A New Family of Quality Loss Functions

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Taguchi first developed the quality loss function to better estimate the economic losses incurred by manufacturers and customers caused by quality characteristics being off-target. The quality loss function measures the quality loss caused by a deviation of a quality characteristic from its defined target value. Several researchers have proposed different revised loss functions for overcoming some flaws of the Taguchi loss function. This paper recommends a new family of quality loss functions, which is very flexible, simple, and easy to implement. Three real case studies demonstrated the usability and capabilities of the proposed new loss function for quantifying and predicting quality losses.

Keywords: *quality loss function, Taguchi method, multi-response optimi–zation, weighted sum method, Pareto frontier*

1. INTRODUCTION

The quality loss function (LF) expresses the economic consequences (in monetary units) when a quality char–acteristic deviates from the nominal (optimal) value.

Many different LFs have been extensively studied over the last few decades and have been used in a wide range of applications, including quality assurance, busi–ness decision-making, reliability settings, as well as process safety assessment, process optimal control, sys–tem performance monitoring, and many other discip–lines [1-3].

The loss function approach has attracted a lot of attention from quality-assurance researchers and practi–tioners after the introduction of Taguchi's quality philo–sophy and his quality monitoring/improvement concept.

Contrary to an abstract concept of quality, Taguchi was the first to introduce the concept of LF, which provides a framework for continuous and effective imp–rovement of products and processes.

The LF suggested by Taguchi was later generally accepted by most researchers and has been used for many decades.

Although the Taguchi LF is the simplest mathe–matical function that possesses the desired qualitative properties and can meaningfully approximate the quality loss in many situations [2], over time, many authors have suggested that the Taguchi LF is inadequate and unrealistic for quality improvement.

To overcome some obvious drawbacks of the quadratic loss function (QLF), Spiring [4] introduced a new LF called reflected normal loss function (RNLF), which provides a more reasonable assessment of the losses. Since the RNLF is essentially an inverted normal probability density function, this LF is also known as the inverted normal loss function (INLF). The INLF is continuous and bounded from the above function, while the introduced shape parameter is proportional to the distance from the target to the point where the maxi–mum loss first occurs. The shape parameter, similar to QLF, defines the general shape of the INFL curve but allows a smooth function rather than a piece-wise func–tion. This author derived a formula for expected loss (risk) associated with the INLF, which is a relatively simple function of its parameters and assumed normal distribution, attaining its minimum at the target for fixed shape parameters and standard deviation. Spiring also created a class of asymmetric INLFs, for situations where the loss is not symmetric around the target. The asymmetric INLF allows practitioners to customize the LF on each side of the target.

Sun et al. [5] developed an augmented RNLF, which is marked as the modified reflected loss function (MRNLF). This LF has a user-determining shape para–meter. By using the different values of the shape para–meter, a family of curves is obtained in a range from uniform LF (with a discontinuity at the target) to quad–ratic LF. Sun et al. [5] also derived the expected loss for the MRLF for normal distribution. Additionally, these authors recommended a non-linear procedure for deter–mining a reasonable value for the shape parameter. Some industrial examples are employed to illustrate the associated properties of the MRLF. The properties of the MRLF to reflect the user's actual loss moved the INLF from a research curiosity to a usable methodology in the area of loss.

Drain and Gough [6] introduced an upside-down normal loss function (UDNLF), which is quite similar to the INLF. A multivariate upside-down normal loss fun–ction (MUDNLF) was also proposed, and a closed-form solution for expected loss for optimizing a product /process with many quality characteristics.

Fathi and Poonthanomsook [7] suggested a quartic loss function (QuLF) in order to overcome some of the disadvantages and inflexibility of the Taguchi loss function. It is a class of continuous symmetric/ asym–metric loss functions with some degrees of flexibility defined by the values of shape parameters. The disad–vantages of this loss function include its complexity and the number of parameters to be determined.

Pan and Wang [8] developed another piece-wise extended variant of INLF based on a request that there

Received: July 2022, Accepted: October 2022
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doi: 10.5937/fme2204701M

is no loss in cases where a quality characteristic is near the desired target value.

Spiring and Yeung [9] suggested the inverted gamma loss function (IGLF), which can reflect pro–cesses with continuous asymmetric losses. The different shape parameters allow researchers and practitioners in the industry to customize the IGLF in order to realistically represent the losses associated with depar–tures from the target. The associated expected loss was derived.

Later, Lueng and Spiring [10] created the inverted beta loss function (IBLF) based on the general pro–erties of the probability density functions (pdfs). The IBLF can be used in asymmetric cases where the loss due to deviation from the target on the one side is diffe–rent from the loss for the same deviation on the opposite side. The IBLF contains two shape parameters with a linear relationship between them. Likewise, the form of the IBLF curve is scale-invariant under a linear trans–formation. In addition, multivariate IBLF was defined according to the multivariate beta probability density function. IGLF and IBLF have some nice properties, but applying these LFs may be problematic for some researchers and practitioners in the industry due to their complexity.

Recently, Abdeen et al. [11] recommended a new loss function, called IWeLF, in the family of the inverted probability loss function (IPLF), based on inversions of probability density functions for the Weibull three-parameter distribution. Unfortunately, the authors did not provide any insight into the properties and usability of this LF.

Overall, most of the abovementioned LFs have similar performances. The selection of the best LF and its associated expected loss in the sense of reflecting the true loss depends, in some measure, on the process characteristic distribution.

In general, each LF based on Spiring's concept of the inverted probability density function depicts the loss quite satisfactory if the quality characteristic follows its conjugate distribution. Otherwise, difficulties may arise in determining the functional form of the expected loss function for some distributions [12].

Finally, it should be mentioned that several rese–archers in the last decade have developed a concept of quality gain-loss function [13, 14], which is based on endowing the constant term in the expansion of the Taylor series with quality compensation.

2. LOSS FUNCTION

An important issue of science and engineering practice is the analysis and quantification of quality losses.

The LFs quantify economic losses associated with departures of a quality characteristic from the desired target value.

The quality characteristic is a product/process res–ponse that is observed for quantifying the quality level for (single) optimization (for several responses, it is multiple optimizations).

In manufacturing, the LFs define a set of economic consequences due to the variation around the target or specification limits, which may be different for different quality characteristics.

Any quality characteristic that satisfies the menti–oned requirement is equally good for manufacturers but not necessarily for customers [2]. According to Ta–guchi's philosophy, the smallest deviation of a quality characteristic from its target value produces a loss in quality, increasing as the distance from the target inc–reases. This is an essential difference from the old tra–ditional approach, which was based on the premise that no loss at all in quality occurs until the quality cha–racteristic drifts within its specification limits.

Taguchi suggested a continuous quadratic loss func–tion (QLF) as follows [1, 2]:

$$L(y) = B(y-T)^2 \tag{1}$$

where $L(y)$ is the actual loss when the quality charac–teristic is equal y, y is the value of the quality charac–teristic (response), T is the target value of the quality characteristic specified by the customer or quality expert, B and is the proportionality constant called the quality loss coefficient.

The Taguchi LF is essentially an analytical approxi–mation of the true LF using Taylor series expansion.

As the QLF is not upper-bounded and therefore ina–dequate for estimating and quantifying losses in the real manufacturing environment, Taguchi developed a mo–dified form of QLF:

$$L(y) = \begin{cases} \dfrac{K}{\Delta^2}(y-T)^2 & for\ |y-T| \le \Delta \\ K & otherwise \end{cases} \tag{2}$$

where K the estimated maximum loss Δ is the distance from the target to the specification limits (points) where the maximum loss first occurs.

The bilateral QLF is plotted in Figure 1. Under the ideal quality, the loss would be zero; consequently, the LF is a non-negative function.

The borders $T+\Delta$, $T-\Delta$ represent the upper specifi–cation limit (USL) and lower specification limit (LSL), respectively. The estimated maximum loss K is a con–stant, representing the additional costs for repair or replacement of the product being outside the speci–fication limits and all other associated losses.

Consequently, one important question in industrial practice has become how to find the flexible LFs that accurately describe the actual economic losses.

