INDEXES OF GENERIC GRASSMANNIANS FOR SPIN GROUPS

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Abstract. Given integers d and m, satisfying $1 \leq m \leq d/2$, and an arbitrary base field, let X_m be the m-th grassmannian of a generic d-dimensional quadratic form of trivial discriminant and Clifford invariant. The index of X_m, defined as the g.c.d. of degrees of its closed points, is a 2-power $2^{\mathrm{i}(m)}$. We find a strong lower bound on the exponent $\mathrm{i}(m)$ which is its exact value for most d, m and which is always within 1 from the exact value.

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1. Introduction

The *index* $\mathrm{ind}(X)$ of a variety X over a field F is the g.c.d. of the degrees of all closed points of X, or equivalently, the g.c.d. of the degrees of all finite field extensions K/F such that X has a point over K.

Let G be an algebraic group over a field F_0. The *torsion index* $\tau(G)$ of G is the l.c.m. of the integers $\mathrm{ind}(E)$ over all G-torsors E over extension fields F of F_0 (see [7]). If G is split semisimple and $B \subset G$ is a Borel subgroup, then $\mathrm{ind}(E) = \mathrm{ind}(E/B)$ (see [6, Lemma 6.5]) so that $\tau(G) = \gcd \mathrm{ind}(E/B)$ over all E as above. Moreover, by a theorem of A. Grothendieck ([3, Théorème 2], see also [6, Theorem 6.4]), $\tau(G) = \mathrm{ind}(E_{\mathrm{gen}}/B)$,

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where E_{gen} is a *generic* G-torsor, that is the generic fiber of the quotient morphism $\mathrm{GL}(N) \to \mathrm{GL}(N)/G$ for an embedding of G into the general linear group $\mathrm{GL}(N)$ for some $N \geq 1$.

Example 1.1. Let $G = \mathrm{SO}(2n+2)$ be the special orthogonal group of a hyperbolic quadratic form of dimension $2n+2$. A G-torsor over a field $F \supset F_0$ is a non-degenerate $(2n+2)$-dimensional quadratic form q of trivial discriminant over F. Write q as an orthogonal sum of binary forms $q_1, \dots, q_{n+1}$. Every form q_i is hyperbolic over at most quadratic extension of F, hence q is split over an extension of degree at most 2^n. In fact, this is a sharp bound: we have $\tau(G) = 2^n$ (see [7, Theorem 3.2]).

The value $\tau(G)$ is the same for $G = \mathrm{SO}(2n+1)$.

Example 1.2. Let $G = \mathrm{Spin}(2n+2)$ be the spin group of a hyperbolic quadratic form of dimension $2n+2$. A G-torsor over a field $F \supset F_0$ yields a non-degenerate $(2n+2)$-dimensional quadratic form q of trivial discriminant and trivial Clifford invariant over F. Since every such form of dimension less that 8 is hyperbolic, q can be split by an extension of degree at most 2^{n-2}. In fact one can do better: in [7] Totaro proved that $\tau(G) = 2^t$, where $t = t(n)$ is equal to

$$n - \left\lfloor \log_2 \left(1 + \frac{n(n+1)}{2} \right) \right\rfloor$$

or that expression plus 1. (Totaro also determined all n when "plus 1" appears.)

The value $\tau(G)$ is the same for $G = \mathrm{Spin}(2n+1)$.

The variety E/B as above is the twist of the split variety of (full) flags G/B by the G-torsor E. We can consider a more general setting replacing B by an arbitrary parabolic subgroup $P \subset G$. The variety E/P is the twist of the split projective homogeneous variety G/P. What is the integer $\gcd \mathrm{ind}(E/P) = \mathrm{ind}(E_{\mathrm{gen}}/P)$?

Example 1.3. Let $G = \mathrm{SO}(2n+2)$ and let q be a non-degenerate quadratic form of trivial discriminant of dimension $2n+2$ over a field $F \supset F_0$ and $m \leq n$ a positive integer. The variety X_m of totally isotropic subspaces of dimension m is a twist of G/P_m by a G-torsor E for a certain parabolic subgroup $P_m \subset G$. The integer $\mathrm{ind}(E/P_m)$ is the g.c.d. of the degrees of all finite field extensions K/F such that the Witt index of q over K is at least m. Using example 1.1, one can show that $\mathrm{ind}(E_{\mathrm{gen}}/P_m) = 2^m$ for a generic quadratic form q.

The answer is the same for $G = \mathrm{SO}(2n+1)$.

In the present paper we consider the setting as in Example 1.3 with G replaced by the spin group $G = \mathrm{Spin}(d)$ with $d = 2n+1$ or $d = 2n+2$. In other words, we would like to compute the integer $\mathrm{ind}(X_m)$ for a generic quadratic form of trivial discriminant and trivial Clifford invariant. This integer is a 2-power; we denote the exponent by $\mathrm{i}(m)$ and call it the *exponent index* of X_m. Note that the integer $\mathrm{i}(m)$ depends on F_0, d and m.

The highest exponent index $\mathrm{i}(\lfloor d/2 \rfloor)$ is Totaro's number t of Example 1.2. For odd $d = 2n+1$, it is the same as for even $d = 2n+2$. This is due to some "exceptional" isomorphisms between grassmannians, described in [1, §2], and, in general, does not hold for other exponent indexes. Moreover, the case of even d turns out to be significantly harder than that of odd d.

It is shown in [2, Theorems 4.2 and 7.2] that $\mathrm{i}(m) = m$ as long as $\dim(X_m) < 2^{n-m}$. This answer is the same as for quadratic forms of arbitrary Clifford invariant (see Example 1.3). In other words, the triviality of the Clifford invariant does not affect the g.c.d. of the degrees of the field extensions making the Witt index $\geq m$ (in the range given by the inequality on $\dim X_m$). What are the integers $\mathrm{i}(m)$ when the inequality does not hold?

In the present paper we show (Theorem 3.2) that for all $m \leq n$, the following inequality holds:

$$\mathrm{i}(m) \geq m - \left\lfloor \log_2 \left(1 + \frac{\dim X_m}{2^{n-m}}\right) \right\rfloor .$$

Note that the mentioned result of [2] and the determination of t in Example 1.2 (up to "plus 1") are the special cases of this inequality.

