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Differential subordination and superordination results for generalized "Srivastava–Attiya" fractional integral operator

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Abstract. In this paper, we derive some subordination and superordination results for the generalized "Srivastava- Attiya" fractional integral operator. Some interesting corollaries for this operator is also obtained.

1. Introduction and preliminaries

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S}(\mathbb{U})$ denote the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions which are also univalent in $\mathbb{U}$. Further let $\mathcal{H}[a, p]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of function of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots, \quad (a \in \mathbb{C},\ p \in \mathbb{N} = \{1, 2, 3, \dots\}).$$

Let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N}). \tag{1}$$

For simplicity, we write $\mathcal{A}_1 := \mathcal{A}$.

Given two functions $f \in \mathcal{H}(\mathbb{U})$ and $g \in \mathcal{H}(\mathbb{U})$, we say that f is subordinate to g or g is superordinate to f in $\mathbb{U}$ and write $f \prec g$, if there exists a Schwarz function w, analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{U}$, such that $f(z) = g(w(z))$ in $\mathbb{U}$. In particular, if $g(z)$ is univalent in $\mathbb{U}$, we have the following equivalence:

$$f(z) \prec g(z), \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad \big[f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U})\big].$$

Supposing that h and k are two analytic functions in $\mathbb{U}$, let $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. If h and $\phi(h(z), zh'(z), z^2h''(z); z)$ are univalent and if h and

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$\phi(h(z), zh'(z), z^2h''(z); z)$ are univalent functions in $\mathbb{U}$ and h satisfies the second-order superordination

$$k(z) \prec \phi(h(z), zh'(z), z^2h''(z); z), \tag{2}$$

then $k(z)$ is said to be a solution of the differential superordination (2). A function $q \in \mathbb{U}$ is called a subordinant of (2), if $q(z) \prec h(z)$ for all the functions h satisfying (2). A univalent subordinant that satisfies $q(z) \prec \tilde{q}(z)$ for all of the subordinants q of (2), is said to be the best subordinant. Recently, Miller and Mocanu [6] obtained the sufficient conditions on the functions k, q and ϕ for which the following implication holds:

$$k(z) \prec \phi(h(z), zh'(z), z^2h''(z); z) \quad \Rightarrow \quad q(z) \prec h(z).$$

Using results of Miller and Mocanu [6], Bulboacã [2] considered certain classes of first order differential superordination as well superordination-preserving integral operators [3] . Ali *et* al. [1] have used the results of Bulboacã [2] to obtain sufficient conditions for normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent function in $\mathbb{U}$. Also, Shanmugam *et* al. [10] obtained sufficient conditions for a normalized analytic $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z),$$
$$q_1(z) \prec \frac{z^2f'(z)}{(f(z))^2} \prec q_2(z),$$

where q_1 and q_2 are given univalent function in $\mathbb{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$. Further subordination results can be found in [7, 8, 11–13].

The fractional integral operator (see [20]) of order $\lambda(\lambda > 0)$ is defined for a funtion f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{3}$$

where f is analytic function in a simply-connected region of z-plane containing the origin and the multiplicity of $(z-t)^{1-\lambda}$ is removed by requiring $\log(z-t)$ to be real, when $\Re(z-t) > 0$.

Recently, Srivastava and Attiya [21] introduced and investigated the linear operator: Now for $f \in \mathcal{A}, b \in \mathbb{C} \backslash \mathbb{Z}_0^-$ and $s \in \mathbb{C}$, we define the function $G_{s,b}(z)$ by

$$G_{s,b}(z) := (1+b)^s \left[\Phi(z, s, b) - b^{-s}\right], \quad (z \in \mathbb{U}). \tag{4}$$

We also denote by

$$J_{s,b}(f) : \mathcal{A} \longrightarrow \mathcal{A}$$

the linear operator defined by

$$(5)\qquad J_{s,b}(f)(z) := G_{s,b}(z) * f(z), \quad (z \in \mathbb{U}; f \in \mathcal{A};\ b \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C})$$

in terms of the Hadamard product (or convolution).

