On the global uniform stability analysis of non-autonomous dynamical systems: A survey

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ABSTRACT. In this survey, we introduce the notion of stability of time varying nonlinear systems. In particular we investigate the notion of global practical exponential stability for non-autonomous systems. The proposed approach for stability analysis is based on the determination of the bounds of perturbations that characterize the asymptotic convergence of the solutions to a closed ball centered at the origin.

1. INTRODUCTION

Asymptotic stability is one of the corner stones of the qualitative theory of dynamical systems and is of fundamental importance in many applications of the theory in virtually all fields where dynamical effects play a role. Stability theory was developed first for systems of ordinary differential equations beginning with Lyapunov [37] in 1892. It is characterized by analyzing the response of a dynamical system to small perturbations in the system states. Lyapunov proved that the existence of a Lyapunov function guarantees asymptotic stability and for linear time-invariant systems also showed the converse statement that asymptotic stability implies the existence of a Lyapunov function. Converse theorems usually are the harder part of the theory and the first general results for nonlinear systems were obtained by Massera and Kurzweil. Converse theorems are interesting because they show the universality of Lyapunov's second method. Specifically an equilibrium point of a dynamical system is said to be stable if, for sufficiently small values of initial disturbances, the perturbed motion remains in an arbitrarily prescribed small region of the state space. More precisely, stability is equivalent to continuity of solutions as a function of the system initial conditions over a neighborhood of the equilibrium point uniformly in time. If, in addition, all solutions of the dynamical system approach the equilibrium point for large values of time, then the equilibrium point is said

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to be asymptotically stable. According to Lyapunov, one can check stability of a system by finding some function V, called the Lyapunov function which is definite along every trajectory of the system, and is such that the total derivative $\frac{dV}{dt}$ is semi definite of opposite sign (or identically 0) along every trajectory of the system. If the function V exists with these properties and admits an infinitely small upper bound, and if $\frac{dV}{dt}$ is definite (with sign opposite to that of V), it can be shown further that every perturbed trajectory which is sufficiently close to the unperturbed motion approaches the latter asymptotically. The intuitive idea is that V can be considered as a generalized energy that is bounded below, and decreasing along solutions. It is well known that there is no general procedure for finding the Lyapunov functions for nonlinear systems, but for linear time invariant systems, the procedure comes down to the problem of solving a linear algebraic equation, called the Lyapunov algebraic equation. This approach is said the Second Method of Lyapunov, also called the Direct Method of Lyapunov because it does work directly on the equation in question instead of on its linearization. Unlike Lyapunov's direct method, which can provide global stability conclusions for an equilibrium point for a nonlinear dynamical system, Lyapunov's indirect method also called the First Method of Lyapunov draws conclusions about local stability of the equilibrium point by examining the stability of the linearized nonlinear system about the equilibrium point in question.

In this survey, a brief review of the theory of continuous autonomous and non-autonomous dynamical systems and some stability concepts of their equilibrium points is given. We will explain why such systems are frequently encountered in science and engineering and why the concept of stability for their equilibrium points is so important ([5, 6, 12–18, 22, 27, 33, 35–54]). Despite the considerable time and effort that has been spent on developing stability theory, important progress has still been made with respect to the theory of Lyapunov functions in recent times. Several authors have introduced the concept of practical stability ([7–11, 19–21, 28–32]), where this notion is important in certain engineering applications. Essentially these applications have one common problem, namely, the existence of external inputs or disturbances, possibly random, time-varying or unbounded in time, that cause instability and tend to produce oscillations. In such a situation, if the system trajectories oscillate around a mathematically unstable course, then the next best course of action would be to ensure that the performance of the system in question is still acceptable in a practical sense. Specifically, a concrete system will be considered stable if, in case the initial values the external disturbances are bounded by suitable constraints, the deviations of the motions from the equilibrium remain within certain bounds determined by the physical situation.

Furthermore, we introduce some generalization of the Gronwall inequality for systematic study of uniform practical asymptotic stability properties, boundedness and convergence to zero properties for the solutions of certain

differential system of equations. To deal with such situations, the concept of practical stability is more useful. Furthermore, we give illustrative examples showing the applicability of the results. To generalize Lyapunov theorems, we establish sufficient conditions for various Lyapunov stability types of perturbed systems to study the problem of uniform practical asymptotic stability and the global uniform practical stability of time varying perturbed system. Notice that the system is not assumed to have an equilibrium point. We use the class $\mathcal{KL}$-function and $\mathcal{K}$-function introduced by [35] in our analysis through the solution of an autonomous and non autonomous scalar differential equation. As a consequence, we obtain upper and lower bounds on a positive definite function in terms of class $\mathcal{K}$-function.

2. Dynamical system

A dynamical system is one which evolves with time. Mathematically, it consists of the space of states of the system together with a rule for determining the state at a future point of time, when present state is given. The mathematical formalism of differential equations has proven useful for describing dynamical systems, that is the evolution of the system with respect to time. There are two main types of differential equations; ordinary and partial differential equations. The first one only include derivatives with respect to one independent variable whereas the second one include derivatives with respect to more than one variable. In this survey, only ordinary differential equations, with time as the independent variable, are considered. All ordinary differential equations can be written as a system of first order derivatives, the state space form:

$$(1) \qquad \dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

where $t \geq t_0 \geq 0$ is the time, (˙) denotes differentiation with respect to time, n is called the dimension of the system and $x_0, x(t) \in \mathbb{R}^n$, often referred to as the state, contains the information about the underlying system that is important to as. The right hand side of this equality is referred to the mapping $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ which expresses how the state changes in time and is piecewise continuous in t, and locally Lipschitz in x. If f is a linear function in x, then the system is said to be a linear system which is used as it is easy to analyse, an approach that has no doubt proven to be very fruitful. However, some physical systems are very poorly described by linear equation and in those cases nonlinear description and thus nonlinear function have to be employed.

When the function f is not explicitly dependent on time, the system is called an autonomous dynamical one. This assumption can however be made without loss of generality since non-autonomous dynamical systems, with function f depend on both t and x, can be transformed to autonomous form by introducing time as an additional state variable. For analytical purposes, let ϕ the flow, represent the state of the system after a time t of

flowing along the trajectory which stats at the point x. Our system can then be expressed in terms of the flow as $\frac{\partial}{\partial \tau}\phi(\tau, t, x) = f(\tau, \phi(\tau, t, x))$ for all $(\tau, t, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$. This mapping satisfies the following properties:

   i) Initial value condition: $\phi(t, t, x) = x$;
   ii) Composition rule: $\phi(t_2, t, x) = \phi(t_2, t_1, \phi(t_1, t, x))$.

A lot of differential equations can not be solved explicitly, or have hardly manageable solutions. With the help of numerical methods the solutions can be approximated very well on fixed time intervals, but more often one is interested in a qualitative behavior of the solutions. A primary topic of this qualitative theory of differential equations is the stability theory. This latter was probably the first question in dynamical systems which dealt with the satisfactory way. Stability question motivated the introduction of new mathematical tools in engineering. Stability theory has been an interest to mathematicians for a long time and had had a stimulating impact on these fields. There are different kinds of stability problems that arise in the study of dynamical systems. We interest, in this work on stability of equilibrium point for some classes of linear and nonlinear systems. This stability is usually characterized in the sense of Lyapunov. In fact, theory of stability in this sense, is new well known and is widely used in concrete problems of the real world. In the following, we define the concept of Lyapunov-stability.

3. STABILITY THEORY

We will be interested in the behavior of the solution of the nonlinear non-autonomous system (1) as time goes to infinity. This is the subject of the stability theory. In order to make things more formal, we need to introduce the concept of an equilibrium point and present a rigorous notion of stability. For $r > 0$, we denote by $B_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ and $U(r) = \{x \in \mathbb{R}^n \mid \|x\| < r\}$.

Definition 1 (Equilibrium point). Let $x^0 \in \mathbb{R}^n$. x^0 is an equilibrium point for (1) if :

$$(2) \qquad\qquad f(t, x^0) = 0, \quad \forall t \geq 0.$$

An equilibrium point has the property that if the state of the system starts at x^0, it will remain there for all future time. Without loss of generality, we can assume that the origin is an equilibrium point of (1) because any equilibrium point can be shifted to the origin via some change of variables. The formal definitions of stability that we are going to be using are as follows.

Definition 2 (Stability). The equilibrium point $x = 0$ is stable if for each $\epsilon > 0$ and any $t_0 \geq 0$, there is $\delta = \delta(t_0, \epsilon) > 0$ such that $\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0$.

That is, an equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise, it is unstable. Let us note that the stability

of systems does not involve the convergence of solutions to the origin, which why the notion of stability alone is not sufficient to study the behavior of solutions. Then, we define the notion of attractiveness.

Definition 3 (Attractiveness)**.** The equilibrium point $x = 0$ is:

i) Attractive if for each $t_0 \in \mathbb{R}_+$, there exist $r(t_0) > 0$ such that

$$\forall x_0 \in U(r(t_0)), \quad \lim_{t \to +\infty} x(t) = 0.$$

ii) Globally attractive if for each $t_0 \in \mathbb{R}_+$ and any $x_0 \in \mathbb{R}^n$:

$$\lim_{t \to +\infty} x(t) = 0.$$

Definition 4 (Asymptotic stability)**.** The equilibrium point $x = 0$ is:

i) Asymptotically stable, if it is stable and attractive.
ii) Globally asymptotically stable, if it is stable and globally attractive.

That is, if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity. Corresponding to different type of stability, we can define concepts of boundedness.

Definition 5 (Uniform boundedness)**.** The solution of (1) is said to be uniformly bounded if there is a nonnegative constant a, such that for all $b \in (0, a)$, there exists $c = c(b) > 0$ such that for each $t_0 \in \mathbb{R}_+$:

$$\|x_0\| < b \Rightarrow \|x(t)\| < c(b), \quad \forall t \geq t_0.$$

It is said to be globally uniformly bounded if the previous property is true for all $b > 0$ ie $a = +\infty$.

Definition 6 (Uniform stability)**.** The equilibrium point $x = 0$ is:

i) Uniformly stable, if for all $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that for each $t_0 \in \mathbb{R}_+$:

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0.$$

ii) Globally uniformly stable, if it is uniformly stable and the solutions of system (1) are globally uniformly bounded.

It is clear that the uniform (global) stability implies the uniform (global) boundedness.

Definition 7 (Uniform attractiveness)**.** The point $x = 0$ is:

i) Uniformly attractive, if there exist $r > 0$ such that $\forall \epsilon > 0$ there is $T := T(\epsilon) > 0$ such that for each $t_0 \in \mathbb{R}_+$ and any $x_0 \in U(r)$:

$$\forall t \geq T + t_0, \quad \|x(t)\| < \epsilon.$$

ii) Globally uniformly attractive, if $\forall \epsilon > 0$ there is $T := T(\epsilon) > 0$ such that for each $t_0 \in \mathbb{R}_+$ and any $x_0 \in \mathbb{R}^n$:

$$\forall t \geq T + t_0, \quad \|x(t)\| < \epsilon.$$

Definition 8 (Uniform asymptotic stability)**.** The equilibrium point $x = 0$ is:

    i) Uniformly asymptotically stable, if it is uniformly stable and uniformly attractive.

    ii) Globally uniformly asymptotically stable, if it is globally uniformly stable and globally uniformly attractive.

It is instructive to note that the definitions of asymptotic stability do not quantify the speed of convergence of trajectories to the origin. Consequently, we use exponential stability.

Definition 9 (Exponential stability)**.** The equilibrium point $x = 0$ is:

    i) Exponentially stable, if it is stable and there exist $r, \lambda_1, \lambda_2 > 0$ such that for each $x_0 \in U(r)$ and any $t_0 \in \mathbb{R}_+$:

$$\|x(t)\| \leq \lambda_1 \|x_0\| \exp(-\lambda_2(t - t_0)), \quad \forall t \geq t_0 \geq 0,$$

    the constant λ_2 is said the convergence rate.

    ii) Globally exponentially stable, if it is stable and there is $\lambda_1, \lambda_2 > 0$ such that for all $x_0 \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}_+$:

$$\|x(t)\| \leq \lambda_1 \|x_0\| \exp(-\lambda_2(t - t_0)), \quad \forall t \geq t_0 \geq 0.$$

Remark 1. (Global) Exponential stability always implies (global) uniform asymptotic stability. The converse is true for linear systems but not for nonlinear systems in general. As an example, the solution of the scalar nonlinear system : $\dot{x} = -x^5$, can be found easily to show asymptotic but not exponential stability.

In general, the question of determining wether the equilibrium point of a nonlinear dynamics is (globally) asymptotically stable can be extremely hard. The main difficulty is that more often that not it is impossible to explicitly write a solution to the differential equation (1). Nevertheless, in some cases, we are still able to make conclusions about stability of nonlinear systems, thanks to a brilliant idea of the famous Russian mathematician Aleksandr Mikhailovich Lyapunov. This method is known as Lyapunov's direct method and was first published in 1892. We devote the next section to present this method.

4. Lyapunov theory

Stability, asymptotic stability and exponential stability can be characterized in terms of special scalar functions know as class $\mathcal{K}, \mathcal{K}_\infty$ and $\mathcal{KL}$ functions.

Definition 10 (Class $\mathcal{K}$ function)**.** A continuous function $\alpha : [0, a) \to [0, +\infty)$ is said to belong to class $\mathcal{K}$, if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = +\infty$ and $\alpha(r) \to +\infty$ as $r \to +\infty$.

Definition 11 (Class $\mathcal{KL}$ function). A continuous function $\beta : [0, a) \times [0, +\infty) \to [0, +\infty)$ is said to belong to class $\mathcal{KL}$, if for each fixed point s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to +\infty$.

The following lemma states some obvious properties of these functions.