Using the inequality of Theorem 3.2, we prove (Theorem 3.6) that

$$\min(m, t) \geq \mathrm{i}(m) \geq \min(m, t) - 1,$$

where $t = t(n)$ is Totaro's number. We prove that $\mathrm{i}(m) = \min(m, t)$ for most of the values of m and we also show that the set of all m such that $\mathrm{i}(m) = \min(m, t) - 1$ is an interval, and we estimate the size of this interval.

We follow the flow of Totaro's paper on the highest m consisting of three major steps. For arbitrary m they are:

(1) reduction to a computation doable (at least in theory) by computer;
(2) finding a strong lower bound on $\mathrm{i}(m)$ which is its exact value for most d, m and which is always within 1 from the exact value;
(3) determination of d and m such that $\mathrm{i}(m)$ coincides with the lower bound obtained in (2).

Step (1) has been fulfilled for odd d in [4]. In this paper we perform step (2) for all d. Surprisingly, this includes all even d even though step (1) is not yet completed for them.

2. Preliminaries

We work with some $d \geq 3$ and we write n for the integral part (i.e., the floor) of $(d-1)/2$ so that d is $2n+1$ or $2n+2$ depending on its parity. We consider a generic d-dimensional quadratic form q of trivial discriminant and Clifford invariant, defined in §1. We recall that the construction of q starts with a choice of some initial field F_0; the field of definition of q is then the function field $F = F_0(\mathrm{GL}(N)/G)$.

For every integer m with $1 \leq m \leq d/2$, we write $\mathrm{i}(m)$ for the exponent index of the m-th grassmannian X_m of q – the variety of totally q-isotropic m-planes. Here we use the "affine" numbering of grassmannians so that X_1 is the projective quadric.

Although we do not indicate it in the notation, besides m, the index $\mathrm{i}(m)$ depends on d and – a priori – of the initial base field F_0. Actually, we expect that $\mathrm{i}(m)$ does not depend on F_0, but so far this is proved (in [4]) for odd d only.

It is convenient to allow $m = 0$ in the notation X_m and $\mathrm{i}(m)$: the variety X_0 is the point $\operatorname{Spec} F$ and $\mathrm{i}(0) = 0$. However, since $\mathrm{i}(0)$ does not need to be determined, we usually have $m \geq 1$ in the statements of our results.

Any non-degenerate quadratic form of trivial discriminant and Clifford invariant is split provided its dimension is < 7. Therefore all exponent indexes vanish for $d < 7$. Besides, for $d \geq 7$ one has $\mathrm{i}(n) = \mathrm{i}(n-1) = \mathrm{i}(n-2)$. Finally, $\mathrm{i}(n) = \mathrm{i}(n+1)$ for any even d. Because of the last equality (and for homogeneity), we usually skip $\mathrm{i}(n+1)$ in the statements of our results.

We note that $\mathrm{i}(n)$ coincides with Totaro's number $t = t(n)$, see §1.

If for some $m = 1, \dots, n$, the variety X_m has a point over a field extension K/F, the variety X_{m-1} also has a K-point. Conversely, if X_{m-1} has a K-point, the variety X_m has an L-point for an at most quadratic field extension L/K. These easy observations yield the inequalities

$$\mathrm{i}(m-1) \leq \mathrm{i}(m) \leq \mathrm{i}(m-1) + 1. \tag{2.1}$$

In particular,

$$0 \leq \mathrm{i}(m) \leq \min(m, t). \tag{2.2}$$

The exponent indexes for $d = 2n+1$ are closely related with the exponent indexes for $d = 2n+2$, The relationship, described below, is sometimes useful. But it is not sufficient to deduce all our results on $d = 2n+2$ from the (simpler) case $d = 2n+1$.

Let us assume that d is even: $d = 2n+2$. Write d' for $d-1 = 2n+1$, and write $\mathrm{i}'(m)$ for the exponent indexes of a d'-dimensional generic quadratic form q' of trivial discriminant and Clifford invariant, constructed out of the same as q initial base field F_0. Note that the field F, over which q is defined, differs from the field F' of definition of q'.

Lemma 2.3. *For any $m = 1, \dots, n$, one has*

$$\mathrm{i}'(m-1) \leq \mathrm{i}(m) \leq \mathrm{i}'(m).$$

Proof. Let q_1 be a d'-dimensional non-degenerate subform in q and let q_1' be a d-dimensional non-degenerate quadratic form containing q'. For any $m = 0, 1, \dots, n$, we write i, i', i_1, i_1' for the corresponding exponent index functions.

If the Witt index of $q_1 \subset q$ over some field $K \supset F$ is at least m, then the Witt index of q over K is at least m. This implies that $\mathrm{i}(m) \leq \mathrm{i}_1(m)$. Since $\mathrm{i}_1(m) \leq \mathrm{i}'(m)$, the second inequality of Lemma 2.3 is proved.

If the Witt index of q_1' over some field $K' \supset F'$ is at least m, then the Witt index of $q' \subset q_1'$ over K' is at least $m-1$. This implies that $\mathrm{i}'(m-1) \leq \mathrm{i}_1'(m) \leq \mathrm{i}(m)$. □

3. Main results

In this section, we list our main results. Most of the proofs are done later on.

We continue to use the settings of §2 and recall the dimension formula:

$$\dim X_m = m(m-1)/2 + m(d-2m). \tag{3.1}$$

Here is our first main result, proved in §4 and §5:

Theorem 3.2. *Given $d = 2n+1$ or $d = 2n+2$ and given some $m \in [1,\ n]$, one has $\mathrm{i}(m) \geq m - r$, where r is the integer satisfying*

$$2^r \leq 1 + \frac{\dim X_m}{2^{n-m}} < 2^{r+1}.$$

Remark 3.3. Theorem 3.2 includes [2, Theorem 4.2] as a particular case: if $\dim X_m < 2^{n-m}$, then $r = 0$, and Theorem 3.2 means $\mathrm{i}(m) = m$.

We recall that we write t for Totaro's number (depending only on n), introduced in Example 1.2.

Example 3.4. For $n = 11$ (and d of any parity), Theorem 3.2 together with [7, Theorem 0.1] determines the value of $\mathrm{i}(m)$ for all m except $m = t = 5$: it tells that $\mathrm{i}(4) \geq 4$ (this is the case covered by [2, Theorem 4.2]) implying that $\mathrm{i}(m) = m$ for $m \leq 4$; it also yields $\mathrm{i}(6) \geq 5$ implying that $\mathrm{i}(m) = 5$ for $m \geq 6$ (the result for $m = 6, 7, 8$ is new).