We note that

$$(6)\quad J_{s,b}f(z) = z + \sum_{k=2}^{\infty}\left(\frac{1+b}{k+b}\right)^s a_k z^k, \quad (z \in \mathbb{U}; b \in \mathbb{C}\backslash\mathbb{Z}_0^-; s \in \mathbb{C}; f \in \mathcal{A}).$$

Remark 1. It follows from (5) and (6) that one can define the operator $J_{s,b}(f)$ for $b \in \mathbb{C}\backslash\mathbb{Z}_0^-$. Therefore, we may use the following limit relationship:

$$(7)\qquad J_{s,0}f(z) := \lim_{b\to 0}\{J_{s,b}(f)(z)\}.$$

Motivated essentially by the above-mentioned "Srivastava-Attiya" operator, Wang [22] introduced the operator for the class $\mathcal{A}_p$.

$$(8)\qquad J_{s,b}^{\alpha,p}(f):\ \mathcal{A}_p \to \mathcal{A}_p,$$

which is defined as

$$(9)\qquad J_{s,b}^{\alpha,p}f(z) = z^p + \sum_{k=1}^{\infty}\frac{(\alpha+p)_k}{k\,!}\left(\frac{p+b}{p+k+b}\right)^s a_{p+k}z^{p+k}, \quad (z \in \mathbb{U}),$$

where $(\nu)_k$ is the Pochhammer symbol defined by

$$(10)\qquad (\nu)_k := \begin{cases} 1, & k = 0, \\ \nu(\nu+1)\cdots(\nu+k-1), & k \in \mathbb{N}. \end{cases}$$

Recently q-extension of "Srivastava-Attiya" operator have been studied in [19], the mathematical applications of q-calculus, fractional q-calculus and the fractional q-derivative operators can be seen in [15]. Srivastava *et* al. [18] also reconnoiter the not-yet-widely-known fact that the so-called (p,q)-variation of classical q-calculus is a rather trivial and inconsequential variation of classical q-calculus. For more detail and related works one can see in ([9, 14, 16, 17]).

Unless otherwise mentioned, we assume throughout this paper that the parameter s, b, p and α are constrained as follows:

$$(11)\qquad s \in \mathbb{C};\ b \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ p \in \mathbb{N}\ \text{and}\ \alpha > -p.$$

From (3) and (9), we get the fractional integral operator $\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z)$ defined as

$$(12)\qquad \begin{aligned}\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z) &= \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)}z^{\lambda+p} \\ &+ \sum_{k=1}^{\infty}\frac{(\alpha+p)_k}{k\,!}\frac{\Gamma(p+k+1)}{\Gamma(\lambda+p+k+1)}\left(\frac{p+b}{p+k+b}\right)^s a_{p+k}z^{p+k+\lambda}\end{aligned}$$

for ($\lambda+p+1>0$, $\alpha+p>0$. Also, it is easily verified from (12) that

$$z\left(\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z)\right)'=(\lambda-\alpha)\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z)+(\alpha+p)\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha+1,p}f(z). \tag{13}$$

Definition 1 (Miller and Mocanu [6]). Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, where

$$E(f)=\{\eta\in\partial\mathbb{U}:\lim_{z\to\eta}f(z)=\infty\},$$

and are such that $f'(\eta)\neq 0$ for $\eta\in\partial U\backslash E(f)$.

To prove our results we shall need the following lemmas.

Lemma 1 (Bulboacã [4]). *Let $q(z)$ be convex univalent in the unit disk $\mathbb{U}$ and θ and ψ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$. Suppose that*

1. $\Re[\theta'(q(z))/\psi(q(z))]>0$ *for* $z\in\mathbb{U}$,
2. $zq'(z)\psi(q(z))$ *is starlike in* $\mathbb{U}$.

