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Effects of the ARA transform method for time fractional problems

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Abstract. The aim of this study is to establish the solutions of time fractional mathematical problems with the aid of new integral transforms called the ARA transform. The fractional derivative is taken in the sense of Liouville-Caputo derivative. The fractional partial differential equations are reduced into ordinary differential equations. Later solving this fractional equation and applying inverse the ARA transform, the solution is acquired. The implementation of this transform for fractional differential equations is very similar to the implementation of the Laplace transform. However, the ARA transform allows us to take the integral transform of some functions for which we can not take the Laplace transform. The illustrated examples justify that the implementation and efficiency of this method are better than any other integral transforms to tackle time fractional differential equations (TFDEs).

1. Introduction

Since fractional differential equations have taken an essential role in the modelling of scientific processes in dynamical systems, fluid flow, biology, electrical networks, reaction, signal processing, and advection–diffusion systems [5, 6, 12, 1], they attract the attention of various scientist in diverse branches increasingly. As a result, a great number of techniques such as [12, 1] have been improved to construct analytical and truncated solutions of fractional differential equations. Integral transform methods, introduced by the mathematician P.S. Laplace [16, 14], have great importance for solving any kind of differential equation. Therefore, they are also utilized in order to acquire the solution of fractional mathematical problems. By these transforms, fractional differential equations are reduced into algebraic equations which are easier to tackle with. In the literature, various integral transforms such as Laplace, Fourier, Sumudu and Shehu, etc. transforms have

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been introduced to deal with differential equations [16, 14, 15, 9]. Some fractional-order kinetic matrix equations are solved by Hadamard fractional integral operator via Mellin integral transform [2]. Moreover, the solutions of higher order fractional differential equations are established by the Shehu integral transform [3].

The novelty of this study is that this is the one of the new studies in which ARA transform method is utilized to established the solution of heat-like fractional differential problem in Liouville–Caputo sense. In this research, we investigate new powerful and versatile integral transform, called the ARA transform, for the establishment of the solution of TFDEs.

The paper is designed as follows: fundamental notions of fractional calculus are given in section 2. The ARA transform of some essential functions, Liouville-Caputo derivative and Riemann-Liouville integral are given in section 3. The algorithm of the method for fractional problems including heat-like and wave-like equations is presented in section 4. Examples are illustrated and analyzed in section 5. The conclusions of the method are presented in final section.

2. Preliminary results

Fundamental definitions and their properties are presented in this subsection [7, 10, 13, 4, 8].

Definition 1. α^{th} order of the time-fractional integral of $u(t)$, a real valued function, in Riemann-Liouville sense is defined as

$$I_t^{\alpha} u\left(t\right) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(t-s\right)^{\alpha-1} u(s)ds.$$

Definition 2. α^{th} order of the fractional derivative of $u(t)$ in Liouville-Caputo sense is introduced as

$$D_t^{\alpha} u\left(t\right) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \left(t-s\right)^{n-\alpha-1} u^{(n)}(s)ds, \quad t \in \left[t_0, t_0+T\right],$$

where $n-1 < \alpha < n$ and $u^{(n)}(t) = \frac{d^n u}{dt^n}$.

Definition 3. The two parameterized Mittag-Leffler function is introduced as follows [13]

$$E_{\alpha,\beta}\left(\lambda\left(t-t_0\right)^{\alpha}\right) = \sum_{k=0}^{\infty} \frac{\left(\lambda\left(t-t_0\right)^{\alpha}\right)^k}{\Gamma\left(\alpha k+\beta\right)}, \quad \alpha,\ \beta > 0,\ \lambda \in \mathbb{R}.$$

For more information refer to [13]. As a result, the following functions are introduced:

$$\sin_\alpha(\lambda t^\alpha) = \frac{E_{\alpha,1}(i\lambda t^\alpha) - E_{\alpha,1}(-i\lambda t^\alpha)}{2i} = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda t^\alpha)^{2k+1}}{\Gamma((2k+1)\alpha+1)}, \tag{1}$$

and

$$\cos_\alpha(\lambda t^\alpha) = \frac{E_{\alpha,1}(i\lambda t^\alpha) + E_{\alpha,1}(-i\lambda t^\alpha)}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda t^\alpha)^{2k}}{\Gamma(2k\alpha+1)}. \tag{2}$$