Here is an example with large "randomly chosen" n, extending the example of [2, §1]:

Example 3.5. For $n = 2021$ (and d of any parity), we get $\mathrm{i}(m) = m$ for $m \leq 2000$ (already by [2, Theorem 4.2]) and $\mathrm{i}(m) = t = 2001$ for $m \geq 2007$ (new for $2007 \leq m \leq 2018$). Here we have $2^6 \leq 1 + 2^{-14} \dim X_{2007} < 2^7$ so that $r = 6$ for $m = 2007$.

Theorem 3.6. *Given $d = 2n + 1$ or $d = 2n + 2$ and given some $m \in [1,\ n]$, one has* $\mathrm{i}(m) \geq \min(m, t) - 1$.

Proof. By Theorem 3.2 and Proposition A.1, we get $\mathrm{i}(t) \geq t - 1$. The rest follows from (2.1) and (2.2). □

For a given $d \geq 1$, we say that m is *deviated*, if $\mathrm{i}(m) = \min(m, t) - 1$. For non-deviated m, we have $\mathrm{i}(m) = \min(m, t)$. It is clear that the set of all deviated m, if non-empty, is an interval containing t; we call it the *deviation interval*.

The deviation interval is empty if $\mathrm{i}(t) = t$. Unfortunately, for $d \geq 13$, Theorem 3.2 would never give us this information (see Proposition A.5). Still we can bound the deviation interval proceeding as follows.

Assume that $d \geq 13$. Let a be the largest $m \geq 1$ for which Theorem 3.2 states $\mathrm{i}(m) \geq m$; we set $a = 0$ in the case there are no such m. Let b be the smallest $m \leq n - 3$ for which Theorem 3.2 states $\mathrm{i}(m) \geq t$; we set $b = n - 2$ in the case there are no such m. The interval $[a + 1,\ b - 1]$ contains t and the deviation interval; we call it the *uncertainty interval*: there are no deviated m outside the uncertainty interval, but for m inside it our results do not determine the value of $\mathrm{i}(m)$.

By Proposition A.4, for an overwhelming majority of d, the left border of the uncertainty interval is t; it is $t - 1$ for the remaining d (see Corollary A.3). In particular, the size of the uncertainty interval divided by n tends to 0 when $n \to \infty$.

The deviation interval is determined for $d \leq 18$ only. It consists of $t = 1$ for $d = 10$, of $t = 4$ and $t + 1 = 5$ for $d = 17, 18$, and is empty otherwise. The determination for $d = 17, 18$, performed in [4], involves computer computations; for $d \leq 16$ see [5].

The uncertainty interval in Example 3.4 consists of t alone; it is $[t,\ t + 5]$ in Example 3.5.

A table of available information on $\mathrm{i}(m)$ for $d \leq 24$ is given in Appendix B.

4. Proof of Theorem 3.2 for odd d

We consider the split spin group $G := \mathrm{Spin}(d)$, where $d = 2n + 1$, and its standard split maximal torus T constructed as in [6, §8.2]. Given some $m = 1, \ldots, n$, let $T \subset P \subset G$ be the standard maximal parabolic subgroup for which the quotient G/P is the m-th orthogonal grassmannian of the split $(2n+1)$-dimensional quadratic form $q_{\rm sp}$ used in the construction of G. For the Weyl group W of P, we consider the homomorphism

$$\varphi\colon S(\hat{T})^W \to \mathrm{CH}(G/P)$$

constructed in [2, §2].

We recall the computation of $S(\hat{T})^W$ performed in [4]. Let us consider the polynomial ring $R = \mathbb{Z}[x_1, \dots, x_m, y_1, \dots, y_l]$ over the integers $\mathbb{Z}$ in the variables $x_1, \dots, x_m$ and $y_1, \dots, y_l$, where $m + l = n$. Let $A := (\mathbb{Z}/2\mathbb{Z})^{\times l}$ be the direct product of l copies of the group $\mathbb{Z}/2\mathbb{Z}$ acting on R as follows: for any $i = 1, \dots, l$, the i-th copy of $\mathbb{Z}/2\mathbb{Z}$ acts by changing the sign of y_i and trivially on the remaining variables.

The Weyl group W' of the spin group $\mathrm{Spin}(2l+1)$ is a semidirect product of A and the symmetric group S_l. The action of W' on R we are interested in is the (unique) extension of the action of A, defined above, and the action of S_l by permutation of $y_1, \dots, y_l$. We will also consider the action of S_m by permutation of $x_1, \dots, x_m$ and the resulting action of $W = S_m \times W'$ on R. The latter action extends (uniquely) to an action of W on $R[z]$, where $R[z]$ is an R-algebra with a generator z subject to the relation

$$2z = x_1 + \dots + x_m + y_1 + \dots + y_l.$$

The ring $S(\hat{T})$ is identified with $R[z]$ and the action of the Weyl group W of P on $S(\hat{T})$ is the action of W on $R[z]$ just defined.

We define an element $\tilde{z} \in R[z]^W$ as the product of all (2^l) elements in the W-orbit of z:

$$\tilde{z} = \prod_{I \subset \{1,\dots,l\}} (z - \sum_{i \in I} y_i).$$

We borrow from [4, §2] the construction of certain elements $f_k \in R[z]$, $k \geq 0$. We set

$$f_0 := 2z - y_1 - \dots - y_l = x_1 + \dots + x_m \in R.$$

Assume that for some $k \geq 0$ the element f_k is already constructed and has the shape

$$f_k = 2z \cdot g_k + a_1 + \dots + a_s, \tag{4.1}$$

where g_k is a polynomial with integer coefficients in $z, y_1, \dots, y_l$ and where $a_1, \dots, a_s$ for some $s \geq 0$ are monomials in $y_1, \dots, y_l$. (More exactly, we assume that $f_k - 2z \cdot g_k$ is a polynomial in $\mathbb{Z}[y_1, \dots, y_l]$ and let $a_1, \dots, a_s$ be the monomials of this polynomial.) Then we define f_{k+1} as one half of the difference

$$f_k^2 - (a_1^2 + \dots + a_s^2) = 2\Big(2z\big(zg_k^2 + (a_1 + \dots + a_s)g_k\big) + \sum_{i<j} a_i a_j\Big). \tag{4.2}$$

Note that the new element f_{k+1} has the shape (4.1) allowing to continue the procedure.

