathbf{B})$, Γ being an n-th order sublattice of Λ, returns the signal $v(\mathbf{x}) = u(\mathbf{x})$, $\mathbf{x} \in \Gamma$, and the relationship between Fourier transforms [3] results as

$$V(\mathbf{f}) = \frac{1}{n} \sum_{\mathbf{a} \in \mathcal{I}} U\left(\mathbf{f} + \mathbf{a}\right) \tag{1}$$

$\mathcal{I}$ being any $\Lambda^\star$-period of $\Gamma^\star$. In order to be able to perfectly reconstruct a signal u defined on Λ from its decimated version v on Γ, it is thus necessary for the support of $U(\mathbf{f})$ to be confined inside some fundamental cell $\mathcal{P}$ of $\Gamma^\star$. In this case the reconstruction can be done by using an interpolating filter from Γ to Λ, having frequency response $H(\mathbf{f}) = n$ in $\mathcal{P}$, and zero elsewhere.

3 Decimation approach

The first step of the procedure we propose consists of estimating the spectral extension of the signal. From such information we determine an upper bound for the index of the sublattices to choose among, by simply computing the spectral occupancy of the estimated spectral extension. The maximum index of decimation k_0 represents the order from which to start looking for suitable decimation grids.

Once the maximum decimation ratio k_0 is available, we generate all distinct k_0-th order sublattices and, among them, we discard all those that are *non-compatible* with the spectral extension. If no compatible subgrids of order k_0 can be found, we decrease the order and repeat the search until some compatible ones are found. For each one of them, we generate the fundamental cell that best fits the elliptical spectral extension, by using a geometrical approach. A selection of the best sublattice-cell pair can finally be performed among the remaining candidates, according to some specific criterion.

As already mentioned in the Introduction, the class of prefilters we are considering has a *convex* passband region. As the only convex regions that tile R^2 are parallelograms and hexagons with central symmetry, the only shape information on the spectral extension we need is given by the direction around which the spectral energy is maximally concentrated and a measure of the energy dispersion about that axis. Quantifying the spectral anisotropy through the second-order energy distribution corresponds to approximating the spectral extension with an ellipse whose shape is decided by the ratio between the inertia moments of the power spectrum, while its size is chosen according to the severity of the spectral truncation we are willing to apply. Spectral extension estimation, as a consequence, consists of finding the inertia ellipse of the power spectrum:

$$\rho_1^2 d_1^2 + \rho_2^2 d_2^2 = r^2 , \tag{2}$$

whose coordinates (d_1, d_2) are referred to the principal axes of the power spectral distribution, while ρ_1 and ρ_2 are the *radii of gyration* (which, in turn, are a function of the inertia moments). The parameter r can be chosen in such a way for the ellipse to enclose a specified portion of the signal energy.

The area $A = \pi\rho_1\rho_2$ of the spectral ellipse can be used as an estimate of the spectral occupancy of the signal. As a consequence, the largest integer k_0 below area$(\mathcal{P})/A$, $\mathcal{P}$ being a fundamental cell of the sampling grid, can be used as an upper bound for the index of the sublattices that could be used for decimation.

Once the maximum decimation ratio k_0 is available, we can generate all k-th order sublattices, with $k \leq k_0$, by determining all matrices in Hermite normal form, with determinant k .

The spectral extension can now be used for deciding whether the sublattice Γ of the original signal support Λ is suitable for decimation. More precisely, we need to check its *compatibility* by verifying that the elliptical spectral replicas, generated by Γ-decimation, does not overlap. In order to do so, we first need to determine the k points of the reciprocal lattice $\Gamma^\star$ that fall inside one $\Lambda^\star$-period of R^2 (centers of replication of the original spectrum). We then should test whether overlappings between any two replicas of the spectral extension occur, which can be done by checking for overlappings between the replica about the origin and the others.

The fact that the spectral extension model is elliptical makes the compatibility check particularly simple. In fact, the limit-region for the replication centers beyond which no overlapping occurs is itself an ellipse (threshold ellipse) whose radii of gyration are twice the spectral ellipse's radii, as shown in Fig. 1.

A sublattice Γ is compatible with the signal spectral extension if none of the replicas overlap with the original extension, i.e. if all points of $\Gamma^\star$ lie outside the threshold ellipse. All compatible subgrids are good candidates for decimation, therefore we need a criterion for deciding among them. As the choice of decimation grid is strongly influenced by the shape of the passband region of the prefilter (fundamental cell of the reciprocal of the sublattice), in order to be able to decide among the compatible subgrids we need a fast method for generating a fundamental cell that well fits the spectral extension, for each compatible sublattice.