When $\alpha = 1$, equations (1) and (2) are $sin(\lambda t)$ and $\cos(\lambda t)$ respectively. Moreover fractional hyperbolic functions are introduced in terms of Mittag-Leffler function as follows:

$$\cosh_\alpha(\lambda t^\alpha) = \frac{E_{\alpha,1}(\lambda t^\alpha) + E_{\alpha,1}(-\lambda t^\alpha)}{2} = \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^{2k}}{\Gamma(2k\alpha+1)},$$

and

$$\sinh_\alpha(\lambda t^\alpha) = \frac{E_{\alpha,1}(\lambda t^\alpha) - E_{\alpha,1}(-\lambda t^\alpha)}{2} = \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^{2k+1}}{\Gamma((2k+1)\alpha+1)}.$$

It is clear above definitions that these functions become hyperbolic functions at $\alpha = 1$.

3. Introduction and features of the ARA transform

Definition 4. The n^{th} order ARA transform of a continuous function $g(t)$ on $(0, \infty)$ is introduced as

$$G_n[g(t)](s) = G(n,s) = s\int_0^\infty t^{n-1} e^{-st} g(t)dt, \quad s > 0.$$

Definition 5. The inverse ARA transform is defined as

$$\begin{aligned} g(t) &= G_{n+1}^{-1}[G_{n+1}[g(t)]] \\ &= \frac{(-1)^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\left((-1)^n\left(\frac{1}{s\Gamma(n-1)}\int_0^s (s-x)^{n-1} G(n+1,x)\,dx\right.\right. \\ &\quad \left.\left. + \sum_{k=0}^{n-1} \frac{s^k}{k!}\frac{\partial^k G(0)}{\partial s^k}\right)\right) ds, \end{aligned}$$

where $G(s) = \int_0^\infty e^{-st} g(t)dt$, is $(n-1)$ times differentiable [11].

The examples of the ARA transform of some fundamental functions are given as follows:

Property 1.

$$\begin{aligned}
G_n\left[t^{\beta-1}E_{\alpha,\beta}\left(\lambda t^{\alpha}\right)\right](s) &= s\int_0^{\infty} t^{n-1}e^{-st}t^{\beta-1}\sum_k^{\infty}\frac{\left(\lambda t^{\alpha}\right)^k}{\Gamma\left(\alpha k+\beta\right)}dt \\
&= \sum_k^{\infty}\frac{\lambda^k}{\Gamma\left(\alpha k+\beta\right)}s\int_0^{\infty} e^{-st}t^{n+\beta-2+\alpha k}dt \\
&= \sum_k^{\infty}\frac{\lambda^k}{\Gamma\left(\alpha k+\beta\right)}s\int_0^{\infty} t^{n-1}e^{-st}t^{\beta-1+\alpha k}dt \\
&= \sum_k^{\infty}\frac{\lambda^k}{\Gamma\left(\alpha k+\beta\right)}\frac{\Gamma\left(\beta-1+\alpha k+n\right)}{s^{\beta-1+\alpha k+n-1}} \\
&= \frac{1}{s^{\beta+n-2}}\sum_k^{\infty}\frac{\lambda^k}{\Gamma\left(\alpha k+\beta\right)}\frac{\Gamma\left(\beta+\alpha k+n-1\right)}{s^{\alpha k}}.
\end{aligned}$$

For $n=1$,

$$\begin{aligned}
G_1\left[t^{\beta-1}E_{\alpha,\beta}\left(\lambda t^{\alpha}\right)\right](s) &= \frac{1}{s^{\beta-1}}\sum_k^{\infty}\frac{\lambda^k}{\Gamma\left(\alpha k+\beta\right)}\frac{\Gamma\left(\beta+\alpha k\right)}{s^{\alpha k}} \\
&= \frac{1}{s^{\beta-1}}\left(\frac{1}{1-\frac{\lambda}{s^{\alpha}}}\right)=\frac{s^{\alpha-\beta+1}}{s^{\alpha}-\lambda}.
\end{aligned}$$

Property 2.