For any $k \geq 0$, the element f_k is W-invariant by [4, Lemma 2.3].

Proposition 4.3 ([4, Proposition 2.4] (see also [2, Proposition 3.3]))**.** *The R^W-algebra $R[z]^W$ is generated by the elements $f_1, \dots, f_{l-1}, \tilde{z}$.*

Note that the ring R^W is generated by the elementary symmetric polynomials in $x_1, \dots, x_m$ and the elementary symmetric polynomials in $y_1^2, \dots, y_l^2$. Also note that all f_k and $\tilde{z}$ are in the W-invariant subring $R_0[z] \subset R[z]$, where $R_0 := \mathbb{Z}[y_1, \dots, y_l]$. Note that z is an independent variable over R_0; it is not an independent variable over R because $2z \in R$.

We write C for the subring in $\mathrm{CH}(G/P)$ generated by the Chern classes of the tautological vector bundle on the m-th grassmannian G/P. We set $e := \varphi(\tilde{z}) \in \mathrm{CH}(G/P)$ (see [4, Proposition 3.4]).

Proposition 4.4. *The element $c := 2e$ is in C.*

Proof. The leading terms of $2\tilde{z}$ and of f_l, viewed as polynomials in z over R_0, are equal to $2z^{2^l}$. Therefore the difference $2\tilde{z} - f_l$ has degree $< 2^l$. It follows by Proposition 4.3 (applied in the case of $m = 1$) that this difference is a polynomial in $f_0, f_1, \dots, f_{l-1}$ over R_0^W. Since $\varphi(f_k) \in C$ for any $k \geq 0$ ([4, Proposition 3.2]) and $\varphi(R_0^W) \subset \varphi(R^W) \subset C$ (([4, Proposition 3.1])), the statement follows. □

Proposition 4.5. *For every $i \geq 1$, c^{2^i} is divisible by 2^{2^i-1} in C.*

Proof. Let B satisfying $T \subset B \subset P \subset G$ be the standard Borel subgroup. The variety G/B is the variety of complete flags of totally isotropic subspaces for the quadratic form q_{sp}. Let $C_B \subset \mathrm{CH}(G/B)$ be the subring generated by the Chern classes of all n (from rank 1 to rank n) tautological vector bundles $\mathcal{T}_1, \dots, \mathcal{T}_n$ on G/B. The pull-back homomorphism $\mathrm{CH}(G/P) \to \mathrm{CH}(G/B)$ with respect to the natural projection $G/B \to G/P$ is injective with a free cokernel (see [2, Proof of Lemma 2.2]). It maps C to C_B, with a free cokernel as well (see [4, Proof of Proposition 3.2]). We identify $\mathrm{CH}(G/P)$ with its image in $\mathrm{CH}(G/B)$. The composition $R[z]^W = \mathcal{S}(\hat{T})^W \xrightarrow{\varphi} \mathrm{CH}(G/P) \hookrightarrow \mathrm{CH}(G/B)$ is a restriction of the homomorphism $R[z] = \mathcal{S}(\hat{T}) \to \mathrm{CH}(G/B)$ mapping $x_1, \dots, x_m, y_1, \dots, y_l$ to the Chern roots of $\mathcal{T}_n$.

In order to show that the element c^{2^i} is divisible by 2^{2^i-1} in C, it suffices to show that it is divisible by 2^{2^i-1} in C_B.

Let us first look at the element c itself. Since $2^{2^l}\tilde{z}$ is in R and is divisible by $2z$ in R, $2^{2^l-1}c$ is divisible in C_B by the first Chern class of $\mathcal{T}_n$. Let $J \subset C_B$ be the ideal generated by the Chern classes $\{c_i(\mathcal{T}_n)\}_{i\geq 1}$. The quotient group C_B/J is free (see [2, Lemma 4.5]). Since $2^{2^l-1}c \in J$, we get that $c \in J$.

By [7, §4], the square of any element of J is divisible by 2 in J. It follows by induction on i that 2^{2^i-1} divides c^{2^i} (as well as 2^i-th power of any other element of J) in J. □

Corollary 4.6. *For every $i \geq 0$, $2e^{2^i} \in C$. More generally, $2^{\alpha(i)}e^i \in C$, where $\alpha(i)$ is the sum of the base 2 digits of i.*

Proof. Since $c^{2^i} = 2^{2^i}e^{2^i}$ is divisible by 2^{2^i-1} in torsion free C, we get that $2e^{2^i} \in C$. Writing i as a sum of $\alpha(i)$ powers of 2, we get the second statement. □

We are ready to prove Theorem 3.2 for odd d. Writing $\bar{X}_m$ for the variety X_m over an algebraic closure of its base field, we have $\mathrm{CH}(\bar{X}_m) = \mathrm{CH}(G/P)$. Let $\bar{\mathrm{CH}}(X_m)$ be the ring $\mathrm{CH}(X_m)$ modulo the ideal of the elements of finite order. The change of field homomorphism $\mathrm{CH}(X_m) \to \mathrm{CH}(\bar{X}_m)$, whose kernel coincides with the above ideal, identifies $\bar{\mathrm{CH}}(X_m)$ with a subring in $\mathrm{CH}(G/P)$ containing C. By [4, Theorem 3.6], the C-algebra $\bar{\mathrm{CH}}(X_m)$ is generated by the single element $e \in \bar{\mathrm{CH}}^{2^l}(X_m)$, where $l = n - m$.

By definition of $\mathrm{i}(m)$, $2^{\mathrm{i}(m)}$ generates the image of the degree homomorphism $\bar{\mathrm{CH}}(X_m) \to \mathbb{Z}$. The degree of any element of $C \subset \bar{\mathrm{CH}}(X_m)$ is divisible by 2^m (see [2, Proof of Theorem

4.2] or Example 1.3). It follows by Corollary 4.6 that $\mathrm{i}(m) \geq m - \alpha(i)$ for the integer i on the interval $[0,\ 2^{-l}\dim X_m]$ with the largest $\alpha(i)$, i.e., for $i = 2^r - 1$. For this i we have $\alpha(i) = r$ so that the lower bound for $\mathrm{i}(m)$ reads as claimed.