Lemma 1. *Let $\alpha_1(s)$ and $\alpha_2(s)$ be class $\mathcal{K}$ functions on $[0, a), \alpha_3(s)$ and $\alpha_4(s)$ be class $\mathcal{K}_\infty$ functions and $\beta(r, s)$ is a class $\mathcal{KL}$ function on $[0, a) \times [0, +\infty)$. Denote $\lim_{x \to a^-} \alpha_i(x)$ by $\alpha_i(a)$ $(i = 1, 2)$, then:*

- i) $\alpha_1^{-1}(s)$ *is defined on $[0, \alpha_1(a))$ and belongs to class $\mathcal{K}$.*
- ii) $\alpha_3^{-1}(s)$ *is defined on $[0, +\infty)$ and belongs to class $\mathcal{K}_\infty$.*
- iii) $\alpha_1 \circ \alpha_2(s)$ *belongs to class $\mathcal{K}$.*
- iv) $\alpha_3 \circ \alpha_4(s)$ *belongs to class $\mathcal{K}_\infty$.*
- v) $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ *belongs to class $\mathcal{KL}$.*

The following result gives equivalent definitions of stability using class $\mathcal{K}$ and $\mathcal{KL}$ functions:

Proposition 1. *The equilibrium point $x = 0$ of (1) is*

- i) *Uniformly stable if and only if there exist a class $\mathcal{K}$ function α and a positive constant c independent of t_0 such that*

$$\|x(t)\| \leq \alpha(\|x_0\|), \ \forall t \geq t_0 \geq 0, \ \forall \|x_0\| < c.$$

- ii) *Globally uniformly stable if and only if the previous inequality is satisfied for any initial state x_0.*
- iii) *Uniformly asymptotically stable if and only if there exist a class $\mathcal{KL}$ function β and a positive constant c independent of t_0 such that*

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0), \quad \forall t \geq t_0 \geq 0, \forall \|x_0\| < c. \tag{3}$$

- iv) *Globally uniformly asymptotically stable if and only if the previous inequality is satisfied for any initial state x_0.*
- v) *Exponentially (resp. globally exponentially) stable if and only if the inequality (3) is satisfied with (resp. $c = +\infty$)*

$$\beta(r, s) = kr \exp(-\gamma s), \quad k > 0, \gamma > 0. \tag{4}$$

Lyapunov's direct method allows us to determine the stability of a system without explicitly integrating the differential equation. This method is a generalization of the idea that if there is an appropriate energy function in a system, then we can study the rate of change of the energy of the system to ascertain stability. To make this precise, we need the following definitions.

Definition 12 (Positive definite function, proper function, decrescent function)**.**

    i) A continuous function $V : \mathbb{R}_+ \times U(r) \longrightarrow \mathbb{R}_+$ is said to be positive definite, if there is a class $\mathcal{K}$ function α_1 such that $V(t, 0) = 0$ and

$$V(t, x) \;\geq\; \alpha_1(\|x\|), \; \forall t \geq 0, \forall x \in U(r).$$

    ii) A continuous function $V : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is said to be proper (radially unbounded), if there is a class $\mathcal{K}_\infty$ function α_2 such that $V(t, 0) = 0$ and

$$V(t, x) \;\geq\; \alpha_2(\|x\|), \; \forall t \geq 0, \forall x \in \mathbb{R}^n.$$

    iii) A continuous function $V : \mathbb{R}_+ \times U(r) \longrightarrow \mathbb{R}_+$ is said to be decrescent, if $V(t, 0) = 0$ and there is a class $\mathcal{K}$ function α_3, such that

$$V(t, x) \leq \alpha_3(\|x\|), \; \forall t \geq 0, \forall x \in U(r).$$

    iv) A continuous function V is said to be negative definite if $-V$ is a positive definite function.

Next, we introduce the Lyapunov function as a generalization of the idea of the "energy" of a system. Then the method studies stability by looking at the rate of change of this "measure of energy".

Definition 13. Let $V : \mathbb{R}_+ \times U(r) \longrightarrow \mathbb{R}_+$ be continuously differentiable function. Then The time derivative of V along the trajectories of system (1) is denoted by $\dot{V}(t, x)$ where

$$\dot{V}(t, x) \;=\; \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x).$$

Note that $\dot{V}$ depends not only on the function V but also on the system (1). The quantity $\dot{V}(t, x)$ can be interpreted as follows: Suppose a solution trajectory of this system passes through x_0 at time t_0. Then, at the instant t_0, the rate of change of the quantity $V(t, x(t))$ is $\dot{V}(t_0, x_0)$, which can be written

$$\dot{V}(t_0, x_0) = \frac{d(V(x(t)))}{dt}|_{t=t_0}.$$

Definition 14 (Lyapunov function)**.** We consider the system (1). Let $r > 0$ and $V : \mathbb{R}_+ \times U(r) \longrightarrow \mathbb{R}$ a continuously differentiable function. V is said to be a Lyapunov function, if it satisfy the two following properties

    i) V is a positive definite function.
    ii) $\dot{V}(t, x) \leq 0, \; \forall t \in \mathbb{R}_+, \; \forall x \in U(r).$

We use the next theorem to deduce the stability of the equilibrium point when the system has a Lyapunov function. Then the stability theorem of Lyapunov can be stated as follows:

Theorem 1 (Stability). *We consider the system* (1).

If this system has a Lyapunov function V on $U(r)$ for $r > 0$, then the origin $x = 0$ is an equilibrium point stable. Moreover, if V is decrescent then $x = 0$ is an equilibrium point uniformly stable.

Finally, If the system (1) *has a Lyapunov function V on $\mathbb{R}^n$, decrescent and radially unbounded, then $x = 0$ is an equilibrium point globally uniformly stable.*

Example 1. Consider the following system:

$$\begin{cases} \dot{x}_1 = -x_1 - \exp(-2t)x_2, \\ \dot{x}_2 = x_1 - x_2. \end{cases}$$

To study the stability of the origin, let $V(t,x) = x_1^2 + (1 + \exp(-2t))x_2^2$. Clearly

$$\alpha_1(\|x\|) = x_1^2 + x_2^2 \leq V(t,x) \leq x_1^2 + 2x_2^2 = \alpha_2(\|x\|),$$

thus, we have that

   i) $V(t,x)$ is positive definite and radially unbounded, since $\alpha_1(\|x\|) \leq V(t,x)$, with α_1 is a class $\mathcal{K}_\infty$ function.

   ii) $V(t,x)$ is decrescent, since $V(t,x) \leq \alpha_2(\|x\|)$, with α_2 is also a class $\mathcal{K}$ function.

Moreover,

$$\begin{aligned} \dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \\ &= -2[x_1^2 - x_1 x_2 + x_2^2(1 + 2\exp(-2t))] \\ &\leq -(x_1^2 + x_2^2) \\ &\leq 0. \end{aligned}$$

Then the origin is globally uniformly stable.

Theorem 2 (Asymptotic stability [35]). *Let $x = 0$ be an equilibrium point of* (1) *and $r > 0$. Let $V : \mathbb{R}_+ \times U(r) \longrightarrow \mathbb{R}$ be a continuously differentiable function such that there exist class functions $\mathcal{K} : \alpha_1(.), \alpha_2(.)$ and $\alpha_3(.)$ defined on $[0, r)$ satisfying $\forall t \geq t_0$, $\forall x \in U(r)$,*

$$(5) \qquad\qquad \alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|),$$

$$(6) \qquad\qquad \dot{V}(t,x) \leq -\alpha_3(\|x\|).$$

Then $x = 0$ is an equilibrium point uniformly asymptotically stable. If $U(r) = \mathbb{R}^n, \alpha_1(.)$ and $\alpha_2(.)$ are class $\mathcal{K}_\infty$ functions, then $x = 0$ is an equilibrium point globally uniformly asymptotically stable.

Theorem 3 (Exponential stability). *We consider the system* (1). *Assume that the system has a Lyapunov function $V(t,x)$ and there exist constants*

$c_1, c_2, c_3, c_4 > 0$ and $r > 0$, such that $\forall\, t \geq t_0, \forall\, x \in U(r)$ we have:

$$\text{(7)} \qquad\qquad c_1\|x\|^2 \;\leq\; V(t,x) \leq c_2\|x\|^2,$$

$$\text{(8)} \qquad\qquad \dot{V}(t,x) \;\leq\; -c_3\|x\|^2.$$

Then $x = 0$ is an equilibrium point exponentially stable. If $U(r) = \mathbb{R}^n$, then the origin $x = 0$ is an equilibrium point globally exponentially stable.

Lyapunov analysis can be used to find conditions for instability of an equilibrium point, for example $x = 0$ must be unstable if V and $\dot{V}$ are both positive definite. These results are known as instability theorems.

Once again we emphasize that Lyapunov's theorems allow stability of the system to be verified without explicitly solving the differential equation. Lyapunov's theorems, in effect, turn the question of determining stability into a search for Lyapunov function. The natural question that immediately arises is if this Lyapunov function always exists? In many situations, the answer is positive and it is due to so-called converse theorems.

5. Converse Lyapunov theorems

The converse Lyapunov theorems, which is the inverse of Lyapunov's theorems, establish the stability of the origin by requiring the existence of an auxiliary function that satisfies certain conditions.

Theorem 4 (Converse Lyapunov theorem)**.** *Assume that the origin is an equilibrium point of the system* (1) *where* $f : [0, +\infty) \times U(r) \longrightarrow \mathbb{R}^n$ *is a continuously differentiable function and the Jacobian matrix* $\left[\dfrac{\partial f}{\partial x}\right]$ *is uniformly bounded on* $U(r)$ *with,* $r, \rho > 0$ *and* k, γ *be constants, such that* $\rho < \frac{r}{k}$. *Assume that all trajectories of the system satisfy*

$$\|x(t)\| \;\leq\; k\|x_0\|\exp(-\gamma(t - t_0)), \quad \forall t \geq t_0 \geq 0, \forall x \in U(\rho).$$

Then, there exist a function $V(.,.) : [0, +\infty) \times U(\rho) \longrightarrow \mathbb{R}$ *and some constants* c_1, c_2, c_3 *and* $c_4 > 0$ *satisfying the following inequalities:*

$$c_1\|x\|^2 \leq V(t,x) \;\leq\; c_2\|x\|^2,$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) \;\leq\; -c_3\|x\|^2,$$

$$\|\frac{\partial V}{\partial x}(t,x)\| \;\leq\; c_4\|x\|.$$

Furthermore, if $r = +\infty$ *and the origin is globally exponentially stable, then* $V(t,x)$ *is defined on* $\mathbb{R}_+ \times \mathbb{R}^n$ *and satisfies the previous inequalities for all* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

If the system is autonomous, V *can be chosen independent of* t.

Theorem 5. *Assume that the origin is an equilibrium point of the system* (1) *where* $f : [0, +\infty) \times U(r) \longrightarrow \mathbb{R}^n$ *is a continuously differentiable function*

and $\left[\dfrac{\partial f}{\partial x}\right]$ *is uniformly bounded on $U(r)$ with $r > 0$. Let β be a class $\mathcal{KL}$* *function and $\rho > 0$ such that $\beta(\rho, 0) < r$. Assume that all trajectories of the* *system satisfy*

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0), \quad \forall t \geq t_0 \geq 0, \forall x \in U(\rho).$$

Then, there exists a function $V : [0, +\infty) \times U(\rho) \longrightarrow \mathbb{R}$ continuously differ- *entiable that satisfies the following inequalities :*

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|),$$
$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)f(t, x) \leq -\alpha_3(\|x\|),$$
$$\|\frac{\partial V}{\partial x}(t, x)\| \leq \alpha_4(\|x\|),$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are class $\mathcal{K}$ functions defined on $[0, \rho)$.

Furthermore, if $r = +\infty$ and the origin is globally uniformly asymptot- *ically stable, then $V(t, x)$ is defined on $\mathbb{R}_+ \times \mathbb{R}^n$ and satisfies the previous* *inequalities for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$.*

If the system is autonomous, V can be chosen independent of t.

Unfortunately, converse theorems are often proven by assuming knowledge of the solutions of (1) and are therefore useless in practice. By this we mean that they offer no systematic way of finding the Lyapunov function. Moreover, little is know about the connection of the dynamics f to the Lyapunov function V. Among the few results in the direction, the case of linear systems is well settled since a stable linear system always admits a quadratic Lyapunov function. It is also known that stable and smooth homogenous systems always have a homogeneous Lyapunov function.

6. Linear systems and linearization

More recently, there has been a significant interest in the study of the stability properties of the solution of linear systems. While the determina- tion of qualitative properties of the solution of linear time-invariant systems (LTI) is relatively simple, the determination of corresponding properties for linear time-varying systems (LTV) is very difficult and complicated since it requires the evolution of the transition system matrix.

For instance, it is known that the stability conditions of (LTI) systems can be obtained by examining the eigenvalues of the system matrix, or by solving some Lyapunov equation. However for (LTV) systems, the stability may not be determined from the eigenvalues of their matrix system. When $A(t)$ is time varying, it was shown that the real parts of the spectrum of $A(t)$ for every t are negative does not imply the asymptotical stability of the time- varying system. In [27] and references therein, some exponential stability conditions are derived for (LTV) systems where $A(t)$ is assumed to be slowly

varying ie $\sup_{t \in \mathbb{R}^+} \|\dot{A}(t)\|$ is sufficiently small. Therefore, finding simple and effective conditions for the qualitative properties of (LTV) systems has been a topic of long-standing interest.

6.1. Linear time-invariant systems.
Consider the linear time invariant system

$$\dot{x} = Ax, \tag{9}$$

where $x \in \mathbb{R}^n$ and A is a constant matrix $(n \times n)$. The system (9) has an equilibrium point at the origin. The equilibrium point is isolated if and only if $\det(A) \neq 0$. If $\det(A) = 0$, the matrix A has a nontrivial null space. Every point in the null space of A is an equilibrium point for the system (9). We are interested to the origin, then stability properties can be characterized by the locations of the eigenvalues of the matrix A. Recall from linear system theory that the solution of (9) for a given initial state $x(0)$ is given by

$$x(t) = \exp(At) x(0),$$

where the matrix exponential is defined by the power series

$$\exp(At) = \sum_{k=0}^{+\infty} \frac{A^k t^k}{k!}.$$

It is well known that the asymptotic behavior of its solutions depends on the eigenvalues of A. More precisely, when all eigenvalues of A have strictly negative real parts, A is called a stability matrix or a Hurwitz matrix. The following theorem characterize the stability properties of the origin.

