

NEW PARTITION IDENTITIES FROM $C_\ell^{(1)}$ -MODULES

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ABSTRACT. In this paper we conjecture combinatorial Rogers-Ramanujan type colored partition identities related to standard representations of the affine Lie algebra of type $C_\ell^{(1)}$, $\ell \geq 2$, and we conjecture similar colored partition identities with no obvious connection to representation theory of affine Lie algebras.

1. INTRODUCTION

The seminal work [15] of J. Lepowsky and R. Wilson on a Lie-theoretic interpretation of the Rogers-Ramanujan identities led to the discovery of numerous new combinatorial identities, like in [6] or [16]. Recently several identities in the style of the Rogers-Ramanujan identities related to the representation theory of affine Lie algebras have appeared, let us mention only [13] and [5]. On the other side, some parts of representation theory lead to Rogers-Ramanujan type *colored* partition identities, let us mention only [7] and [18] for this vein of research.

In this paper we conjecture combinatorial Rogers-Ramanujan type colored partition identities related to standard representations of the affine Lie algebra of type $C_\ell^{(1)}$, $\ell \geq 2$, and we conjecture similar colored partition identities with no obvious connection to representation theory of affine Lie algebras.

In Section 3 we start with the array of natural numbers \mathcal{N}_5 composed as a multiset of two copies of the set of natural numbers \mathbb{N} and the additional set of odd numbers, arranged in 5 rows, with diagonals of width $w = 5$. Such an array appears naturally in the representation theory of the affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$ of type $C_2^{(1)}$ and it is expected that colored partitions

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on \mathcal{N}_5 , satisfying certain difference and initial conditions, should parametrize bases of standard $\widehat{\mathfrak{g}}$ -modules—see Remark 3.4 below. In Conjecture 3.3 we guess, inspired by [20], a possible form of these colored partitions and, by a computing experiment, we conjecture the corresponding Rogers-Ramanujan type colored partition identities. Numerical evidence supports the conjecture for colored partitions on the array $\mathcal{N}_{2\ell+1}$, related to all standard modules for affine Lie algebra of type $C_\ell^{(1)}$, $\ell \geq 2$.

In Section 4 we start with the array of natural numbers \mathcal{N}_4 composed as a multiset of two copies of the set of natural numbers \mathbb{N} , arranged in 4 rows, with diagonals of width $w = 4$. In analogy with the “ w odd case”, in Conjecture 4.1 we conjecture similar colored partition identities for all $w = 2\ell$, $\ell \geq 2$, with no obvious connection to representation theory of affine Lie algebras, but again supported by numerical evidence.

The arrays of natural numbers

$$\mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5, \mathcal{N}_6, \mathcal{N}_7, \dots$$

form a natural sequence and for each $\mathcal{N}_{2\ell}$ and $\mathcal{N}_{2\ell+1}$, $\ell \geq 1$, and all nonnegative integers k_0, k_1, \dots, k_ℓ , $k = k_0 + k_1 + \dots + k_\ell > 0$, we have a class of $(k_0, k_1, \dots, k_\ell)$ -admissible colored partitions (defined in (3.3)–(3.5) and (4.3)–(4.5) below) for which we conjecture Rogers-Ramanujan type combinatorial identities. For $w = 2\ell = 2$ and $k = 1$ we have the two Rogers-Ramanujan identities, and for $w = 2\ell + 1 = 3$ and $k = 1$ we have two identities (in some sense) equivalent to the two Capparelli identities¹.

In Section 5 we describe an algorithm for constructing admissible colored partitions, and in the Appendix we give a Python code for counting admissible colored partitions (available at <https://github.com/aprimc/new-partition-identities>).

2. LEPOWSKY’S PRODUCT FORMULA

In this section we give the Lie theoretic origin of the product expression in Conjecture 3.3 below. Note however that Conjecture 3.3 is a purely combinatorial statement.

By $C_\ell^{(1)}$ we denote the affine Lie algebra of type $C_\ell^{(1)}$, see e.g. [12], Table Aff 1, page 54 and (7.2.1), page 98. By $L(k_0, \dots, k_\ell)$ we denote the irreducible highest weight module $L(\Lambda)$ with highest weight $\Lambda = k_0\Lambda_0 + \dots + k_\ell\Lambda_\ell$, see e.g. [12], page 147. The principally specialized character $\text{ch}_q L(k_0, \dots, k_\ell)$ is defined in [12], page 152 together with §10.9, page 181. Associated to $L(k_0, \dots, k_\ell)$ set $k = k_0 + k_1 + \dots + k_\ell$.