If $p(z)\in\mathcal{H}[q(0),1]\cap Q$ with $p(\mathbb{U})\subseteq\mathbb{D}$ and $\theta(p(z))+zp'(z)\psi(p(z))$ is univalent in $\mathbb{U}$ and

$$\theta(q(z))+zq'(z)\psi(q(z))\prec\theta(p(z))+zp'(z)\psi(p(z)). \tag{14}$$

then $q(z)\prec p(z)$ and q is the best subordinant of (14).

Lemma 2 (Frasin [5]). *Let the function $p(z)$ and $q(z)$ be analytic in $\mathbb{U}$ and suppose that $q(z)\neq 0$ $(z\in\mathbb{U})$ is also univalent in $\mathbb{U}$ and that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{U}$. If $q(z)$ satisfies*

$$\Re\left(1+\frac{c_1}{\beta}q(z)+\frac{2c_2}{\beta}(q(z))^2+\cdots+\frac{nc_n}{\beta}(q(z))^n-\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right)>0 \tag{15}$$

and

$$\begin{aligned}&c_0+c_1p(z)+c_2(p(z))^2+\cdots+c_n(p(z))^n+\beta\frac{zp'(z)}{p(z)}\\&\prec c_0+c_1q(z)+c_2(q(z))^2+\cdots+c_n(q(z))^n+\beta\frac{zq'(z)}{q(z)},\\&(z\in\mathbb{U};c_0,c_1,c_2,\ldots,c_n,\beta\in\mathbb{C};\beta\neq 0),\end{aligned} \tag{16}$$

then $p(z)\prec q(z)$ $(z\in\mathbb{U})$ and q is the best dominant.

We now first prove the following subordination result involving the operator $\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z)$.

2. SUBORDINATION RESULTS FOR ANALYTIC FUNCTIONS

Theorem 1. *Let the function $q(z)$ be analytic and univalent in $\mathbb{U}$ such that $q(z)\neq 0$, $(z\in\mathbb{U})$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{U}$ and the* *inequality* (15) *holds true. Let*

$$\begin{aligned}
&\Omega_j^m(c_0, c_1, c_2, ...c_n, \beta, \alpha, \lambda, p, f)\\
&= c_0 + c_1\left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)}\frac{\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z)}{z^{\lambda+p}}\right) + c_2\left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)}\frac{\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z)}{z^{\lambda+p}}\right)^2\\
&+\cdots + c_n\left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)}\frac{\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z)}{z^{\lambda+p}}\right)^n + \beta(\alpha+p)\left(\frac{\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha+1,p}f(z)}{\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z)} - 1\right).
\end{aligned} \tag{17}$$

If $q(z)$ *satisfies*

$$\begin{aligned}
&\Omega_j^m(c_0, c_1, c_2, ...c_n, \beta, \alpha, \lambda, p, f)\\
&\prec c_0 + c_1 q(z) + c_2(q(z))^2 + \cdots + c_n(q(z))^n + \beta\frac{zq'(z)}{q(z)},\\
&(z \in \mathbb{U}; c_0, c_1, c_2, ...c_n, \beta \in \mathbb{C}; \beta \neq 0),
\end{aligned} \tag{18}$$

then

$$\left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)}\frac{\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z)}{z^{\lambda+p}}\right) \prec q(z), \quad (z \in \mathbb{U}\backslash\{0\}),$$

and q *is the best dominant.*

Proof. Define the function $h(z)$ by

$$h(z) = \frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)}\frac{\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z)}{z^{\lambda+p}}, \quad (z \in \mathbb{U}\backslash\{0\}).$$

Then a computation shows that

$$\frac{zh'(z)}{h(z)} = \frac{z\mathfrak{D}_z^{-\lambda}(J_{s,b}^{\alpha,p}f(z))'}{\mathfrak{D}_z^{-\lambda}(J_{s,b}^{\alpha,p}f(z))} - (\lambda+p).$$

By using the identity (13), we obtain

$$\frac{zh'(z)}{h(z)} = (\alpha+p)\left(\frac{\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha+1,p}f(z)}{\mathfrak{D}_z^{-\lambda}J_{s,b}^{\alpha,p}f(z)} - 1\right),$$

which, in light of hypothesis (16), yields the following subordination

$$\begin{aligned}
&c_0 + c_1 h(z) + c_2(h(z))^2 + ... + c_n(h(z))^n + \beta\frac{zh'(z)}{h(z)}\\
&\prec c_0 + c_1 q(z) + c_2(q(z))^2 + \cdots + c_n(q(z))^n + \beta\frac{zq'(z)}{q(z)},
\end{aligned}$$

and Theorem 1 follows by an application of Lemma 2.