$$\begin{aligned}
G_1\left[\cos_{\alpha}\left(\lambda t^{\alpha}\right)\right](s) &= G_1\left[\frac{E_{\alpha,1}\left(i\lambda t^{\alpha}\right)+E_{\alpha,1}\left(-i\lambda t^{\alpha}\right)}{2}\right](s)=\frac{s^{2\alpha}}{s^{2\alpha}+\lambda^2}. \\
G_1\left[\sin_{\alpha}\left(\lambda t^{\alpha}\right)\right](s) &= G_1\left[\frac{E_{\alpha,1}\left(i\lambda t^{\alpha}\right)-E_{\alpha,1}\left(-i\lambda t^{\alpha}\right)}{2i}\right](s)=\lambda\frac{s^{\alpha}}{s^{2\alpha}+\lambda^2}. \\
G_1\left[\cosh_{\alpha}\left(\lambda t^{\alpha}\right)\right](s) &= G_1\left[\frac{E_{\alpha,1}\left(\lambda t^{\alpha}\right)+E_{\alpha,1}\left(-\lambda t^{\alpha}\right)}{2}\right](s)=\frac{s^{2\alpha}}{s^{2\alpha}-\lambda^2}. \\
G_1\left[\sinh_{\alpha}\left(\lambda t^{\alpha}\right)\right](s) &= G_1\left[\frac{E_{\alpha,1}\left(\lambda t^{\alpha}\right)-E_{\alpha,1}\left(-\lambda t^{\alpha}\right)}{2}\right](s)=\lambda\frac{s^{\alpha}}{s^{2\alpha}-\lambda^2}.
\end{aligned}$$

Property 3. The ARA transform of $t^{p\alpha}$ for $p\in\mathbb{N}$ is defined as follows:

$$G_n\left[t^{p\alpha}\right](s)=s\int_0^{\infty}t^{n-1}e^{-st}t^{p\alpha}dt=\Gamma(p\alpha+n)\left(\frac{1}{s}\right)^{p\alpha+n}s=\frac{\Gamma(p\alpha+n)}{s^{p\alpha+n-1}}.$$

Property 4. The ARA transform of Riemann-Liouville integral is defined as follows:

$$G_n\left[{}_0^{RL}I_t^{\alpha}f(t)\right](s)=G_n\left[\frac{1}{\Gamma(\alpha)}\left[t^{\alpha-1}*f(t)\right]\right](s).$$

Property 5. The ARA transform of Liouville-Caputo derivative is defined as follows:

$$G_n\left[{}_0^C D_t^\alpha f(t)\right](s) = G_n\left[{}_0^{RL} I_t^{m-\alpha} f^{(m)}(t)\right](s).$$

For the substantial features of ARA transform refer to [4, 8, 11].

4. The implementation of the ARA transform for some time fractional mathematical problems

4.1. **The implementation of the ARA transform for heat-like equation.** Let us consider the following heat-like equation with initial and homogenous Dirichlet boundary conditions:

$$ {}_0^C D_t^\alpha\left(u(x,t)\right) = D_x^2\left(u(x,t)\right), \tag{3}$$

$$\begin{aligned} u(0,t) &= u(1,t) = 0, \\ u(x,0) &= \varphi(x), \end{aligned} \tag{4}$$

where ${}_0^C D_t^\alpha$ denotes the time fractional derivative in Liouville-Caputo sense, $0 < \alpha \leq 1, 0 < x < l, 0 < t \leq T$. Utilizing ARA transform for the equation (3) leads to the following:

$$D_x^2\left[G_1\left[u(x,t)\right](s)\right] - s^\alpha G_1\left[u(x,t)\right](s) = -s^\alpha \varphi(x). \tag{5}$$

Therefore, this differential equation has the following characteristic equation

$$r^2 - s^\alpha = 0,$$

which leads to the solution of homogenous part of it:

$$G_{1c}\left[u(x,t)\right](s) = c_1 e^{-\sqrt{s^\alpha}x} + c_2 e^{\sqrt{s^\alpha}x}.$$