In general, there exists a non-numerable multitude of fundamental cells for a sublattice, and the degrees of

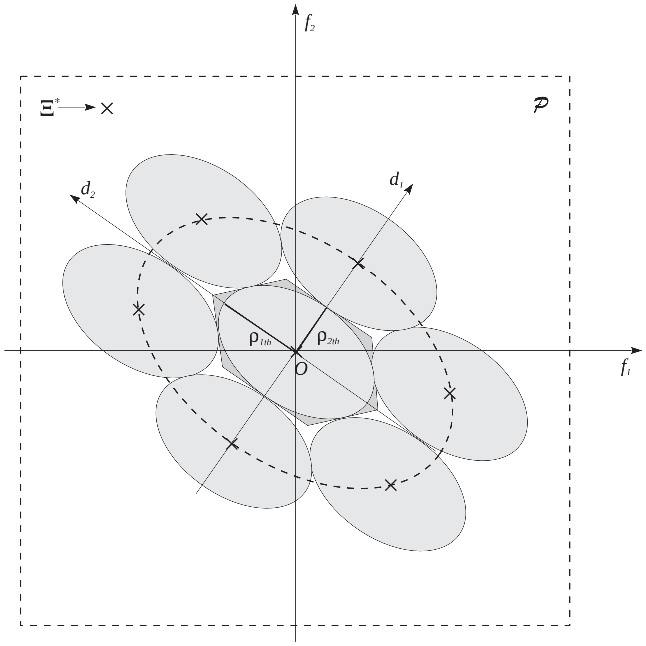

Figure 1: Threshold ellipse for checking the compatibility of a sublattice $\Gamma \subset \Lambda$.

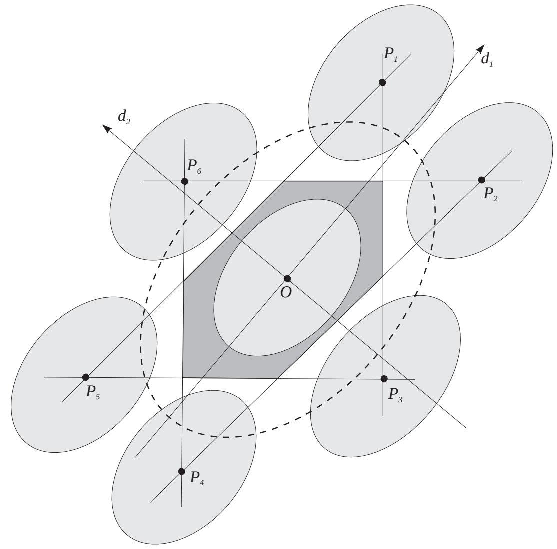

Figure 2: Construction of the prefilter: the hexagon is given by the intersection between the two triangles $\{P_1, P_3, P_5\}$ and $\{P_2, P_4, P_6\}$ built on the six closest points of the lattice.

freedom in their shape are enough to make the search extremely difficult. Restricting the class of fundamental cells to the convex ones, however, greatly simplifies the situation as, according to the results of Section 2, all convex fundamental cells of a two-dimensional lattice are hexagons with central symmetry.

A method for determining a hexagonal fundamental cell of a given *compatible* sublattice which entirely encircles the elliptical spectral extension, is described in Fig. 2. The method consists of determining the 6 points of $\Gamma^\star$ that are closest to the threshold ellipse, building two triangles by using two triplets of alternate points, and determining the hexagonal cell as the intersection of the two triangles. Notice that, when all six points used for constructing the hexagon lie on the threshold ellipse, the above method generates the tangent hexagon of Fig. 1. The fundamental cell we obtain can be used for constructing both the anti-aliasing and the reconstruction filters. In fact, both of them will have a pass-band region that resembles the fundamental cell determined above.

Last step of the decimation procedure consists of choosing the best decimation setup among the compatible subgrid/prefilter pairs. Notice that all the available candidates are acceptably good, therefore the choice must be made according to some criterion of optimality that takes into account some measure of the *fitness* between fundamental cell and spectral extension. In practice, we select the sublattice whose prefilter has minimum impact on the principal axis of the power spectral distribution. More precisely, we can choose the prefilter that minimizes the angle between the principal axis of the non-prefiltered spectrum and that of the prefiltered spectrum.

4 Examples of application

The method proposed in this article has been implemented into a completely automatic computer procedure and tested over a series of real images.

