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Coordinated maintenance in a multi-component system with compound Poisson deterioration

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ABSTRACT

This paper proposes a coordinated maintenance model in a multi-component system with compound Poisson deterioration. The main contribution is a policy-iteration approach for Semi-Markov processes that optimizes the threshold at which the component is eligible for preventive maintenance if another component requires corrective maintenance. The methodology is novel as we develop explicit expressions for the policy evaluation and prove these expressions to satisfy the set of linear equations which characterize traditional policy evaluation. By doing so, long-run average cost savings are achieved, since setup costs can be shared.

1. Introduction

Corrective maintenance is maintenance that is carried out after a failure detection to restore the condition of an asset such that it can perform its tasks again. *Preventive maintenance*, on the other hand, is carried out regularly to decrease the likelihood of failures. The combination of these two maintenance types is called *opportunistic maintenance*, by means of which savings in set-up costs can be obtained. As the term suggests, corrective maintenance of a certain component results in an opportunity for preventive maintenance of another component. This is usually referred to as economic dependence between components [1,2] and show commonalities of methods with can-order (S, c, s) policies in [3–5]. The general setup for this paper is the following one.

One wishes to design an optimization policy that takes interdependencies between components into account. The deterioration of components will be generated by independent compound Poisson processes, which causes the deterioration to be of random size and to happen at random moments in time. Once maintenance is being carried out, a machine's production will come to a halt. The resulting loss is seen as the major setup cost incurred when carrying out a maintenance action. The major setup cost is proportional to the time it takes to open up a machine and perform maintenance. Thus, when a component is undergoing corrective maintenance with the incurred major setup cost, another component has the opportunity to undergo preventive maintenance with only minor setup costs incurred. Furthermore, it is also possible that a component has been so severely damaged that it also damages other parts of a machine, which in turn incurs higher maintenance costs.

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1.1. Highlights of contributions

Motivated by coordinated maintenance we develop a renewal model which involves deterioration and maintenance by exploiting the analogy with renewal inventory models. The deterioration process can be viewed as the inventory decay process and maintenance operations correspond to reordering actions. The main contribution of this paper is a novel (S, c, s, F)-policy and a constructive method to design the must-repair s and can-repair c thresholds to minimize the long-run average maintenance costs. A failure-level F is defined which resembles the state of a component in which it damages surrounding parts and thus incurs higher costs. We frame our approach within the area of dynamic programming and policy iteration. Policy iteration consists of iterated sequences of policy evaluation and policy improvement. As for the policy evaluation we provide an explicit expression for the long-run average cost under a given rule. We prove that such expression satisfies the traditional set of linear equations obtained from the Bellman's equation. The result is used to design an algorithm that is able to find the optimal maintenance policy for one component in a multi-component system with compound Poisson deterioration.

1.2. Related literature

Several studies have been conducted on opportunistic maintenance. For example, the authors in [6] considered a system in which the only possible action is a repair or replacement of all components. Thus, corrective maintenance is carried out on all failed components and preventive maintenance is simultaneously carried out on all non-failed components. This action is performed if more than m components have failed or time T has passed [6].

Other studies require immediate maintenance once a component has failed [2,7–13]. The most recent example of these studies has been conducted by *Zhu et al. (2018)*, who considered a dynamic inspection interval as well as dynamic thresholds. Thus, the inspection interval can decrease or increase based on the state of the components. Additionally, the thresholds can be altered based on the state of the components after maintenance [2]. More recent studies focus on condition-based maintenance optimization problems of multi-component systems with stochastic deterioration [14–16]. In these studies, considering the economic dependence between components, the purpose is to determine the optimal policy that minimizes the total cost. In [16] the authors use a stochastic programming approach to determine which components require maintenance. In [15] the authors determine the optimal control limit at which preventive maintenance of a single component should be performed by developing an approximate evaluation procedure. The maintenance occurs when the deterioration level of the component is above the control limit and there is another component getting failure-based maintenance or periodic maintenance. In the work by Poope et al. [14] a similar maintenance model as in [15], with two optimal control limits to perform maintenance, is studied. Similarly to the works mentioned above, in this research we also consider the economic dependence between components and we analyse the case of opportunistic maintenance for a single component in a multi-component system where the deterioration is determined by a Poisson process. However, the main difference with respect to previous research is that we propose a new (S, c, s, F)-policy and we apply a policy iteration algorithm to determine the optimal can-repair level c and must-repair level s that minimize the cost in the long-term. Additional studies regarding opportunistic maintenance with an economic dependence approach can be found in [17–19]. For further references related to condition-based maintenance models of multi-component systems, in [20] the authors present an extensive review of the literature. They provide a classification based on the dependencies between the components and the impacts that it has on the optimal maintenance policy.

Previous research relates maintenance to degradation modeling [21–23]. These maintenance models are based on how the deterioration of the components influences the way in which the system operates, and how by taking into account the deterioration we can improve the optimization and the operation of the system. Differently from those papers, in our work we focus on the impact that the optimal policy has on the cost minimization in the long term rather than on the performance of the system.

Models and methods in this research have a resemblance with can-order policies, also known as (s, c, S)-policies developed in the context of coordinated replenishment [3–5,24–27]. The setup of this paper is motivated by the procedure that has been developed by *Federgruen et al. (1984)*, in which optimal 'can-order' policies for multi-item inventory systems are determined. By means of these so called (S, c, s)-policies, an item is reordered up to a certain level 'S' once its inventory level is at or below 's'. Furthermore, if another item is reordered because its inventory level is at or below 's', an item whose inventory level is at or below 'c' is also replenished up to level 'S'. Thus, to extend this concept to this research, a component requires corrective maintenance once its state is at or below a level 's' and preventive maintenance once its state is at or below a level 'c' when corrective maintenance is carried out on another component. In addition, compared to the ideas presented in the work by Federgruen et al. [5], we determine not only the "can-repair" level c but also the "must-repair" level s in the (S, c, s, F)-policy that minimize the long-run average cost in a multi-component system. To develop the policy iteration algorithm that determines the optimal policy we use an explicit expression of the long-run average cost.

This paper is organized as follows. In Section 2 we develop the model and formulate the problem. In Section 3 we introduce the policy-iteration method and develop an explicit form for the policy-evaluation and prove analogies with the traditional Bellman's equation. In Section 4 we provide simulations. In Section 5 we provide conclusions and future works.

2. Model and problem formulation

Let $x(t)$ and $u(t)$ be the state (deterioration condition) and control (repair/norepair) of a machine at time t. Let t_k be the time of the kth transition where $t_0 = 0$. The state and control stay constant between transitions, namely, $x_k = x(t_k)$; $x(t) = x_k$ for $t_k \leq t \leq t_{k+1}$, and $u_k = u(t_k)$; $u(t) = u_k$ for $t_k \leq t \leq t_{k+1}$.

Control $u(t)$ is obtained from an (S, c, s, F)-type control policy $R : I \to A(i)$,where I is the state space and $A(i)$ is the control space. More specifically $R : i \mapsto a$ which maps state i into action a as follows. In an (S, c, s, F)-policy corrective maintenance occurs every time the component is below a "must-repair" level s. When the component has reached the failure level $F < s$ corrective maintenance is performed with a higher cost. Preventive maintenance is executed when the state of the component is below the "can-repair" level $c > s$ and there is at least another component that requires corrective maintenance. Every time a repair occurs the component goes back to the "repair-up-to" level S. Three parameters are specified similarly, namely: the 'repair-up-to' level S, the 'can-repair' level c and the 'must-repair' level s, where in this research $s < c < S$. The repair-up-to threshold S will be fixed, since a component's state is always ought to be fully restored. Additionally, a certain state of the component will be defined as a state at which failure has occurred, F, which has to be lower than the 'must-repair' level s. It is assumed that reaching this state incurs more costs, which will be further referred to as penalty cost.

The size of the deterioration is a non-negative variable with common discrete probability distribution. The probability of the size of the deterioration being j is denoted by $\phi(j)$, $j \geq 0$, and the state of the component is known at any moment in time. Deterioration follows a compound Poisson process with rate λ. There are different techniques that can be applied to estimate the rate parameter λ based on past data, such as exponential [28] smoothing or Poisson regression [29]. However, these techniques are outside of the scope of this research.

There are two possible repair opportunities, namely when the component under review has deteriorated, or when another component has reached its 'must-repair' level. The first case is referred to as a 'normal' repair opportunity with setup costs K whereas the second case is referred to as a 'special' repair opportunity with setup costs κ, where $\kappa < K$. These special repair opportunities occur at epochs that are generated by a Poisson process, in this case with rate μ. These special repair opportunities are calculated by multiplying the amount of other components, n, by the rate of repair of the other components, η. This is done under the assumption that these components can only undergo corrective maintenance. Thus, μ is an approximation to the superposition of the preventive maintenance actions triggered by corrective maintenance on any other component and is calculated by

$$\eta = \frac{\lambda}{S_{other} - s_{other}}, \quad \mu = n * \eta, \tag{1}$$

where $S_{other} - s_{other}$ is the difference between the fully repaired state and the state that requires corrective maintenance of all other components.

When deterioration on the component under review has occurred leaving the component's state at or below the failure state F, a penalty cost P is incurred, where $P \geq K$. Additionally, the lead time of a repair is assumed to be zero. Lastly, the time between two consecutive decision epochs is independent and exponentially distributed with mean $\frac{1}{\lambda+\mu}$. Thus, the probability that the next decision epoch is generated by the component's own deterioration is $\frac{\lambda}{\lambda+\mu}$, whereas the probability that it is generated by another component requiring maintenance is $\frac{\mu}{\lambda+\mu}$.

The *expected time until the following decision epoch* if action a is chosen in state i, $\tau_i(a)$, is determined by the deterioration rate of the component under review, λ, and the approximation of the superposition of the maintenance opportunities triggered by the other components, μ. Thus, we have,

$$\tau_i(a) = \frac{1}{\lambda + \mu}, \quad \forall \ i \in I, a \in A. \tag{2}$$

Since the expected time until the following decision epoch does not depend on action a, it will be further referred to as τ.

To obtain the *transition probability matrix*, consider that the probability that the system will be in state j at the next decision epoch when action a is chosen in state i, $p_{ij}(a)$, is determined by calculating the transition probabilities for three possible state ranges, namely $F \leqslant i \leqslant s$, $s < i \leqslant c$ and $c < i \leqslant S$. Let $\beta := \frac{\lambda}{\lambda+\mu}$, and $\gamma := \frac{\mu}{\lambda+\mu}$.