Theorem 6.
    i) *The origin of (9) is globally asymptotically stable if and only if it is exponentially stable.*
    ii) *The origin of (9) is globally asymptotically stable if and only if all eigenvalues of A have strictly negative real parts ie A is a Hurwitz matrix.*
    iii) *The origin of (9) is stable if and only if all eigenvalues of A satisfy $\Re \lambda_i \leq 0$ and every eigenvalue with $\Re \lambda_i = 0$ has an associated Jordan block of order one ie simple eigenvalue.*

Example 2. Determine the stability of the system with state matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ with } x_0 = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}.$$

By using Theorem 6, we conclude that the origin is stable. Moreover, we have:

$$x(t) = \begin{pmatrix} x_{01} \cos(t) - x_{02} \sin(t) \\ x_{01} \sin(t) + x_{02} \cos(t) \end{pmatrix}.$$

Using the 2-norm leads to

$$\|x(t)\| = \|x_0\|.$$

Therefore the system is not asymptotically stable (since in general $\lim_{+\infty} \|x(t)\| = \|x_0\| \neq 0$).

Asymptotic stability of the origin can be investigated using Lyapunov's method. We can consider as a Lyapunov function candidate $V(x) = x^T P x$ where P is a real symmetric positive matrix. The following proposition characterize the asymptotic stability of the origin in terms of the solution of the Lyapunov equation.

Proposition 2 ([35]). *A matrix A is a Hurwitz matrix, if and only if for any given positive definite symmetric matrix Q there exists a unique positive definite symmetric matrix P that satisfies the following Lyapunov equation*

$$PA + A^T P = -Q.$$

Remark 2. A more common technique to obtain a (P, Q) is:
  i) Choose a P symmetric matrix that is positive definite, compute Q, and check if it is positive definite symmetric matrix. This is not a smart approach, since A is stable, then not every positive definite and symmetric P will yield a positive definite and symmetric Q.
  ii) If A is stable, any positive definite symmetric matrix Q will yield a positive definite symmetric matrix P. The usual approach is to set $Q = I_n$, then solve for P.

Example 3. Determine the stability of the system with state matrix $A = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$ using the Lyapunov equation. The system is clearly stable by inspection the matrix A ($\lambda_{1,2} = -2, -3$).

Let $P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and consider the Lyapunov equation $PA + A^T P = -I_2$. It can be written:

$$\begin{pmatrix} 0 & 12 & 0 \\ 1 & -5 & -6 \\ 0 & -2 & 10 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The elements of the matrix P must satisfy $a > 0$ and $ac - b^2 > 0$. Then, the unique solution of this equation is given by $P = \begin{pmatrix} \frac{67}{60} & \frac{1}{12} \\ \frac{1}{12} & \frac{7}{60} \end{pmatrix}$.

Note: Choosing $P = I_2$ will not work! No conclusion.

In general, the Lyapunov equation can be used to test whether or not a matrix A is a stability matrix, as an alternative to calculating the eigenvalues of A. However, there is no computational advantage in solving the Lyapunov equation over calculating the eigenvalues of the matrix A. Besides, the eigenvalues provide more direct information about the response of the linear system. The interest in the Lyapunov equation is not in its use as a stability test for linear systems; rather, it is in the fact that it provides a procedure

for finding a Lyapunov function for any linear system when A is a stability matrix. The mere existence of a Lyapunov function will allow us to draw conclusions about the system when the right-hand side of (9) is perturbed, whether such perturbation is a linear perturbation in the coefficients of A or a nonlinear perturbation.

6.2. **Linear time-varying systems.** Stability analysis for linear time-varying systems is of increasing interest theory. One reason is the growing importance of adaptive controllers for which the underlying closed-loop adaptive system often is time-varying and linear which can be modeled as:

$$(10) \qquad \dot{x} = A(t)x, \quad x(t_0) = x_0,$$

where A is an $n \times n$ matrix whose entries are all real valued piecewise continuous functions of $t \in \mathbb{R}^+$. The space of solutions of (10) is of dimension n. A basis of the space of solutions of the system (10), i.e., a set $\{x_1, \ldots, x_n\}$ of linearly independent solutions, is called a fundamental set of solutions. A matrix $\Psi(t) = [x_1(t) \quad \cdots \quad x_n(t)]$ whose columns are the vectors of a basis of the solution space of (10) is called a fundamental matrix of (10). A fundamental matrix of (10) is solution of the matrix equation

$$(11) \qquad \dot{\Psi}(t) = A(t)\Psi(t),$$

and conversely, any nonsingular solution of (11) is a fundamental matrix of (10).

Definition 15. Let $\Psi(t)$ be a fundamental matrix of (10). Then

$$(12) \qquad \Phi(t, t_0) = \Psi(t)\Psi^{-1}(t_0), \quad \forall t \geq t_0$$

is called the state transition matrix of (10).

Notice that the above definition is consistent in the sense that $\Phi(t, t_0)$ is uniquely defined by $A(t)$ and independent of the particular choice of $\Psi(t)$. Indeed, for two different fundamental matrices $\Psi_1(t)$ and $\Psi_2(t)$, there exists a nonsingular matrix $P(t)$ such that $\Psi_2(t) = \Psi_1(t)P(t)$. Thus, following the Definition, we get:

$$\Phi(t, t_0) = \Psi_1(t)\Psi_1^{-1}(t_0) = \Psi_2(t)P^{-1}(t)P(t)\Psi_2^{-1}(t_0) = \Psi_2(t)\Psi_2^{-1}(t_0).$$

The transition matrix has the following properties:

Proposition 3. *If $\Phi(t, t_0)$ is the state transition matrix of the system* (10), *then:*

i) $\frac{\partial \Phi}{\partial t}(t, t_0) = A(t)\Phi(t, t_0);$

ii) $\frac{\partial \Phi}{\partial t_0}(t, t_0) = -\Phi(t, t_0)A(t_0);$

iii) $\Phi(t, t) = I_n;$

iv) $\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0);$

v) $\Phi^{-1}(t_1, t_2) = \Phi(t_2, t_1);$

vi) $\det \Phi(t, t_0) = \exp(\int_{t_0}^{t} Tr(A(u))\, d\, u).$

The solution of (10) with initial conditions $x(t_0) = x_0$ is

(13) $$x(t) = \Phi(t, t_0)x_0, \forall t \geq t_0.$$

Formula (13) can be directly checked using the definition relation (12). It shows that the state transition matrix is a linear transformation which maps the initial condition x_0 into the state x at time t.

Remark 3.

    i) If the system is time-invariant $(A(t) = A)$, we can define

$$\Phi(t, t_0) = \exp(A(t - t_0)).$$

    ii) If $A(t)$ is not constant and the matrices $A(t)$ and $A(s)$ commute for all (t, s), then

$$\Phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(s)\,\mathrm{d}\,s\right).$$

Example 4. Let $A(t) = \begin{pmatrix} -1 & -t \\ t & -1 \end{pmatrix}$, then the transition matrix $\Phi(t, t_0)$ satisfies:

$$\begin{aligned} \Phi(t, t_0) &= \exp((t - t_0)A) \\ &= \exp(-(t - t_0))\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \end{aligned}$$

We get some further characterization for asymptotic stability.

Theorem 7 (Transition matrix and stability). *The trivial solution of the homogeneous system* (10) *is:*

    i) *Stable if and only if*

$$\forall t_0 \geq 0, \quad \exists \beta > 0, \quad \forall t \geq t_0 : \|\Phi(t, t_0)\| \leq \beta.$$

    ii) *Uniformly stable if and only if*

$$\exists \beta > 0, \quad \forall t_0 \geq 0, \quad \forall t \geq t_0 : \|\Phi(t, t_0)\| \leq \beta.$$

    iii) *Asymptotically stable if and only if*

$$\forall t_0 \geq 0 : \lim_{t \to \infty} \Phi(t, t_0) = 0.$$

    iv) *Globally uniformly asymptotically stable if and only if there exist positive constants* k *and* γ *such that*

$$\|\Phi(t, t_0)\| \leq k \exp(-\gamma(t - t_0)), \quad \forall t \geq t_0 \geq 0.$$

Proof. see [35]. □

Corollary 1. *The system* (10) *is:*

i) *Uniformly asymptotically stable if and only if it is globally uniformly asymptotically stable.*

ii) *Uniformly asymptotically stable if and only if it is exponentially stable.*

Corollary 2. *The trivial solution of the homogeneous system* (10) *is:*

i) *Globally asymptotically stable if and only if it is asymptotically stable.*

ii) *Globally exponentially stable if and only if it is exponentially stable.*

Corollary 3. *The system* (10) *is:*

i) *Uniformly stable if,*

$$\|x(t)\| \le \gamma \|x(t_0)\|, \quad \forall t \ge t_0$$

for some $\gamma > 0$.

ii) *Uniformly asymptotically stable if,*

$$\|x(t)\| \le \gamma \exp(-\lambda(t - t_0))\|x(t_0)\|, \quad \forall t \ge t_0$$

for some positive constants $\gamma, \lambda > 0$.

Remark 4. When we turn to linear time-varying system, the study of stability is much complicated. In view of Theorem 6 , it might be thought that if all eigenvalues of $A(t)$ have negative real parts for all $t \ge t_0$, then the origin of (10) would be asymptotically stable. Unfortunately, this conjecture is not true. Therefore uniform stability or asymptotic stability cannot be characterized by the locations of the eigenvalues of $A(t)$ as the following examples shows.

Example 5. Consider the linear system $\dot{x}(t) = A(t)x$, where

$$A(t) = \begin{pmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t \cos t \\ -1 - 1.5\sin t \cos t & -1 + 1.5\sin^2 t \end{pmatrix}.$$

We have the characteristic equation: $\lambda^2 + \frac{1}{2}\lambda + \frac{1}{2} = 0 \Rightarrow \lambda_{1,2} = \frac{-1 \pm i\sqrt{7}}{4}$. Thus, the eigenvalues are independent of t and lie in the open left-half plane. Yet, the origin is unstable. We can verify that

$$\Phi(t,0) = \begin{pmatrix} \exp(0.5t)\cos t & e^{-t}\sin t \\ -\exp(0.5t)\sin t & \exp(-t)\cos t \end{pmatrix},$$

which shows that there are initial states $x(0)$ arbitrarily close to the origin, for which the solution is unbounded and escape to infinity.

Example 6. Consider the linear system $\dot{x}(t) = A(t)x$, where

$$A(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

We have

$$\Phi(t, t_0) = \exp(\lambda_1) \begin{pmatrix} \cos \lambda_2 & \sin \lambda_2 \\ -\sin \lambda_2 & \cos \lambda_2 \end{pmatrix},$$

where $\lambda_1 = \sin t - \cos t_0$ and $\lambda_2 = \cos t_0 - \cos t$. Then,

$$\|x(t)\| \le \exp(2)\|x(t_0)\|, \quad \forall t \ge t_0,$$

and consequently the system is uniformly stable. It is not uniformly asymptotic stable because $\exp(\sin t) \cos(1 - \cos t) \not\to 0$ as $t \to \infty$.

Interestingly, if we freeze time and compute the eigenvalues of $A(t)$ at that instant, the results do not tell us whether the system is stable, except in special cases wherein the eigenvalues are changing "slowly." Also, when a time-varying system has an unstable eigenvalue, that does not necessarily imply that the system is unstable overall!! (unless all of them are unstable).

Notice that for linear systems, uniform asymptotic stability is equivalent to exponential stability. Also, in the general case of nonlinear systems, an equilibrium point is globally uniformly or exponentially stable if it is the only equilibrium of the system.

Now we investigate the asymptotic stability of system (10) by the use of a Lyapunov-like approach. The following lemma states the Lyapunov-equation and gives a solution under certain conditions.

Lemma 2 (Solution of the Lyapunov-equation).
Let $A, Q : [0, +\infty) \to \mathbb{R}^{n^2}$ be continuous. If the integral

$$P(t) := \int_t^{+\infty} \Phi(s, t)^T Q(s) \Phi(s, t) \, \mathrm{d} s$$

exists for all $t \ge 0$, then the time-varying Lyapunov-equation

(14) $$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$$

has the continuously differentiable solution $P : [0, +\infty) \to \mathbb{R}^{n^2}$.

When the linear system (10) is time invariant, that is, when A is constant, then for a constant matrix Q, the matrix $P(t)$ is given by

$$\begin{aligned} P(t) \quad &:= \quad \int_t^{+\infty} \exp[(s-t)A^T] Q \exp[(s-t)A] \, \mathrm{d} s \\ &= \quad \int_0^{+\infty} \exp(sA^T) Q \exp(sA) \, \mathrm{d} s, \end{aligned}$$

which is independent of t.

Now, using Lyapunov approach, we suppose that there exists a continuously differentiable bounded, positive definite, symmetric matrix $P(t)$ that is,

$$c_1 I \le P(t) \le c_2 I, \quad \forall t \ge 0 \ (c_1 > 0, \ c_2 > 0),$$

which satisfies the Lyapunov-equation (14) where $Q(t)$ is continuously, positive definite, symmetric matrix, that is,

$$Q(t) \geq c_3 I, \quad \forall t \geq 0 \ (c_3 > 0).$$

Consider the Lyapunov function candidate

$$V(t, x) = x^T P(t) x.$$

This function satisfies

$$c_1 \|x\|^2 \leq V(t, x) \leq c_2 \|x\|^2,$$

Moreover, this function is proper since $c_1 \|x\|^2$ is a class $\mathcal{K}_\infty$ function. the derivative of $V(t, x)$ along the trajectories of system (10) is given by

$$
\begin{aligned}
\dot{V}(t, x) &= x^T \dot{P}(t) x + x^T P(t) \dot{x} + \dot{x}^T P(t) x \\
&= x^T (\dot{P}(t) + P(t) A(t) + A^T(t) P(t)) x \\
&= -x^T Q(t) x \\
&\leq -c_3 \|x\|^2.
\end{aligned}
$$

Then $\dot{V}(t, x)$ is negative definite. All assumptions of Theorem 2 are satisfied with $\alpha_i(r) = c_i r^2$, then, the origin is globally uniformly exponentially stable.

Theorem 8 ([35]). *Suppose that the origin is an equilibrium point uniformly asymptotically stable of system* (10). *Suppose also that $A(t)$ is continuous and bounded. Then, for any matrix $Q(t)$ continuous, positive definite, symmetric, and bounded, there exists continuously differentiable bounded, positive definite, symmetric matrix $P(t)$ that satisfies* (14). *It follows that $V(t, x) = x^T P(t) x$ is a Lyapunov function of the system that satisfies assumptions of Theorem 2.*

Remark 5. Asymptotic stability of the system (10) is not sufficient in general, to guarantee existence of solution to Lyapunov-equation, since there is a need for a certain convergence rate of the solution of (10) to zero. Furthermore, the following example illustrates that asymptotic and exponential stability are not equivalent, as it is in case, when A is constant.