A new loss function proposed in this paper, the polynomial loss function (henceforth referred to as PLF), is defined in general form:

$$L(y) = \begin{cases} K_1 & for\ y < T-\Delta_1 \\ K_1\left\{1-\left[1-\left(\dfrac{y-(T-\delta_1)}{\Delta_1-\delta_1}\right)^2\right]^\beta\right\} & for\ T-\Delta_1 \le y < T-\delta_1 \\ 0 & for\ T-\delta_1 \le y \le T+\delta_2 \\ K_2\left\{1-\left[1-\left(\dfrac{y-(T+\delta_2)}{\Delta_2-\delta_2}\right)^2\right]^\beta\right\} & for\ T+\delta_2 < y \le T+\Delta_2 \\ K_2 & for\ y > T+\Delta_2 \end{cases} \tag{3}$$

where δ_1, and δ_2 are the lower and the upper inner dis–tance from the target value, respectively, β is the pre–determined shape parameter ($\beta \in R^m \mid \beta > 0$).

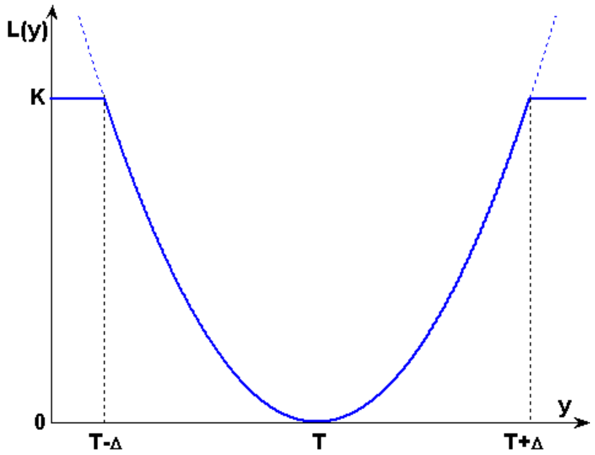


Figure 1. Taguchi loss function (QLF)

The parameters K_1, K_2 in PLF have the same meaning as in the Taguchi LF (see Eq. (2)).

When a quality characteristic falls within the target interval $[\delta_1, \delta_2]$, the quality loss will not incur from the customer's point of view [8]. However, it should be noted that the size of the inner distances does not have the same meaning for the manufacturer. In this sense, they should satisfy the desire of the customer and the need of the manufacturer; in other words, inner tole–rances should be chosen so as not to cause high addi–tional costs for the manufacturer without the customer being able to notice any difference in product quality due to the variation of the given quality characteristic within the target interval.

So, the proposed PLF is zero at the target point, T or target interval ("optimal platou"), and asymptotically reaches the maximum losses, as can be seen from Figure 2.

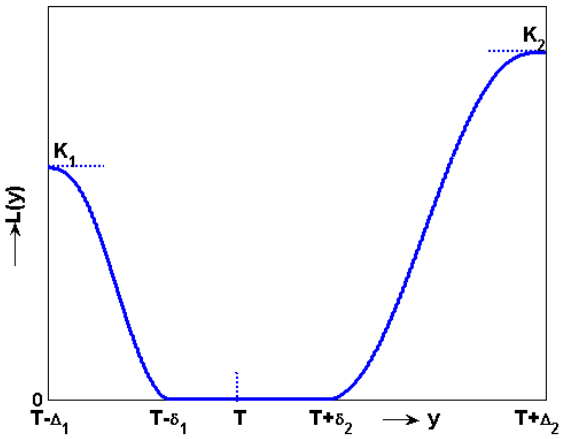


Figure 2. General form of the proposed polynomial loss function (PLF)

The shape parameter β adjusts the penalty for de–partures of a quality characteristic from the desired target T.

Note: In the following sections of this paper, it will be assumed that for responses outside the acceptable specification limits (LSL, USL), is always $L(y) = K_1$ and $L(y) = K_1$, where $K_1 < K_2$ or $K_1 > K_2$.

In the general form, the PLF is an asymmetric LF. It means that the maximum losses and/or shape parameter β are different on each side of the target and/or the target is not at the center of specification limits.

Formula (3) is easily applied to possible sub-variants of asymmetric PLF, for instance:

a) $\delta_1 = \delta_2 = \delta,\ \Delta_1 \neq \Delta_2,\ K_1 \neq K_2$;
b) $\delta_1 \neq \delta_2,\ \Delta_1 = \Delta_2 = \Delta,\ K_1 \neq K_2$;
c) $\delta_1 \neq \delta_2,\ \Delta_1 \neq \Delta_2,\ K_1 = K_2 = K$.

A fully symmetric PLF with a 'target interval' is often expected in practice. Its analytical expression is represented by equation (4), while the proposed family of PLFs with a 'target point' can be expressed in the form of equation (5), respectively:

$$L(y)=\begin{cases} K\left\{1-\left[1-\left(\dfrac{y-(T-\delta)}{\Delta-\delta}\right)^2\right]^\beta\right\} & \text{for } T-\Delta \le y < T-\delta \\ 0 & \text{for } T-\delta \le y \le T+\delta \\ K\left\{1-\left[1-\left(\dfrac{y-(T+\delta)}{\Delta-\delta}\right)^2\right]^\beta\right\} & \text{for } T+\delta < y \le T+\Delta \end{cases} \tag{4}$$

$$L(y)=\begin{cases} K_1\left\{1-\left[1-\left(\dfrac{y-T}{\Delta_1}\right)^2\right]^\beta\right\} & \text{for } T-\Delta_1 \le y \le T \\ K_2\left\{1-\left[1-\left(\dfrac{y-T}{\Delta_2}\right)^2\right]^\beta\right\} & \text{for } T \le y \le T+\Delta_2 \end{cases} \tag{5}$$

A graphical interpretation of Eq. (5) is given in Figure 3.

Equation (5) can easily be applied to the following subvariants:

a) $\Delta_1 = \Delta_2 = \Delta$, $K_1 \neq K_2$;

b) $\Delta_1 \neq \Delta_2$, $K_1 = K_2 = K$;

In practice, some quality characteristics (such as the amount of soft drink in a bottle, drug dose, etc.) have an asymmetric loss with respect to their target values. In such cases, Eq. (5) or its subvariants should be applying.

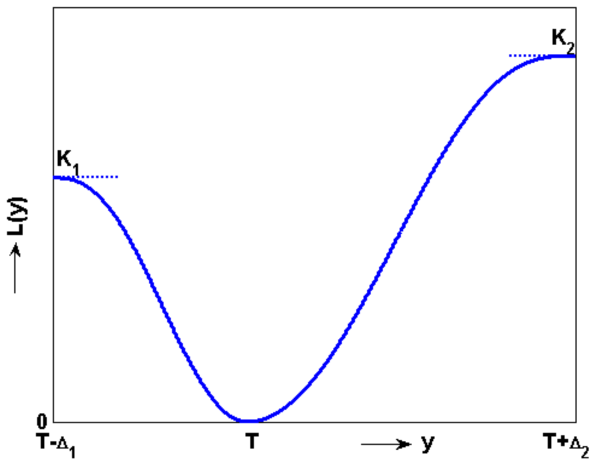


Figure 3. Typical form of the proposed asymmetric PLF

A completely symmetric PLF from this family is shown in Figure 4.

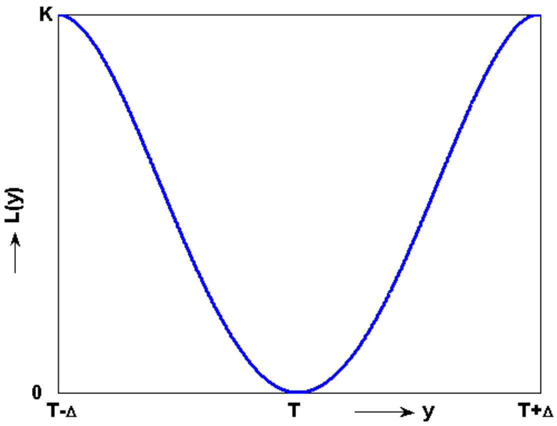


Figure 4. Typical form of the proposed symmetric PLF

This type of loss function is expressed mathe–matically by a simple formula:

$$L(y) = K\left\{1-\left[1-\left(\frac{y-T}{\Delta}\right)^2\right]^\beta\right\} \; for \; T-\Delta \le y \le T+\Delta \quad (6)$$

The symmetric LFs are special cases of asymmetric LFs and, thus, simpler for practical engineering rese–arch.