First of all, if the state of the component is at or below the must-repair threshold, $i \leqslant s$, action a implies a mandatory repair. Thus, the probability of the system being in state j at the next decision epoch when action a is chosen in state i, for $i \leqslant s$, has the same transition probabilities as state $i = S$. This is due to the fact that there is zero lead-time, which implies that a component whose state is at or below s is immediately repaired up to state $i = S$. Then we have

$$p_{ij}(a) = \begin{cases} \gamma + \beta\phi(0), & F \leqslant i \leqslant s, \quad j = S, \\ \beta\phi(S - j), & F \leqslant i \leqslant s, \quad F < j < S, \\ \beta \displaystyle\sum_{t=S-F}^{\infty} \phi(t), & F \leqslant i \leqslant s, \quad j = F. \end{cases} \tag{3}$$

Secondly, if the state of the component is above the must-repair threshold and at or below the can-repair threshold, action a implies a preventive repair if another component requires corrective maintenance. Thus, with probability $\frac{\mu}{\lambda+\mu}$ the state of the component will be $i = S$ at the next decision epoch. Additionally, with the probability of the component deteriorating, $\frac{\lambda}{\lambda+\mu}$, multiplied by the probability of the size of the deterioration, $\phi(i-j)$, $i-j \geq 0$, the state of the component at the next decision epoch is j, for $j > F$. The probability of the component reaching the failure state, F, is equal to $\frac{\lambda}{\lambda+\mu}\sum_{j=i-F}^{\infty}\phi(j)$, $\forall\ s < i \leqslant c$. This yields

$$p_{ij}(a) = \begin{cases} \beta\phi(i-j), & s < i \leqslant c, \quad F < j \leqslant c, \quad i \geqslant j, \\ \gamma, & s < i \leqslant c, \quad j = S, \\ \beta \displaystyle\sum_{t=i-F}^{\infty} \phi(t), & s < i \leqslant c, \quad j = F. \end{cases} \tag{4}$$

Lastly, if the state of the component is above the can-repair threshold and at or below state $i = S$, action a implies that no repair has to be conducted. Thus, with probability $\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\phi(0)$ the state of the component will remain the same. Furthermore, with the probability of the component deteriorating, $\frac{\lambda}{\lambda+\mu}$, multiplied by the probability of the size of the deterioration, $\phi(i-j)$, $i-j > 0$, the state of the component at the next decision epoch is j, for $j > F$. The probability of the component reaching the failure state, F, is equal to $\frac{\lambda}{\lambda+\mu}\sum_{j=i-F}^{\infty}\phi(j)$, $\forall\ c < i \leqslant S$. Then we can write

$$p_{ij}(a) = \begin{cases} \gamma + \beta\phi(0), & c < i \leqslant S, \quad F < j \leqslant S, \quad i = j, \\ \beta\phi(i-j), & c < i \leqslant S, \quad F < j \leqslant S, \quad i > j, \\ \beta \displaystyle\sum_{j=i-F}^{\infty} \phi(j), & c < i \leqslant S, \quad j = F, \\ 0, & otherwise. \end{cases} \tag{5}$$

The transitions between states is governed by the transition distribution

$$Q_{ij}(\tau, a) = \mathbb{P}\{t_{k+1} - t_k \leq \tau, x_{k+1} = j | x_k = i, u_k = a\} = p_{ij}(a)(1 - e^{-\nu_i(a)\tau}),$$

where the $p_{ij}(a)$ are transition probabilities as in (3)–(5) and $\nu_i(a)$ is the transition rate which equals $\lambda + \mu$.

As for the *expected costs until next decision epoch*, note that the expected incurred costs until the following decision epoch if action a is chosen in state i, $c_i(a)$, is dependent on the action a which is prescribed by rule R. When the state of the component is $c < i \leq S$, the expected incurred costs are 0, since no repair will be done. If the state of the component is $s < i \leqslant c$, a preventive repair of costs κ will be performed with the probability of another component requiring a mandatory repair, $\frac{\mu}{\lambda+\mu}$. Thus, the expected incurred costs are $\frac{\mu}{\lambda+\mu}\kappa$. Furthermore, if the state of the component is $F < i \leqslant s$, a mandatory repair is performed of costs K. Lastly, if the state of the component is at the failure level, F, the costs of repair are P. Thus, the expected incurred costs until the next decision epoch can be formulated as

$$c_i(a) = \begin{cases} 0, & c < i \leq S, \\ \dfrac{\mu}{\lambda+\mu}\kappa, & s < i \leqslant c, \\ K, & F < i \leqslant s, \\ P, & i = F. \end{cases} \tag{6}$$

The problem we wish to solve is to find the optimal thresholds to minimize

$$J_R(i) = \lim_{N\to\infty} \frac{1}{\mathbb{E}\{t_N | x_0 = i, R\}} \mathbb{E}\left\{\sum_{k=0}^{N-1} c_{x_k}(R_{x_k}) | x_0 = i\right\}.$$

It is well known that the above problem can be assimilated to a stochastic shortest path problem (SSP) by dividing the trajectory into cycles characterized by successive visits to a generic predefined node, say node n, associated in our example to state S. The cost at state i and under action a is $c_i(a) - \lambda^*\tau_i(a)$, where λ^* is the optimal expected cost per unit time. We view each cycle as a trajectory of an associated SSP problem with node n as the termination state. So the Bellman's equations for the average cost problem is given by:

$$v_i^* = \min_{a\in A(i)} \left\{ c_i(a) - \lambda^*\tau_i(a) + \sum_{j\in I} p_{ij}(a) v_j^* \right\}. \tag{7}$$

3. Policy-iteration

The policy-iteration algorithm starts by initializing a stationary policy R, where R is of the (S, c, s, F)-type in this research. The state of the component at which the costs of maintenance are the highest due to the component incurring damage to other parts of the machine is denoted as F. The initial state of the component, which resembles the component being in its initial and fully functioning state, is denoted by S. The optimal 'can-repair' threshold c and the optimal 'must-repair' threshold s are to be determined by the policy-iteration algorithm which consists in two steps performed iteratively: a policy-evaluation and a policy-improvement step.

After the initialization of the stationary policy R, *policy-evaluation* is performed by determining the long-run average cost $g(R)$ and the relative values $\upsilon_i(R)$. For the chosen rule R, $g(R)$ and $\upsilon_i(R)$, $i \in I$, are computed as the unique solution of the linear equations

$$\begin{aligned} \upsilon_i &= c_i\left(R_i\right) - g\tau_i\left(R_i\right) + \sum_{j\in I} p_{ij}\left(R_i\right)\upsilon_j, \quad i \in I \\ \upsilon_s &= 0, \end{aligned} \tag{8}$$

in which s is arbitrarily chosen such that the normalization equation $\upsilon_s = 0$.

To solve the linear programming problem which is stated in Eq. (8), $c_i\left(R_i\right)$, $\tau_i(R_i)$ and $p_{ij}\left(R_i\right)$ need to be determined.

Once the stationary policy R has been evaluated, *policy-improvement* is performed. Policy-improvement is performed by determining action a for every state $i \in I$ that yields the minimum in

$$\min_{a\in A(i)}\left\{c_i(a) - g(R)\tau + \sum_{j\in I} p_{ij}(a)\upsilon_j(R)\right\} \tag{9}$$

where $g(R)$ and $\upsilon_j(R)$ are obtained from the policy-evaluation step. Furthermore, $c_i(a)$ and $p_{ij}(a)$ are recalculated for every action $a \in A$. The one action a, which is prescribed by the values of the thresholds s and c, that minimizes Eq. (9) for all states i, determines the new policy $\bar{R} = (S, a, F) = (S, c, s, F)$.

If the new policy $\bar{R} = R$, the algorithm is stopped and $\bar{R}$ is the optimal policy. We also have $g(\bar{R}) = \lambda^*$. Otherwise, the algorithm is repeated with R replaced by $\bar{R}$.

In the following we develop explicit formulas to obtain the long-run average cost. These explicit formulas represent an equivalent and alternative way to solving the set of linear equations (8).

3.1. Validation model policy-evaluation

The expected long-run average costs $g(R)$ for a policy R can be equivalently obtained by calculating the total expected costs from the initial state S to the state right after a repair and dividing these expected costs by the expected time it takes from the initial state S to the state right after a repair has taken place. We prove that this is the same value as the value of $g(R)$ which is obtained from the policy-evaluation step.

For the system which is controlled by rule R and is currently in state i, the expected time it takes to reach the next epoch at which a repair is conducted and the probability that the component's next repair is triggered by its own deterioration, are denoted by $t_R(i)$ and $q_R(i)$, respectively. For the determination of $t_R(i)$ and $q_R(i)$, the must-repair threshold s is the lowest state that is considered, since any state $i < s$, will trigger a mandatory repair similarly to state $i = s$. Thus, the probabilities of reaching state $i \leqslant s$ are all assigned to state s. Furthermore, the expected penalty costs incurred until the next repair is conducted are denoted by $h_R(i)$, which is calculated by multiplying the probability of going from state i to state $i = F$ by the penalty costs P. The determination of $t_R(i)$, $q_R(i)$ and $h_R(i)$ will be clarified in the following.