Example 7. Consider the scalar differential equation

$$(15) \qquad \dot{x}(t) = -\frac{1}{1+t} x(t), \quad t \in [0, +\infty), \ x(t_0) = x_0.$$

Then, for any $x_0 \in \mathbb{R}, t_0 \in \mathbb{R}^+$, the scalar differential equation (15) has a unique global solution

$$x : [0, +\infty) \to \mathbb{R}, \quad t \mapsto \frac{1 + t_0}{1 + t} x_0,$$

$\forall \varepsilon > 0$, take $\delta = \varepsilon$. When $|x_0| < \delta$, $|x(t, t_0, x_0)| \leq |x_0| < \varepsilon \quad \forall t \geq t_0$ holds, so the zero solution is uniformly stable, and $\lim_{t \to +\infty} x(t, t_0, x_0) = 0$. Hence,

the zero solution is asymptotically stable. But for each $T > 0$ and any $t_0 \geq 0$, we can choose $t = t_0 + T$, so we obtain for all $x_0 \neq 0$:

$$x(t, t_0, x_0) = x_0 \frac{t_0 + 1}{t_0 + T + 1} \to x_0 \neq 0, \quad \text{as } t \to +\infty.$$

This means that the zero solution is not uniformly attractive, and therefore, the zero solution is not uniformly asymptotically stable ie not exponentially stable.

Theorem 9.

    i) *Let $A(t)$ be bounded and $Q(t)$ is continuously, positive definite, symmetric matrix, that is*

$$Q(t) \geq c_3 I > 0, \quad \forall t \geq 0 \ (c_3 > 0).$$

    If (1.5.2) is exponentially stable, then there exists a continuously differentiable solution $P(t)$ is a continuously differentiable bounded, positive definite, symmetric matrix, that is,

$$0 < c_1 I \leq P(t) \leq c_2 I, \quad \forall t \geq 0 \ (c_1 > 0, c_2 > 0),$$

    to (1.5.3).

    ii) *If there exist $P(t), Q(t)$ continuously, positive definite, symmetric matrix, such that $P(t)$ is continuously differentiable, bounded and (14) holds, then (1.5.2) is exponentially stable.*

Many authors have obtained additional restrictions for the variation of the elements of $A(.)$ that are imposed in order to obtain sufficient conditions for exponential stability.

6.3. **Lyapunov's indirect method.** The indirect method of Lyapunov uses a linearization of a system to determine the local stability of the original system. So, let us go back to the system (1) where $f : [0, +\infty) \times U(r) \longrightarrow \mathbb{R}^n$ is continuously differentiable function and $r > 0$. Suppose that the origin is an equilibrium point of system (1). Assume also that the Jacobian matrix $\left[\frac{\partial f}{\partial x} \right]$ is uniformly bounded and Lipschitz on $U(r)$, that is, there exist k and $L > 0$ such that

$$\| \frac{\partial f}{\partial x}(t, x) \| \leq k, \quad \forall x \in U(r), \ \forall t \geq t_0,$$

$$\| \frac{\partial f}{\partial x}(t, x_1) - \frac{\partial f}{\partial x}(t, x_2) \| \leq L \|x_1 - x_2\|, \quad \forall x_1, x_2 \in U(r), \ \forall t \geq t_0.$$

We can write $f(t, x)$ in the form

$$f(t, x) = f(t, 0) + \frac{\partial f}{\partial x}(t, z)x,$$

where $z \in (0, x)$. Since $f(t, 0) = 0$, then

$$\begin{aligned}
f(t,x) &= \frac{\partial f}{\partial x}(t,z)x \\
&= \frac{\partial f}{\partial x}(t,0)x + \left[\frac{\partial f}{\partial x}(t,z) - \frac{\partial f}{\partial x}(t,0)\right]x \\
&= A(t)x + g(t,x),
\end{aligned}$$

where $A(t) = \frac{\partial f}{\partial x}(t,0)$ and $g(t,x) = \left[\frac{\partial f}{\partial x}(t,z) - \frac{\partial f}{\partial x}(t,0)\right]x$.

Therefore, in a small neighborhood of the origin, we may approximate the nonlinear system (1) by its linearization about the origin. For small deviations from the equilibrium point, the performance of the system is approximately governed by the linear terms. These terms dominate and thus determine stability provided that the linear terms do not vanish.

The following theorem states Lyapunov's indirect method for showing the uniform exponential stability of the origin.

Theorem 10. *Suppose that the origin is an equilibrium point of the nonlinear system* (1) *where* $f : [0, +\infty) \times U(r) \longrightarrow \mathbb{R}^n$ *is continuously differentiable function and* $\left[\frac{\partial f}{\partial x}\right]$ *is uniformly bounded and Lipschitz on* $U(r)$. *Let*

$$(16) \qquad\qquad A(t) = \frac{\partial f}{\partial x}(t,0).$$

If the origin is an equilibrium point exponentially stable of the linear system (10), *then it is an equilibrium point exponentially stable of the system* (1).

The following theorem shows that the exponential stability of the linearized system is a necessary and sufficient assumption for exponential stability of the origin of the nonlinear system.

Theorem 11. *Suppose that the origin is an equilibrium point of the nonlinear system* (1) *where* $f : [0, +\infty) \times U(r) \longrightarrow \mathbb{R}^n$ *is continuously differentiable function and* $\left[\frac{\partial f}{\partial x}\right]$ *is uniformly bounded and Lipschitz on* $U(r)$. *Then the origin is an equilibrium point exponential stable of nonlinear system* (1) *if and only if it is an equilibrium point exponential stable of the system* (10).

Suppose now that the nonlinear system (1) is autonomous ie $f(t,x) = f(x)$, then the matrix $A(t)$ is constant ($A(t) = A$) and we have the following theorem.

Theorem 12. *Let* $x = 0$ *be an equilibrium point of the nonlinear system* (1) *where* $f : [0, +\infty) \times U(r) \longrightarrow \mathbb{R}^n$ *is continuously differentiable function. Let* (λ_i) *denote the eigenvalues of the matrix* A.

1. *If* $\Re \lambda_i < 0$ *for all* λ_i *then* $x = 0$ *is asymptotically stable for the nonlinear system* (1).

2. *If $\Re\lambda_i > 0$ for one or more λ_i then $x = 0$ is unstable for the nonlinear system* (1).

3. *If $\Re\lambda_i \leq 0$ for all λ_i and at least one $\Re\lambda_j = 0$ then $x = 0$ may be either stable, asymptotically stable or unstable for the nonlinear system* (1).

7. Perturbed systems

Many dynamics are described by perturbed systems as in the following form:

$$(17) \qquad \dot{x} = f(t,x) + g(t,x),$$

where $f, g : \mathbb{R}^+ \times U(r) \to \mathbb{R}^n$ are piecewise continuous on t, locally Lipschitz on x and $r > 0$. Such system was seen as a perturbation of the nominal system:

$$(18) \qquad \dot{x} = f(t,x).$$

Here, we represent the perturbation as an additive term on the right-hand side of the state equation. The perturbation term $g(t,x)$ could result from errors in modeling the nonlinear system, aging of parameters, or uncertainties and disturbances which exist in any realistic problem. In a typical situation, we do not know $g(t,x)$, but we know some information about it, like knowing an upper bound on $\|g(t,x)\|$. The trivial question posed here is if the nominal system present one of the type of stability, the perturbed one keeps the same behavior or not? A natural approach is to use a Lyapunov function for the nominal system as a Lyapunov function for the perturbed system. It is used in [35] to prove the exponential stability of the perturbed system. This approach, called indirect Lyapunov method, is based on the use of the linearized $A(t) = \frac{\partial f}{\partial x}(t,0)$ (if the function f is continuously differentiable). But, its disadvantage lies in that the perturbation term could be more general than whose in the case of linearization. We have two cases: the first one is when $g(t,0) = 0$ that is the perturbed system has an equilibrium point at the origin, then we analyze the stability behavior of the origin as an equilibrium point of the perturbed system. While the second case, which is the more general case, is when we do not know that $g(t,0) = 0$. Therefore, we can no longer study the problem as a question of stability of equilibria and some new concepts were introduced in [35] to study the exponential stability of the perturbed system. The latter case will be treated in section 1.7. Let us start with the case $g(t,0) = 0$.

Theorem 13. *Suppose $x = 0$ is exponentially stable equilibrium point of the nominal system* (18), *and let $V(t,x)$ be a Lyapunov function that satisfies, for all $(t,x) \in \mathbb{R}^+ \times U(r)$:*

$$(19) \qquad c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2,$$

$$(20) \qquad \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) \;\leq\; -c_3\|x\|^2,$$

$$(21) \qquad \|\frac{\partial V}{\partial x}(t,x)\| \;\leq\; c_4\|x\|,$$

for some positive constants c_1, c_2, c_3 and c_4. If the perturbation term $g(t,x)$ satisfies the linear growth bound

$$(22) \qquad \|g(t,x)\| \leq \gamma\|x\|,$$

$$(23) \qquad \gamma < \frac{c_3}{c_4},$$

then the origin is exponentially stable equilibrium point of the perturbed system (17). Moreover, if all the assumptions hold globally, then the origin is globally exponentially stable.

Remark 6. One can obtain exponential convergence to zero for system (1) especially, where

$$g(t,x) = B(t)x$$

under the conditions $B(t)$ is continuous and $B(t) \to 0$ as $t \to +\infty$. Similar conclusions can be obtained where

$$\int_0^{+\infty} \|B(t)\|\,\mathrm{d}\,t < +\infty$$

or

$$\int_0^{+\infty} \|B(t)\|^2\,\mathrm{d}\,t < +\infty.$$

The more interesting case occurs when $g(t,0)$ is not necessarily zero. In this case $x = 0$ is no longer an equilibrium point. The best we can do is find a bound on the size of $g(t,x)$ that ensures $x(t)$ remains close to the origin. This case will be treated here, by examining the differential system:

$$(24) \qquad \dot{x}(t) \;=\; A(t)x + h(t,x),$$

where $A(t)$ is piecewise continuous matrix and $h(t,x)$ is defined on $\mathbb{R}_+ \times \mathbb{R}^n$, piecewise continuous in t, and locally Lipshitz in x. This system is a perturbed one of the linear system (10).

In this survey, we give some new results on the globally uniformly asymptotically (or exponentially) stability of certain classes of perturbation of nonlinear systems of the form (24).

8. Practical stability

There are some systems that may be unstable and yet these systems may oscillate sufficiently near this state that its performance is acceptable. To deal with this situations, we need a notion of stability that is more suitable in several situation than Lyapunov stability such a concept is called practical stability. This stability, introduced by Lasalle and Lefschetz, is

concerned with quantitative analysis as opposed to Lyapunov analysis which is qualitative in nature. So, unlike Lyapunov stability, the study of the practical stability led back to the study of stability of a ball centered at the origin. That is why we start by given the definition of the uniformly stability and the uniformly attractiveness of a bounded ball $B_r = \{x \in \mathbb{R}^n | \; \|x\| \leq r\}$.

Definition 16 (Uniformly stability of B_r)**.**
  i) B_r is uniformly stable, if for all $\epsilon > r$, there exists $\delta = \delta(\epsilon)$ such that, for all $t_0 \geq 0$,
$$\|x_0\| < \delta \; \Rightarrow \; \|x(t)\| < \epsilon, \quad \forall t \geq t_0.$$
  ii) B_r is globally uniformly stable, if it is uniformly stable and the solutions of (1) are globally uniformly bounded.

Definition 17 (Uniformly attractiveness of B_r)**.** B_r is uniformly attractive (resp. uniformly attractive on a ball B_R, with $r \leq R < \infty$), if for all $\epsilon > r$, there exists $T(\epsilon) > 0$ such that for each $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$ (resp. $x_0 \in B_R$)
$$\|x(t)\| < \epsilon, \quad \forall t \geq t_0 + T(\epsilon).$$

Definition 18 (Practical stability)**.**
  i) The system is said uniformly practically asymptotically stable if there exists a ball $B_r \subset \mathbb{R}^n$ such that B_r is uniformly stable and uniformly attractive ie uniformly asymptotically stable.
  ii) The system is said globally uniformly practically asymptotically stable if there exists a ball $B_r \subset \mathbb{R}^n$ such that B_r is globally uniformly stable and globally uniformly attractive ie globally uniformly asymptotically stable.

Definition 19. B_r is globally uniformly exponentially stable if there exist $\gamma > 0$ and $k > 0$, such that for all $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$,
$$(25) \qquad \|x(t)\| \leq k\|x_0\| \exp(-\gamma(t - t_0)) + r, \quad \forall t \geq t_0.$$
System (1) is globally practically uniformly exponentially stable if there exists $r \geq 0$, such that B_r is globally uniformly exponentially stable. In particular, if $r = 0$, the system (1) is globally uniformly exponentially stable.

Proposition 4. *If there exists a class $\mathcal{K}$-function α, and a constant $r \geq 0$ such that, given any initial state x_0, the solution satisfies*
$$\|x(t)\| \leq \alpha(\|x_0\|) + r, \quad \forall t \geq t_0,$$
then the system (1) *is globally uniformly practically stable.*

Proposition 5. *If there exist a class $\mathcal{KL}$-function β, a constant $r \geq 0$ such that, given any initial state x_0, the solution satisfies*
$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + r, \quad \forall t \geq t_0,$$
then the system (1) *is globally uniformly practically asymptotically stable.*

Theorem 14. *Consider the system* (1). *Assume there exist a class* $\mathcal{C}^\infty$ *function* $V : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$, *two class* $\mathcal{K}_\infty$ *functions* α_1 *and* α_2, *a class* $\mathcal{K}$ *function* α_3 *and a positive real* r *small enough such that the following inequalities are satisfied for any* $t \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^n$,

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|),$$

$$\frac{\partial}{\partial t}V(t,x) + \frac{\partial}{\partial x}V(t,x)f(t,x) \leq -\alpha_3(\|x\|) + r.$$

Then the system (1) *is globally practically stable with*

$$B_r = \left\{ x \in \mathbb{R}^n \ : \ \|x\| \leq \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(r) \right\}.$$

Theorem 15. *Consider the system* (1). *Assume there exist a class* $\mathcal{C}^\infty$ *function* $V : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *and some positive constants* a, b, c_1, c_2 *and* c_3 *such that the following inequalities are satisfied for any* $t \in \mathbb{R}_+$ *et* $x \in \mathbb{R}^n$:

(26)
$$c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2 + a,$$

(27)
$$\frac{\partial}{\partial t}V(t,x) + \frac{\partial}{\partial x}V(t,x)f(t,x) \leq -c_3 V(t,x) + b.$$

Then the ball B_α *is globally uniformly exponential stable with* $\alpha = \sqrt{\frac{2ac_3 + bc_2}{c_1 c_3}}$.