However, in some situations, it is quite necessary to employ asymmetric LFs, for instance, when the devi–ation of the quality characteristic from the target in one direction is more harmful than in the opposite direction and/or the estimated maximum losses on specification limits are really different. In manufacturing practice, the costs associated with rework and scrap may not be the same. As a rule, the repair of a product causes fewer costs than its replacement. For instance, if the diameter of a bearing is less than the LSL, it can still be reworked relatively easily and cheaply and brought to its target /nominal value. Conversely, if the diameter is greater than USL, the product may be functionally defective or must be discarded as scrap.

Figure 5 illustrates the flexibility of the PLF and the effect of the shape parameter β by showing the fa–mily of the curves β ranging from 0.5 to 6 (from bottom to top).

As the shape parameter decreases, the PLF beco–mes increasingly convex and asymptotically approac–hes the quadratic LF (β = 1) and then the so-called step LF (β = 0). Therefore, it is reasonable and expe–dient to choose the shape parameter in the interval $1 \le \beta \le 5$.

Everything previously shown indicates the high flexibility of this LF (which will be confirmed in the analysis that follows).

As seen from Figure 5, small β yields small eco–nomic losses for slight departures from the target, while large β causes larger losses for the same departures from the target. In other words, the shape parameter value adequately adjusts the penalty for each deviation from the desired target.

Likewise, from this figure, it is apparent that the curve for the shape parameter β = 2 separates the set of concave from the set of convex PLFs, so in that sense, it can be adopted as the initial ('default') shape parameter value for each calculation.

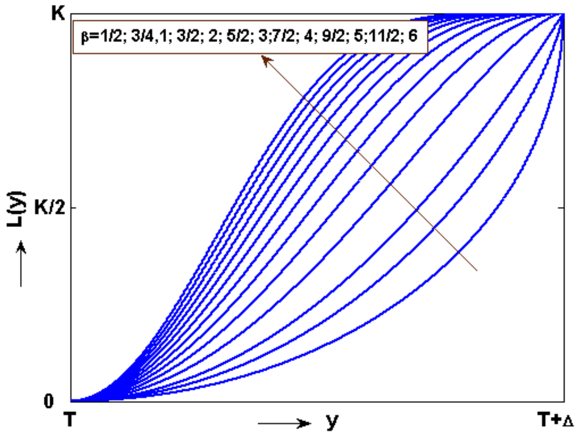


Figure 5. The appearance of the PLF depends on the shape parameter values (only for the right-hand side)

2.1 Determination of the shape parameter

The most frequent situation that occurs in the engine–ering practice is the case where the maximum loss and target are known in advance.

The shape of the curve that graphically displays the LF can be predicted by the appropriate choice of the shape parameter. Namely, assuming the target to be fixed, various shapes of the LF can be created for various shape parameter values.

The shape parameter allows researchers and/or practitioners to customize an LF in order to accurately reflect economic losses associated with characteristic quality deviations from the target.

The shape parameter β can be chosen by the sub–jective judgment of the decision maker (DM). However, this parameter can be determined less subjectively.

Suppose the target T, the maximum loss K, and the actual loss L_i for the given quality characteristic (response) y_i are known. In that case, the value of the unknown shape parameter can be easily computed from the simple formula:

$$\beta = \frac{\log\left(1-\frac{L_i}{K}\right)}{\log\left[1-\left(\frac{y_i-T}{\Delta}\right)^2\right]} \quad (7)$$

Namely, the LF is completely defined by just one point on the corresponding curve (see Figure 6). That point, say $M_i\{y_i, L_i\}$, represents the value of a response and its actual loss.

However, when two or more points are known (which is certainly a better option), the value of the shape parameter should be determined using the ordinary least squares method in the following way:

$$\min SS = \min \sum_{i=1}^{n} \left[L_i - L(y_i)\right]^2 \quad (8)$$

Equation (8) can be solved by using any traditional optimization method or a graphical approach.

Two above-described possibilities for determining β will be shown in an example discussed by Taguchi [15] and Sun et al. [5].

In this example, the following data are known: T = 10, K = 150, Δ =4 (as primary information), also (y_1,L_1) = (7, 125), and (y_2,L_2) = (9,17) (as secondary infor–mation).

By applying Eq. (7) were obtained:

- β =2.167, for point M_1 (7, 125);
- β =1.864, for point M_2 (9, 170).

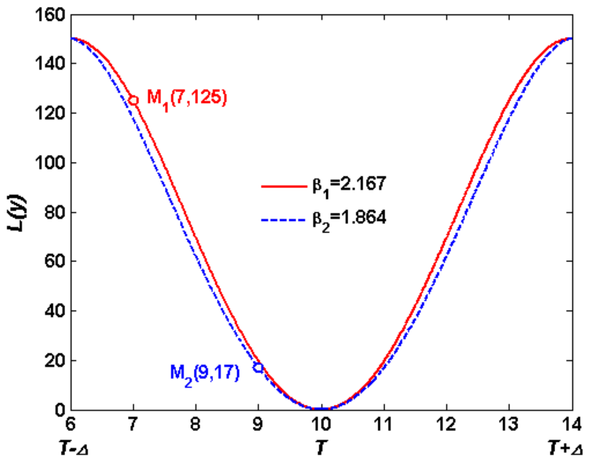


Figure 6. Influence of secondary points on the form of PLF curve

A more realistic result is obtained by using both points. In this case, as could be expected, an intermediate value for the shape parameter was obtained, namely β = 2.126. Figure 7 shows a graphical approach for determining the shape parameter based on Eq. (8).

For the same input data, the proposed PLF and other well-known LFs are plotted in Figure 8, along with two secondary points.

It is worthwhile to note that the points are very close to, but not exactly, the fitted curve.

As can be seen, the proposed PLF fits the data best, while the Taguchi and other LFs underestimate or over–estimate the losses at the given reference points. Consequently, in this sense, the proposed PLF is supe–rior to the other LFs.

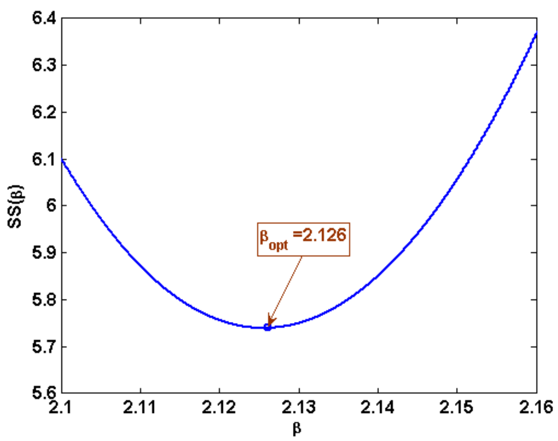


Figure 7. Graphical method for determining the shape parameter

A high value of the response (far from the target value) corresponds to a high value of the loss, and vice versa. In any way, the different shape parameters for different loss functions will be obtained.

Certainly, the different values of shape parameters produce different forms of the loss function curve, as shown in Figure 5.

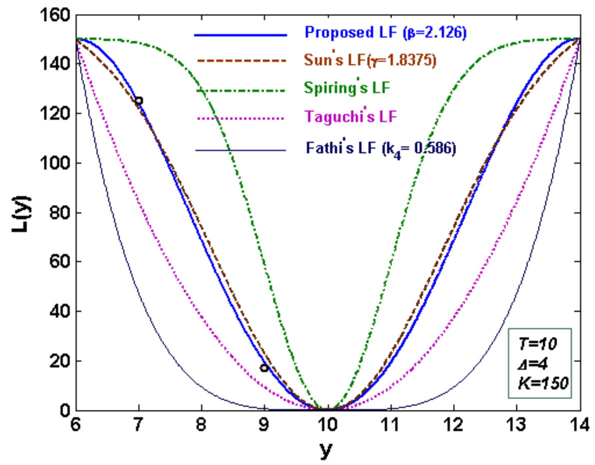


Figure 8. Comparison of the proposed PLF with some known LFs drawn for the same data

3. EXPECTED LOSS

It is well known that over time the quality charac–teristics of a product may vary from unit to unit. A pro–bability distribution function can represent these vari–ations.