5. Proof of Theorem 3.2 for even d

Although the exponent indexes for $d = 2n+1$ are very close to the exponent indexes for $d = 2n+2$ (see, e.g., Lemma 2.3), the analysis of the case $d = 2n+1$, as we will see below, is closer to the case $d = 2n$. So, also following [2, Part II], we change the notation for this section and assume that $d = 2n$ (instead of the agreement $d = 2n+2$ we had until now). After the end of this section we return to our usual agreement $d = 2n+2$ or $d = 2n+1$.

We consider the split spin group $G := \mathrm{Spin}(2n)$ with the standard split maximal torus T constructed as in [6, §8.4]. We recall that q is the quadratic form of dimension $2n$, trivial discriminant, and trivial Clifford invariant, given by a generic G-torsor E over a certain field F. For $1 \leq m \leq n-2$, the variety X_m can be identified with the quotient E/P_m by the standard maximal parabolic subgroup $P_m \supset T$ of G corresponding to the m-th vertex of the Dynkin diagram of G. The situation with $m = n-1$ and $m = n$ is messier: $E/P_n \simeq E/P_{n-1} \simeq X_n$ and X_{n-1} is the quotient of E by the parabolic subgroup corresponding to the union of the vertices number $n-1$ and n. But since $\mathrm{i}(n-1) = \mathrm{i}(n) = t$ is already computed, we don't care about these two highest values of m.

We set $P := P_m$ for some $1 \leq m \leq n-2$. As in the previous section, we identify $\bar{\mathrm{CH}}(E/P)$ with a subring of $\mathrm{CH}(G/P)$ and notice that it is contained in the image of the homomorphism

$$\varphi\colon S(\hat{T})^W \to \mathrm{CH}(G/P),$$

where W is the Weyl group of P, and $S(\hat{T})^W$ is the ring of W-invariant elements of the symmetric ring $S(\hat{T})$ of the group $\hat{T}$ of characters of T. In fact, $\bar{\mathrm{CH}}(E/P)$ is exactly the image of the composition

$$\mathrm{CH}(BP) \longrightarrow S(\hat{T})^W \xrightarrow{\ \varphi\ } \mathrm{CH}(G/P),$$

out of the Chow ring of the classifying space of P, but this information is of moderate help because most of the time we are not able to compute $\mathrm{CH}(BP)$. However, we are able to find a reasonable system of generators for $S(\hat{T})^W$.

5a. **Invariants.** In order to determine $S(\hat{T})^W$, we modify the considerations of §4.

As in §4, the ring $S(\hat{T})$ is identified with the ring $R[z]$. The Weyl group W of P acts on $R[z]$ the same way as in §4, however the Weyl group itself is smaller: it is generated by the symmetric groups S_m and S_l together with the subgroup A' of A consisting of the elements with the trivial sum of components.

We define an element $\check{z} \in R[z]^W$ as the product of all (2^{l-1}) elements in the W-orbit of z:

$$\check{z} = \prod_{I \subset \{1,\dots,l\},\ \#I \text{ is even}} (z - \sum_{i\in I} y_i),$$

where $\#I$ stands for the number of elements in the set I.

We continue to use the elements $f_k \in R[z]$, $k \geq 0$. They are W''-invariant, where $W'' \supset W$ is the larger Weyl group from §4. In particular, they are W-invariant.

The following fact is obtained exactly as [4, Proposition 2.4] (see also [2, Proposition 6.1]):

Proposition 5.1. *The R^W-algebra $R[z]^W$ is generated by the elements $f_1, \dots, f_{l-2}, \check{z}$.* □

Note that the ring R^W is generated by the elementary symmetric polynomials in $x_1, \dots, x_m$, the elementary symmetric polynomials in $y_1^2, \dots, y_l^2$, and the product $y_1 \cdots y_l$. Note that the last generator (and only it) did not show up in §4.

Summarizing, we get the following list of generators for the ring $R[z]^W$. We divide them into four groups numbered according to the order of their treatment in the next subsection:

Corollary 5.2. *The ring $R[z]^W$ is generated by the following four groups of elements (all together):*

(1) *the elementary symmetric polynomials in $x_1, \dots, x_m$ and the elementary symmetric polynomials in $y_1^2, \dots, y_l^2$;*
(2) $f_1, \dots, f_{l-2}$*;*
(3) $\check{z}$*;*
(4) $y_1 \cdots y_l$. □

5b. **Images of invariants.** We are considering the four groups (1)–(4) of generators, listed in Corollary 5.2, and investigate their images under φ.

Let $B \subset G$ be the standard Borel subgroup; we have $T \subset B \subset P$. Let $C \subset \bar{\mathrm{CH}}(E/P)$ be the subring generated by the Chern classes of the tautological (rank m) vector bundle $\mathcal{T}$ on $X_m = E/P$.

The groups (1) – (3) of generators also appear in the case of odd d treated in [4]. The groups (1) and (3) of generators are treated here exactly as in [4]. The group (2) is treated by reduction to what has been proved on this group in [4].

Proposition 5.3. *The image of (1) lies in $C \subset \bar{\mathrm{CH}}(E/P)$.*

Proof. As in [4], the images of $x_1, \dots, x_m$ in $\mathrm{CH}(G/B)$ are the roots of the vector bundle $\mathcal{T}$ (pulled back to G/B along the projection $G/B \to G/P$). The roots of the vector bundle $\mathcal{T}^\perp$, given by the orthogonal complement, are the images of $x_1, \dots, x_m$ along with the images of $\pm y_1, \dots, \pm y_l$. Finally, the roots of the trivial vector bundle V given by the vector space of definition of q are the images of all $\pm x_1, \dots, \pm x_m$, $\pm y_1, \dots, \pm y_l$ together.