For uniform asymptotic stability of a ball B_r on a ball B_R, with $0 \leq r < R < \infty$, in [2], a converse theorem is established when the origin is not an equilibrium but there exists a nonnegative constant f_0 such that $\|f(t,0)\| \leq f_0, \ \forall t \geq 0$.

Theorem 16 ([2]). *Consider the nonlinear system* (1) *and assume there exists a nonnegative constant* f_0 *such that* $\|f(t,0)\| \leq f_0, \ \forall t \geq 0$, *and that* f *is a class* $\mathcal{C}^\infty$ *function and that* $\left[\dfrac{\partial f}{\partial x}\right]$ *is bounded on* $\mathbb{R}^n$, *uniformly in* t.
Assume that the trajectories of the system satisfy (25) *for all* $t_0 \in \mathbb{R}_+$ *and* $x_0 \in \mathbb{R}^n$, *For some positive constants* k, γ *and* r.

Then, there is a function $V : [0, +\infty[\times \mathbb{R}^n \longrightarrow \mathbb{R}$ *that satisfies the inequalities:*

$$
\begin{aligned}
c_1\|x\|^2 \leq V(t,x) &\leq c_2\|x\|^2 + a \\
\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) &\leq -c_3\|x\|^2 + \rho \\
\left\|\frac{\partial V(t,x)}{\partial x}\right\| &\leq c_4\|x\| + b
\end{aligned}
$$

for some positive constants $c_1, c_2, c_3, c_4, a, \rho$ *and* b.

9. Gronwall's Lemma

The Gronwall type integral inequalities play a very important role in the qualitative theory of differential equations. There exist many Lemmas which carry the name of Gronwall's Lemma. A main class may be identified is the

integral inequality. The original Lemma proved by Gronwall in 1919 [22], is the following.

Lemma 3 (Gronwall). *Let $z : [a, a + h] \to \mathbb{R}$ be a continuous function that satisfies the inequality*

$$0 \leq z(x) \leq \int_a^x (A + Mz(s)) \, \mathrm{d}s$$

for all $a \leq x \leq a + h$, where $A, M \geq 0$ are constants. Then

$$0 \leq z(x) \leq Ah \exp(Mh)$$

for all $a \leq x \leq a + h$.

The above Lemma can be formulated by the following famous inequality, which is called the Gronwall inequality:

Let $u(t)$ be a continuous function defined on the interval $[t_0, t_1]$ and

$$u(t) \leq a + b \int_{t_0}^t u(s) \, \mathrm{d}s,$$

where a and b are nonnegative constants. Then, for all $t \in [t_0, t_1]$, we have

$$u(t) \leq a \exp(b(t - t_0)).$$

After more than 20 years, Bellman [6] extended the last inequality, which reads in the following:

Let a be a positive constant, $u(t)$ and $b(t)$, $t \in [t_0, t_1]$ be real-valued continuous functions, $b(t) \geq 0$, satisfying

$$u(t) \leq a + \int_{t_0}^t b(s)u(s) \, \mathrm{d}s, \quad t \in [t_0, t_1].$$

Then, for all $t \in [t_0, t_1]$, we have

$$u(t) \leq a \exp\left(\int_{t_0}^t b(s) \, \mathrm{d}s\right).$$

Next, we present some generalizations of Gronwall Lemma type.

9.1. **Some generalizations of Gronwall's inequality.** In 1919, Gronwall [22] proved a remarkable inequality which has attracted and continues to attract considerable attention in literature.

Theorem 17. *Let $u(t)$, $a(t)$ and $b(t)$ be real continuous functions defined in $[\alpha, \beta)$, such that $b(t) \geq 0$, $\forall t \in [\alpha, \beta)$. We suppose that on $[\alpha, \beta)$ we have the inequality*

$$u(t) \leq a(t) + \int_\alpha^t b(s)u(s) \, \mathrm{d}s,$$

then

$$u(t) \leq a(t) + \int_\alpha^t b(s)a(s) \exp\left(\int_s^t b(\sigma)d\sigma\right) \mathrm{d}\,s, \quad \forall t \in [\alpha, \beta).$$

In particular cases: (*a* is non-decreasing on $[\alpha, \beta)$) and (*a* is differentiable on $[\alpha, \beta)$), we obtain the following theorems.

Theorem 18. *Let* $u(t)$, $a(t)$ *and* $b(t)$ *be real continuous functions defined in* $[\alpha, \beta)$, *such that* $b(t) \geq 0$ *and* $a(t)$ *is positive and non-decreasing on* $[\alpha, \beta)$. *We suppose that on* $[\alpha, \beta)$ *we have the inequality*

$$u(t) \leq a(t) + \int_\alpha^t b(s)u(s)\,\mathrm{d}\,s,$$

then

$$u(t) \leq a(t) \exp\left(\int_\alpha^t b(s)\,\mathrm{d}\,s\right), \quad \forall t \in [\alpha, \beta).$$

Theorem 19. *Let* $u(t)$, $a(t)$ *and* $b(t)$ *be real continuous functions defined in* $[\alpha, \beta)$, *such that* $b(t) \geq 0$ *and* $a(t)$ *is differentiable on* $[\alpha, \beta)$. *We suppose that on* $[\alpha, \beta)$ *we have the inequality*

$$u(t) \leq a(t) + \int_\alpha^t b(s)u(s)\,\mathrm{d}\,s,$$

then

$$u(t) \leq a(\alpha) \exp\left(\int_\alpha^t b(s)\,\mathrm{d}\,s\right) + \int_\alpha^t \dot{a}(s) \exp\left(\int_s^t b(\sigma)\,\mathrm{d}\,\sigma\right) \mathrm{d}\,s, \quad \forall t \in [\alpha, \beta).$$

Proof. Let

$$z(t) = a(t) + \int_\alpha^t b(s)u(s)\,\mathrm{d}\,s,$$

it follows that z is differentiable and $z \geq u$. We have

$$\dot{z} = \dot{a} + bu, \quad z(\alpha) = a(\alpha).$$

Let $v = z - u$, then

$$\dot{z} = \dot{a} + bz - bv,$$

whose state transition matrix is

$$\Phi(t, \tau) = \exp\left(\int_\tau^t b(s)\right) \mathrm{d}\,s.$$

Therefore,

$$z(t) = \Phi(t, \alpha)z(\alpha) + \int_\alpha^t \Phi(t, \tau)\left[\dot{a}(\tau) - b(\tau)v(\tau)\right] \mathrm{d}\,\tau.$$

Because

$$\int_\alpha^t \Phi(t, \tau)b(\tau)v(\tau)d\tau \geq 0,$$

resulting from $\Phi(t,\tau)$, $b(\tau)$, $v(\tau)$ being nonnegative, we have

$$z(t) \leq \Phi(t,\alpha)z(\alpha) + \int_\alpha^t \Phi(t,\tau)\dot{a}(\tau)\,\mathrm{d}\,\tau.$$

Using the expression for $\Phi(t,\alpha)$ in the above inequality, we have

$$u(t) \leq z(t) \leq a(\alpha)\exp\Big(\int_\alpha^t b(s)\,\mathrm{d}\,s\Big)$$
$$+ \int_\alpha^t \dot{a}(s)\exp\Big(\int_s^t b(\sigma)\,\mathrm{d}\,\sigma\Big)\,\mathrm{d}\,s, \quad \forall t \in [\alpha,\beta),$$

and the proof is complete. $\qquad\square$

There are various generalizations of Gronwall's inequality involving an unknown function of a single variable.

Theorem 20. *Let $u(t)$, $f(t)$, $g(t)$ and $h(t)$ be nonnegative continuous functions defined on $[\alpha,\beta]$, and*

$$(28) \qquad u(t) \leq f(t) + g(t)\int_\alpha^t h(s)u(s)\,\mathrm{d}\,s, \quad \forall t \in [\alpha,\beta].$$

Then

$$(29) \quad u(t) \leq f(t) + g(t)\int_\alpha^t h(s)f(s)\exp\Big(\int_s^t h(\tau)g(\tau)d\tau\Big)\,\mathrm{d}\,s, \quad \forall t \in [\alpha,\beta].$$

Proof. Define a function $z(.)$ by

$$z(t) = \int_\alpha^t h(s)u(s)\,\mathrm{d}\,s,$$

then $z(\alpha) = 0$, $u(t) \leq f(t) + g(t)z(t)$ and

$$(30) \qquad z'(t) = h(t)u(t) \leq h(t)f(t) + h(t)g(t)z(t).$$

Multiplying (30) by the integrating factor $\exp\big(-\int_\alpha^t h(\tau)g(\tau)\,\mathrm{d}\,\tau\big)$, we have

$$(31) \qquad \frac{\mathrm{d}}{\mathrm{d}\,t}\Big[z(t)\exp\Big(-\int_\alpha^t h(\tau)g(\tau)\,\mathrm{d}\,\tau\Big)\Big]$$
$$\leq h(t)f(t)\exp\Big(-\int_\alpha^t h(\tau)g(\tau)\,\mathrm{d}\,\tau\Big).$$

By setting $t = s$ in (31) and integrating it with respect to s from α to t, we get

$$(32) \qquad z(t)\exp\Big(-\int_\alpha^t h(\tau)g(\tau)\,\mathrm{d}\,\tau\Big)$$
$$\leq \int_\alpha^t h(s)f(s)\exp\Big(-\int_\alpha^s h(\tau)g(\tau)\,\mathrm{d}\,\tau\Big)\,\mathrm{d}\,s.$$

Using the bound on $z(t)$ from (32) in (28), we obtain the required inequality in (29). The proof of the Theorem is complete. $\qquad\square$

A fairly general version of Gronwall Theorem is given in the following theorem.

Theorem 21. *Let $a(t)$, $b(t)$, $c(t)$ and $u(t)$ be continuous functions in $[\alpha, \beta]$, let $b(t)$ and $c(t)$ be nonnegative in $[\alpha, \beta]$, and suppose*

$$(33) \qquad u(t) \leq a(t) + \int_\alpha^t \Big[b(s)u(s) + c(s) \Big] \, \mathrm{d}\, s, \quad t \in [\alpha, \beta].$$

Then

$$(34) \qquad u(t) \leq \Big[\sup_{s \in [\alpha,t]} a(s) + \int_\alpha^t c(s) \, \mathrm{d}\, s \Big] \exp\Big(\int_\alpha^t b(s) \, \mathrm{d}\, s \Big), \quad t \in [\alpha, \beta].$$

The following theorem gives the best possible estimate for a function $u(t)$ satisfying (33).

Theorem 22 (Chandirov, 1970). *Let $a(t)$, $b(t)$, $c(t)$ and $u(t)$ be continuous functions in $[\alpha, \beta]$, let $b(t)$ be nonnegative in $[\alpha, \beta]$, and suppose*

$$(35) \qquad u(t) \leq a(t) + \int_\alpha^t \Big[b(s)u(s) + c(s) \Big] \, \mathrm{d}\, s, \quad t \in [\alpha, \beta].$$

Then

$$(36) \quad u(t) \leq a(t) + \int_\alpha^t \Big[a(s)b(s) + c(s) \Big] \exp\Big(\int_s^t b(r) \, \mathrm{d}\, r \Big) \, \mathrm{d}\, s, \quad t \in [\alpha, \beta].$$

In particular, if $a(t) = a$ is a constant, then

$$(37) \qquad \begin{aligned} u(t) \leq {} & a \exp\Big(\int_\alpha^t b(s) \, \mathrm{d}\, s \Big) \\ & + \int_\alpha^t c(s) \exp\Big(\int_s^t b(r) \, \mathrm{d}\, r \Big) \, \mathrm{d}\, s, \quad t \in [\alpha, \beta]. \end{aligned}$$

We have also the following generalization of Gronwall Theorem.

Theorem 23. *Let $u(t)$, $p(t)$, $q(t)$, $f(t)$ and $g(t)$ be nonnegative continuous functions defined on $[\alpha, \beta]$, and*

$$(38) \qquad u(t) \leq p(t) + q(t) \int_\alpha^t \Big[f(s)u(s) + g(s) \Big] \, \mathrm{d}\, s, \quad \forall t \in [\alpha, \beta].$$

Then

$$(39) \qquad \begin{aligned} u(t) \leq {} & p(t) + q(t) \int_\alpha^t \Big[f(s)p(s) + g(s) \Big] \cdot \\ & \cdot \exp\Big(\int_s^t f(\tau)q(\tau) \, \mathrm{d}\, \tau \Big) \, \mathrm{d}\, s, \quad \forall t \in [\alpha, \beta]. \end{aligned}$$

We have also the following result.

Theorem 24 ([17])**.** *Let $A, B, C : [\alpha, \beta] \longrightarrow \mathbb{R}_+$; $L, M : [\alpha, \beta] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be continuous functions and*

$$(40) \qquad 0 \leq L(t, u) - L(t, v) \leq M(t, v)(u - v), \quad t \in [\alpha, \beta], 0 \leq v \leq u.$$

Then for every nonnegative continuous function $x : [\alpha, \beta] \to [0, +\infty)$ satisfying the inequality

$$(41) \qquad x(t) \leq A(t) + B(t) \int_\alpha^t C(s) L(s, x(s)) \, \mathrm{d}\, s, \quad \forall t \in [\alpha, \beta],$$

we have the estimation

$$(42) \qquad
\begin{aligned}
x(t) \leq A(t) + B(t) \int_\alpha^t C(u) L(u, A(u)) \cdot \\
\cdot \exp\left(\int_u^t M(s, A(s)) B(s) C(s) \, \mathrm{d}\, s \right) \mathrm{d}\, u.
\end{aligned}$$

Proof. Let us consider the function

$$y(t) = \int_\alpha^t C(s) L(s, x(s)) \, \mathrm{d}\, s, \quad \forall t \in [\alpha, \beta].$$

Then y is differentiable on $[\alpha, \beta]$, $y'(t) = C(t) L(t, x(t))$ and $y(\alpha) = 0$.