As a result, the general solution of equation (5) becomes

$$G_1\left[u(x,t)\right](s) = c_1 e^{-\sqrt{s^\alpha}x} + c_2 e^{\sqrt{s^\alpha}x} + G_{1p}\left[u(x,t)\right](s),$$

where $G_{1p}\left[u(x,t)\right](s)$ is the special solution of equation (5). In order to determine the coefficients c_1 and c_2, we utilize the ARA transform of boundary conditions:

$$\begin{aligned} u(0,t) &= 0 \quad \Rightarrow \quad G_1\left[u(0,t)\right](s) = 0 \\ &\Rightarrow \quad c_1 + c_2 + G_{1p}\left[u(0,t)\right](s) = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} u(1,t) &= 0 \quad \Rightarrow \quad G_1\left[u(1,t)\right](s) = 0 \\ &\Rightarrow \quad c_1 e^{-\sqrt{s^\alpha}} + c_2 e^{\sqrt{s^\alpha}} + G_{1p}\left[u(1,t)\right](s) = 0. \end{aligned} \tag{7}$$

Using equations (6) and (7), the coefficients c_1 and c_2 are determined. At this stage to establish the solution of fractional mathematical problem, the inverse ARA transform G_1^{-1} are applied:

$$u(x,t) = G_1^{-1}\left[c_1 e^{-\sqrt{s^\alpha}x} + c_2 e^{\sqrt{s^\alpha}x} + G_{1p}\left[u(x,t)\right](s)\right].$$

4.2. **The implementation of the ARA transform for wave-like equation.** Let us consider the following fractional problem including wave-like equation

$$\begin{aligned} {}_0^C D_t^{2\alpha}\left(u(x,t)\right) &= D_x^2\left(u(x,t)\right)+F(x), \\ u(0,t) &= u(1,t)=0, \\ u\left(x,0\right) &= u_t\left(x,0\right)=0, \end{aligned} \tag{8}$$

where ${}_0^C D_t^{2\alpha}\left(u(x,t)\right) = {}_0^C D_t^{\alpha}\left({}_0^C D_t^{\alpha}\left(u(x,t)\right)\right), 1<\alpha\le 2, 0<x<l, 0<t\le T$. Utilizing ARA transform for the equation (8) leads to the following equation

$$D_x^2\left[G_1\left[u\left(x,t\right)\right](s)\right]-s^{2\alpha}G_1\left[u(x,t)\right](s)=-F(x). \tag{9}$$

Therefore, this differential equation has the following characteristic equation

$$r^2-s^{2\alpha}=0,$$

which leads to the solution of homogenous part of it:

$$G_{1c}\left[u\left(x,t\right)\right](s)=c_1e^{-s^\alpha x}+c_2e^{s^\alpha x}.$$

As a result, the general solution of equation (9) becomes

$$G_1\left[u\left(x,t\right)\right](s)=c_1e^{-s^\alpha x}+c_2e^{s^\alpha x}+G_{1p}\left[u\left(x,t\right)\right](s),$$

where $G_{1p}\left[u\left(x,t\right)\right](s)$ is the special solution of equation (9). In order to determine the coefficients c_1 and c_2, we utilize the ARA transform of boundary conditions:

$$\begin{aligned} u(0,t)=0 \quad &\Rightarrow \quad G_1\left[u\left(0,t\right)\right](s)=0 \\ &\Rightarrow c_1+c_2+G_{1p}\left[u\left(0,t\right)\right](s)=0, \end{aligned} \tag{10}$$

$$\begin{aligned} u\left(l,t\right)=0 \quad &\Rightarrow \quad G_1\left[u\left(1,t\right)\right](s)=0 \\ &\Rightarrow \quad c_1e^{-s^\alpha}+c_2e^{s^\alpha}+G_{1p}\left[u\left(l,t\right)\right](s)=0. \end{aligned} \tag{11}$$

Using equations (10) and (11), the coefficients c_1 and c_2 are determined. At this stage to establish the solution of fractional mathematical problem, the inverse ARA transform G_1^{-1} are applied:

$$u\left(x,t\right)=G_1^{-1}\left[c_1e^{-s^\alpha x}+c_2e^{s^\alpha x}+G_{1p}\left[u\left(x,t\right)\right](s)\right].$$