The images in $\mathrm{CH}(G/P)$ of the elementary symmetric polynomials in $x_1, \dots, x_m$ are the Chern classes of $\mathcal{T}$. The images of the elementary symmetric polynomials in $y_1^2, \dots, y_l^2$ are the Chern classes of the quotient $\mathcal{T}^\perp/\mathcal{T}$. The isomorphism $V/\mathcal{T}^\perp = \mathcal{T}^\vee$, where $\mathcal{T}^\vee$ is the dual vector bundle, shows that the Chern classes of $\mathcal{T}^\perp$ are polynomials in the Chern classes of $\mathcal{T}$. □

Proposition 5.4. *The image of (2) also lies in $C \subset \bar{\mathrm{CH}}(E/P)$. More generally, for any $k \geq 0$, the image of f_k lies in C.*

Proof. Let q'' be a non-degenerate $(2n+1)$-dimensional quadratic form containing q as a subform. Its m-th orthogonal grassmannian X''_m contains X_m as a closed subvariety (of codimension m). The pull-back to X_m of the tautological vector bundle $\mathcal{T}''$ on X''_m is $\mathcal{T}$.

Therefore the pull-back of the subring $C'' \subset \mathrm{CH}(X''_m)$, generated by the Chern classes of $\mathcal{T}''$, is C.

For any $k \geq 0$, let f''_k be the analogue of f_k for the spin group $G'' := \mathrm{Spin}(2n+1)$. The image in $\bar{\mathrm{CH}}(X_m)$ of f_k is the pull-back of the image in $\bar{\mathrm{CH}}(X''_m)$ of f''_k. The latter image is in C'' by [4, Proposition 3.2]. □

The following statement is not needed for our purposes; we include it for completeness. It is proven exactly the same way as [4, Proposition 3.4].

Proposition 5.5. *The image of (3) lies in* $\bar{\mathrm{CH}}(E/P)$. □

As for the generator (4), we do not claim that its image under φ belongs to $\bar{\mathrm{CH}}(E/P)$. One can show that it does for $l \leq 5$ and it doesn't (at least in characteristic 0) for $l \geq 6$,[1] but we are not going to use this information here.

5c. **Bounding** $\mathrm{i}(m)$**.** The exact value of $\mathrm{i}(m)$ is given by $\bar{\mathrm{CH}}_0(X_m)$. Unlike the situation with odd d, where the ring $\bar{\mathrm{CH}}(X_m)$ coincides with $\varphi\big(S(\hat{T})^W\big)$, we are not able to determine $\bar{\mathrm{CH}}(X_m)$ for even d. We prove the lower bound of Theorem 3.2 on $\mathrm{i}(m)$ using the upper bound on $\bar{\mathrm{CH}}(X_m)$ given by $\varphi\big(S(\hat{T})^W\big) \supset \bar{\mathrm{CH}}(X_m)$.

The C-algebra $\varphi\big(S(\hat{T})^W\big)$ has two generators: $e := \varphi(\check{z})$ and $y := \varphi(y_1 \cdots y_l)$. Since $y^2 = \varphi(y_1^2 \cdots y_l^2) \in C$, the C-module $\varphi\big(S(\hat{T})^W\big)$ is generated by e^i and $e^i \cdot y$, $i \geq 0$.

The proof of the following statement is similar to Proposition 4.4.

Proposition 5.6. *The element* $c := 2e$ *is in* $C[y]$.

Proof. The leading terms of $2\check{z}$ and of f_{l-1}, viewed as polynomials in z over R_0, are equal to $2z^{2^{l-1}}$. Therefore the difference $2\check{z} - f_{l-1}$ has degree $< 2^{l-1}$. It follows by Proposition 5.1 (applied with $m = 1$) that this difference is a polynomial in $f_0, f_1, \dots, f_{l-2}$ over R_0^W. Since $\varphi(f_k) \in C$ for $k \geq 0$ by Proposition 5.4 and $\varphi(R_0^W) \subset C[y]$, the statement follows. □

The (involutory) automorphism of the polynomial ring

$$R = \mathbb{Z}[x_1, \dots, x_m, y_1, \dots, y_l],$$

changing the sign of y_l and identical on the remaining variables, extends (uniquely) to an (involutory) automorphism of $R[z]$. We write σ for this extension.

Lemma 5.7. $\tilde{z} = \check{z} \cdot \sigma(\check{z})$.

Proof. Since $\sigma(z) = z - y_l$, $\sigma(\check{z})$ is the product of $z - \sum_{i \in I} y_i$ over all subsets $I \subset \{1, \dots, l\}$ with odd $\#I$. □

Proposition 5.8. *For any* $i \geq 0$*, one has* $2e^{2^i} \in C[y]$.

Proof. We induct on i starting with the case $i = 0$ already proved (Proposition 5.6).

Assume that $2e^{2^i} \in C[y]$ for some $i \geq 0$ and write $2e^{2^i} = a + by$ with $a, b \in C$. Then

$$C[y] \ni 4e^{2^{i+1}} = (a+by)^2 \equiv (a+by)(a-by) = \varphi(4\check{z}^{2^i}\sigma(\check{z}^{2^i})) = 2\varphi(2\tilde{z}^{2^i}),$$

[1] Thanks to A. Vishik for providing a computation with symmetric operations [8] on a 12-dimensional quadric allowing one to show this.

where the congruence is modulo $2C[y]$ and the last equality holds by Lemma 5.7. Using the argument as in the proof of Proposition 5.4, we deduce from Corollary 4.6 that $\varphi(2\tilde{z}^{2^i}) \in C$. Since the additive group of the ring $C[y]$ is the direct sum $C \oplus Cy$, it is (torsion) free and the statement follows. □

We are ready to prove Theorem 3.2 for even d. As we know, $\bar{\mathrm{CH}}(X_m)$ is contained in the C-algebra $\varphi(\mathsf{S}(\hat{T})^W)$ generated by the element $y \in \bar{\mathrm{CH}}^l(G/P)$ (satisfying $y^2 \in C$) and the element $e \in \bar{\mathrm{CH}}^{2^{l-1}}(G/P)$. As C-module, $\varphi(\mathsf{S}(\hat{T})^W)$ is generated by e^i and $e^i y$ with

$$0 \le i \le \frac{\dim X_m}{2^{l-1}}.$$

By definition of $\mathrm{i}(m)$, $2^{\mathrm{i}(m)}$ generates the image of the degree homomorphism $\bar{\mathrm{CH}}(X_m) \to \mathbb{Z}$. We get a lower bound on $\mathrm{i}(m)$ using the image of the degree homomorphism $\mathrm{CH}(\bar{X}_m) \to \mathbb{Z}$ on $\varphi(\mathsf{S}(\hat{T})^W)$, sitting between $\mathrm{CH}(\bar{X}_m)$ and $\bar{\mathrm{CH}}(X_m)$:

$$\mathrm{CH}(\bar{X}_m) \supset \varphi(\mathsf{S}(\hat{T})^W) \supset \bar{\mathrm{CH}}(X_m).$$