The expected loss is defined as the average loss one would expect over a long period of a stable working process [6].

The expected loss is determined mathematically by evaluating the definite integral of the product of the LF as a random variable and the probability density func–tion of its distribution as follows:

$$E[L(\mathbf{y})] = \int_{-\infty}^{+\infty} L(y)\, f(y)\, dy \qquad (9)$$

The expected loss can predict and quantify the eco–nomic loss as a consequence of typical manufacturing disturbances.

The application of LFs combines the double requi–rement, the desire of the customer L(y, and the need of the manufacturing f(y), which are all combined by computing the expected loss value of the LF with respect to the probability distribution that represents the performance of a product or process.

In general, the expected loss associated with a par–ticular LF can be evaluated for most distributions, such as uniform, normal, and others, that the quality characteristic may follow.

Usually, it is assumed that the product or process quality characteristics follow the normal distribution, denoted as $N(\mu,\sigma^2)$. In such cases, the expected loss function can be expressed analytically. On the other hand, numerical integration can always determine the expected loss.

The expected loss for any LF, by definition, is keeping a process characteristic on target and, at the same time, reducing its variability.

For the proposed PLF, if the shape parameter β is a positive integer, the expected loss function (here denoted simply as L_β) can be expressed in closed form with a finite number of terms, as follows:

$$E_1 = \frac{K}{\Delta^2}\left[\sigma^2 + (\mu - T)^2\right] \tag{10a}$$

$$E_2 = 2E_1 - \frac{K}{\Delta^4}\left[3\sigma^4 + 6\sigma^2(\mu - T)^2 + (\mu - T)^4\right] \tag{10b}$$

$$E_3 = 3(E_2 - E_1) + \frac{K}{\Delta^6}\left[15\sigma^6 + 45\sigma^4(\mu - T)^2 + 15\sigma^2(\mu - T)^4 + (\mu - T)^6\right] \tag{10c}$$

$$E_4 = 2(2E_3 - 3E_2 + 2E_1) - \frac{K}{\Delta^8}\begin{bmatrix} 105\sigma^8 + 420\sigma^6(\mu - T)^2 + 210\sigma^4(\mu - T)^4 + \\ 28\sigma^2(\mu - T)^6 + (\mu - T)^8 \end{bmatrix} \tag{10d}$$

$$E_5 = 5(E_4 - 2E_3 + 2E_2 - E_1) + \frac{K}{\Delta^{10}}\begin{bmatrix} 945\sigma^{10} + 4725\sigma^8(\mu - T)^2 + 3150\sigma^6(\mu - T)^4 + \\ 630\sigma^4(\mu - T)^6 + 45\sigma^2(\mu - T)^8 + (\mu - T)^{10} \end{bmatrix} \tag{10e}$$

4. PROPERTIES OF THE PROPOSED LOSS FUNCTION

The proposed PLF has the following properties:

- It is a continuous and differentiable function over the whole defined domain;
- It has a minimum of zero at the target point for all target values within the specification limits,
- It is defined for all types of LFs (two-sided (NTB) and one-sided (LTB or STB)), for symmetric or asymmetric shapes, as well as for LFs that have a target interval or only a single target point; this functional form can be employed to represent either symmetric or asymmetric PLFs,
- For a known maximum loss and tolerance, the pro–posed PLF is fully defined by only one parameter,

Since the proposed PLF is a continuous function, it can be approximated by its Taylor series expansion about the target up to the desired number of terms. Taylor's polynomial with five terms is quite an appro–priate choice in this case. For instance, after omitting higher-order terms, the expected loss for the proposed PLF with rational values (half-integers) of the shape parameter can be expressed as:

$$E_{1/2} \cong (140E_1 - 70E_2 + 28E_3 - 5E_4)/128 \tag{10f}$$

$$E_{3/2} \cong (60E_1 + 90E_2 - 20E_3 + 3E_4)/128 \tag{10g}$$

As stated earlier, the proposed PLF for $\beta = 1$ dege–nerates in Taguchi's quadratic loss function, so the same is true for its expected loss function.

Quality experts always aim to eliminate quality loss. If the main goal is to reduce the expected loss to zero, then the basic task is to reduce the variation to zero. That is why variation reduction has become synony–mous with quality improvement [1,16].

According to the so-called 3σ rule, the tolerance range (allowable manufacturing deviation) for conside–red quality characteristics in a properly designed and realized product/process should satisfy the following relation: $t = 2\Delta = 6\sigma$.

By tightening the tolerance, manufacturing costs usually increase rapidly, while at the same time, the amount of scrap increases. Therefore, quality and cost should be well balanced [1].

- It is very flexible; in this sense, it can simulate the properties of most known LFs (Step LF, Taguchi's LF, Fathi's LF, Spiring's LF, Sun's LF, etc.),
- In terms of its performance, it does not lag behind the previously proposed LFs;
- In optimization problems, adjusting the shape para–meter value, it can improve the initial solution,
- Unlike some other LFs, the proposed PLF is sim–ple, easy to implement and interpret, and likewise understandable for practitioners with a basic know–ledge of mathematics (statistics).

Equations (10) guarantee good and realistic results if tolerance is chosen around the indicated limit.

Figure 9 shows the expected loss associated with the PLF for various levels of ratio σ^2/Δ^2, for the case where the quality characteristic is centered at the target (i.e., μ=T).

Obviously, Figure 9 confirms the fact that the QLF is only a special case of the PLF family (for β=1) because the expected loss for the QLF is linear in $(\sigma/\Delta)^2$. At the same time, the curves in this figure indicate a significant effect of variance reduction in the sense of expected loss reduction, especially in the domain of its small values.

Spiring [4] and Sun et al. [5] derived similar conclusions for their loss functions (INLF, MINLF).

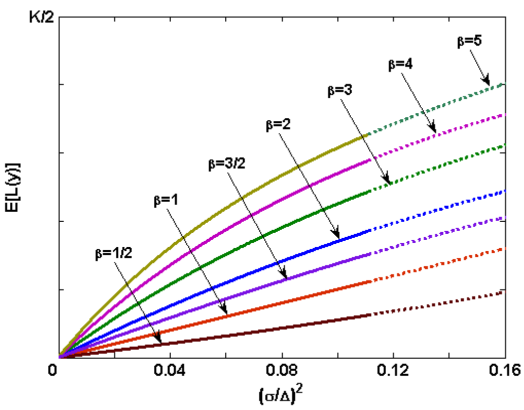


Figure 9. Expected loss for the PLF vs. relative variance

5. MULTI-RESPONSE OPTIMIZATON

The choice of the optimization method and executive algorithm depends on the particular problem.

For the multi-response optimization problems, the total loss function, as an aggregation function, may be defined as the weighted sum of the individual loss functions:

$$L_T\left[\mathbf{y}(\mathbf{x})\right]=\sum_{j=1}^{m}\omega_j L_j\left[y_j(\mathbf{x})\right] \quad (11)$$

where m is the number of responses, $y_j(\mathbf{x})$ is the j-th response, ω_j is the j-th relative weight ($\omega_j \in R^m \middle| \omega_j > 0, \sum \omega_j = 1$), L_j and is the j-th indivi–dual LF.

When it is a case of a dual-response problem (m = 2), then $\omega_1 = \omega$ and $\omega_2 = 1 - \omega$ can be written.

The choice of relative weight is based on the impor–tance of the various responses to be simulta–neously optimized. In real circumstances, it is quite likely that there is a difference in the importance deg–rees (prio–rities) of various responses, i.e., that one response has a greater impact on the system/ product/ process in comparison with the others.

Aggregation means formulating a single-response optimization problem such that optimal solutions to the single-response optimization problem are so-called Pareto optimal solutions to the multi-response optimization pro–blem. In other words, this approach converts the original optimization problem with multiple res–ponses into an optimization problem with a single response.