Lemma 1. *It holds*

$$t_R(i) = \frac{1 + \lambda\sum_{j=1}^{i-s-1} t_R(i-j)\phi(j)}{\lambda + \mu - \lambda\phi(0) - \delta(i-c)\mu}, \quad i > s. \tag{10}$$

Proof. *Calculation of $t_R(i)$ for $s < i \leq c$*

The first situation that is elaborated upon, is the determination of $t_R(i)$ for $s < i \leq c$. In Fig. 1, the probabilities of moving from $s+1$ to all other states are shown. With the probability of another machine requiring a repair, $\frac{\mu}{\lambda+\mu}$, the machine under control is also repaired. Furthermore, with the probability that deterioration occurs and is greater than or equal to 1, $\frac{\lambda}{\lambda+\mu}\sum_{j=1}^{\infty}\phi(j)$, the machine under control is also repaired. In both cases, the time it takes to reach the next epoch at which a repair is conducted, is $\frac{1}{\lambda+\mu}$. Lastly, with the probability that a deterioration of 0 takes place, the machine under control remains in the state $s+1$. In this case, the time it takes to reach the next epoch at which a repair is conducted, is $\frac{1}{\lambda+\mu} + t_R(s+1)$. Evidently, it can be concluded that the time it takes to reach the next epoch at which a repair is conducted, is at least $\frac{1}{\lambda+\mu}$. With probability $\frac{\lambda}{\lambda+\mu}\phi(0)$, it takes $t_R(s+1)$ longer. Thus, $t_R(s+1)$ is defined as

$$t_R(s+1) = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\Big(\phi(0)t_R(s+1)\Big). \tag{11}$$

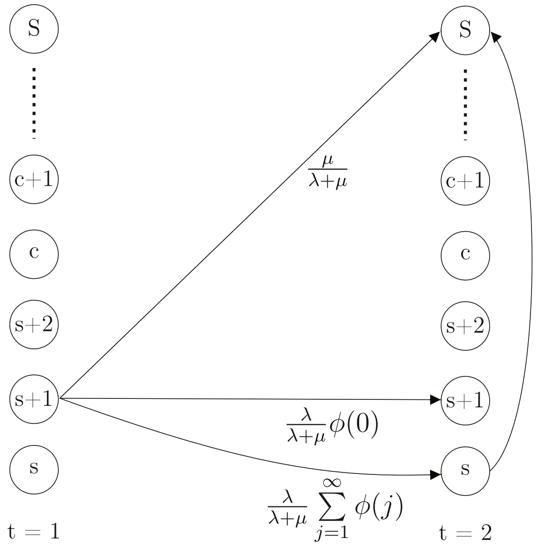


Fig. 1. Probabilities of going from $s+1$ to another state.

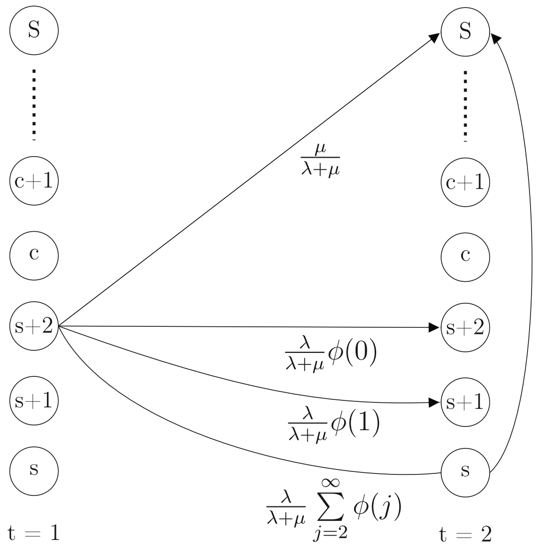


Fig. 2. Probabilities of going from $s+2$ to another state.

In Fig. 2, the probabilities of moving from $s+2$ to all other states are shown. Similarly to the situation in Fig. 1, it takes at least $\frac{1}{\lambda+\mu}$ to reach the next epoch at which a repair is conducted. However, in this case, with probability $\frac{\lambda}{\lambda+\mu}\phi(0)$ it takes $t_R(s+2)$ longer and with probability $\frac{\lambda}{\lambda+\mu}\phi(1)$ it takes $t_R(s+1)$ longer. Thus, $t_R(s+2)$ is defined as

$$t_R(s+2) = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\Big(\phi(0)t_R(s+2) + \phi(1)t_R(s+1)\Big). \tag{12}$$

It readily follows that $t_R(i)$ for $s < i \leq c$ can be defined as

$$t_R(i) = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\sum_{j=0}^{i-s-1} t_R(i-j)\phi(j). \tag{13}$$

Calculation of $t_R(i)$ for $i > c$

The second situation that is elaborated upon, is the determination of $t_R(i)$ for $i > c$. In Fig. 3, the probabilities of moving from $c+1$ to all other states are shown. Similarly to $t_R(i)$ for $s < i \leq c$, it takes at least $\frac{1}{\lambda+\mu}$ to reach the next epoch

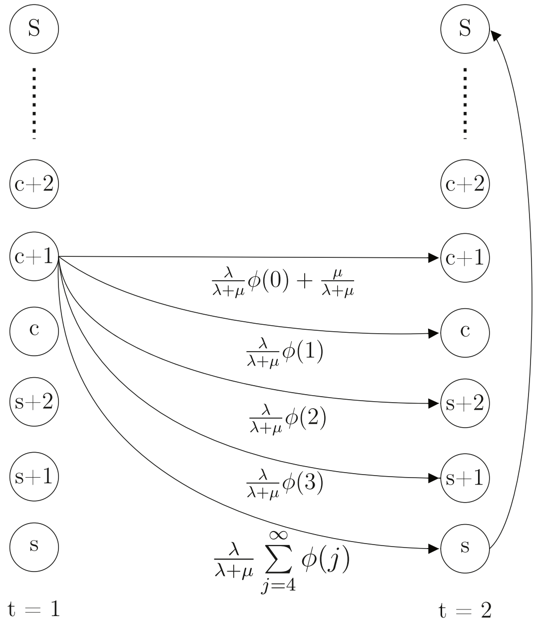


Fig. 3. Probabilities of going from $c+1$ to another state.

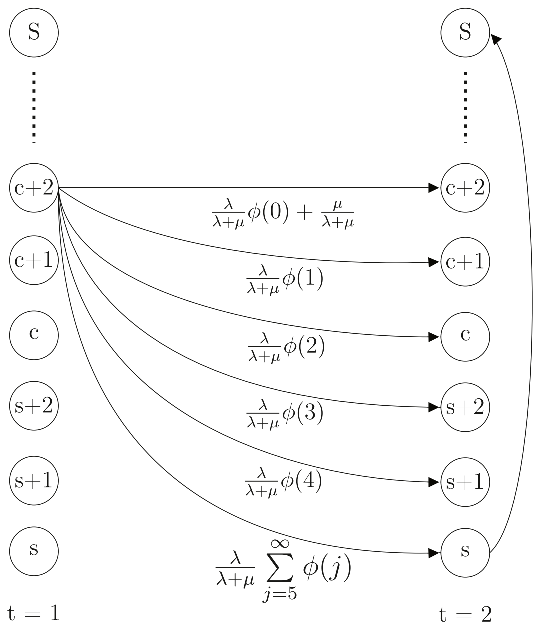


Fig. 4. Probabilities of going from $c+2$ to another state.

at which a repair is conducted. In this case, with the probability of deterioration occurring and being 0, $\frac{\lambda}{\lambda+\mu}\phi(0)$, and the probability of another machine breaking down, $\frac{\mu}{\lambda+\mu}$, it takes $t_R(c+1)$ longer. With the probability of deterioration occurring and being 1, $\frac{\lambda}{\lambda+\mu}\phi(1)$, it takes $t_R(c)$ longer. Additionally, with the probability of deterioration occurring and being 2 or 3, it takes $t_R(s+2)$ or $t_R(s+1)$ longer, respectively. Thus, $t_R(c+1)$ is defined as

$$t_R(c+1)=\frac{1}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}t_R(c+1)+\frac{\lambda}{\lambda+\mu}\Big(\phi(0)t_R(c+1)+\phi(1)t_R(c)+\phi(2)t_R(s+2)+\phi(3)t_R(s+1)\Big). \tag{14}$$

In Fig. 4, the probabilities of moving from $c+2$ to all other states are shown. Similarly to the aforementioned situations, it takes at least $\frac{1}{\lambda+\mu}$ to reach the next epoch at which a repair is conducted. However, in this case, with the probability of deterioration occurring and being 0, $\frac{\lambda}{\lambda+\mu}\phi(0)$, it takes $t_R(c+2)$ longer. Furthermore, with the probability of deterioration occurring and being 1, 2, 3 or 4, it takes $t_R(c+1)$, $t_R(c)$, $t_R(s+2)$ or $t_R(s+1)$ longer, respectively. Thus, $t_R(c+2)$ is defined as

$$t_R(c+2)=\frac{1}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}t_R(c+2)+\frac{\lambda}{\lambda+\mu}\Big(\phi(0)t_R(c+2)+\phi(1)t_R(c+1)+\phi(2)t_R(c)+\phi(3)t_R(s+2)+\phi(4)t_R(s+1)\Big). \quad (15)$$

It readily follows that $t_R(i)$ for $i>c$ can be defined as

$$t_R(i)=\frac{1}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}t_R(i)+\frac{\lambda}{\lambda+\mu}\sum_{j=0}^{i-s-1}t_R(i-j)\phi(j). \quad (16)$$

Calculation of $t_R(i)$ for $i>s$

By combining Eqs. (13) and (16), $t_R(i)$ for $i>s$ is obtained as

$$t_R(i)=\frac{1}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}t_R(i)\delta(i-c)+\frac{\lambda}{\lambda+\mu}\sum_{j=0}^{i-s-1}t_R(i-j)\phi(j), \quad (17)$$

where $\delta(i-c)=1$ if $i>c$ and $\delta(i-c)=0$ otherwise.