The degree of any element of $C \subset \bar{\mathrm{CH}}(X_m)$ is divisible by 2^m (see [2, Proof of Theorem 7.2] or Example 1.3). We claim that the degree homomorphism vanishes on $Cy \subset C[y]$ so that the degree of any element of $C[y]$ is divisible by 2^m as well. To prove the claim, let q_{sp} be the standard split $(2n)$-dimensional quadratic form used in the construction of $G = \mathrm{Spin}(2n)$ and let us consider the (involutory) automorphism of q_{sp} given by the switch of the vectors of the hyperbolic basis of its last hyperbolic plane. This automorphism yields an automorphism of G, fixing P and T, and an automorphism of $\bar{X}_m$. The automorphism of $\mathsf{S}(\hat{T}) = R[z]$ we get this way is the involution σ considered above. The automorphism of $\mathrm{CH}(\bar{X}_m)$ induced by the automorphism of $\bar{X}_m$ commutes with the degree homomorphism, is identical on C, and changes the sign of y. The claim follows.

We finish the proof of Theorem 3.2 for even d exactly the same way as for odd d in §4. Writing $\alpha(i)$ for the sum of the base 2 digits of an integer $i \ge 0$, we get by Proposition 5.8 that $\mathrm{i}(m) \ge m - \alpha(i)$ for $i \in [0,\ (\dim X_m)/2^{l-1}]$ with the largest $\alpha(i)$, i.e., for $i = 2^r - 1$. For this i we have $\alpha(i) = r$ so that the lower bound for $\mathrm{i}(m)$ reads as claimed. (We recall that $d = 2n$ now whereas $d = 2n + 2$ in the formulation of Theorem 3.2.)

Appendix A. Some dimension estimates

For a given $n \ge 1$, let r be the integer satisfying $2^r \le 1 + n(n+1)/2 < 2^{r+1}$. We say that n is *T-regular*, if Totaro's number t for $d = 2n+1$ and $d = 2n+2$ is equal to $n - r$. Otherwise, $t = n - r + 1$ and we say that n is *T-exceptional*.

Proposition A.1. $\dim X_t < 3 \cdot 2^{n-t}$.

Proof. By (3.1), $\dim X_t$ is larger for larger d. Therefore we may assume that d is $2n+2$, not $2n+1$.

Case 1: n is T-regular.

Since

$$\frac{3}{2} \cdot \left(1 + \frac{n(n+1)}{2}\right) < 3 \cdot 2^r$$

and $r = n - t$, it suffices to show

$$\dim X_t < \frac{3}{2} \cdot \left(1 + \frac{n(n+1)}{2}\right).$$

After plugging in the formula (3.1) for $\dim X_m$ with $d = 2n+2$ and $m = t = n - r$, the difference between the right and the left sides times 4 equals

$$n^2 - (3+4r) \cdot n + (6r^2 + 6r + 6).$$

This is positive since the discriminant

$$(3+4r)^2 - 4(6r^2 + 6r + 6) = -8r^2 - 15$$

is negative.

Case 2: n is T-exceptional.

By [7, Theorem 0.1], since n is T-exceptional, it has the form $n = 2^a + b$ with some integers $a, b \geq 0$ satisfying $b \leq a - 3$. It follows that $t = n - 2a + 2$ and therefore

$$\dim X_t = t \cdot (t-1)/2 + 2t \cdot (2a-1) = t \cdot (t + 8a - 5)/2 = (n - 2a + 2) \cdot (n + 6a - 3)/2.$$

We will show that

$$2(n - 2a + 2) \cdot (n + 6a - 3) < 3 \cdot 2^{2a} = 3(n-b)^2.$$

The difference between the right and the left sides equals

$$n^2 - 2n \cdot (4a + 3b - 1) + (24a^2 + 3b^2 - 36a + 12)$$

and is quadratic in n. The constant term $24a^2 + 3b^2 - 36a + 12$ is positive since $a \geq 3$. Therefore the value of the difference is positive if $n \geq 2(4a + 3b - 1)$.

If $n \geq 14a - 20$, then

$$n \geq 6(a-3) + (8a - 2) \geq 8a + 6b - 2 = 2(4a + 3b - 1)$$

and we are done.

Since the inequality $n \geq 14a - 20$ holds for $a \geq 6$, it remains to consider $a \leq 5$. For such a we have only four values of n: $8, 16, 32, 33$. The statement of Proposition A.1 is checked for each of them by a direct computation:

For $n = 8$, we have $t = 4$ and $\dim X_t = 46 < 48 = 3 \cdot 2^4$.

For $n = 16$, we have $t = 10$ and $\dim X_t = 185 < 192 = 3 \cdot 2^6$.

For $n = 32$, we have $t = 24$ and $\dim X_t = 708 < 768 = 3 \cdot 2^8$.

For $n = 33$, we have $t = 25$ and $\dim X_t = 750 < 768 = 3 \cdot 2^8$. □

Proposition A.2. *For $d = 2n + 2$ and $1 \leq m < n$, one has*

$$\dim X_{m-1} < \frac{4}{3} \dim X_{m+1}.$$

Proof. For a fixed m, the minimum of the difference

$$8 \dim X_{m+1} - 6 \dim X_{m-1} = -3m^2 + 4mn - 39m + 28n + 18$$

is taken at $n = m + 1$ and equals

$$-3m^2 + 4m(m+1) - 39m + 28(m+1) + 18 = m^2 - 7m + 46 > 0.$$ □

Corollary A.3. *Given $d = 2n + 1$ or $d = 2n + 2$ with $n \geq 6$ (to ensure that $t \geq 2$), one has* $\dim X_{t-2} < 2^{n-(t-2)}$.