According to the above, it is clear that the DM (design/process engineers, quality control experts, end users, etc.) plays an important role in conducting the optimization procedure.

When the DM has preference information about the importance of individual responses, then the optimal solution can be determined by solving Eq. (11) directly. If the DM does not have any information in advance (*a priori*), he/she should be supported to subsequently (*a posteriori*) choose the most appropriate optimal solu–tion, according to his/her subjective judgment. For this purpose, it is necessary to determine a larger number of optimal solutions for different relative weights in proportion to the importance (priority) of the response, that is, to collect a representative set of Pareto optimal solutions. Then the DM must choose one of them.

When the optimization procedure is performed without the DM articulating any preferences among the responses, then researchers, as a rule, attribute the same importance to all responses, i.e., choose the same relative weights for each response. In this case, all Pareto optimal solutions are treated as equally good.

The Pareto optimal approach for multi-response optimization problems means solving such a problem is more complex than it is for a traditional single-response optimization problem.

6. ILLUSTRATIVE EXAMPLES

Three real examples appearing in the literature were employed to illustrate the properties and possibilities of the proposed PLF.

6.1 Example 1: Roman-style catapult study

A well-known and well-studied experiment called the Roman-style catapult has been studied by several researchers [17-22].

This experiment aimed to investigate the influence of three design factors, arm length (x_1), stop angle (x_2), and pivot height (x_3), on the prediction of the distance (y) from the point where the projectile landed to the position of the Roman-style catapult.

A central composite design (CCD) with three replicates at each factor combination and six replicates at the center point were selected and conducted.

Data from this experiment necessary for further calculation are given in Table 1.

Table 1. Experimental results for the Roman-style catapult problem

n	$y_{\mu 1}$	$y_{\mu 2}$	$y_{\mu 3}$	$\bar{y}_{\mu}$	y_{σ}
1	39	34	42	38.3	4.0
2	80	71	91	80.7	10.0
3	52	44	45	47.0	4.4
4	97	68	60	75.0	19.5
5	60	53	68	60.3	7.5
6	113	104	127	114.7	11.6
7	78	64	65	69.0	7.8
8	130	79	75	94.7	30.7
9	59	51	60	56.7	4.9
10	115	102	117	111.3	8.1
11	50	43	57	50.0	7.0
12	88	49	43	60.0	24.4
13	54	50	60	54.7	5.0
14	122	109	119	116.7	6.8
15	87	78	89	84.7	5.9
16	86	79	85	83.3	3.8
17	88	81	87	85.3	3.8
18	89	82	87	86.0	3.6
19	86	79	88	84.3	4.7
20	88	79	90	85.7	5.9

The fitted second-order regression functions from Kim and Lin [18] are as follows:

$$\begin{aligned}\hat{y}_{\mu} &= 84.88+15.29\,x_1+0.24\,x_2+\\ &18.80\,x_3-0.52\,x_1^2-11.80\,x_2^2+0.39\,x_3^2\\ &+0.22\,x_1x_2+3.60\,x_1x_3-4.42\,x_2x_3\end{aligned} \quad (12)$$

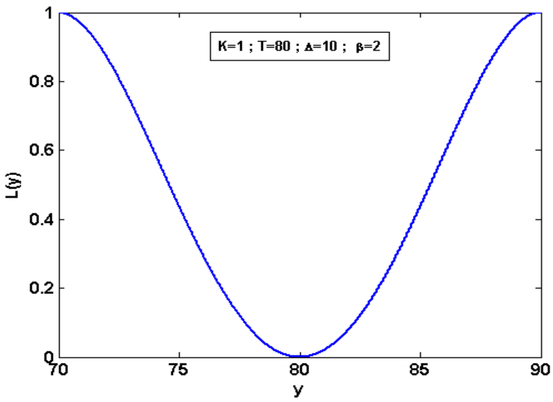
$$\begin{aligned}\hat{y}_{\sigma} &= 4.53+1.84\,x_1+4.28\,x_2+3.73\,x_3\\ &+1.16\,x_1^2+4.40\,x_2^2+0.94\,x_3^2+\\ &1.20\,x_1x_2+0.73\,x_1x_3+3.49\,x_2x_3\end{aligned} \quad (13)$$

Note that the subscripts μ σ in Eqs. (12) and (13) represent the sample mean and sample standard devi–ation, respectively.

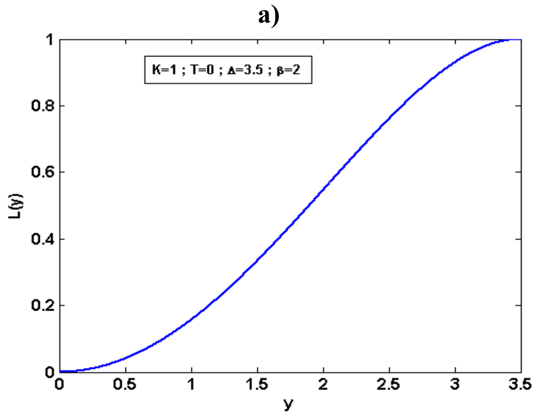
The considered problem requires that the target value for the mean response be 80, while the desired value for the standard deviation should be at most 3.5. (Theoretically, the minimum value of standard deviation is always zero).

The response y_μ y_σ is nominal-the-best (NTB) and smaller-the-better (STB) type responses, respectively.

In Figure 10 are displayed the corresponding LFs.



a)



b)

Figure 10. Loss functions for Example 1: a) bilateral LF, b) unilateral LF

Assuming the regression functions are adequate, this paper puts more emphasis on the optimization proce–dure and the pure finding of a potentially satisfactory solution.

In this example, by using the proposed PLF under given conditions and constraints, the dual-response optimization problem can be stated as:

$$\begin{aligned} &\textit{Minimize } L_T(\mathbf{y}) = \omega L_\mu(\mathbf{y}) + (1-\omega) L_\sigma(\mathbf{y}) \\ &\textit{subject to } \quad 70 \le y_\mu \le 90 \\ &\qquad 0 \le y_\sigma \le 3.5 \\ &\qquad -1.68 \le x_i \le 1.68,\ i = 1,2,3. \\ &\qquad (K=1;\ T=80;\ \beta=2) \end{aligned} \quad (14)$$

By using Eq. (14), the simultaneous dual-response optimization procedure (by taking the various value of shape parameter ω) yields the optimal factor setting $\mathbf{x}^*$, with corresponding responses $\hat{y}_\mu^*$ and $\hat{y}_\sigma^*$, and the re–sulting loss values $L_T^*(\mathbf{y})$.

A subset of Pareto optimal (non-dominated) solutions for the Roman-style catapult problem is shown in Table 2.

Note that by using the symmetric LF the constant K is generally ignored because it does not affect the optimization result.

According to Table 2, both responses $\hat{y}_\mu$ and $\hat{y}_\sigma$ increase as the relative weight ω grows. A trade-off curve (in the cases of two or three responses) that connects the non-dominated (efficient) points on the border of the feasible region are often referred to as the Pareto frontier. The decision of which of the two adjacent solutions on the Pareto frontier is favorable must be made by the DM, since it is known that there is no unique optimum solution for all responses simultaneously (in fact, there are potentially infinitely many optimal solutions).

On the other hand, a large number of Pareto optimal solutions may cause an excessive burden to the DM in his/her attempts to choose the most preferred solution among a set of equivalent alternatives. To facilitate the decision-making process, the obtained (near) optimal solutions in this example are arranged in the Pareto frontier in the responses space, which is displayed in Figure 11.

In general, such plots can be very useful since they present the trade-offs between two different responses.

It is worth noting here that evenly distributed rela–tive weights must not necessarily produce a uniformly distributed presentation of the Pareto optimal points.

The graph shown in this figure suggests that there is a conflict between the two responses. As the standard deviation decreases, the predicted distance proportio–nally moves away from its target value and vice versa.

In this example, the Pareto frontier can be divided into two zones since this graph has the shape of a stra–ight line (in the first zone) and the shape of a bent line (in the second zone). In the first zone, the predicted mean response (distance) is too far from the target va–lue, while the standard deviation has its lowest values.