However, for modelling purposes, $t_R(i)$ needs to be isolated in Eq. (17). This is done in the following way

$$t_R(i)\left(\frac{\lambda+\mu-\mu\delta(i-c)-\lambda\phi(0)}{\lambda+\mu}\right)=\frac{1+\lambda\sum_{j=1}^{i-s-1}t_R(i-j)\phi(j)}{\lambda+\mu}, \quad (18)$$

which yields (10) and this concludes our proof. ■

Lemma 2. *It holds*

$$q_R(i)=\frac{\lambda\Big(1-\sum_{j=0}^{i-s-1}\phi(j)+\sum_{j=1}^{i-s-1}q_R(i-j)\phi(j)\Big)}{\lambda+\mu-\lambda\phi(0)-\mu\delta(i-c)},\quad i>s. \quad (19)$$

Proof. *Calculation of $q_R(i)$ for $s<i\le c$*

The first situation that is elaborated upon, is the determination of $q_R(i)$ for $s<i\le c$. The probabilities of moving from $s+1$ to all other states are shown in Fig. 1 in the previous section. In this case, with the probability of deterioration occurring and being 0, $\frac{\lambda}{\lambda+\mu}\phi(0)$, the probability is again $q_R(s+1)$. Furthermore, with the probability of a deterioration occurring and being greater than or equal to 1, $\frac{\lambda}{\lambda+\mu}\sum_{j=1}^{\infty}\phi(j)$, the repair is triggered by the machine's own deterioration. Thus, $q_R(s+1)$ is defined as

$$q_R(s+1)=\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=1}^{\infty}\phi(j)+q_R(s+1)\phi(0)\Big). \quad (20)$$

The probabilities of moving from $s+2$ to all other states are shown in Fig. 2 in the previous section. In this case, with the probability of deterioration occurring and being 0, $\frac{\lambda}{\lambda+\mu}\phi(0)$, the probability of deterioration triggering repair is again $q_R(s+2)$. Additionally, with the probability of deterioration occurring and being 1, $\frac{\lambda}{\lambda+\mu}\phi(1)$, the probability of deterioration triggering repair is $q_R(s+1)$. Lastly, with the probability of a deterioration occurring and being greater than or equal to 2, $\frac{\lambda}{\lambda+\mu}\sum_{j=2}^{\infty}\phi(j)$, the repair is triggered by the machine's own deterioration. Thus, $q_R(s+2)$ is defined as

$$q_R(s+2)=\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=2}^{\infty}\phi(j)+q_R(s+2)\phi(0)+q_R(s+1)\phi(1)\Big). \quad (21)$$

It readily follows that $q_R(i)$ for $s<i\le c$ can be defined as

$$q_R(i)=\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=i-s}^{\infty}\phi(j)+\sum_{j=0}^{i-s-1}q_R(i-j)\phi(j)\Big). \quad (22)$$

Calculation of $q_R(i)$ for $i>c$

The second situation that is elaborated upon, is the determination of $q_R(i)$ for $i>c$. The probabilities of moving from $c+1$ to all other states are shown in Fig. 3 in the previous section. In this case, with the probability of another machine breaking down, $\frac{\mu}{\lambda+\mu}$, the probability of deterioration triggering repair is again $q_R(c+1)$. Additionally, with the probability of deterioration occurring and being 0, $\frac{\lambda}{\lambda+\mu}\phi(0)$, the probability of deterioration triggering repair is also

$q_R(c+1)$. Furthermore, with the probability of deterioration occurring and being 1, 2 or 3, the probability of deterioration triggering repair is $q_R(c)$, $q_R(s+2)$ or $q_R(s+1)$, respectively. Lastly, with the probability of deterioration occurring and being greater than or equal to 4, $\frac{\lambda}{\lambda+\mu}\sum_{j=4}^{\infty}\phi(j)$, the repair is triggered by the machine's own deterioration. Thus, $q_R(c+1)$ is defined as

$$q_R(c+1)=\frac{\mu}{\lambda+\mu}q_R(c+1)+\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=4}^{\infty}\phi(j)+q_R(c+1)\phi(0)+q_R(c)\phi(1)+q_R(s+2)\phi(2)+q_R(s+1)\phi(3)\Big). \quad (23)$$

The probabilities of moving from $c+2$ to all other states are shown in Fig. 4 in the previous section. In this case, with the probability of another machine breaking down, $\frac{\mu}{\lambda+\mu}$, the probability of deterioration triggering repair is again $q_R(c+2)$. With the probability of deterioration occurring and being 0, $\frac{\lambda}{\lambda+\mu}\phi(0)$, the probability of deterioration triggering repair is also $q_R(c+2)$. Furthermore, with the probability of deterioration occurring and being 1, 2, 3 or 4, the probability of deterioration triggering repair is $q_R(c+1)$, $q_R(c)$, $q_R(s+2)$ or $q_R(s+1)$, respectively. Lastly, with the probability of deterioration occurring and being greater than or equal to 5, $\frac{\lambda}{\lambda+\mu}\sum_{j=5}^{\infty}\phi(j)$, the repair is triggered by the machine's own deterioration. Thus, $q_R(c+2)$ is defined as

$$q_R(c+2)=\frac{\mu}{\lambda+\mu}q_R(c+2)+\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=5}^{\infty}\phi(j)+q_R(c+2)\phi(0)+q_R(c+1)\phi(1)+q_R(c)\phi(2)+q_R(s+2)\phi(3)+q_R(s+1)\phi(4)\Big). \quad (24)$$

It readily follows that $q_R(i)$ for $i>c$ can be defined as

$$q_R(i)=\frac{\mu}{\lambda+\mu}q_R(i)+\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=i-s}^{\infty}\phi(j)+\sum_{j=0}^{i-s-1}q_R(i-j)\phi(j)\Big). \quad (25)$$

Calculation of $q_R(i)$ for $i>s$

By combining Eqs. (22) and (25), $q_R(i)$ for $i>s$ is obtained as

$$q_R(i)=\frac{\mu}{\lambda+\mu}q_R(i)\delta(i-c)+\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=i-s}^{\infty}\phi(j)+\sum_{j=0}^{i-s-1}q_R(i-j)\phi(j)\Big), \quad (26)$$

where $\delta(i-c)=1$ if $i>c$ and $\delta(i-c)=0$ otherwise. However, for modelling purposes, $q_R(i)$ needs to be isolated in Eq. (26). To do this let us rewrite as follows:

$$q_R(i)\Big(\frac{\lambda+\mu-\mu\delta(i-c)-\lambda\phi(0)}{\lambda+\mu}\Big)=\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=i-s}^{\infty}\phi(j)+\sum_{j=1}^{i-s-1}q_R(i-j)\phi(j)\Big), \quad (27)$$

From the above we obtain

$$q_R(i)=\frac{\lambda\Big(\sum_{j=i-s}^{\infty}\phi(j)+\sum_{j=1}^{i-s-1}q_R(i-j)\phi(j)\Big)}{\lambda+\mu-\lambda\phi(0)-\mu\delta(i-c)}, \quad (28)$$

which yields (19) and this concludes our proof. ■

Lemma 3. *It holds*

$$h_R(i)=\frac{P\lambda\Big(1-\sum_{j=0}^{i-F-1}\phi(j)+\sum_{j=1}^{i-F-1}F_R(i-j)\phi(j)\Big)}{\lambda+\mu-\lambda\phi(0)-\mu\delta(i-c)}, \quad i>s. \quad (29)$$

Proof. *Calculation of $h_R(i)$*

The expected penalty costs incurred from state i until the next repair, $h_R(i)$, are calculated by multiplying the probability of reaching the failure state F from every state i by the penalty cost P. Let $F_R(i)$ be the probability of reaching the failure state from state i.

For $s<i\le c$, the probabilities of moving from $s+1$ to all other relevant states are shown in Fig. 5. In this case, with the probability of deterioration occurring and being 0, $\frac{\lambda}{\lambda+\mu}\phi(0)$, the probability of reaching the failure state is again $F_R(s+1)$. Additionally, with the probability of a deterioration occurring and being greater than or equal to $s+1-F$, $\frac{\lambda}{\lambda+\mu}\sum_{j=s+1-F}^{\infty}\phi(j)$, the component reaches its failure state F. Hence, $F_R(s+1)$ is defined as

$$F_R(s+1)=\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=i-F}^{\infty}\phi(j)+F_R(s+1)\phi(0)\Big). \quad (30)$$

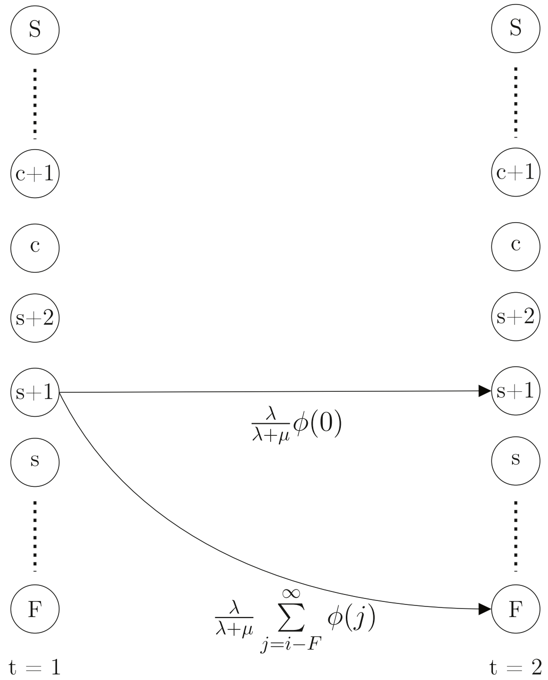


Fig. 5. Probabilities of going from $s+1$ to another state.

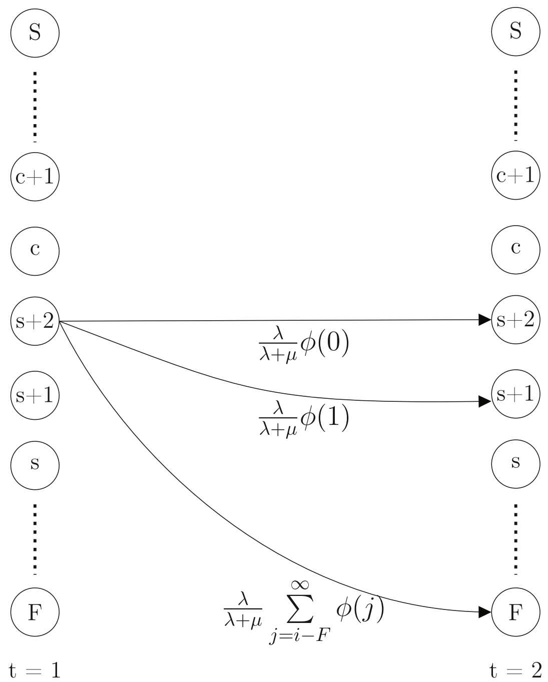


Fig. 6. Probabilities of going from $s+2$ to another state.