5. Illustrative examples

Example 1. Let us take the following time fractional problem including heat-like equation

$$\begin{aligned} {}_0^C D_t^{\alpha}\left(u(x,t)\right) &= D_x^2\left(u(x,t)\right), \quad 0<\alpha\le 1,\ 0<x<1,\ 0<t\le T, \\ u(0,t) &= u(1,t)=0, \quad 0<t\le T, \\ u\left(x,0\right) &= \sin\left(2\pi x\right), \quad 0<x<1, \end{aligned} \tag{12}$$

which has the following ordinary differential equation after applying the ARA transform

$$D_x^2\left[G_1\left[u\left(x,t\right)\right]\left(s\right)\right]-s^{\alpha}G_1\left[u\left(x,t\right)\right]\left(s\right)=-s^{\alpha}\varphi(x). \tag{13}$$

As it explained in subsection 4.1, the solution of the homogenous part of equation (12) is obtained as:

$$G_{1c}\left[u\left(x,t\right)\right]\left(s\right)=c_1e^{-\sqrt{s^{\alpha}}x}+c_2e^{\sqrt{s^{\alpha}}x}.$$

Based on the right hand side function, the special solution of equation (12) take the following form:

$$G_{1p}\left[u\left(x,t\right)\right]\left(s\right)=\lambda_1\sin\left(2\pi x\right)+\lambda_2\cos\left(2\pi x\right). \tag{14}$$

Substituting (14) into equation (12), the coefficients λ_1 and λ_2 are determined as:

$$\lambda_2=0,\lambda_1=\frac{s^{\alpha}}{4\pi^2+s^{\alpha}}.$$

As a result, the special solution of equation (12) becomes

$$G_{1p}\left[u\left(x,t\right)\right]\left(s\right)=\frac{s^{\alpha}}{4\pi^2+s^{\alpha}}\sin\left(2\pi x\right).$$

Consequently, the general solution of equation (12) take the following form:

$$G_1\left[u\left(x,t\right)\right]\left(s\right)=c_1e^{-\sqrt{s^{\alpha}}x}+c_2e^{\sqrt{s^{\alpha}}x}+\frac{s^{\alpha}}{4\pi^2+s^{\alpha}}\sin\left(2\pi x\right).$$

As follows from subsection 4.1, the coefficients c_1 and c_2 are computed as

$$c_1=0=c_2.$$

Hence, the following solution is obtained

$$G_1\left[u\left(x,t\right)\right]\left(s\right)=\frac{s^{\alpha}}{4\pi^2+s^{\alpha}}\sin\left(2\pi x\right). \tag{15}$$

The inverse ARA transform of equation (15) leads to the solution of fractional mathematical problem in the following form:

$$u\left(x,t\right)=E_{\alpha,1}\left(-4\pi^2t^{\alpha}\right)\sin\left(2\pi x\right),$$

where the following inverse ARA transform is used

$$G_1^{-1}\left[\frac{s^{\alpha-\beta+1}}{s^{\alpha}-\lambda}\right]=t^{\beta-1}E_{\alpha,\beta}\left(\lambda t^{\alpha}\right).$$

It is evident from Figures 1 and 2 that the rate of decreasing of the analytical solutions decreases as the order of the fractional derivative α decreases. As a result, it can be deduced that the diffusion rate decreases slower as α decreases.

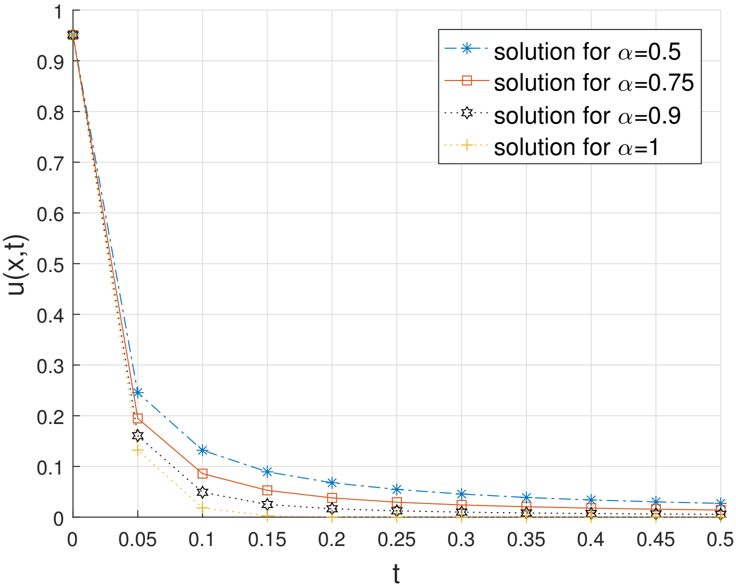

FIGURE 1. Analytic solutions of example 1 for various values of α at $x = 0.2$ in $2D$.