By the relation (40) and (41), it follows that for any $t \in [\alpha, \beta]$:

$$(43) \qquad
\begin{aligned}
y'(t) &\leq C(t) L(t, A(t) + B(t) y(t)) \\
&\leq C(t) L(t, A(t)) + M(t, A(t)) B(t) C(t) y(t).
\end{aligned}$$

Putting $s(t) := y(t) \exp(-\int_\alpha^t M(s, A(s)) B(s) C(s) \, \mathrm{d}\, s)$, then from (43), we obtain the following integral inequality

$$s'(t) \leq C(t) L(t, A(t)) \exp\left(- \int_\alpha^t M(s, A(s)) B(s) C(s) \, \mathrm{d}\, s \right), \quad \forall t \in [\alpha, \beta].$$

By integration between α and t, we get

$$
\begin{aligned}
s(t) \leq \int_\alpha^t C(u) L(u, A(u)) \cdot \\
\cdot \exp\left(- \int_\alpha^u M(s, A(s)) B(s) C(s) \, \mathrm{d}\, s \right) \mathrm{d}\, u, \quad \forall t \in [\alpha, \beta],
\end{aligned}$$

which implies that

$$
\begin{aligned}
y(t) \leq \int_\alpha^t C(u) L(u, A(u)) \cdot \\
\cdot \exp\left(\int_u^t M(s, A(s)) B(s) C(s) \, \mathrm{d}\, s \right) \mathrm{d}\, u, \quad \forall t \in [\alpha, \beta],
\end{aligned}$$

from where results the estimation (42). $\qquad \square$

There are also wide nonlinear generalizations of Gronwall's inequality, we can see several variants in particular in [5] and [17].

9.2. **Some applications.** Gronwall lemma has found wide applications in ordinary differential equations. First, it is used to obtain estimation results for the solution of differential equation and to get sufficient conditions of boundedness, uniqueness and differentiability for the solution of these equations. Therefore, the first use of the Gronwall's inequality to establish boundedness and stability is due to R.Bellman. Second, it is used to give some results of uniform stability, uniform asymptotic stability, global exponential stability and global asymptotic stability.

Let us consider the system of differential equation

$$(44) \qquad \dot{x}(t) = f(t, x(t)), \ t \in [0, +\infty)$$

where f is defined on $[0, +\infty) \times \mathbb{R}^n$, piecewise-continuous in t, locally Lipshitz in x, $f(t, 0) \equiv 0$ and f satisfies the following condition:

$$(45) \qquad \|f(t, x)\| \leq L(t, \|x\|), \ (t, x) \in [0, +\infty) \times \mathbb{R}^n.$$

The main purpose of this subsection is to give some theorems of uniform stability for the trivial solution of the above equation.

Theorem 25 (Theorem of Uniform stability). *If L and M satisfies the relation* (40)*, $L(t, 0) \equiv 0$ for all $t \in [0, +\infty)$ and the following condition*

$$\exists \delta_0 > 0, \ \sup_{0 \leq \delta \leq \delta_0} \int_0^\infty M(s, \delta) \, \mathrm{d}s < +\infty$$

holds, then the trivial solution $x \equiv 0$ of (44) *is uniformly stable.*

Now, let us consider the non-homogenous system

$$(46) \qquad \dot{x}(t) = A(t)x + f(t, x(t)), \ t \in [0, +\infty)$$

and the linear system

$$(47) \qquad \dot{x}(t) = A(t)x, \ t \in [0, +\infty),$$

where $A : [0, +\infty) \to \mathbb{R}^{n^2}$, $f : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ are piecewise continuous in t, locally Lipshitz in x and $f(t, 0) \equiv 0$ for all $t \in [0, +\infty)$.

We denote $R(t, t_0)$ the transition matrix of solution of the linear system (47).

Theorem 26 (Theorem of Uniform asymptotic stability). *If the trivial solution of* (47) *is uniformly asymptotically stable, the function f satisfies the relation* (45)*, $L(t, 0) \equiv 0$ for all $t \in [0, +\infty)$ and there exists $\delta_0 > 0$ such that*

$$\sup_{0 \leq \delta \leq \delta_0} \int_0^\infty M(s, \delta) \, \mathrm{d}s < +\infty,$$

then the trivial solution $x \equiv 0$ of the system (46) *is uniformly asymptotically stable.*

Proof. Let $x(., t_0, x_0)$ be the solution of (46), such that $x(t_0) = x_0$. Then $x(., t_0, x_0)$ verifies the integral equation

$$x(t, t_0, x_0) = R(t, t_0)x_0 + R(t, t_0) \int_{t_0}^{t} R(t_0, s)f(s, x(s, t_0, x_0))\, \mathrm{d}\, s$$

for all $t \geq t_0$.

Passing at norms, we obtain

$$\|x(t, t_0, x_0)\| \leq \|R(t, t_0)\|\|x_0\| + \|R(t, t_0)\| \int_{t_0}^{t} \|R(t_0, s)\|L(s, \|x(s, t_0, x_0)\|)\, \mathrm{d}\, s,$$

for all $t \geq t_0$.

Since the trivial solution of the linear system (47) is uniformly asymptotically stable, then there exists $\beta > 0, m > 0$ such that

$$\|R(t, t_0)\| \leq \beta \exp(-m(t - t_0)), t \geq t_0.$$

It follows that

$$\begin{aligned}
\|x(t, t_0, x_0)\| \quad \leq \quad & \beta \exp(-m(t - t_0))\|x_0\| \\
& + \beta \exp(-m(t - t_0)) \int_{t_0}^{t} \beta e^{-m(t_0 - s)}L(s, \|x(s, t_0, x_0)\|)\, \mathrm{d}\, s,
\end{aligned}$$

for all $t \geq t_0$.

Applying Theorem 24, we obtain

$$\begin{aligned}
\|x(t, t_0, x_0)\| \quad \leq \quad & \beta \exp(-m(t - t_0))\|x_0\| + \beta \exp(-m(t - t_0)) \\
& \times \int_{t_0}^{t} \Big[\beta \exp(-m(t_0 - s))L(s, \beta e^{-m(s - t_0)}\|x_0\|) \\
& \times \exp \Big(\int_{s}^{t} \beta^2 M(u, \beta \exp(-m(u - t_0))\|x_0\|)\, \mathrm{d}\, u \Big) \Big]\, \mathrm{d}\, s,
\end{aligned}$$

for all $t \geq t_0$.

By the condition (40), we have

$$\begin{aligned}
& \beta \exp(-m(t_0 - s))L(s, \beta e^{-m(s - t_0)}\|x_0\|) \\
\leq \quad & \beta \exp(-m(t_0 - s))M(s, 0)\beta \exp(-m(s - t_0))\|x_0\| \\
= \quad & \beta^2 M(s, 0)\|x_0\|, \quad \forall s \geq t_0.
\end{aligned}$$

We obtain

$$\begin{aligned}
\|x(t, t_0, x_0)\| \quad \leq \quad & \beta \exp(-m(t - t_0))\|x_0\| + \beta^3 \exp(-m(t - t_0))\|x_0\| \cdot \\
& \cdot \int_{0}^{t} \Big[M(u, 0) \times \exp \Big(\beta^2 \int_{0}^{t} M(s, \beta \exp(-m(s - t))\|x_0\|)\, \mathrm{d}\, s \Big) \Big]\, \mathrm{d}\, u.
\end{aligned}$$

If $\|x_0\| < \frac{\delta_0}{\beta}$, we have for all $t \geq t_0$

$$\|x(t, t_0, x_0)\| \leq \Big(\beta + \beta^3 \tilde{M} \exp \beta^2 \tilde{M} \Big) \exp(-m(t - t_0))\|x_0\|,$$

where $\tilde{M} = \sup_{0 \leq \delta \leq \delta_0} \int_0^\infty M(s, \delta)\, \mathrm{d}\, s$, which means that the trivial solution of (46) is uniformly asymptotically stable. $\qquad\square$

Another result is embodied in the following Theorem.

Theorem 27 (Theorem of Uniform exponential stability)**.** *If the trivial solution of* (47) *is uniformly asymptotically stable, f satisfies the relation* (45), $L(t, 0) \equiv 0$ *for all $t \geq 0$ and*

$$\sup_{\delta \geq 0} \int_0^\infty M(s, \delta)\, \mathrm{d}\, s < +\infty,$$

then the trivial solution $x \equiv 0$ of the system (46) *is globally exponentially stable.*

Proof. Let $x(., t_0, x_0)$ be the solution of (46), such that $x(t_0) = x_0$.
By similar computation we have the estimate for all $x_0 \in \mathbb{R}^n$ and $t \geq t_0$

$$
\begin{aligned}
\|x(t, t_0, x_0)\| &\leq \beta \exp(-m(t - t_0))\|x_0\| + \beta^3 \exp(-m(t - t_0))\|x_0\| \cdot \\
&\quad \cdot \int_0^t \left[M(u, 0) \exp\left(\int_0^t \beta^2 M(s, \beta e^{-m(s-t_0)}\|x_0\|)\, \mathrm{d}\, s \right) \right] \mathrm{d}\, u \\
&\leq \beta \exp(-m(t - t_0))\|x_0\| + \beta^3 \exp(-m(t - t_0))\|x_0\| \tilde{M} \exp(\beta^2 \tilde{M}) \\
&= \left(\beta + \beta^3 \tilde{M} \exp(\beta^2 \tilde{M}) \right) \exp(-m(t - t_0))\|x_0\|,
\end{aligned}
$$

where $\tilde{M} = \displaystyle\sup_{0 \leq \delta \leq \delta_0} \int_0^\infty M(s, \delta)\, \mathrm{d}\, s$. $\qquad\square$

10. On the global uniform asymptotic stability of time-varying dynamical systems

We present in this section some results concerning the global uniform asymptotic stability of a class of time-varying perturbed systems, see [24].

10.1. Problem statement. Let's consider the following system

$$(48) \qquad \qquad \dot{x} = f(t, x) + g(t, x),$$

where $f, g : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ are piecewise continuous in t, locally Lipschitz in x, such that $f(t, 0) = g(t, 0) = 0$ for all $t \geq 0$. Suppose that the origin of the nominal system

$$(49) \qquad \qquad \dot{x} = f(t, x),$$

is globally uniformly asymptotically stable with V as an associate Lyapunov function, then, if we calculate its derivative along the trajectories of (48), one can conclude the negative definiteness of $\dot{V}$ by imposing some restrictions on g ([2, 13, 16, 23, 27, 29, 35, 36, 47]). However, we cannot usually conclude the behavior of the solutions of the perturbed system (48) by using $V(t, x)$ as a Lyapunov function candidate. This fact can be viewed from the following example.

Example 8.

(50)
$$\dot{x} = -a(t)x + \delta(t)\frac{x|x|}{1 + \sqrt{|x|}},$$

where $a(.)$ is a bounded continuous function and $\delta(.)$ is a positive continuous unbounded integrable function. The nominal system $\dot{x} = -a(t)x$ is globally uniformly asymptotically stable with a Lyapunov function $V(t,x) = x^2$. Nevertheless, if we use $V(t,x)$ as a Lyapunov function for the perturbed system (50), we cannot conclude the behavior of its solutions. Indeed, the derivative of $V(t,x)$ along the trajectories of (50) is given by

$$\dot{V}(t,x) = 2\big(\delta(t)\frac{|x|}{1 + \sqrt{|x|}} - a(t)\big)x^2 \geq 2\big(\delta(t) - a(t)\big)x^2,$$

for all $x \in S = \{x \in \mathbb{R}/|x| \geq 1 + \sqrt{|x|}\}$. Since, $\delta(.) - a(.)$ is a continuous unbounded function, then, there exists a bounded interval I, such that

$$\delta(t) - a(t) \geq 1, \quad \text{for all } t \in I.$$

It follows that $\dot{V}(t,x) \geq 2x^2$, for all $t \in I$ and $x \in S$, although system (50) is globally uniformly asymptotically stable.

Our goal is to study the global uniform asymptotic stability of a class of time-varying perturbed system under the following assumption:

$\mathcal{A}$: There exist a class $\mathcal{K}_\infty$ function α and a positive integrable function δ such that
$$\|g(t,x)\| \leq \delta(t)\alpha(\|x\|).$$
Thus, we have addressed the problem in two different ways. The first one is to consider the time-varying cascaded system of the form

(51)
$$\begin{cases} \dot{x}_1 = f_1(t,x_1) + g(t,x)x_2, \\ \dot{x}_2 = f_2(t,x_2), \end{cases}$$

which can be regarded as time-varying perturbed system. The second one is to construct a new Lyapunov function for the perturbed system (48).

10.2. **Cascaded systems.** For instance, in [46], the authors established sufficient conditions for the global uniform asymptotic stability of (51) based on a similar linear growth condition as in [34] and an integrability assumption on the input x_2, while in [47] they assume that the interconnection term $g(t,x)$ satisfies the following condition

$$\|g(t,x)\| \leq \alpha(\|x\|)$$

and prove that the integrability of the solutions of

(52)
$$\dot{x}_2 = f_2(t,x)$$

is sufficient to obtain the global uniform asymptotic stability of (51). Here, the integrability condition is imposed on the function δ that bounds $g(t,x)$

and is not imposed on the state of the system. Thus, we have assumed the following hypothesis:

(H_1) There exist an integrable continuous function $\delta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and class $\mathcal{K}_\infty$ functions γ and θ, such that for all $t \geq 0$ and $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$ we have

$$\|g(t,x)\| \leq \delta(t)\theta(x_2)\gamma(x_1).$$

(H_2) There exist a continuous differentiable function $V(t,x_1)$, class $\mathcal{K}_\infty$ functions γ_i, $i = 1, 2, 3$, and a constant $\lambda > 0$, such that for $t \geq 0$ and $x_1 \in \mathbb{R}^n$ we have

$$\gamma_1(\|x_1\|) \leq V(t,x_1) \leq \gamma_2(\|x_1\|),$$

$$\frac{\partial V}{\partial t}(t,x_1) + \frac{\partial V}{\partial x_1}f(t,x_1) \leq -\lambda V(t,x_1).$$

$$\left\| \frac{\partial V}{\partial x_1}f(t,x_1) \right\| \leq \gamma_3(\|x_1\|).$$

(H_3) There exists a $\mathcal{KL}$-function β such that for each initial condition $x_2 \in \mathbb{R}^m$, the solutions of (52) satisfy

$$\|\phi_2(s,t,x_2)\| \leq \beta(\|x_2\|, s - t), \quad \text{for all} \ \ s \geq t.$$

Therefore, we have obtained the following Lemma.