Table 2. Final results from the optimization procedure

ω_j	x_1	x_2	x_3	$\hat{y}_\mu$	$\hat{y}_\sigma$
0.00	0.047633	-0.100816	-0.809467	70.00000	2.08102
0.25	0.105415	-0.234039	-0.430801	77.14227	2.83286
0.50	0.123869	-0.273729	-0.317209	79.35916	3.07890
0.60	0.125752	-0.277926	-0.305183	79.59449	3.10535
0.70	0.127226	-0.280594	-0.297516	79.74860	3.12271
0.75	0.128126	-0.281464	-0.294907	79.80698	3.12929
0.80	0.128235	-0.282456	-0.292109	79.85708	3.13494
0.85	0.130246	-0.282409	-0.291411	79.89987	3.13978
0.90	0.121100	-0.286462	-0.281494	79.93897	3.14427
0.92	0.135905	-0.280995	-0.292860	79.96150	3.14682
0.94	0.124736	-0.285686	-0.282814	79.96981	3.14769
0.96	0.127109	-0.285296	-0.284125	79.97983	3.14880
0.98	0.125700	-0.286594	-0.282249	79.98735	3.14967
1.00	0.124992	-0.237370	-0.300000	80.00000	3.16123

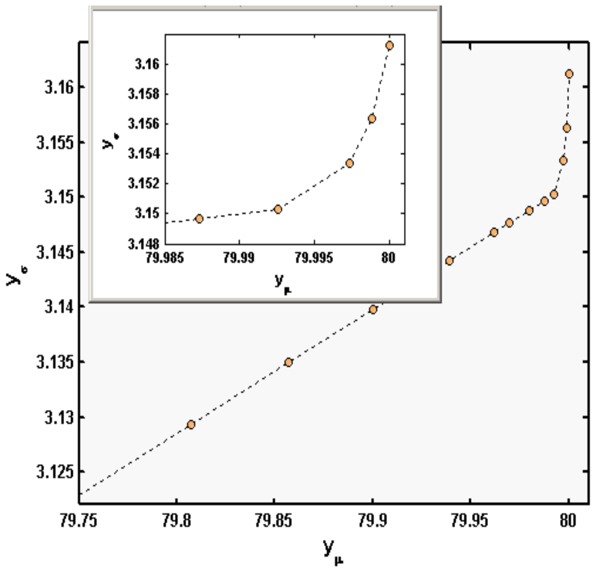


Figure 11. Pareto frontier obtained by following optimization of the two responses (The inserted box shows the Pareto frontier in the second zone)

It seems unlikely the decision-maker can accept these solutions. In the second zone, the mean response reaches its topmost values, while the standard deviation also increases but acceptably.

The DM may need help in choosing the best compromise solution among the responses considered.

In this example, although both responses are supplemented to each other, bearing in mind the type of problem at hand, it is appropriate to attach greater importance to the mean response.

Table 3. Comparison of the optimal results based on expected loss for the catapult problem

Method	Köksoy - Fan [19] [Expected loss based on INLF (σ_L=17)] [a]	Proposed method [Expected loss based on PLF] [b]
x_1	0.126215	0.126709
x_2	-0.27890	-0.27871
x_3	-0.30238	-0.30279
$\hat{y}_\mu$	79.64979	79.64998
$\hat{y}_\sigma$	3.11157	3.11159

[a] MSE= 9.804534 ; [b] MSE= 9.804506

In real manufacturing practice, a system operating behavior over a long period of time can only be predicted by employing the expected loss approach.

The Roman-style catapult problem is a suitable example to illustrate the application of the expected loss approach.

By using the expected loss E_2 proposed in this paper and the constraints in Eq. (14), the simultaneous optimization procedure yields the following optimal solution $\hat{\mathbf{y}}^* = \{\hat{y}_\mu^*,\ \hat{y}_\sigma^*\} = \{79.64989,\ 3.11158\}$.

The existing optimization methods, among them also the methods based on loss function, can predict the appropriate optimal conditions in the actual state of a system (product or process).

In such circumstances, the DM could seek a suitable final solution in the second zone of the Pareto frontier (with any weight close to but not equal to one). That seems to be a rational choice.

Regardless of the satisfactory result, as in this example, the DM may require one potentially better solution. An improved solution can be found by using the shape-based approach.

Namely, the expected loss functions according to the method recommended in this paper are derived for different shape parameters (see equations (10)).

In this example, the shape-based approach in the tightening mode was implemented; that is, the expected loss E_3 was chosen.

After repeating the optimization procedure, the new vector of optimal responses $\hat{\mathbf{y}}^* = \{\hat{y}_\mu^*,\ \hat{y}_\sigma^*\} = \{79.64998,\ 3.11159\}$ was obtained.

It can be said that the new solution is somewhat better than the initial one. Also, it seems that in this case, a significantly better solution cannot be obtained.

Using the expected loss approach, Köksoy and Fun [19] found a similar optimal solution. Both solutions are summarized in Table 3.

Mean squared error (MSE) [21] can be used as a criterion for comparing the obtained results. In this case, it is evident that the proposed method provides a very good balance between the two considered responses.

Figure 12 displays the overlaid contour plot of the PLF estimated mean and standard deviation responses when keeping the value of x_3 (pivot height) at its optimal level.

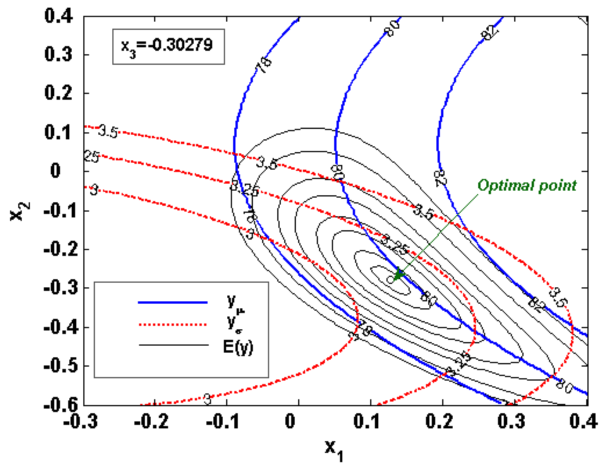


Figure 12. Contour plot for the estimated mean and standard deviation for Example 1

Once again, the catapult study requires preconditions according to which the target value should be at 80 and the standard deviation less than 3.5.

This graphical presentation may be helpful for a better understanding of the situation.

It is important to note that there is a serious drawback in the application of the expected loss based on INLF [4], related to the choice of the so-called scale factor (σ_L), which can significantly affect the optimization result [19]. In fact, there are no theoretically based recommendations for choosing the scale factor [6, 9, 19].

Other researchers have also studied the Roman-style catapult problem. Their results can hardly be com–pared in a straightforward way since they have applied different methods and optimization criteria and different underlying assumptions.

Since each optimization method has its own merits and limitations, one can employ different methods to find the potentially best optimal solution and then choose the favored alternative. In this sense, the results of different optimization methods can be compared without any specified criterion, as done in Table 4 for Example 1.

It is interesting that (approximately) all the different optimal solutions from Table 4 can be found in Table 2. Based on the relative weight values, it can be noticed that some optimization methods favor the response of sample mean , while others favor the response of sample variance .

Table 4. Comparison of the optimal results obtained by different methods

Approach	x_1	x_2	x_3	$\hat{y}_\mu$	$\hat{y}_\sigma$
Köksoy - Fan [19]	0.126215	-0.27890	-0.30238	79.64979	3.111573
Kim-Cho [22]	0.115	-0.256	-0.369	78.352	2.966
Costa [26]	0.152	-0.321	-0.285	80.00000	3.158
Kim-Lin [18]	0.12	-0.27	-0.32	79.2300	3.0600
Ding-Lin-Wei [20]	0.1290	-0.2848	-0.2856	79.9813	3.1490
Lin-Tu [21]	0.12621	-0.27890	-0.30238	79.64964	3.111557
Vining–Myers [23]	0.129129	-0.28511	-0.28461	80.0000	3.151077
Proposed method	0.126709	-0.27871	-0.30279	79.64998	3.111590

6.2 Example 2: Television process

In this example, Spiring and Yeung [9] considered the expected loss in the production of a television set in Factory A (located in Japan) and in Factory B (located in America).