For the choices $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ and $q(z) = \left(\frac{1+z}{1-z}\right)^\mu$, $0 \leq \mu \leq 1$ in Theorem 1, we get Corollaries 1 and 2 below. □

Corollary 1. *Assume that* (15) *holds true. If* $f \in \mathcal{A}_p$ *and*

$$\begin{aligned}&\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \alpha, \lambda, p, f)\\ &\prec c_0 + c_1\left(\frac{1+Az}{1+Bz}\right) + c_2\left(\frac{1+Az}{1+Bz}\right)^2 + \cdots\\ &\quad + c_n\left(\frac{1+Az}{1+Bz}\right)^n + \beta\frac{(A-B)z}{(1+Az)(1+Bz)},\\ &\quad (z \in \mathbb{U};\ c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0),\end{aligned}$$

where $\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \alpha, \lambda, p, f)$ *is as defined in equation* (17), *then*

$$\left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)}\frac{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha,p} f(z)}{z^{\lambda+p}}\right) \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Corollary 2. *Assume that* (15) *holds true. If* $f \in \mathcal{A}_p$ *and*

$$\begin{aligned}&\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \alpha, \lambda, p, f)\\ &\prec c_0 + c_1\left(\frac{1+z}{1-z}\right)^{\mu} + c_2\left(\frac{1+z}{1-z}\right)^{2\mu} + \ldots\\ &\quad + c_n\left(\frac{1+z}{1-z}\right)^{2n\mu} + \frac{2\beta\mu z}{1-z^2},\\ &\quad (z \in \mathbb{U}; c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0),\end{aligned}$$

where $\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \alpha, \lambda, p, f)$ *is as defined in equation* (17), *then*

$$\left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)}\frac{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha,p} f(z)}{z^{\lambda+p}}\right) \prec \left(\frac{1+z}{1-z}\right)^{\mu},$$

and $\frac{1+z}{1-z}$ *is the best dominant.*

For $q(z) = e^{\epsilon A z}$, $(|\epsilon A| < \pi)$, in Theorem 1, we get the following result.

Corollary 3. *Assume that* (15) *holds true. If* $f \in \mathcal{A}_p$ *and*

$$\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \alpha, \lambda, p, f) \prec c_0 + c_1 e^{\epsilon Az} + c_2 e^{2\epsilon Az} + c_n e^{n\epsilon Az} + \beta\epsilon Az,$$

where $\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \alpha, \lambda, p, f)$ *is as defined in equation* (17), *then*

$$\left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)}\frac{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha,p} f(z)}{z^{\lambda+p}}\right) \prec e^{\epsilon Az}, \quad (z \in \mathbb{U}\backslash\{0\}),$$

and $e^{\epsilon Az}$ *is the best dominant.*

3. Superordination for analytic functions

Next, applying Lemma 1, we obtain the following two theorems.