The probabilities of moving from $s+2$ to all other relevant states are shown in Fig. 6. In this case, with the probability of deterioration occurring and being 0, $\frac{\lambda}{\lambda+\mu}\phi(0)$, the probability of reaching the failure state is again $F_R(s+2)$. Furthermore, with the probability of deterioration occurring and being 1, $\frac{\lambda}{\lambda+\mu}\phi(1)$, the probability of reaching the failure state is $F_R(s+1)$. Lastly, with the probability of a deterioration occurring and being greater than or equal to $s+2-F$, $\frac{\lambda}{\lambda+\mu}\sum_{j=s+2-F}^{\infty}\phi(j)$,

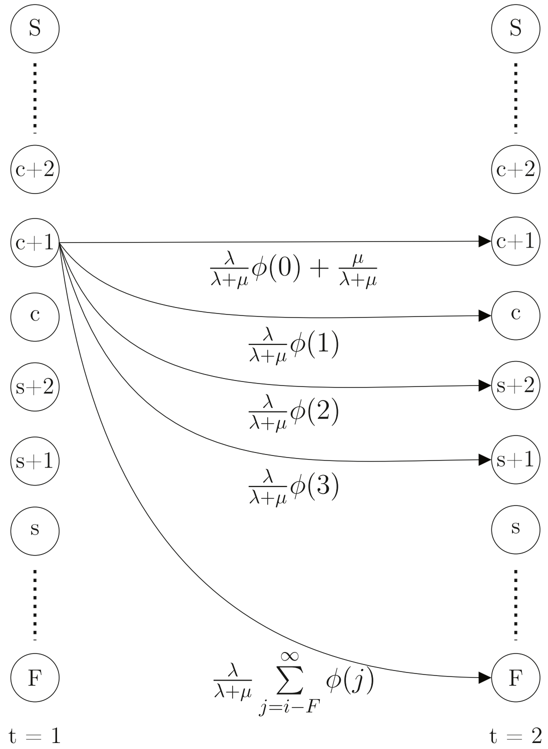


Fig. 7. Probabilities of going from $c + 1$ to another state.

the component reaches its failure state F. Thus, $F_R(s+2)$ is defined as

$$F_R(s+2) = \frac{\lambda}{\lambda+\mu}\Big(\sum_{j=s+2-F}^{\infty}\phi(j) + F_R(s+2)\phi(0) + F_R(s+1)\phi(1)\Big). \tag{31}$$

It readily follows that $F_R(i)$ for $s < i \leq c$ can be defined as

$$F_R(i) = \frac{\lambda}{\lambda+\mu}\Big(\sum_{j=i-F}^{\infty}\phi(j) + \sum_{j=0}^{i-s-1}F_R(i-j)\phi(j)\Big). \tag{32}$$

The second situation that is elaborated upon, is the determination of $F_R(i)$ for $i > c$. The probabilities of moving from $c+1$ to all other relevant states are shown in Fig. 7. In this case, with the probability of another machine breaking down, $\frac{\mu}{\lambda+\mu}$, the probability of reaching the failure state is again $F_R(c+1)$. With the probability of deterioration occurring and being 0, $\frac{\lambda}{\lambda+\mu}\phi(0)$, the probability of reaching the failure state is also $F_R(c+1)$. Additionally, with the probability of deterioration occurring and being 1, 2 or 3, the probability of reaching the failure state is $F_R(c)$, $F_R(s+2)$ or $F_R(s+1)$, respectively. Lastly, with the probability of deterioration occurring and being greater than or equal to $c+1-F$, $\frac{\lambda}{\lambda+\mu}\sum_{j=c+1-F}^{\infty}\phi(j)$, the component reaches its failure state F. Thus, $F_R(c+1)$ is defined as

$$F_R(c+1) = \frac{\mu}{\lambda+\mu}F_R(c+1) + \frac{\lambda}{\lambda+\mu}\Big(\sum_{j=c+1-F}^{\infty}\phi(j) + F_R(c+1)\phi(0) + F_R(c)\phi(1) + F_R(s+2)\phi(2) + F_R(s+1)\phi(3)\Big). \tag{33}$$

The probabilities of moving from $c+2$ to all other relevant states are shown in Fig. 8. In this case, with the probability of another machine breaking down, $\frac{\mu}{\lambda+\mu}$, the probability of reaching the failure state is again $F_R(c+2)$. With the probability of deterioration occurring and being 0, $\frac{\lambda}{\lambda+\mu}\phi(0)$, the probability of reaching the failure state is also $F_R(c+2)$. Furthermore, with the probability of deterioration occurring and being 1, 2, 3 or 4, the probability of reaching the failure state is $F_R(c+1)$, $F_R(c)$, $F_R(s+2)$ or $F_R(s+1)$, respectively. Lastly, with the probability of deterioration occurring and being greater than or

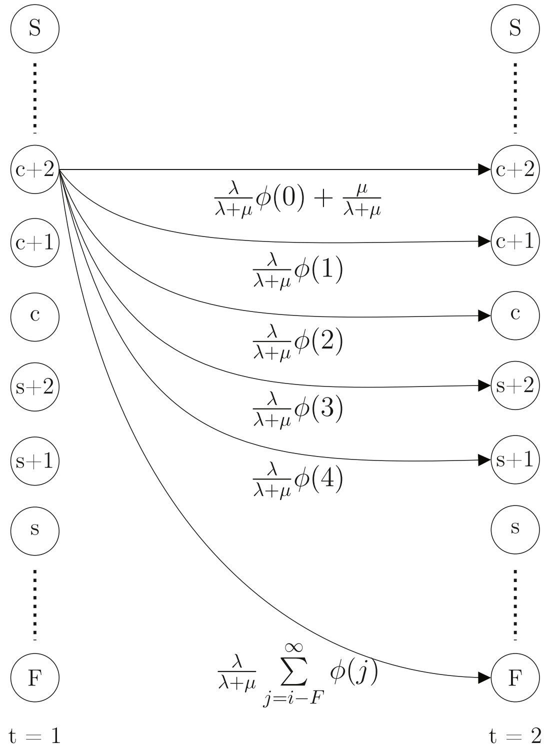


Fig. 8. Probabilities of going from $c+2$ to another state.

equal to $c+2-F$, $\frac{\lambda}{\lambda+\mu}\sum_{j=c+2-F}^{\infty}\phi(j)$, the component reaches its failure state F. Thus, $F_R(c+2)$ is defined as

$$F_R(c+2)=\frac{\mu}{\lambda+\mu}F_R(c+2)+\frac{\lambda}{\lambda+\mu}\left(\sum_{j=c+2-F}^{\infty}\phi(j)\right.$$
$$\left.+F_R(c+2)\phi(0)+F_R(c+1)\phi(1)+F_R(c)\phi(2)+F_R(s+2)\phi(3)+F_R(s+1)\phi(4)\right). \tag{34}$$

It readily follows that $F_R(i)$ for $i>c$ can be defined as

$$F_R(i)=\frac{\mu}{\lambda+\mu}F_R(i)+\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=i-F}^{\infty}\phi(j)+\sum_{j=0}^{i-F-1}F_R(i-j)\phi(j)\Big). \tag{35}$$

By combining Eqs. (32) and (35), $F_R(i)$ for $i>s$ is obtained as

$$F_R(i)=\frac{\mu}{\lambda+\mu}F_R(i)\delta(i-c)+\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=i-F}^{\infty}\phi(j)+\sum_{j=0}^{i-F-1}F_R(i-j)\phi(j)\Big), \tag{36}$$

where $\delta(i-c)=1$ if $i>c$ and $\delta(i-c)=0$ otherwise.

However, for modelling purposes, $F_R(i)$ needs to be isolated in Eq. (36). This is done in the following way

$$F_R(i)\Big(\frac{\lambda+\mu-\mu\delta(i-c)-\lambda\phi(0)}{\lambda+\mu}\Big)=\frac{\lambda}{\lambda+\mu}\Big(\sum_{j=i-F}^{\infty}\phi(j)+\sum_{j=1}^{i-F-1}F_R(i-j)\phi(j)\Big), \tag{37}$$

$$F_R(i)=\frac{\lambda\left(\sum_{j=i-F}^{\infty}\phi(j)+\sum_{j=1}^{i-F-1}F_R(i-j)\phi(j)\right)}{\lambda+\mu-\lambda\phi(0)-\mu\delta(i-c)}, \tag{38}$$

$$F_R(i)=\frac{\lambda\left(1-\sum_{j=0}^{i-F-1}\phi(j)+\sum_{j=1}^{i-F-1}F_R(i-j)\phi(j)\right)}{\lambda+\mu-\lambda\phi(0)-\mu\delta(i-c)},\quad i>s. \tag{39}$$

Finally, the expected penalty costs incurred from state i until the next repair, $h_R(i)$, are calculated by multiplying $F_R(i)$ by the penalty costs P. Thus, for $h_R(i)$ we obtain (29) and this concludes our proof. ■

The total expected costs of going to the repaired state from state i, when using rule R, are denoted by $k_R(i)$ as

$$k_R(i) = h_R(i) + Kq_R(i) + \kappa\,(1 - q_R(i))\,, \quad i > s. \tag{40}$$

Analogous to the model by *Federgruen et al. (1984)*, the long-run average cost per unit time when using rule R is denoted by

$$g_R = k_R(S)/t_R(S). \tag{41}$$

The relative costs of going to the regeneration state from state i when using rule R, $\upsilon_R(i)$, are defined by

$$\upsilon_R(i) = \begin{cases} k_R(i) - g_R t_R(i), & i > s, \\ K, & F < i \leqslant s, \\ P, & i = F. \end{cases} \tag{42}$$

Theorem 1. *Let rule R be given. The long-run average cost g_R and the relative costs $\upsilon_R(i)$ for all $i \in I$ satisfy the set of linear Eq. (8).*

Proof. From the definition of $k_R(i)$ in (40) we have that

- for all $j = 0, \dots, i - s - 1$ with probability $\frac{\lambda}{\lambda+\mu}\phi(j)$ the transition is to state $i-j$ where the cost is $k_R(i-j)$,
- for all $j = i - s, \dots, i - F - 1$ with probability $\frac{\lambda}{\lambda+\mu}\phi(j)$ the transition is to a state $F + 1 \le i-j \le s$ where the cost $k_R(i-j) = K$, and
- for all $j = j = i-F, \dots, \infty$ with probability $\frac{\lambda}{\lambda+\mu}\phi(j)$ the transition is to a state $i-j \le F$ where the cost is $k_R(i-j) = P$,
- with probability $\frac{\mu}{\lambda+\mu}$ a discounted repair occurs which leaves the state unchanged if $i \ge c$ or has a cost k if $i < c$.