Proof. We may assume that $d = 2n+2$. First by Proposition A.2 and then by Proposition A.1,

$$\dim X_{t-2} < \frac{4}{3} \dim X_t < \frac{4}{3} \cdot 3 \cdot 2^{n-t} = 2^{n-t+2}. \qquad \square$$

Proposition A.4. *Given $d = 2n + 1$ or $d = 2n + 2$, assume that*

$$n \in [2^s + s - 2, 2^{s+\frac{1}{2}} - 2s - \frac{3}{2}) \cup (2^{s+\frac{1}{2}}, 2^{s+1} - 2s - \frac{5}{2}]$$

for some positive integer s. Then $\dim X_{t-1} < 2^{n-(t-1)}$.

Proof. It is sufficient to prove the statement for $d = 2n + 2$. We set $m := t - 1$.

Suppose first that $n \in [2^s + s - 2, 2^{s+\frac{1}{2}} - 2s - 3/2)$. If n is T-exceptional, it has the form $n = 2^s + b$ with $0 \leq b \leq s - 3$; hence $n \leq 2^s + s - 3$, a contradiction. Therefore n is T-regular meaning that

$$t = n - \lfloor \log_2 (1 + n(n + 1)/2) \rfloor = n - 2s + 1.$$

We have $n - m = 2s$. We show that the inequality

$$2^{2s} = 2^{n-m} > \dim(X_m) = \frac{m(m-1)}{2} + 2m(n - m + 1) = \frac{(n - 2s)(n + 6s + 3)}{2}$$

holds. Indeed,

$$(n - 2s)(n + 6s + 3) < n^2 + (4s + 3)n < \left(n + 2s + \frac{3}{2}\right)^2 < (2^{s+\frac{1}{2}})^2 = 2^{2s+1}.$$

Now let $n \in (2^{s+\frac{1}{2}}, 2^{s+1} - 2s - 5/2]$. Clearly, n is T-regular so that

$$t = n - \lfloor \log_2 (1 + n(n + 1)/2) \rfloor = n - 2s.$$

We have $n - m = 2s + 1$. We show that the inequality

$$2^{2s+1} = 2^{n-m} > \dim(X_m) = \frac{m(m-1)}{2} + 2m(n - m + 1) = \frac{(n - 2s - 1)(n + 6s + 6)}{2}$$

holds. Indeed,

$$(n - 2s - 1)(n + 6s + 6) < n^2 + (4s + 5)n < \left(n + 2s + \frac{5}{2}\right)^2 \leq (2^{s+1})^2 = 2^{2s+2}. \qquad \square$$

Proposition A.5. *For $d \geq 13$, one has* $\dim X_t \geq 2^{n-t}$.

Proof. We may assume that d is odd: $d = 2n + 1$. For $n = 6, 7, 8, 9$, the inequality is checked directly. Below we assume that $n \geq 10$.

Lemma A.6. *Under the assumptions just made, let r be a positive integer non-exceeding* $\log_2(1 + n(n + 1)/2)$. *Then* $\dim X_{n-r} \geq 1 + n(n + 1)/2$.

Proof. The inequality is equivalent to $n \geq (3r^2+r+2)/(2r)$. This holds for $r=1$ so that we may assume $r \geq 2$ for the rest. It is easy to see that $2^{2n-2} \geq (1+n(n+1)/2)^3$ for $n \geq 10$. It follows that $2n-2 \geq 3 \cdot \log_2(1+n(n+1)/2)$, hence

$$n \geq \frac{3}{2}\log\left(1+\frac{n(n+1)}{2}\right)+1 \geq \frac{3}{2}r+1 \geq \frac{3r^2+r+2}{2r}. \qquad \square$$

If n is T-regular, then $t = n-r$, where $r = \lfloor \log_2(1+n(n+1)/2) \rfloor$. By Lemma A.6, $\dim X_t \geq 1+n(n+1)/2 \geq 2^r = 2^{n-t}$.

If n is T-exceptional, then $t=n-r$, where $r = \lfloor \log_2(1+n(n+1)/2)\rfloor - 1$. By Lemma A.6, $\dim X_t \geq 1+n(n+1)/2 \geq 2^{r+1} > 2^r = 2^{n-t}$. $\square$

Appendix B. Table of exponent indexes

Here is the table of all $\mathrm{i}(m)$ for $d \leq 24$ determined so far. The d-th line provides information on the values $\mathrm{i}(1), \mathrm{i}(2), \ldots, \mathrm{i}(\lfloor d/2 \rfloor)$. The exact values are written in regular size. The numbers written in smaller size are lower bounds, where the actual value can be one higher. The uncertainty intervals (starting from $d=13$) are underlined.

$d=3$:	0											
$d=4$:	0	0										
$d=5$:	0	0										
$d=6$:	0	0	0									
$d=7$:	1	1	1									
$d=8$:	1	1	1	1								
$d=9$:	1	1	1	1								
$d=10$:	0	1	1	1	1							
$d=11$:	1	1	1	1	1							
$d=12$:	1	1	1	1	1	1						
$d=13$:	1	$\underline{2}$	$\underline{2}$	2	2	2						
$d=14$:	1	$\underline{2}$	$\underline{2}$	2	2	2	2					
$d=15$:	1	2	$\underline{3}$	$\underline{3}$	3	3	3					
$d=16$:	1	2	$\underline{3}$	$\underline{3}$	3	3	3	3				
$d=17$:	1	2	3	$\underline{3}$	$\underline{3}$	4	4	4				
$d=18$:	1	2	3	$\underline{3}$	$\underline{3}$	4	4	4	4			
$d=19$:	1	2	3	$\underline{3}$	$\underline{\scriptstyle 3}$	$\underline{\scriptstyle 3}$	4	4	4			
$d=20$:	1	2	3	$\underline{3}$	$\underline{\scriptstyle 3}$	$\underline{\scriptstyle 3}$	4	4	4	4		
$d=21$:	1	2	3	4	$\underline{\scriptstyle 4}$	$\underline{\scriptstyle 4}$	$\underline{\scriptstyle 4}$	5	5	5		
$d=22$:	1	2	3	4	$\underline{\scriptstyle 4}$	$\underline{\scriptstyle 4}$	$\underline{\scriptstyle 4}$	5	5	5	5	
$d=23$:	1	2	3	4	$\underline{\scriptstyle 4}$	5	5	5	5	5	5	
$d=24$:	1	2	3	4	$\underline{\scriptstyle 4}$	5	5	5	5	5	5	5

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