Lemma 4. *If assumptions (H_1), (H_2) and (H_3) are satisfied and the solutions of (51) are globally uniformly bounded then (51) is globally uniformly asymptotically stable.*

Consequently, the question which can be arises here: what condition that ensures the uniform bounded of the solutions of (51)? Thus, we have obtained the following theorem.

Theorem 28. *If assumptions (H_1), (H_2) and (H_3) are satisfied such that there exist a constant $c > 0$ such that*

$$\int_c^{+\infty} \frac{\mathrm{d}\,s}{\gamma_1(\gamma_1^{-1}(s))\gamma(\gamma_1^{-1}((s))} = \infty,$$

then, (51) is globally uniformly asymptotically stable.

Thus, we have extended the result given in [48] for the autonomous case to a class of time-varying perturbed systems which can be unbounded in time.

10.3. **Construction of a strict Lyapunov function.** We have used almost the same idea as in ([1, 34]) to construct a new Lyapunov function for (48) that ensures the global uniform asymptotic stability of the equilibrium point, without the bounded hypothesis with respect to time. We have then considered the following function:

(53)
$$W(t,x) = \begin{cases} V(t,x)\exp(\varphi(t,x), & \text{if } x \neq 0, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Our goal is to seek a suitable function φ which can compensate the perturbation term. Therefore, if we consider the derivative of W along the trajectories of the system (48), we get

$$\dot{W}(t,x) = \dot{V}(t,x)\exp\Big(\varphi(t,x)\Big) + \dot{\varphi}(t,x)V(t,x)\exp\Big(\varphi(t,x)\Big)$$
$$= \Big[\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x)\Big]\exp\Big(\varphi(t,x)\Big)$$
$$+ \frac{\partial V}{\partial x}(t,x)g(t,x)\exp\Big(\varphi(t,x)\Big).$$

The first term of the right-hand side constitutes the derivative of W along the trajectories of the nominal system. The second term is the effect of the perturbation, while the third term is the derivative of $\exp(\varphi)$ multiplied by V. In order to guarantee that $\dot{W}$ is a negative definite function, we shall choose

$$\varphi(t,x) = \int_t^{+\infty} \frac{1}{V(s,\phi(s,t,x))} \frac{\partial V}{\partial x}(s,\phi(s,t,x))g(s,\phi(s,t,x))\,\mathrm{d}\,s.$$

This implies with this choice that

$$\dot{W}(t,x) = \Big[\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x)\Big]\exp\Big(\varphi(t,x)\Big).$$

This equality has used for the analysis of (48). To this end, we have imposed some conditions in view to prove that the new Lyapunov function is continuous positive definite radially unbounded and decreasing along the trajectories of solutions of (48), for more details see [24].

11. Lyapunov function with indefinite derivative

We present in this section some results on the indefinite derivative of Lyapunov function, see [25].

11.1. **Problem statement.** Lyapunov direct approach, which is also known as the Lyapunov second approach, is a powerful tool for stability analysis and design of control systems [35]. By this method, if a positive definite function of the state can be found such that its time-derivative along the trajectories of the considered system is negative definite, it is claimed that

the system is stable. Moreover, by imposing different positive definiteness assumptions and different negative definiteness assumptions on the Lyapunov function and its time-derivative, respectively, different stability properties of the considered system can be deduced. Generally, for time-invariant systems, the negative definiteness of the time-derivative of the Lyapunov function can be relaxed as negative semi-definiteness, for which the so-called Lasalle invariant principle can be utilized. For time-varying systems, expect for some special cases, the Lasalle invariant principle is either not valid or difficult to use [49]. Hence, some researchers attempt to use the available Lyapunov function, whose time-derivative is not strictly negative definite, to construct a new Lyapunov function whose time-derivative is negative definite ([1,3,4,34,39]). One could hope that a method for proving the existence of a Lyapunov function might carry with it a constructive method for obtaining this function. This hope has not been realized. Therefore, constructing a Lyapunov function is a very hard problem.

We have showed in this work that a new Lyapunov function can be designed for the system (51) where its time derivative is neither required to be negative definite nor required to be negative semi-definite [25].

11.2. **Converse theorem.** Let's start by introducing the concept of stable functions proposed in [55]. Consider the following scalar linear time-varying system

$$\dot{y}(t) = \mu(t)y(t), \tag{54}$$

where $t \in \mathbb{R}_+$ and $\mu \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$. It is not hard to see that the state transition matrix for system (54) is given by

$$\phi(s,t) = \exp\left(\int_t^s \mu(\tau)\,\mathrm{d}\,\tau\right), \quad \text{for all } s \geq t.$$

Definition 20. The function $\mu \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ is said to be:
- Globally asymptotically stable if system (54) is globally asymptotically stable.
- Globally uniformly exponentially stable if system (54) is globally uniformly exponentially stable.

Lemma 5. *The function* $\mu \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ *is:*
- *Globally asymptotically stable if and only if*

$$\int_t^s \mu(\tau)\,\mathrm{d}\,\tau = -\infty.$$

- *Globally uniformly exponentially if and only if*

$$\int_t^s \mu(\tau)\,\mathrm{d}\,\tau \leq -\alpha(s-t) + \beta.$$

Now, let's consider the following time-varying system

(55)
$$\dot{x}(t) = F(t, x)$$

with initial condition $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, is denoted by $\phi(., t, x)$, such that $\phi(t, t, x) = x$.

The next theorem spells out conditions under which system (55) is globally asymptotically stable.

Theorem 29. *If there exist $V : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}_+$ a continuously differentiable function, two $\mathcal{NK}_\infty$ functions $\alpha_i, i = 1; 2$, and a scalar function $\mu \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$, such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$,*

$$k_1(t, \|x\|) \leq V(t, x) \leq k_2(t, \|x\|),$$

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) F(t, x) \leq \mu(t) V(t, x),$$

then, system (55) is globally asymptotically stable if μ is globally asymptotically stable.

Note that, if there exist $m > 0$, $k_i > 0$, $i = 1; 2$ such that $k_i(t, s) = k_i s^m$, $i = 1; 2$, then (55) is globally uniformly exponentially stable if μ is globally uniformly exponentially stable.

This result deserves the following question: if system (55) is globally uniformly exponentially stable, is there a function V which satisfies the hypothesis of the above Theorem 29?

Therefore we have established an extension of a well known converse stability result concerning exponential stability for systems (55) with indefinite derivative and under the lack of the bounded hypothesis for their dynamics F with respect to time. In order to give the precise assumption imposed for F, I have introduced the following subclass of $\mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$.

Definition 21. We say that a function $L \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ is of class $\mathcal{BC}$, if there exist a function $M \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ and a function $\delta \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\int_t^{t+\delta(t)} L(s)\, \mathrm{d}\, s \leq M(\delta(t)), \quad \text{for all } t \geq 0, \ \delta \geq 0.$$

Remark 7. Let $L \in L^P([0, +\infty[)$, $p \geq 1$, then by applying holder's inequality, we have $M(\delta(t)) = \|L\|_p \delta(t)^{\frac{p-1}{p}}$.

Consequently, we have supposed the following assumptions.

(H_1') There exists a function $L(.) \in \mathcal{BC}$ such that

$$\|F(t, x)\| \leq L(t)\|x\|,$$

F is continuously differentiable, and

$$\left\|\frac{\partial F}{\partial x}(t, x)\right\| \leq L(t).$$

(H_2') There exist $\gamma > 0$ and $k \geq 0$ such that for all $x \in \mathbb{R}^n$, the solution of (55) satisfies

$$\|\phi(s,t,x)\| \leq k\|x\| \exp(-\gamma(s-t)) \;\; \text{for all } s \geq t \geq 0.$$

Then, we have stated the following theorem.

Theorem 30. *Under assumptions (H_1') and (H_2'), with $M(\delta(t)) < \lambda\delta(t)^r$, where $0 < \lambda < \gamma$ and $r \in]0,1[$, there exist $\mathcal{C}^1$ function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, bounded functions $k_i(.)$, $i = 1,2,3$, $m > 1$ and scalar function $\mu \in \mathcal{PC}(\mathbb{R}_+,\mathbb{R})$ satisfying the following inequalities for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$:*

$$k_1(t)\|x\|^m \leq V(t,x) \leq k_2(t)\|x\|^m,$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)F(t,x) \leq \mu(t)V(t,x),$$

$$\left\|\frac{\partial V}{\partial t}(t,x)\right\| \leq k(t)\|x\|^{m-1},$$

such that μ is globally uniformly practically exponentially stable.

Proof. The considered function has the following form

$$V(t,x) = \int_t^{t+\delta(t)} \|\phi(s,t,x)\|^m \, ds,$$

where $\delta : t \in \mathbb{R}_+ \longrightarrow e^t$, which satisfies $\delta(t) > t$, for all $t \geq 0$. $\qquad\square$

11.3. **Indefinite Lyapunov function.** Using the same idea given in the previously section I have considered the following assumptions.

(H_1'') There exist $V_1 : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}_+$ a continuously differentiable function, positive constants a_i, $i = 1;2$, $m_1 > 0$ and a scalar functions $\mu_1 \in \mathcal{PC}(\mathbb{R}_+,\mathbb{R})$, such that, for all $t \geq 0$ and $x_1 \in \mathbb{R}^n$,

$$a_1\|x_1\|^{m_1} \leq V_1(t,x_1) \leq a_2\|x_1\|^{m_1},$$

$$\frac{\partial V_1}{\partial t}(t,x_1) + \frac{\partial V_1}{\partial x_1}(t,x_1)f_1(t,x_1) \leq \mu_1(t)V_1(t,x_1),$$

(H_2'') There exist $V_2 : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}_+$ a continuously differentiable function, positive constants b_i, $i = 1;2$, $m_2 > 0$ and a scalar functions $\mu_2 \in \mathcal{PC}(\mathbb{R}_+,\mathbb{R})$, such that, for all $t \geq 0$ and $x_2 \in \mathbb{R}^m$,

$$b_1\|x_2\|^{m_2} \leq V_2(t,x_2) \leq b_2\|x_2\|^{m_2},$$

$$\frac{\partial V_2}{\partial t}(t,x_2) + \frac{\partial V_2}{\partial x_2}(t,x_2)f_2(t,x_2) \leq \mu_2(t)V_2(t,x_2),$$

(H_3'') There exist a function $S \in \mathcal{BC}$ with $M(\delta(t)) \in L^1[0,+\infty[$ such that

$$\left\|\frac{\partial V_1}{\partial t}(t,x_1)g(t,x)\right\| \leq \delta(t)S(t)V(t,x_1), \quad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m.$$

Remark 8. One can see that under assumptions (H_1'') and (H_3'') with $\delta S \in L^1([0; +\infty[)$ the perturbed system (48) is globally uniformly exponentially stable. Indeed, the derivative of V along the trajectories of system (48) is given by

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) + \frac{\partial V}{\partial x}(t, x) \leq \big(\mu(t) + \delta(t)S(t)\big)V(t, x),$$

with

$$\int_t^s \big(\mu(\tau) + \delta(\tau)S(\tau)\big)\mathrm{d}\tau \leq -\alpha(s - t) + \beta + \|\delta.S\|_p, \quad \text{for all } s \geq t \geq 0.$$

Theorem 31. *Under assumptions (H_1''), (H_2''), and (H_3''), the cascaded system (51) is globally asymptotically stable if the scalar functions μ_1 and μ_2 are globally uniformly exponentially stable.*

Proof. The considered Lyapunov function with indefinite derivative has the following form

$$W(t, x) = \begin{cases} V(t, x_1)\exp(\varphi(t, x_1)) + V_2(t, x_2), & \text{if } x_1 \neq 0, \\ \\ 0, & \text{otherwise,} \end{cases}$$

where,

$$\begin{aligned}
\varphi(t, x_1) = & \int_t^{t+\delta(t)} \int_t^u \frac{1}{\delta(s)V_1(s, \phi_1(s, t, x_1))} \frac{\partial V_1}{\partial x_1}(s, \phi_1(s, t, x_1)) \\
& \times g(s, \phi(s, t, x))\phi_2(s, t, x_2)\,\mathrm{d}s\,\mathrm{d}u \\
& + \int_{t+\delta(t)}^{+\infty} \int_u^{u+\delta(u)} \frac{1}{\delta(s)V_1(s, \phi_1(s, t, x_1))} \\
& \times \frac{\partial V_1}{\partial x_1}(s, \phi_1(s, t, x_1))g(s, \phi(s, t, x))\phi_2(s, t, x_2)\,\mathrm{d}s\,\mathrm{d}u. \quad \square
\end{aligned}$$

12. Practical stability of time-varying continuous systems

We present in this section the practical stability analysis of some classes of time-varying systems, see [26].

12.1. **Problem statement.** Let's consider the following scalar linear time-varying system

$$(56) \qquad\qquad \dot{x} = A(t)x + \pi(t),$$

where $t \in \mathbb{R}_+$, $A(t) = t\cos(t^2) - \alpha$, with $\alpha > 0$, and the scalar function π is continuously differentiable. With the help of the notion of stable functions, the author in [55] showed that system (56) is uniformly exponentially stable in the case when $\pi(t) = 0$, for all $t \geq 0$. However, one can see that if $\pi(t) \neq 0$, the origin of the system (56) is not an equilibrium point and then we cannot conclude the behavior of its solutions using the results of Lyapunov stability

given in ([55, 56]). The challenge is then, can we extend the notion of the stable functions to the practical stability case?

12.2. **Practical asymptotic and exponential stability.**

Definition 22. Let $\mu, \pi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$. The function μ is $\pi-$globally uniformly practically exponentially stable if there exist $\theta > 0$, $\lambda \geq 0$ and $\rho > 0$, such that, for all $s \geq t$,

$$\int_t^s \mu(\tau)\,\mathrm{d}\tau \leq -\theta(s-t) + \lambda,$$

and

$$\int_t^s |\pi(\tau)|\psi(s,\tau)\,\mathrm{d}\tau \leq \rho,$$

where $\psi(s,t) = \exp\Big(\int_t^s \mu(\tau)\,\mathrm{d}\tau\Big)$.