We can summarize the above cases in the following equation

$$\begin{aligned} k_R(i) &= \tfrac{\lambda}{\lambda+\mu}\textstyle\sum_{j=0}^{i-s-1} k_R(i-j)\phi(j) + K\tfrac{\lambda}{\lambda+\mu}\textstyle\sum_{j=i-s}^{i-F-1}\phi(j) \\ &+ P\frac{\lambda}{\lambda+\mu}\sum_{j=i-F}^{\infty}\phi(j) + \frac{\mu}{\lambda+\mu}\{\delta(i-c)k_R(i) + (1-\delta(i-c))k\}, \end{aligned} \tag{43}$$

which represents a recursive expression of the total expected costs of going to the repaired state from state i. As regards the expected time of going to the repaired state from state i we can use (17) which we write below again

$$t_R(i) = \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}t_R(i)\delta(i-c) + \frac{\lambda}{\lambda+\mu}\sum_{j=0}^{i-s-1} t_R(i-j)\phi(j). \tag{44}$$

The stage cost of the assimilated SSP problem is $k_R(i) - g_R t_R(i)$ which can be obtained by subtracting g_R times (44) from (43). Thus we have

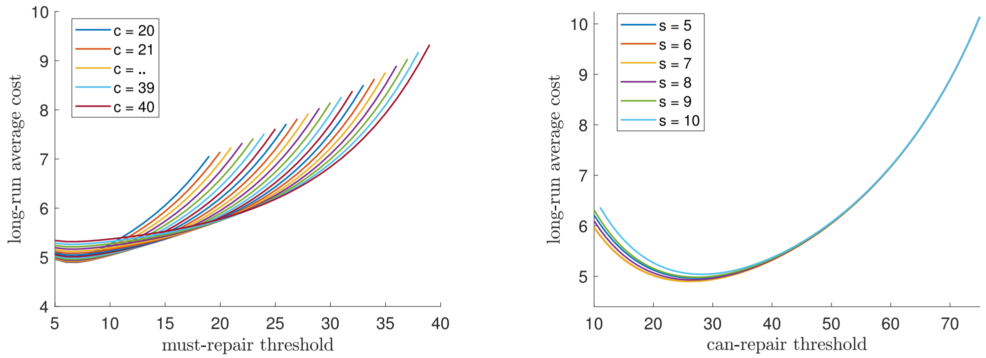
$$\begin{aligned} &k_R(i) - g_R t_R(i) \\ &= -g_R\frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\sum_{j=0}^{i-s-1}\phi(j)\Big(k_R(i-j) - g_R t_R(i-j)\Big) \\ &+ K\frac{\lambda}{\lambda+\mu}\sum_{j=i-s}^{i-F-1}\phi(j) + P\frac{\lambda}{\lambda+\mu}\sum_{j=i-F}^{\infty}\phi(j) \\ &+ \frac{\mu}{\lambda+\mu}\delta(i-c)\Big(k_R(i) - g_R t_R(i)\Big) + \frac{\mu}{\lambda+\mu}(1-\delta(i-c))k. \end{aligned} \tag{45}$$

Table 1
Simulation parameters for Figs. 9 and 10.

S	F	λ	λ	S_{other}	s_{other}	η	n	μ	K	κ	P
100	1	2	3	100	9	0.044	15	0.66	100	60	300

Table 2
Simulation parameters for Figs. 9 and 10.

S	F	λ	λ	S_{other}	s_{other}	η	K	P
100	1	2	3	100	9	0.044	100	300



(a) g_R over s for $20 \leqslant c \leqslant 40$. (b) g_R over c for $5 \leqslant s \leqslant 10$.

Fig. 9. Long-run average cost over thresholds.

Setting $l := \max\{i - j, F\}$ we can rewrite the above as

$$
\begin{aligned}
& k_R(i) - g_R t_R(i) \\
& \quad = -g_R \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \sum_{j=i}^{s+1} p_{il} \Big(k_R(i-j) - g_R t_R(i-j) \Big) \\
& \quad + K \sum_{l=s}^{F+1} p_{il} + P p_{iF} + c_i(a) \\
& = c_i(R) - g_R \frac{1}{\lambda+\mu} + \sum_{l=F}^{S} p_{il} v_R(l).
\end{aligned}
\tag{46}
$$

Note that $v_R(l) = v_R(S)$ for all $l \leq s$. This concludes our proof. ∎

4. Simulations

In this section, simulations are performed to extend the theoretical results which are developed in Section 3.1. The numerical studies show convexity of the long-run average cost with respect to s and c. Furthermore, the influence of the amount of other components, n, is simulated with regards to the optimal threshold levels. Hereafter, the evolution of the state of the component under review is simulated over a period of two years. Subsequently, the influence of the costs of preventive maintenance, corrective maintenance and failure is simulated. The cost-savings that can be achieved by means of coordinated maintenance are also acquired.

4.1. Convexity

In order to ensure that the algorithm which is stated in Section 3 provides a global optimum, the long-run average cost is plotted for several combinations of the must-repair and can-repair thresholds. In Fig. 9(a) the long-run average cost is plotted over the must-repair threshold, $5 \leqslant s \leqslant 39$, for policies with the can-repair threshold, $20 \leqslant c \leqslant 40$, where $s < c$. In Fig. 9(b) the long-run average cost is plotted over the can-repair threshold, $10 \leqslant c \leqslant 75$, for policies with the must-repair threshold, $5 \leqslant s \leqslant 10$, where $s < c$. The simulation parameters are listed in Table 1.

In the 2D plots in Fig. 9, it can be seen that the long-run average cost function is convex with respect to the must-repair and can-repair thresholds. In Fig. 10 the long-run average cost is plotted over the must-repair and can-repair thresholds

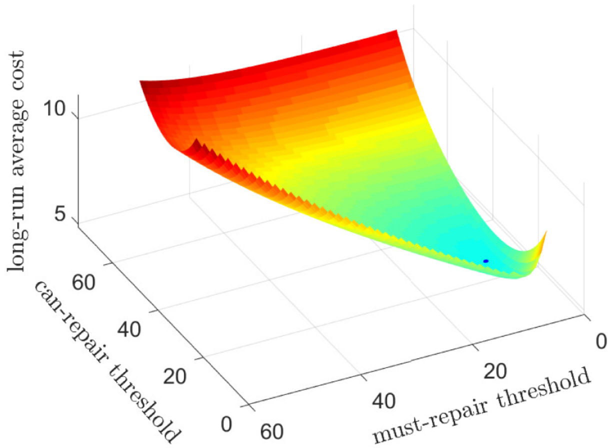


Fig. 10. 3D plot of long-run average cost over the must-repair and can-repair thresholds.

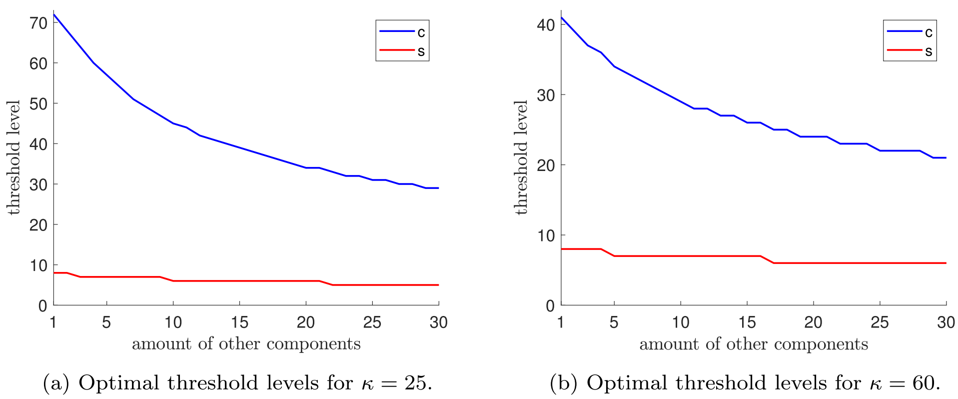


Fig. 11. Optimal must-repair and can-repair thresholds levels for different values of n.

in 3D, where the global minimum is indicated by the blue dot. The convexity of the long-run average cost is evident in this figure. It can thus be concluded that the optimal must-repair and can-repair thresholds obtained by the algorithm are global optima.

4.2. Influence of amount of other components on optimal thresholds

The influence of the amount of other components that can trigger preventive maintenance of the component under review on the must-repair and can-repair thresholds is simulated by determining the optimal values of the thresholds for only one other machine up to thirty other machines, $1 \leqslant n \leqslant 30$. The simulation parameters are listed in Table 2.

In Figs. 11(a) and 11(b), the optimal must-repair and can-repair thresholds are shown for $\kappa = 25$, and $\kappa = 60$, respectively. The thresholds are monotonically non-increasing for an increasing n. Also, the can-repair threshold is affected the most by an increase in the amount of other components. If the value of μ is rather high, the can-repair threshold should be relatively low. On the contrary, if the value of μ is low, the risk of the component being repaired while it is still in a rather good state is relatively low.

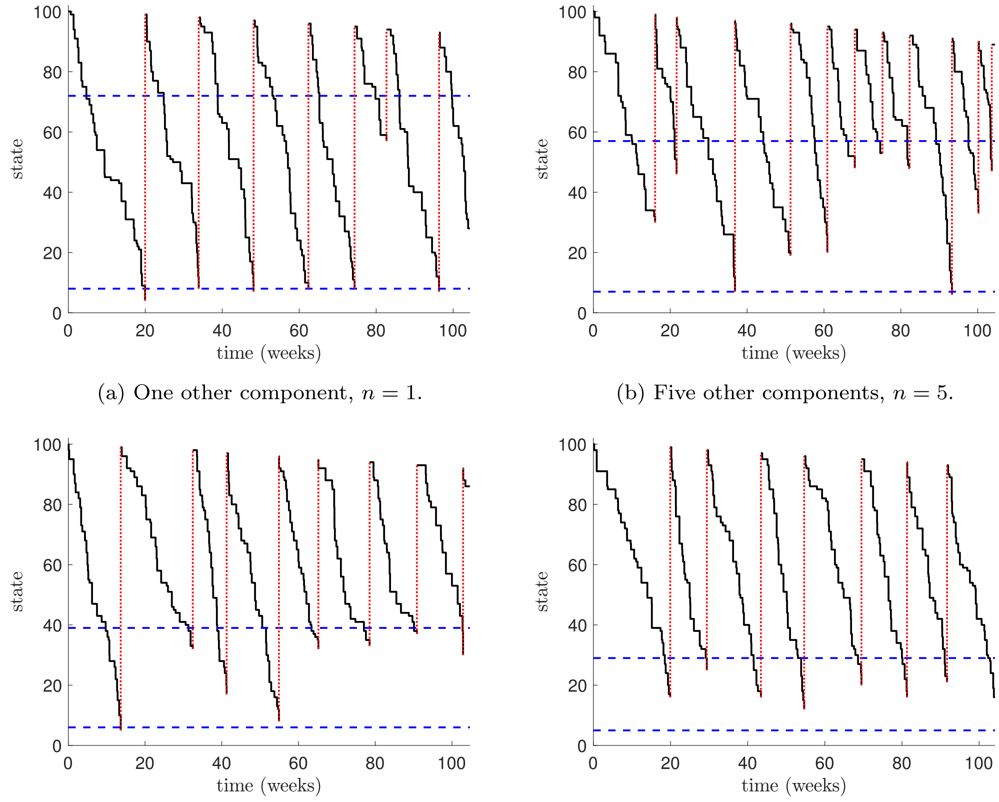
The value of the optimal can-repair threshold is also influenced by κ. For a low κ as in Fig. 11(a), preventive maintenance becomes highly desirable. Thus, the time window during which the component is eligible for such a repair should be rather high, which corresponds to a relatively high can-repair threshold.