An example of application is shown in in Fig. 3*a*. The spectrum of this image (Fig. 3*b*), exhibits a certain anisotropy. The principal axes of the elliptical spectral extension have been chosen to be 3.5 times the inertia axes of the power spectrum samples. The maximum order of decimation in which some compatible sublattices can be found is $k = 15$. Among the hexagonal fundamental cells that are associated to all compatible 15-th order sublattices, the one whose principal axes are closest to those of the elliptical spectral extension is chosen to define the passband region of the image prefilter (see Fig. 3*b*), while the relative sublattice is the corresponding decimation grid. At this point the image can be prefiltered through spectral windowing, by using a smoothened version (in order to prevent ringing from occurring) of the ideal prefiltered obtained above. Decimating the image over the selected subgrid causes the truncated power spectrum to replicate like in Fig. 3*c* where it is quite apparent how the elongation of the power spectrum due to the prevalence of some edges along a specific direction, causes subgrid and prefilter to preserve the spectrum in that direction.

The same filter used for avoiding aliasing is now used for reconstructing the original image from the decimated one. The reconstruction results of the test image are reported in Fig. 3*d*, where a comparison is made between corresponding zoomed-in details of original (left) and reconstructed (right) images. As we can see, the blurring due to the low-pass anti-aliasing filtering is still acceptable despite a reduction of 15 times in the amount of samples that are actually being used for describing the image itself. This result would not be possible with more traditional decimation methods.

5 Conclusions

In this paper we presented a new technique for decimating discrete 2D signals, which is capable of considerably reducing the spectral redundancy while suppressing the least amount of spectral energy. Spectral characteristics of the signal are taken into account for determining both decimation grid and anti-alias filter.

We have implemented our decimation technique into a fully-automatic computer procedure and tested it over a series of images. The quality of the reconstruction results have proven the effectiveness of the method, showing that decimation factors between 10 and 20 can be reached with acceptable blurring, in spite of the fact that the resulting sampling grids are uniform.

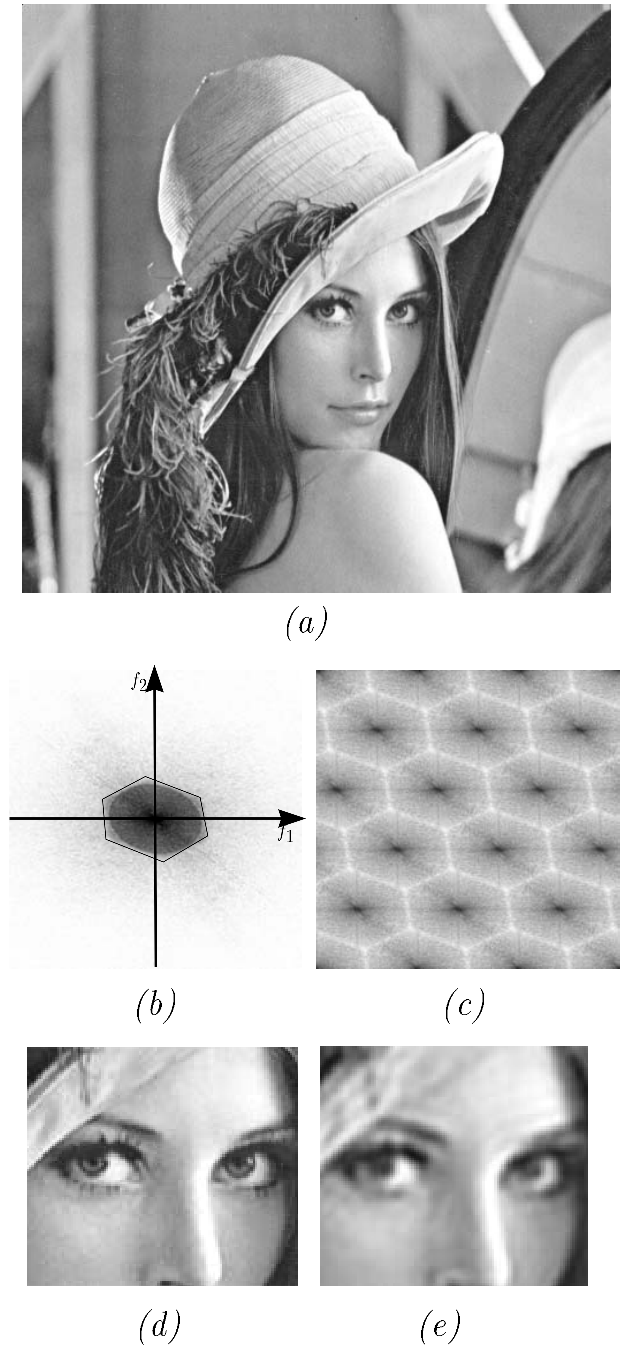

Figure 3: Decimation of the image "Lenna" with the 15^{th}-order decimation basis $\mathbf{v}_1 = [5\ 0]^T$, $\mathbf{v}_2 = [2\ 3]^T$: *a)* original image, *b)* power spectrum of the original image (log scale), elliptical extension and optimal prefilter, *c)* spectrum of the decimated image, zoomed-in details of the original *(d)* and the reconstructed *(e)* images.

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