Let $s_0, s_1, \dots, s_\ell \in \mathbb{N}$, $s = s_0 + s_1 + \dots + s_\ell$. We define the following triangular scheme of natural numbers:

¹See Remarks 3.4 and 4.3 below.

$$\begin{aligned}
 D(s_0, s_1, \dots, s_\ell) &= \{s_0, \dots, 2s - s_0\} \quad \text{has } 2\ell + 1 \text{ ascending numbers with} \\
 &\quad \text{increments } s_1, s_2, \dots, s_\ell, s_\ell, \dots, s_2, s_1, \\
 D(s_1, \dots, s_\ell) &= \{s_1, \dots\} \quad \text{has } 2\ell - 1 \text{ ascending numbers with increments} \\
 &\quad s_2, \dots, s_\ell, s_\ell, \dots, s_2, \\
 D(s_2, \dots, s_\ell) &= \{s_2, \dots\} \quad \text{has } 2\ell - 3 \text{ ascending numbers with increments} \\
 &\quad s_3, \dots, s_\ell, s_\ell, \dots, s_3, \\
 &\quad \vdots \\
 D(s_{\ell-1}, s_\ell) &= \{s_{\ell-1}, s_{\ell-1} + s_\ell, s_{\ell-1} + 2s_\ell\}, \\
 D(s_\ell) &= \{s_\ell\}
 \end{aligned}$$

EXAMPLE 2.1. For example, for $(s_0, s_1, s_2, s_3, s_4) = (3, 2, 1, 1, 2)$ we have

$$\begin{array}{rcccccccc}
 D(3, 2, 1, 1, 2) & & 3 & 5 & 6 & 7 & 9 & 11 & 12 & 13 & 15 \\
 D(2, 1, 1, 2) & & & 2 & 3 & 4 & 6 & 8 & 9 & 10 & \\
 D(1, 1, 2) & = & & & 1 & 2 & 4 & 6 & 7 & & . \\
 D(1, 2) & & & & & 1 & 3 & 5 & & & \\
 D(2) & & & & & & & 2 & & &
 \end{array}$$

Note that we can obtain $D(s_{r+1}, \dots, s_\ell)$ from $D(s_r, \dots, s_\ell)$ by eliminating endpoints in $D(s_r, \dots, s_\ell)$ and then subtracting s_r from the remaining elements. We define *the congruence triangle* as the multiset

$$\Delta(s_1, \dots, s_\ell) = D(s_1, \dots, s_\ell) \cup D(s_2, \dots, s_\ell) \cup \dots \cup D(s_\ell)$$

and we denote by $\{0\}^\ell$ the multiset consisting of ℓ copies of 0.

Now we can state *Lepowsky's product formula* for $C_\ell^{(1)}$, $\ell \geq 2$ (cf. [14], [4]):

$$\text{ch}_q L_{C_\ell^{(1)}}(k_0, k_1, \dots, k_\ell) = \frac{\prod_{a \in \{0\}^\ell \cup D(k_0+1, k_1+1, \dots, k_\ell+1); b \in \Delta(k_1+1, \dots, k_\ell+1); j \equiv a, \pm b \pmod{2\ell+2k+2}} (1 - q^j)}{\prod_{j \text{ odd}} (1 - q^j) \prod_{j \in \mathbb{N}} (1 - q^j)^\ell}.$$

EXAMPLE 2.2. By using Example 2.1 we see that the principally specialized character $\text{ch}_q L_{C_4^{(1)}}(2, 1, 0, 0, 1)$ is equal to

$$\prod_{j \text{ odd}; j \equiv 1, 1, 2, 3, 4, 4, 5, 5, 6, 7, 7, 8, 8, 9, 10, 10, 11, 11, 12, 13, 13, 14, 14, 15, 16, 17, 17 \pmod{18}} (1 - q^j)^{-1}.$$

Note that in this product the factor $(1 - q^1)^{-1}$ appears three times.

3. ARRAYS WITH ODD WIDTH $w \geq 5$

Let $\mathcal{N} = \mathcal{N}_5 = \mathcal{N}_{C_2^{(1)}}$ be the array of natural numbers

$$(3.1) \quad \begin{array}{cccccccc} 1 & 3 & 5 & 7 & 9 & & & \\ & 2 & 4 & 6 & 8 & 10 & & \\ 1 & 3 & 5 & 7 & 9 & \dots & & \\ & 2 & 4 & 6 & 8 & 10 & & \\ 1 & 3 & 5 & 7 & 9 & & & \end{array}$$