4.3. Time evolution of state of component

The time evolution of the state of the component is simulated over two years by firstly determining the optimal must-repair and can-repair thresholds. During the simulations the state of the component once repaired is ϵ less than it was after the previous repair. The simulation parameters are listed in Table 3.

Table 3
Simulation parameters for Figs. 12 and 13.

S	F	λ	λ	S_{other}	s_{other}	η	K	P	ϵ
100	1	2	3	100	9	0.044	100	300	1



(a) One other component, $n = 1$.
(b) Five other components, $n = 5$.
(c) Fifteen other components, $n = 15$.
(d) Thirty other components, $n = 30$.

Fig. 12. Evolution of the state of the component for different values of n, $\kappa = 25$.

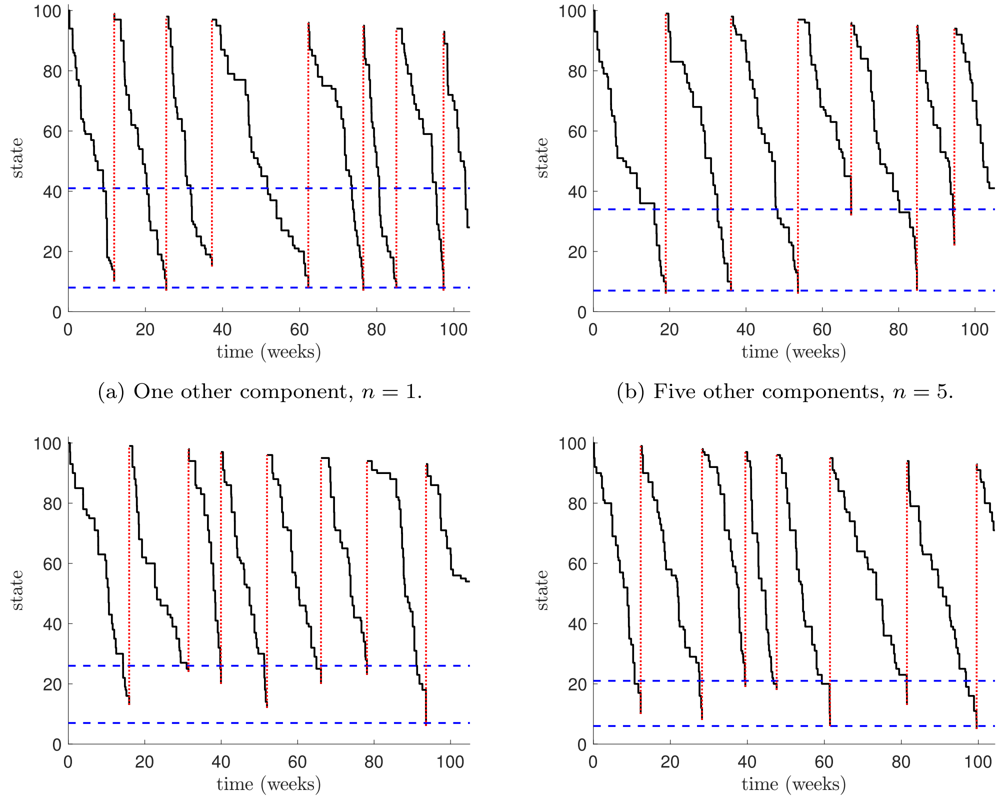
In Fig. 12, the time evolution of the state of the component is plotted (black solid) for $n = 1, 5, 15$ and 30. The figure depicts the must-repair and can-repair thresholds (blue dashed) and the instant repair (red dotted).

When μ is low (only one other component), the machine's repair is mostly triggered by its own deterioration but can also undergo a preventive repair while it is still in a good state. Actually, in Fig. 12(a) a preventive repair of cost $\kappa = 25$ was conducted while the component's state was 57. Also, the optimal thresholds are positioned in such a way that the probability of preventive maintenance for the relatively low costs $\kappa = 25$ is increasing, for an increasing amount of other components. Moreover, in Fig. 12(d), all maintenance actions are triggered by another component's corrective maintenance, thus resulting in all repairs only costing $\kappa = 25$.

It can be seen that the thresholds in Fig. 13 are similar to the optimal thresholds as shown in Fig. 11(b). The difference between the plots in Figs. 12 and 13 are the costs of preventive maintenance, $\kappa = 25$ and $\kappa = 60$, respectively. As it has been explained in Section 4.2, the optimal thresholds are non-increasing for an increasing amount of other components. This results in lower thresholds for the plots in Fig. 13 compared to the plots in Fig. 12. Resulting from this is the fact that, on average, the greater the ratio κ:K is, the more maintenance actions will be corrective instead of preventive for a greater amount of components. Lastly, it is concluded that the must-repair threshold is positioned in such a way that the probability of reaching the rather expensive failure state, $F = 1$, is very low.

4.4. Influence of costs

The influence of the costs of corrective and preventive maintenance as well as the costs of failure, K, κ and P, respectively, is simulated by determining the optimal values of the thresholds for the case of fifteen other components, $n = 15$. The simulation parameters are listed in Table 4.



(a) One other component, $n = 1$.

(b) Five other components, $n = 5$.

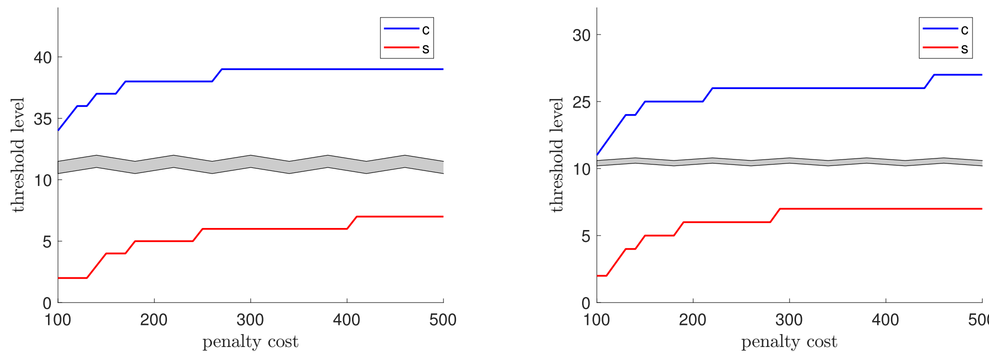
(c) Fifteen other components, $n = 15$.

(d) Thirty other components, $n = 30$.

Fig. 13. Evolution of the state of the component for different values of n, $\kappa = 60$.

Table 4
Simulation parameters for Figs. 14–16.

S	F	λ	λ	S_{other}	s_{other}	η	n	μ	K	κ	P
100	1	2	3	100	9	0.044	15	0.66	100	60	300

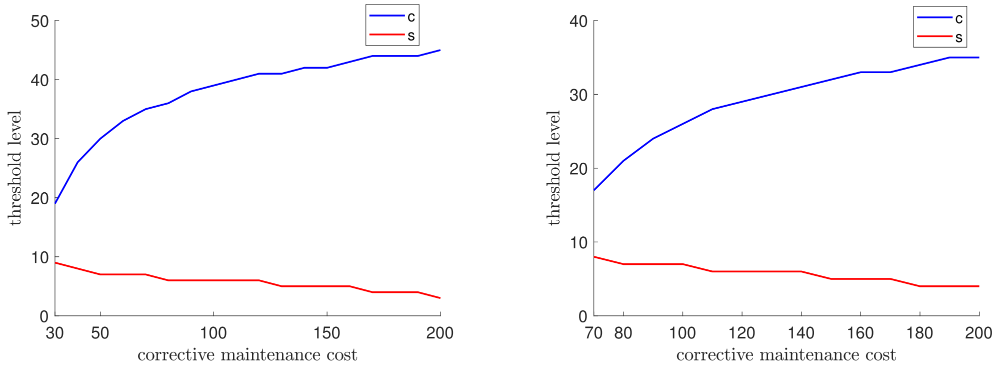


(a) Optimal threshold levels for $\kappa = 25$.

(b) Optimal threshold levels for $\kappa = 60$.

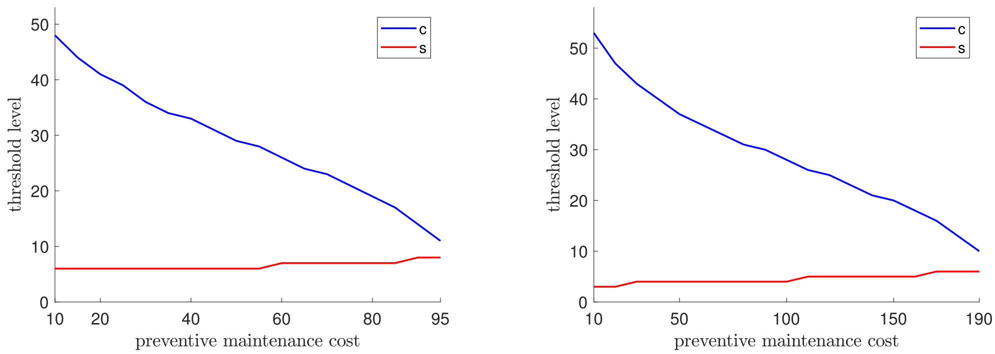
Fig. 14. Optimal must-repair and can-repair threshold levels for different penalty costs.