Theorem 2. *Let q be analytic and convex univalent in $\mathbb{U}$ such that $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{U}$. Suppose also that*

$$\Re\left(\frac{c_1}{\beta}q(z) + \frac{2c_2}{\beta}(q(z))^2 + ... + \frac{nc_n}{\beta}(q(z))^n\right) > 0, \quad (19)$$
$$(z \in \mathbb{U}; c_0, c_1, c_2, ...c_n, \beta \in \mathbb{C}; \beta \neq 0).$$

If $f \in \mathcal{A}_p$

$$\left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)}\frac{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha,p} f(z)}{z^{\lambda+p}}\right) \in \mathcal{H}[q(0),1] \cap Q$$

and $\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \alpha, \lambda, p, f)$ defined in (17) is univalent in $\mathbb{U}$, then the following superordination:

$$c_0 + c_1 q(z) + c_2(q(z))^2 + \cdots + c_n(q(z))^n + \beta\frac{zq'(z)}{q(z)} \quad (20)$$
$$\prec \Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \alpha, \lambda, p, f),$$
$$(z \in \mathbb{U}; c_0, c_1, c_2, ...c_n, \beta \in \mathbb{C}; \beta \neq 0),$$

implies that

$$q(z) \prec \left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)}\frac{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha,p} f(z)}{z^{\lambda+p}}\right), \quad (z \in \mathbb{U}\backslash\{0\}),$$

and $q(z)$ is the best subordinant.

Proof. Let

$$\theta(\omega) = c_0 + c_1\omega + c_2\omega^2 + ...c_n\omega^n \text{ and } \psi(\omega) := \beta\frac{\omega'}{\omega}.$$

Then, we observe that $\theta(\omega)$ is analytic in $\mathbb{C}$, $\psi(\omega)$ is analytic in $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and that $\psi(\omega) \neq 0$ $(\omega \in \mathbb{C}^*)$. Since q is a convex univalent in $\mathbb{U}$, it follows that

$$\Re\left(\frac{\theta'(q(z))}{\psi(q(z))}\right) = \Re\left(\frac{c_1}{\beta}q(z) + \frac{2c_2}{\beta}(q(z))^2 + \cdots + \frac{nc_n}{\beta}(q(z))^n\right) > 0,$$
$$(z \in \mathbb{U}; c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0).$$

Theorem 2 follows as an application of Lemma 1. □

Combining the results of differential subordination and superordination, we state that the following sandwich result.

Theorem 3. *Let* q_1 *be convex univalent and* q_2 *be univalent in* $\mathbb{U}$ *such that* $q_1(z) \neq 0$ *and* $q_2(z) \neq 0$ $(z \in \mathbb{U})$. *Suppose also that* q_2 *satisfies* (19) *and* q_1 *satisfies* (15). *If* $f \in \mathcal{A}_p$,

$$\left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)} \frac{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha,p} f(z)}{z^{\lambda+p}}\right) \in \mathcal{H}[q(0),1] \cap Q$$

and

$$c_0 + c_1 \left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)} \frac{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha,p} f(z)}{z^{\lambda+p}}\right) + c_2 \left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)} \frac{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha,p} f(z)}{z^{\lambda+p}}\right)^2 +$$
$$+ \cdots + c_n \left(\frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)} \frac{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha,p} f(z)}{z^{\lambda+p}}\right)^n + \beta(\alpha+p) \left(\frac{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha+1,p} f(z)}{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha,p} f(z)} - 1\right),$$
$$(z \in \mathbb{U};\ c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0)$$

is univalent in $\mathbb{U}$, *then the subordination given by*

$$\begin{aligned} & c_0 + c_1 q_1(z) + c_2 (q_1(z))^2 + \cdots + c_n (q_1(z))^n + \beta \frac{z q_1'(z)}{q_1(z)} \\ & \prec \Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \alpha, \lambda, p, f) \\ & \prec c_0 + c_1 q_2(z) + c_2 (q_2(z))^2 + \cdots + c_n (q_2(z))^n + \beta \frac{z q_2'(z)}{q_2(z)}, \\ & \quad (z \in \mathbb{U}; c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0), \end{aligned} \tag{21}$$

implies that

$$q_1(z) \prec \frac{\Gamma(\lambda+p+1)}{\Gamma(p+1)} \frac{\mathfrak{D}_z^{-\lambda} J_{s,b}^{\alpha,p} f(z)}{z^{\lambda+p}} \prec q_2(z),$$

and q_1 *and* q_2 *are respectively, the best subordinant and the best dominant of* (21).

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