Both factories made television sets using identical designs and tolerances. Besides, these factories belong to the same manufacturing company.

The distribution of quality characteristic of interest (color density) for Factory A was approximately normal, while the distribution for Factory B was approximately uniform [1, 2] .

The maximum cost (loss) of repairing a failed television set in the factories was \$2 per unit.

This cost was incurred once the quality characteristic exceeded the two-sided tolerance (Δ) of ±5 units from its target value equal to zero.

For Factory A, it is assumed that the quality characteristic follows the normal distribution with $\mu_R = 0$ and $\sigma_R^2 = 4$.

The expected loss associated with INLF [4, 9] has a form of the exponential function (15), while the expec–ted loss associated with INLF [9] and a uniform distri–bution is expressed as an erf function (16), respectively:

$$E[L(\mathbf{y})] = K\left\{1 - \frac{\sigma_L}{\sqrt{\sigma_R^2 + \sigma_L^2}} \exp\left[-\frac{(\mu_R - T)^2}{2(\sigma_R^2 + \sigma_L^2)}\right]\right\} \quad (15)$$

$$E[L(\mathbf{y})] = K\left\{1 - \sqrt{\frac{\pi}{2}}\left(\frac{\sigma_L}{a-b}\right)\left[erf\left(\frac{a-T}{\sqrt{2}\sigma_L}\right) - erf\left(\frac{b-T}{\sqrt{2}\sigma_L}\right)\right]\right\} \quad (16)$$

where $erf(z) = 2/\sqrt{\pi}\int_0^z \exp(-t^2)\,dt$

Using Eq. (15), the expected loss per television set for K = 2, and σ_L = 1.25 would be \$0.94.

For Factory B, it is assumed that the quality charac–teristic follows the uniform distribution over the interval (a, b), with a = -5 and b = 5.

Using Eq. (16), the expected loss per television set for all the same conditions would be \$1,373.

Relative expected loss (cost) can be defined as the ratio:

$$R_E = E[L(\mathbf{y})]/K \quad (17)$$

In this example, by using the UNLF the following ratios are obtained:

- R_E = 0.94 / 2 = 0.47 (for Factory A),
- R_E = 1.373 / 2 = 0.687 (for Factory B).

Using the proposed PLF and associated expected loss (see equation (10b)), the expected value of the loss caused by deviation in the production of the television set in Factory A would be \$0.486.

The expected loss associated with the proposed PLF and uniform distribution for response is expressed as a simple function:

$$E[L(\mathbf{y})] = \frac{K}{b-a}\left[\frac{2}{3}\frac{(b-T)^3 - (a-T)^3}{\Delta^2} - \frac{1}{5}\frac{(b-T)^5 - (a-T)^5}{\Delta^4}\right] \quad (18)$$

Using Eq. (18), the expected value of the loss caused by deviation in the production of the television set in Factory B would be \$0.933.

In this example, by using the proposed PLF the following ratios are obtained:

- $R_E = 0.486/2 = 0.243$ (for Factory A),
- $R_E = 0.933/2 = 0.467$ (for Factory B).

The computation results obtained by employing two different LFs (INF, PLF) lead to the same general conc–lusion, except that the proposed PLF predicts smaller losses for both factories and explicitly favors Factory A compared to Factory B.

The different results for the two factories are due to two different perceptions of quality; in Factory B, the engineers paid all their attention to meeting the tole–ran–ces, whereas in Factory A, they were focused on meeting the target.

6.2 Example 3: Printing ink process

In this example, the proposed method is illustrated using the printing ink problem taken from Box and Draper [23]. Many other authors also discussed the printing ink problem [18,20,21,24-28] as an ideal example for pre–senting the properties of their approaches.

The selected experiment in this study was the three-level full factorial design (FFD), with three replicates at each run, i.e., combination of design factor levels (low, intermediate, high).

This type of factorial design was discussed exten–sively by Ozoemena et al. [29]. Box-Behnken design (BBD) [30] or Taguchi's orthogonal array L_{16} [31] are also very useful for this type and size of experiment sin–ce they provide a good balance between the cost of the experiment (including time) and the required accuracy of the results.

The original experimental data set for the printing ink problem is displayed in Table 5.

Table 5. Experimental results for the printing ink process

n	$y_{\mu 1}$	$y_{\mu 2}$	$y_{\mu 3}$	$\bar{y}_{\mu}$	y_{σ}
1	34	10	28	24.0	12.49
2	115	116	130	120.3	8.39
3	102	186	263	213.7	42.80
4	82	88	88	86.0	3.46
5	4	178	188	136.7	80.41
6	322	350	350	340.7	16.17
7	141	110	86	112.3	27.57
8	259	251	259	256.3	4.62
9	290	280	245	271.7	23.63
10	81	81	81	81.0	0.00
11	90	122	93	101.7	17.67
12	319	376	376	357.0	32.91
13	180	180	154	171.3	15.01
14	372	372	372	372.0	0.00
15	541	568	396	501.7	92.50
16	288	192	312	264.0	63.50
17	432	336	513	427.0	88.61
18	713	725	754	730.7	21.08
19	364	99	199	220.7	133.80
20	232	221	266	239.7	23.46
21	408	415	443	422.0	18.52
22	182	233	182	199.0	29.45
23	507	515	434	485.3	44.64
24	846	535	640	673.7	158.20
25	236	126	168	176.7	55.51
26	660	440	403	501.0	138.90
27	878	991	1161	1010.0	142.50

The purpose of this experiment was to examine the effect of speed (x_1), pressure (x_2), and distance (x_3) on the ability of the printing machine, as an experimental response (y), to apply colored inks to package labels.

Supposing second-order models were adequate, the fitted responses for the mean and standard deviation of the characteristic of interest were given by Vining and Myers [24]:

$$\hat{y}_{\mu} = 327.6 + 177.0 x_1 + 109.4 x_2 + 131.5 x_3 + 32.0 x_1^2 - 22.4 x_2^2 - 29.1 x_3^2 + 66.0 x_1 x_2 + 75.5 x_1 x_3 + 43.6 x_2 x_3 \quad (19)$$

$$\hat{y}_{\sigma} = 34.9 + 11.5 x_1 + 15.3 x_2 + 29.2 x_3 + 4.2 x_1^2 - 1.3 x_2^2 + 16.8 x_3^2 + 7.7 x_1 x_2 + 5.1 x_1 x_3 + 14.1 x_2 x_3 \quad (20)$$

In both the fitted models, the group means and the group standard deviations as responses were used (rat–her than using all the experimental data).

This process requires that the target value for the mean be 500 and a standard deviation less than 60.

The optimization procedure was completely the same as in Example 1. Using the default expected loss E_2 and bearing in mind given data ($T_{\mu} = 500$, $K = 1$, $\Delta = 100$), the optimal solution $\hat{\mathbf{y}}^* = \left\{\hat{y}_{\mu}^*, \hat{y}_{\sigma}^*\right\} = \{494.778, 44.475\}$ was obtained.

The results of the proposed method, based on a cu–boidal experimental region, are summarized and com–pared with those of other authors in Table 6. As can be seen in Table 6, the optimal results obtained by different methods are similar and comparable, with the exception of the results presented by Vining and Myers [24].

Figure 13 displays the overlaid contour plot of the PLF estimated mean and standard deviation responses when keeping the value of x_2 (pressure) at its optimal level.