In Fig. 14, the optimal must-repair and can-repair thresholds for different penalty costs are shown for $\kappa = 25$ and $\kappa = 60$. It can be seen that in both cases the optimal must-repair and can-repair thresholds are monotonically non-decreasing for an increasing penalty cost. It can also be seen that both optimal thresholds follow roughly the same pattern. The monotonically non-increasing must-repair threshold can be explained by the fact that, for an increasing penalty cost,



(a) Optimal threshold levels for $\kappa = 25$. (b) Optimal threshold levels for $\kappa = 60$.

Fig. 15. Optimal must-repair and can-repair threshold levels for different amounts of corrective maintenance costs.



(a) Optimal thresholds for $K = 100$. (b) Optimal thresholds for $K = 200$.

Fig. 16. Optimal must-repair and can-repair threshold levels for different amounts of preventive maintenance costs.

the probability of having to pay the penalty cost should decrease. Thus, the must-repair threshold is higher when the ratio K:P is lower. The can-repair threshold is monotonically non-increasing due to the fact that the must-repair threshold is forced to increase due to the increasing penalty cost, which in turn forces the can-repair threshold to increase. As can be seen, the difference between the optimal thresholds for $\kappa = 25$ and $\kappa = 60$ is mainly the level of the can-repair threshold, which appears to be higher for a lower preventive maintenance cost. This is explained by the fact that, for a lower preventive maintenance cost, it is beneficial to have a high probability of the component undergoing preventive maintenance instead of corrective maintenance. Thus, the difference $c - s$ is greater when the ratio κ:K is lower.

In Fig. 15, the optimal must-repair and can-repair thresholds for different amounts of corrective maintenance costs are shown for $\kappa = 25$ and $\kappa = 60$. It can be seen that in both cases the optimal must-repair threshold is monotonically non-increasing, whereas the optimal can-repair threshold is monotonically non-decreasing for an increasing corrective maintenance cost. The optimal must-repair threshold is monotonically non-increasing due to the fact that it is mainly influenced by the penalty cost P and its ratio to the corrective maintenance cost, as has been explained in the previous paragraph. The optimal can-repair threshold is monotonically non-decreasing due to the fact that, for an increasing corrective maintenance cost, the probability of the component requiring corrective maintenance should be decreased. Furthermore, for a higher corrective maintenance cost, the probability of the component undergoing the relatively cheap preventive maintenance is higher, since the difference $c - s$ is larger. Lastly, analogously to the last argument, it can be seen that the average value of the can-repair threshold is higher when the ratio κ:K is lower.

In Fig. 16, the optimal must-repair and can-repair thresholds for different amounts of preventive maintenance costs are shown for $K = 100$ and $K = 200$. In both cases the optimal must-repair threshold is monotonically non-decreasing whereas the optimal can-repair threshold is monotonically non-increasing for an increasing preventive maintenance cost. The optimal must-repair threshold is monotonically non-decreasing due to the fact that, for a lower can-repair threshold, the probability of reaching the failure state is greater. To reduce the probability of the system to reach the failure state, the must-repair threshold needs to be higher. The optimal can-repair threshold is monotonically non-increasing due to the fact that the ratio κ:K is increasing for an increasing preventive maintenance cost.

Table 5
Simulation parameters for Figs. 17 and 18.

S	F	λ	λ	S_{other}	s_{other}	η	n	μ	K	κ	P
100	1	2	3	100	9	0.044	15	0.66	100	60	500

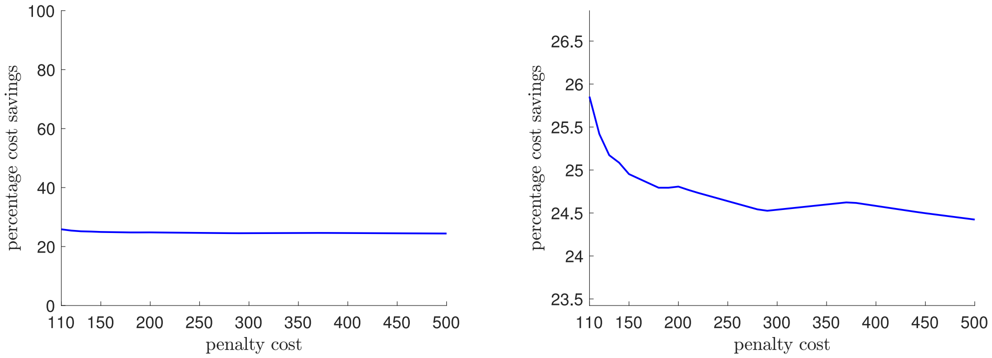
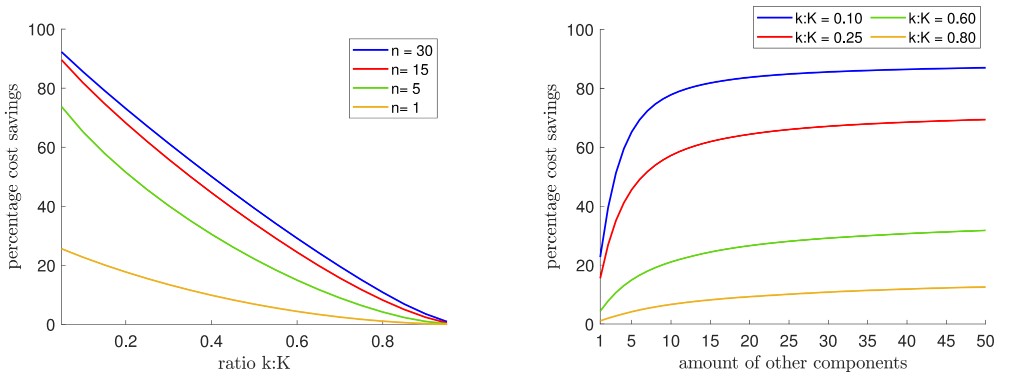


Fig. 17. Cost savings in percentages over the penalty costs.



(a) Cost savings in percentages over the ratio $\kappa : K$ for different amounts of other components.

(b) Cost savings in percentages over the amount of other components for specific ratios of $\kappa : K$.

Fig. 18. Cost savings in percentages over ratios κ:K and amounts of other components.

4.5. Cost savings

The optimal coordinated (S, c, s, F)-policy that is obtained in this research is compared to the optimal independent (S, s, F)-policy where all components are controlled independently assuming that the cost of corrective maintenance is K, similar to the cost of corrective maintenance of the coordinated control problem. The influence of the penalty costs, ratio κ:K and the amount of other machines is simulated by determining the long-run average cost for both the coordinated and independent control problem, where all parameters are similar. The simulation parameters are listed in Table 5.

In Fig. 17, the percentage of cost savings is plotted over different values of the penalty costs. It can be seen that the influence that the penalty costs have on the cost savings are marginal. This is mainly due to the fact that the optimal policies reduce the probability of the component reaching the failure state to a very small amount. For this reason, the penalty costs will be set to $P = 500$ for the rest of the simulations in this section. In this way, the influence that the ratio κ:K and the amount of other components have on the cost savings can be analysed accurately.

In Fig. 18(a), the percentage of cost savings is plotted over different values of the ratio κ:K for four different amounts of other components, namely $n = 1, 5, 15$ and 30. In Fig. 18(b), the percentage of cost savings is plotted over different amounts of components for four different ratios κ:K, namely $\kappa : K = 0.10, 0.25, 0.60$ and 0.80. First of all, it can be seen that for small ratios κ:K the cost savings are large. This is mainly due to the fact that for the coordinated control rule there is a possibility of having reduced setup costs. As has been explained in previous sections, for small ratios κ:K the optimal coordinated control rule ensures that the component under review is mainly repaired by means of preventive maintenance with reduced setup cost κ. Evidently, for an independent control rule, there is no possibility of having reduced setup costs and the setup costs are always K. Secondly, it can be seen that cost savings are higher for a greater amount of other components. This is mainly due to the fact that for a larger amount of other components, the chance of a maintenance action being preventive is larger.

5. Conclusions and future research

In this research, an algorithm has been developed that optimizes maintenance thresholds to minimize long-run average costs in a multi-component system with compound Poisson deterioration.

The policy-iteration used in this research consists of three steps, namely value-determination, policy-improvement and the convergence test. Firstly, the value of the current policy is evaluated by determining the relative values and long-run average costs. Secondly, the policy is improved by means of the policy-improvement step. Finally, a convergence test is performed where the optimal policy has been achieved if the newest policy $\bar{R}$ is equal to the previous policy R.

The main insights gained from this analysis are that the percentage of cost savings that can be achieved is mainly dependent on the relative difference between the preventive cost κ and corrective maintenance cost K. In particular, the higher the percentage cost savings the lower the ratio κ:K. Furthermore, the cost savings increases with the number of components in the system. It is also concluded that the must-repair threshold is inverse proportional to the ratio K:P. Lastly, it is concluded that the lower the ratio κ:K is, the greater the difference between the optimal must-repair and can-repair thresholds is.

For further research, it is suggested to extend the model by incorporating non-zero lead-time. Furthermore, instead of approximating the superposition of the maintenance opportunities triggered by another machine's corrective maintenance action, a model could be created to account for different deterioration rates and setup costs. Note that in this work the deterioration rate λ is fixed over time, and as a result the rate that determines the special repair opportunities μ is also fixed. Therefore, as we one can see in the Simulation section, the thresholds of the (S, c, s, F)-policy do not change over time. An interesting extension for further research is to analyse a dynamic behaviour of λ and how it affects the (S, c, s, F)-policy and the long-run average cost.

CRediT authorship contribution statement

Pim Rombouts: Conceptualization, Development of the model as well as the theoretical and numerical analysis. **Stefanny Ramirez:** Critical appraisal of the literature, Collocation of the paper within the existing literature, Standardization of the nomenclature and language, Development of the theoretical analysis and clarification and interpretation of the numerical findings. **Dario Bauso:** Conceptualization, Model development, Theoretical analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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