This array consists of two copies of the set of natural numbers \mathbb{N} and the additional set of odd numbers, arranged in 5 rows, with diagonals of width $w = 5$. Numbers increase by one going to the right on any diagonal. We shall consider elements in these sets as different, say “colored” by their position in the array. For example, the number 7 appears three times on three different places of the array \mathcal{N} and we consider 7 in the first row different from the other two. We say that two elements in an array are *adjacent* if they are simultaneously on two adjacent rows and two adjacent diagonals. For example, 6 and 8 in the second row are adjacent to 7 in the first row and, just as well, adjacent to 7 in the third row. We say that the set $\{a_1, a_2, a_3, \dots\}$ is a downward path \mathcal{Z} in an array if a_i is in the i -th row and if (a_i, a_{i+1}) is a pair of two adjacent elements for all i . For example, $\mathcal{Z} = \{7, 6, 5, 4, 5\}$ is a downward path in \mathcal{N} and there are altogether 2^4 downward paths through 7 in the first row. In Section 5 we shall consider downward paths which start from the top row, but need not reach the bottom row. $\mathcal{Z} = \{7, 6, 5, 4\}$ is an example of such downward path—we shall say that this $\mathcal{Z} = \{7, 6, 5, 4\}$ ends at 4 in the fourth row.

We consider colored partitions

$$(3.2) \quad n = \sum_{a \in \mathcal{N}} f_a \cdot a,$$

where f_a is the frequency of the part $a \in \mathcal{N}$ in the colored partition (3.2) of n . Let $k_0, k_1, k_2 \in \mathbb{N}_0$, $k = k_0 + k_1 + k_2 > 0$. We say that an array of frequencies \mathcal{F}

$$(3.3) \quad \begin{array}{cccccccc} f_1 & & f_3 & & f_5 & & f_7 & \\ & f_2 & & f_4 & & f_6 & & f_8 \\ f_1 & & f_3 & & f_5 & & f_7 & \dots \\ & f_2 & & f_4 & & f_6 & & f_8 \\ f_1 & & f_3 & & f_5 & & f_7 & \end{array}$$

is (k_0, k_1, k_2) -admissible if the extended array of frequencies $\mathcal{F}^{(k_0, k_1, k_2)}$

$$(3.4) \quad \begin{array}{cccccc} k_2 & f_1 & f_3 & f_5 & f_7 & \\ & 0 & f_2 & f_4 & f_6 & f_8 \\ k_1 & f_1 & f_3 & f_5 & f_7 & \dots \\ & 0 & f_2 & f_4 & f_6 & f_8 \\ k_0 & f_1 & f_3 & f_5 & f_7 & \end{array}$$

satisfies the *difference condition*

$$(3.5) \quad \sum_{m \in \mathcal{Z}} m \leq k$$

for all downward paths \mathcal{Z} in $\mathcal{F}^{(k_0, k_1, k_2)}$. So, for example, f_1 in the first row must be $\leq k_2$ because of (3.5) for the downward path $\mathcal{Z} = \{f_1, 0, k_1, 0, k_0\}$. We say that colored partitions (3.2) with (k_0, k_1, k_2) -admissible arrays of frequencies (3.3) are (k_0, k_1, k_2) -admissible colored partitions.

As explained, we consider the natural numbers at different places in the array (3.1) as different. Likewise, in the array of frequencies (3.3) the entry f_1 in the first row denotes the frequency of the part 1 in the first row, which may be different from the entry f_1 in the third row denoting the frequency of the part 1 in the third row. However, sometimes we need to write explicitly the coloring of elements in the array (3.1) and change the notation of frequency arrays (3.3) and extended frequency arrays (3.4) accordingly, like in the following example.

EXAMPLE 3.1. Here we list all $(2, 0, 0)$ -admissible colored partitions for $n \leq 8$. First we write explicitly one possible coloring of elements in the array (3.1)

$$(3.6) \quad \begin{array}{cccccc} 1_1 & 3_1 & 5_1 & 7_1 & 9_1 & \\ & 2_1 & 4_1 & 6_1 & 8_1 & 10_1 \\ 1_2 & 3_2 & 5_2 & 7_2 & 9_2 & \dots \\ & 2_2 & 4_2 & 6_2 & 8_2 & 10_2 \\ 1_3 & 3_3 & 5_3 & 7_3 & 9_3 & \end{array}$$

and change the notation for the extended array of frequencies (3.4) accordingly

$$(3.7) \quad \begin{array}{cccccc} 0 & f_{1_1} & f_{3_1} & f_{5_1} & f_{7_1} & \\ & 0 & f_{2_1} & f_{4_1} & f_{6_1} & f_{8_1} \\ 0 & f_{1_2} & f_{3_2} & f_{5_2} & f_{7_2} & \dots \\ & 0 & f_{2_2} & f_{4_2} & f_{6_2} & f_{8_2} \\ 2 & f_{1_3} & f_{3_3} & f_{5_3} & f_{7_3} & \end{array}$$