Remark 9. Let $\mu \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ and $\pi \in L^p[0, +\infty[$, $p > 1$. If there exist $\theta > 0$, and $\lambda \geq 0$ such that, for all $s \geq t$,

$$\int_t^s \mu(\tau)\,\mathrm{d}\tau \leq -\theta(s-t) + \lambda,$$

then, using holder inequality, we have for all $s \geq t$,

$$\int_t^s |\pi(\tau)|\psi(s,\tau)\,\mathrm{d}\tau \leq \frac{e^\lambda \|\pi\|_p}{(\theta q)^{\frac{1}{q}}},$$

where $q = \frac{p}{p-1}$.

Remark 10. Let $\mu, \pi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$. One can see that if the function μ is $\pi-$globally uniformly practically exponentially stable then the system

$$\dot{y}(t) = \mu(t)y(t) + \pi(t)$$

is globally uniformly practically exponentially stable.

The next theorems spells out conditions under which system (55) is globally uniformly practically asymptotically and exponentially stable. Notice that differently from the existing results ([2, 21]), the function μ is not required to be positive for all t and the the function π is not be a constant.

12.2.1. *Practical asymptotic stability.* For the asymptotic case, we have obtained the following theorem.

Theorem 32. *Assume that there exist $V : [0, +\infty[\times\mathbb{R}^n \to \mathbb{R}_+$ a continuously differentiable function, two $\mathcal{K}_\infty$ functions $\alpha_i, i = 1; 2$, a constant $a \geq 0$ and scalar functions $\mu, \pi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$, such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$,*

(57) $$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|) + a,$$

(58) $$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)F(t,x) \leq \mu(t)V(t,x) + \pi(t),$$

then, system (55) is globally uniformly practically asymptotically stable if μ is $\pi-$globally uniformly practically exponentially stable.

12.2.2. *Practical exponential stability.* For the exponential case, we have obtained the following theorem.

Theorem 33. *Assume that there exist $V : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}_+$ a continuously differentiable function, constants $a \geq 0$, $b \geq 0$, $\sigma > 0$, $m \geq 1$ and scalar functions $\mu, \pi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$, such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$,*

$$(59) \qquad c_1(t)\|x\|^m \leq V(t,x) \leq c_2(t)\|x\|^m + a(t),$$

$$(60) \qquad \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)F(t,x) \leq \mu(t)V(t,x) + \pi(t),$$

where $c_1(.) > 0$, $c_2(.) \geq 0$, and $a(.) \geq 0$ are bounded functions with $\frac{c_2(.)}{c_1(.)} \leq b$ and $\frac{a(.)}{c_1(.)} \leq a$, then, system (55) is globally uniformly practically exponentially stable if μ is $\pi-$globally uniformly practically exponentially stable with $\frac{\rho}{c_1(.)} \leq \sigma$.

12.3. **Converse theorem.** The authors in ([2, 21]) have established a converse stability theorem for (55), when the origin is not an equilibrium point where the function μ is a negative constant and π is a constant. Here, we have present a converse stability result for (55), whose dynamics are in general unbounded with respect to time and satisfies almost the same conditions as theorem (33), where the function π is not required to be a positive constant for all t. In order to give the precise assumption imposed for $F(.,.)$, we need to introduce the following subclasses of $\mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$, see [21].

Definition 23. We say that a function $L \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ is of class $\mathcal{BN}$, if there exists a function $M \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\int_t^{t+\delta} L(s)\,\mathrm{d}s \leq M(\delta), \quad \text{for all } t \geq 0,\ \delta \geq 0.$$

Definition 24. We say that a function $L \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ is of class $\mathcal{BN}^2$, if there exists a function $N \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\int_t^{t+\delta} L^2(s)\,\mathrm{d}s \leq N(\delta), \quad \text{for all } t \geq 0,\ \delta \geq 0.$$

Now, in order to prove a converse theorem, we have supposed the following assumptions.

(H_1''') There exist functions $R(.) \in \mathcal{BN}$ and $K(.) \in \mathcal{BN}^2$, such that

$$\|F(t,x)\| \leq L(t)\|x\| + K(t),$$

where

$$L(t) = L + R(t),$$

with L is a positive constant.

(H_2''') There exist $\gamma > 0$, $r > 0$ and $k \geq 0$ such that for all $x \in \mathbb{R}^n$, the solution of (55) satisfies

$$\|\phi(s,t,x)\| \leq k\|x\| \exp(-\gamma(s-t)) + r \ \text{ for all } s \geq t \geq 0.$$

Then, one can state the following theorem.

Theorem 34. *Under assumptions (H_1''') and (H_2'''), there exist $\mathcal{C}^1$ function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, bounded functions $c_1(.)$, $c_2(.)$, $a(.)$, an integer $p \geq 2$, and scalar functions $\mu, \pi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$ satisfying the following inequalities for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$:*

$$c_1(t)\|x\|^p \leq V(t,x) \leq c_2(t)\|x\|^p + a(t),$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)F(t,x) \leq \mu(t)V(t,x) + \pi(t),$$

such that μ is $\pi-$globally uniformly practically exponentially stable.

Proof. The considered Lyapunov function has the following form

$$V(t,x) = \int_t^{t+\delta} \eta(s)\Big(\|\phi(s,t,x)\|^p + \Big[\frac{e^{2M(\delta)}}{2L}\int_t^s K^2(\tau)d\tau\Big]^{\frac{p}{2}}\Big)\mathrm{d}\,s,$$

where $\eta \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ is a decreasing function and $\delta > \frac{\ln(2^{p-1}k^p)}{p\gamma}$. $\square$

12.4. **Perturbed systems.** We have considered the perturbed system of the form (48). Consequently, the obtained results can be applied to cascaded system (51), for more details see [26].

12.4.1. *Practical asymptotic stability.* We have considered the following assumptions:

(H_1'''') There exist $V : [0, +\infty[\times\mathbb{R}^n \to \mathbb{R}_+$ a continuously differentiable function, two $\mathcal{K}_\infty$ functions $\xi_i, i = 1; 2$, a constant $\tilde{a} \geq 0$ and scalar functions $\tilde{\mu}, \tilde{\pi} \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$, such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$,

$$\xi_1(\|x\|) \leq V(t,x) \leq \xi_2(\|x\|) + \tilde{a},$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) \leq \tilde{\mu}(t)V(t,x) + \tilde{\pi}(t),$$

(H_2'''') There exist $S \in \mathcal{BN}$, such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$,

$$\Big|\frac{\partial V}{\partial x}(t,x)g(t,x)\Big| \leq S(t)V(t,x).$$

Theorem 35. *Under assumption (H_1'''') and (H_2'''') the perturbed system (48) is globally uniformly practically asymptotically stable if $\tilde{\mu}$ is $\tilde{\pi}-$globally uniformly practically exponentially stable.*

Proof. We have considered the following function:

$$W_\delta(t,x) = V(t,x) \exp\Big(\varphi_\delta(t,x)\Big),$$

where

$$\varphi_\delta(t,x) = \int_t^{t+\delta} \int_t^s \frac{1}{\delta V(\tau, \phi(\tau,t,x))} \frac{\partial V}{\partial x}(\tau, \phi(\tau,t,x)) g(\tau, \phi(\tau,t,x)) \, \mathrm{d}\tau \, \mathrm{d}s.$$

where $\delta > 0$. □

12.4.2. *Practical exponential stability.* For the exponential case, we have considered the following assumption.

(H_3'''') There exist $V : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}_+$ a continuously differentiable function, constants $a \geq 0$, $b \geq 0$, $m \geq 1$ and scalar functions $\mu, \pi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$, such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$,

$$c_1(t)\|x\|^m \leq V(t,x) \leq c_2(t)\|x\|^m + a(t),$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)F(t,x) \leq \tilde{\mu}(t)V(t,x) + \tilde{\pi}(t),$$

where $c_1(.) > 0$, $c_2(.) \geq 0$, and $a(.) \geq 0$ are bounded functions with $\frac{c_2(.)}{c_1(.)} \leq c$ and $\frac{a(.)}{c_1(.)} \leq a$.

Theorem 36. *Under assumption (H_2'''') and (H_3'''') the perturbed system (48) is globally uniformly practically exponentially stable if $\tilde{\mu}$ is $\tilde{\pi}-$globally uniformly practically exponentially stable.*

Proof. The same arguments as the proof of the theorem 36. □

13. A NUMERICAL EXAMPLE

Consider the following system

(61) $$\ddot{q} + c(t)\dot{q} + k(t)q = 0,$$

which represents a non-linear mass-spring-damper system where both the damping coefficient, $c(t)$, and the elastic constant, $k(t)$, are time-varying ([55]). The variable $q \in \mathbb{R}$ represents the position of the mass with respect to its rest position. We use the notation $\dot{q}$ to denote the derivative of q with respect to time (i.e., the velocity of the mass) and $\ddot{q}$ to represent the second derivative (acceleration). Such a model is natural to use for celestial mechanics, because it is difficult to influence the motion of the planets. In many examples, it is useful to model the effects of external disturbances or controlled forces on the system. One way to capture this is to replace equation (61) by

(62) $$\ddot{q} + c(t)\dot{q} + k(t)q = u,$$

where u represents the effect of external influences. The model (62) is called a forced or controlled differential equation. It implies that the rate of change

of the state can be influenced by the input u. Adding the input makes the model richer and allows new questions to be posed. For example, we can examine what influence external disturbances have on the trajectories of a system. Or, in the case when the input variable is something that can be modulated in a controlled way, we can analyze whether it is possible to steer the system from one point in the state space to another through proper choice of the input.

Let $\dot{q} = x$. Then system (62) can be rewritten as

$$(63) \qquad \dot{X} = A(t)X(t) + u(t, X(t)),$$

with $X^T = (q, x)$, $A(t) = \begin{bmatrix} 0 & 1 \\ c(t) & k(t) \end{bmatrix}$ and $u : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is a continuously differentiable function.

Let's $c(t) = -(2 - \alpha \sin(t))$ and $k(t) = -(2 - \alpha \cos(t))$, where the scalar α is a constant parameter that accounts for the variability [38]. When $u = 0$, [55] showed that the system (63) is uniformly exponentially stable with $V(t, X) = X^T P(t) X$ as a Lyapunov function that satisfies $V(t, X) \leq \mu(t) V(t, X)$, where

$$(t) = \begin{bmatrix} 1 + \frac{2}{5}\gamma \cos(t) & \frac{1}{5} + \gamma(\frac{17}{5}\cos(t) - \frac{1}{5}\sin(t)) \\ \frac{1}{5} + \gamma(\frac{17}{5}\cos(t) - \frac{1}{5}\sin(t)) & \frac{17}{50} - \frac{17}{25}\gamma \sin(t) \end{bmatrix}$$

is positive define when the constant $\gamma \in [-0.5, 04480]$ and

$$\mu(t) = \frac{1}{2}\lambda_{\max}\big((A(t)^T P(t) + P(t)A(t) + \dot{P}(t))P^{-1}(t)\big),$$

with the constant parameter $\alpha = 2.292$.

Now, let's suppose that there exist a scalar function $\pi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$, such that

$$X^T P(t)u(t, X) + u(t, X)^T P(t)X \leq \pi(t).$$

We consider $W(t, X) = V(t, X) + \psi(t)$, where $\psi(t) = \exp(\int_0^t \mu(s)\,\mathrm{d}\,s)$, as a Lyapunov function for the system (63) which satisfies the inequalities given in (59) with $a(t) = \psi(t)$. The derivative of W along the trajectories of system (63) satisfies

$$\dot{W}(t, X) \leq \mu(t)W(t, X) + \pi(t).$$

Therefore, if μ is $\pi-$uniformly practically exponentially stable, then the closed-loop system (63) is uniformly practically exponentially stable.

For simulation, we choose $\alpha = 2.125$, $\gamma = 0$, $\mu(t) = \frac{1}{2}\lambda_{\max}\big((A(t)^T P(t) + P(t)A(t))P^{-1}(t)\big)$, $\pi(t) = \frac{1}{1+t^2}$ and

$$u(t, q, x) = \Big[\frac{\pi(t)}{(\frac{8}{5}q - \frac{7}{25}x)^2 + 1}, -\frac{\pi(t)}{(\frac{8}{5}q - \frac{7}{25}x)^2 + 1}\Big]^T.$$

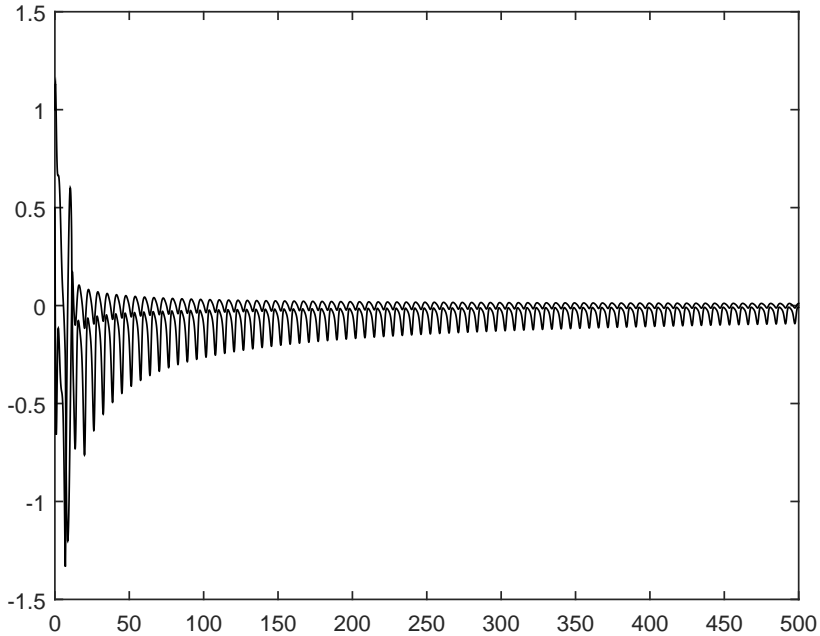We select $(q = 1, x = 0.5)$ as initial condition. Then, we obtain the following simulation result (see Figure 1).

Figure 1. Time evolution of the solution
$(q(t), x(t))$ of system (62).

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