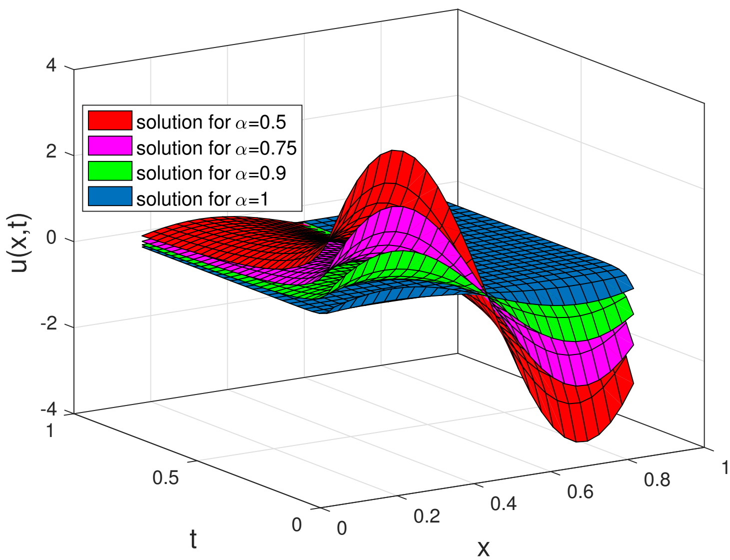

FIGURE 2. Analytic solutions of example 1 for various values of α in $3D$.

Example 2. Let us take the following time fractional problem including wave-like equation into consideration

$$
\begin{aligned}
(16)\qquad & {}_0^C D_t^{2\alpha}\left(u(x,t)\right) = D_x^2\left(u(x,t)\right) + \sin(\pi x), \quad 1 < \alpha \le 2, \\
& u(0,t) = u(1,t) = 0, \quad 0 < t \le T, \\
& u\left(x,0\right) = u_t\left(x,0\right) = 0, \quad 0 < x < 1,
\end{aligned}
$$

which has the following ordinary differential equation after applying the ARA transform

$$
(17)\qquad D_x^2\left[G_1\left[u\left(x,t\right)\right](s)\right] - s^{2\alpha} G_1\left[u(x,t)\right](s) = -\sin(\pi x).
$$

As it explained in subsection 4.2, the solution of the homogenous part of equation (16) is obtained as:

$$
G_{1c}\left[u\left(x,t\right)\right](s) = c_1 e^{-s^{\alpha}x} + c_2 e^{s^{\alpha}x}.
$$

Based on the right hand side function, the special solution takes the following form:

$$
(18)\qquad G_{1p}\left[u\left(x,t\right)\right](s) = \lambda_1 \sin\left(\pi x\right) + \lambda_2 \cos\left(\pi x\right).
$$

Substituting (18) into equation (16), the coefficients λ_1 and λ_2 are determined as:

$$
\lambda_2 = 0, \lambda_1 = \frac{1}{s^{2\alpha} + \pi^2}.
$$

As a result, the special solution of equation (16) becomes

$$
G_{1p}\left[u\left(x,t\right)\right](s) = \frac{1}{s^{2\alpha} + \pi^2}\sin(\pi x).
$$

As a result, equation (16) have the following solution

$$
G_1\left[u\left(x,t\right)\right](s) = c_1 e^{-s^{\alpha}x} + c_2 e^{s^{\alpha}x} + \frac{1}{s^{2\alpha} + \pi^2}\sin(\pi x).
$$

As follows from subsection 4.2, the coefficients c_1 and c_2 are computed as

$$
c_1 = 0 = c_2,
$$

which leads to the following

$$
(19)\qquad G_1\left[u\left(x,t\right)\right](s) = \frac{1}{s^{2\alpha} + \pi^2}\sin(\pi x) = \frac{1}{\pi^2}\left(1 - \frac{s^{2\alpha}}{s^{2\alpha} + \pi^2}\right)\sin(\pi x).
$$