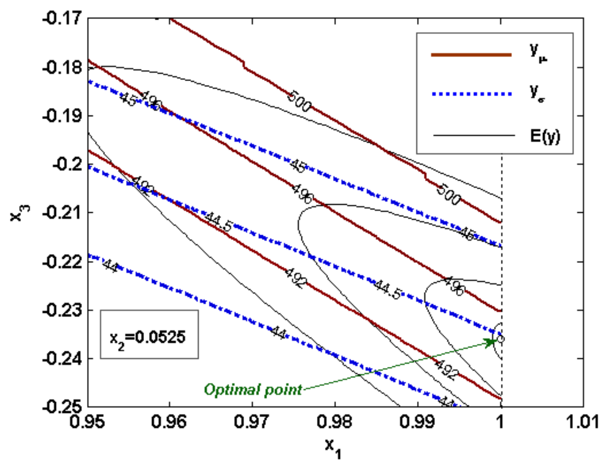


Figure 13. Contour plot for the estimated mean and standard deviation for Example 2

There are many different criteria for comparing the optimization results. For that purpose, the relative target deviation (RTD) can also be used [32,33]. The RTD is a test of the "goodness-of-target prediction" type. The RTD value should be as small as possible (zero is the most favorable value). It can be easily proved that the application of this test confirms the rank of each method listed in Table 6.

7. DISCUSSION

The most important step in any multiple-response opti–mization procedure is to establish a functional relation–

ship between the responses and the design factors. Based on the fitted models, many optimization schemes have been proposed by many researchers.

The dual response (DRS) approach is a special case of the response surface methodology (RSM) with two responses, in this case, the mean and the standard deviation of the characteristic of interest. The DRS approach first builds the empirical models for both responses separately and then optimizes one of these responses subject to an adequate constraint on the other's value (typically keeping the standard deviation below a specified value).

Unfortunately, the DRS approach has certain limitations and drawbacks. For instance, this approach always assumes a second-order mathematical model for both the mean and standard deviation, and accordingly, fitted models are derived. Also, a normal distribution is assumed in the DRS approach. However, such assumptions may not be valid in practice. Lastly, the DRS approach cannot be applied to unreplicated factorial designs.

Following these considerations, the printing ink process has been the subject of an investigation by several researchers (Das [34], Zeybek et al. [35], Boylan and Cho, [36]). These authors found in their studies that a normal distribution cannot properly model printing ink process data. This process data rather follows skew-normal distribution or (approximately) right-skewed gamma distribution.

Zeybek et al. [35] fitted both the mean and standard deviation responses for the printing ink process also in the form of quadratic (second-order) models as follows:

$$\hat{y}_\mu = 327.6 + 177.0\, x_1 + 109.4\, x_2 + 131.5\, x_3 + 32.0\, x_1^2 - 22.4\, x_2^2 - 29.1\, x_3^2 + 66.0\, x_1 x_2 + 75.5\, x_1 x_3 + 43.6\, x_2 x_3 \tag{21}$$

$$\hat{y}_\sigma = 20.1 + 6.0\, x_1 + 8.11\, x_2 + 15.17\, x_3 + 1.8\, x_1^2 - 1.6\, x_2^2 + 9.0\, x_3^2 + 4.29\, x_1 x_2 + 2.62\, x_1 x_3 + 7.24\, x_2 x_3 \tag{22}$$

Table 6. Comparison of the optimal results obtained by different methods

Approach	x_1	x_2	x_3	$\hat{y}_\mu$	$\hat{y}_\sigma$	MSE
Kim-Lin [18]	1.000	0.0860	-0.2540	496.080	44.630	2007.1
Vining-Myers [23]	0.614	0.2280	0.1000	500.000	51.778	2681.0
Ding-Lin-Wei [20]	1.000	0.0890	-0.2550	496.473	44.671	2007.9
Lin-Tu [21]	1.000	0.0740	-0.2520	494.440	44.430	2005.1
Copeland-Nelson [25]	0.975	0.0589	-0.2139	495.000	44.730	2025.8
Castilo- Montgomery [24]	1.000	0.1184	-0.2590	500.000	45.097	2033.7
Costa [26]	1.000	0.2049	-0.3180	500.000	45.132	2036.9
Köksoy-Doganaksoy [27]	1.000	0.1643	-0.3085	495.980	44.650	2009.8
Proposed method	1.000	0.0525	-0.2358	494.778	44.475	2005.3

Applying the optimization procedure described previously, and the new equations (21) and (22), a completely different optimal solution was obtained. It is evident from Table 7, that the proper modeling of experimental data yields a larger mean and remarkably lower variance.

Table 7. The optimal solution for the printing ink problem with improved modeling by Zeybek et al. [35]

Approach	Proposed method
x_1	1.00000
x_2	-0.08743
x_3	-0.10231
$\hat{y}_\mu$	500.001
$\hat{y}_\sigma$	25.142

Das [34] has recommended the use of generalized linear models (GLMs) [37], which are known as joint GLMs (JGLMs), to derive the joint mean and variance models instead of separate mean and variance models as in the DRS approach. In general, the GLM can improve the fit of the mathematical model to the given data and produce less variance (shorter confidence interval on the mean response).

This author applied GLM approach in order to derive the non-linear models, instead of the quadratic polynomials, for both the mean and the standard deviation. In this case, these models have the form of simple exponential functions, namely:

$$\hat{y}_\mu = \exp(5.51 + 0.62x_1 + 0.42x_2 + 0.46x_3 - 0.10x_1x_2 - 0.07x_1x_3 - 0.10x_2x_3 + 0.25x_1x_2x_3) \tag{23}$$

$$\hat{y}_\sigma = \exp(-2.45 - 0.76x_1 - 0.37x_3 - 0.23x_1x_3) \tag{24}$$

The optimal solution for new conditions is given in Table 8. In this case, it is clear that the proposed PLF and appropriate mathematical models can ensure that the process characteristic of interest reaches the target value with a negligibly small standard deviation.

Table 8. The optimal solution for the printing ink problem with improved modeling by Das [34]

Approach	Proposed method
x_1	0.91190
x_2	-0.27970
x_3	0.64155
$\hat{y}_\mu$	499.999
$\hat{y}_\sigma$	0.030

The real problem considered in the present example suggests that the different mathematical models may give completely different estimated responses and, consequently, the optimal settings even when applying the same optimization approach. These observations should be taken into account when applying and analyzing the optimization methods presented in this paper.

From the above examples, it is clear that the inherent performance of the proposed method enables an adequate response to the quality of the model predictions.

8. CONCLUSION

The new family of the symmetric and asymmetric poly–nomial loss functions proposed in this paper represents the realistic and representative LFs that provide a reaso–nable assessment and prediction of the actual losses associated with the variability of the process/product quality characteristics.

The proposed LFs have some good properties and capabilities. The main advantage of this family of LFs is flexibility in the sense that it allows a researcher or qua–lity practitioner to choose a specific LF that can best reflect the problem at hand.

In contrast, Taguchi LF and some others do not ref–lect actual losses within specification limits [LSL, USL] (when these data are known). That's why these LFs al–ways underestimate or overestimate the actual losses. Therefore, the choice of some LF should be based on the process/product behavior and availability of loss data.

Additionally, the developed PFLs provide a chance to find a potentially improved solution compared to the initial one.

It is also shown that the method based on a new family of PLFs is insensitive (robust) with respect to empirical data, which do not deviate significantly from the normal distribution.

The application of the proposed PLF has been de–monstrated in three well-known examples that have been analyzed in previously published papers.

The results show that the new LFs can be compe–titive with other known LFs and serve as an alternative choice that quality experts, industrial statisticians, and quality practitioners may utilize to solve particular optimization problems.

Certainly, the family of LFs proposed in this paper covers only some of the spectrum of potentially new and different loss functions.

The focus in this work was on symmetric PLF, while the properties of asymmetric PLF were not considered to a sufficient extent. Future research is going to address this issue.

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НОВА ФАМИЛИЈА ФУНКЦИЈА ГУБИТКА КВАЛИТЕТА

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Функција губитка квалитета је мера губитка квалитета изазваног одступањем карактеристике квалитета од њене дефинисане циљне вредности. Тагучи је први развио функцију губитка квалитета како би обезбедио бољу процену економских губитака које би претрпели произвођачи и купци због одступања карактеристика квалитета од циљне вредности. Неколико истраживача је предложило различите ревидиране функције губитка за превазилажење неких недостатака Тагучијеве функције губитка. У овом раду се препоручује нова фамилија функција губитка квалитета, која је веома флексибилна, једноставна и лака за имплементацију. Три стварне студије случаја су показале употребљивост и могућности предложене нове функције губитка за квантификацију и предвиђање губитака квалитета.