Taking the inverse ARA transform of equation (19), we determine the solution of fractional mathematical problem in the following form:

$$
u\left(x,t\right) = G_1^{-1}\left[\frac{1}{\pi^2}\left(1 - \frac{s^{2\alpha}}{s^{2\alpha} + \pi^2}\right)\sin(\pi x)\right] = \frac{1}{\pi^2}\left(1 - \cos_{\alpha}\left(\pi t^{\alpha}\right)\right)\sin(\pi x).
$$

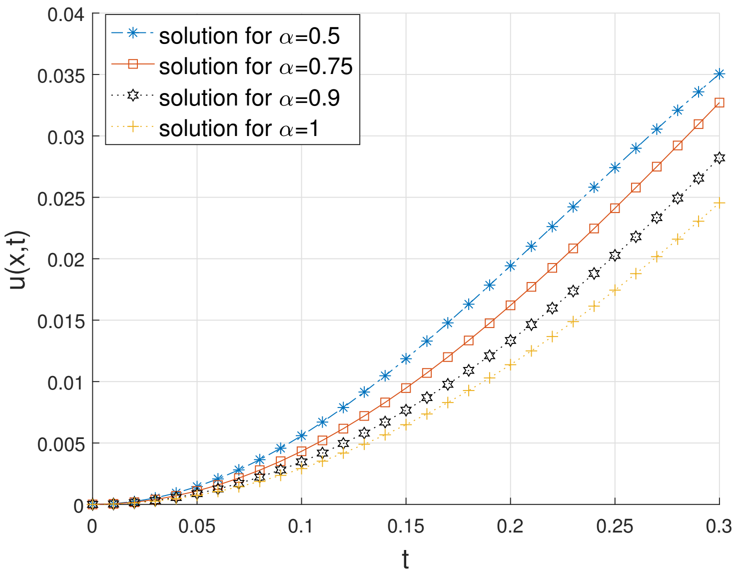

FIGURE 3. Analytic solutions of example 2 for various values of α at $x = 0.2$ in $2D$.

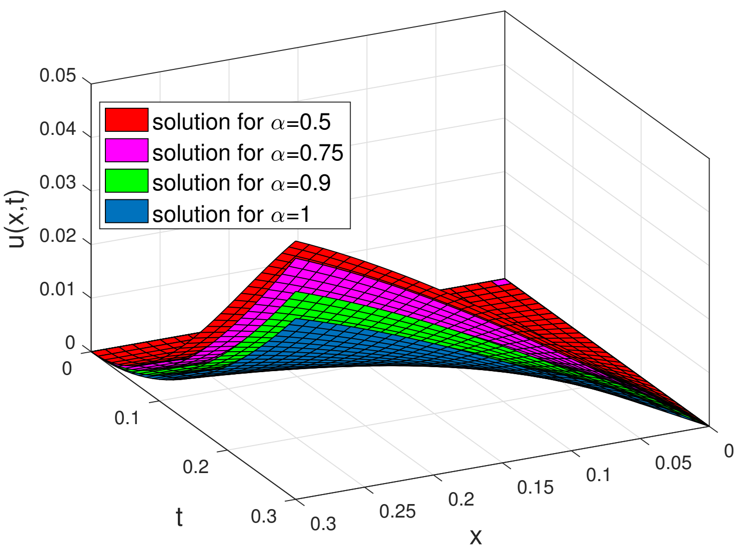

FIGURE 4. Analytic solutions of example 2 for various values of α in $3D$.

It is obvious from Figures 3 and 4 that the rate of increasing of the analytical solutions increases as the order of the fractional derivative α decreases.

As a result, it can be deduced that the diffusion rate increases faster as α decreases.

It is clear from Figures 1-4 that as α tends to 1, the solutions of time fractional problems tend to the solutions of mathematical problems which has ordinary differential equations. This results indicate that the ARA transform method works for time fractional mathematical problems effectively.

6. Conclusion

In our present research, the fundamental properties and implementation of the ARA transform for time fractional mathematical problems including heat-like and wave-like equations are given. As in the case of all integral transform, the problem is converted into a simpler form to tackle. Later, solving the reduced problem, inverse ARA transform is utilized to obtain the solution of fractional mathematical problem. It is clear from illustrative examples that this method is a powerful and versatile method.

In the future studies, modified and improved version of this method will be developed and utilized for the fractional mathematical problems.

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