Then we have $(2, 0, 0)$ -admissible colored partitions for $n \leq 8$:

$$\begin{aligned}
 1 &= 1_3 \\
 2 &= 2_2 = 1_3 + 1_3 \\
 3 &= 3_2 = 3_3 = 2_2 + 1_3 \\
 4 &= 4_1 = 4_2 = 3_2 + 1_3 = 3_3 + 1_3 = 2_2 + 2_2 \\
 5 &= 5_1 = 5_2 = 5_3 = 4_1 + 1_3 = 4_2 + 1_3 = 3_2 + 2_2 = 3_3 + 2_2 = 3_3 + 1_3 + 1_3 \\
 6 &= 6_1 = 6_2 = 5_1 + 1_3 = 5_2 + 1_3 = 5_3 + 1_3 = 4_1 + 2_2 = 4_2 + 2_2 = 4_2 + 1_3 + 1_3 \\
 7 &= 7_1 = 7_2 = 7_3 = 6_1 + 1_3 = 6_2 + 1_3 = 5_1 + 2_2 = 5_2 + 2_2 = 5_3 + 2_2 \\
 &= 5_2 + 1_3 + 1_3 = 5_3 + 1_3 + 1_3 = 4_1 + 3_2 = 4_1 + 3_3 = 4_2 + 3_2 = 4_2 + 3_3 \\
 &= 4_2 + 2_2 + 1_3 = 3_2 + 3_3 + 1_3 = 3_3 + 3_3 + 1_3 \\
 8 &= 8_1 = 8_2 = 7_1 + 1_3 = 7_2 + 1_3 = 7_3 + 1_3 = 6_1 + 2_2 = 6_2 + 2_2 = 6_1 + 1_3 + 1_3 \\
 &= 6_2 + 1_3 + 1_3 = 5_1 + 3_2 = 5_1 + 3_3 = 5_2 + 3_2 = 5_2 + 3_3 = 5_3 + 3_2 = 5_3 + 3_3 \\
 &= 5_2 + 2_2 + 1_3 = 5_3 + 2_2 + 1_3 = 4_1 + 4_1 = 4_1 + 4_2 = 4_2 + 4_2 = 4_1 + 3_3 + 1_3 \\
 &= 4_2 + 3_3 + 1_3 = 4_2 + 3_2 + 1_3 = 4_2 + 2_2 + 2_2 = 3_3 + 3_3 + 1_3 + 1_3 .
 \end{aligned}$$

Note that for any $(2, 0, 0)$ -admissible colored partition $n = \sum_{a \in \mathcal{N}} f_a \cdot a$ difference conditions (3.5) imply

$$f_{1_1} = f_{2_1} = f_{3_1} = f_{1_2} = 0.$$

Also note that, for example, $8 = 3_3 + 3_3 + 1_3 + 1_3$ is a $(2, 0, 0)$ -admissible colored partition, and $8 = 3_2 + 3_2 + 1_3 + 1_3$ is not since difference condition (3.5) is violated:

$$f_{3_2} + f_{1_3} = 2 + 2 = 4 > 2.$$

REMARK 3.2. The partitions in the above example are (essentially) the partitions in Example 3 in [19]; only the coloring of the array (3.1) is different.

We extend these notions for $C_\ell^{(1)}$, $\ell \geq 1$, by starting with the array $\mathcal{N} = \mathcal{N}_{2\ell+1} = \mathcal{N}_{C_\ell^{(1)}}$ of ℓ copies of the set of natural numbers \mathbb{N} and the additional set of odd numbers, arranged in $2\ell + 1$ rows and diagonals of width $w = 2\ell + 1$; with numbers in \mathcal{N} increasing by one going to the right on any diagonal. For example, for $\ell = 3$ we have the extended array of frequencies $\mathcal{F}^{(k_0, k_1, k_2, k_3)}$

$$\begin{array}{cccccc}
 k_3 & f_1 & f_3 & f_5 & f_7 & & \\
 & 0 & f_2 & f_4 & f_6 & f_8 & \\
 k_2 & f_1 & f_3 & f_5 & f_7 & & \\
 & 0 & f_2 & f_4 & f_6 & f_8 & \dots \\
 k_1 & f_1 & f_3 & f_5 & f_7 & & \\
 & 0 & f_2 & f_4 & f_6 & f_8 & \\
 k_0 & f_1 & f_3 & f_5 & f_7 & &
 \end{array}
 \tag{3.8}$$

and the corresponding notion of (k_0, k_1, k_2, k_3) -admissible colored partitions on the array \mathcal{N}_7 .

CONJECTURE 3.3. *Let $\ell \geq 2$. The principally specialized character*

$$\frac{\prod_{a \in \{0\}^\ell \cup D(k_0+1, k_1+1, \dots, k_\ell+1); b \in \Delta(k_1+1, \dots, k_\ell+1); j \equiv a, \pm b \pmod{2\ell+2k+2}} (1 - q^j)}{\prod_{j \text{ odd}} (1 - q^j) \prod_{j \in \mathbb{N}} (1 - q^j)^\ell}$$

is the generating function for the number of $(k_0, k_1, \dots, k_\ell)$ -admissible partitions

$$n = \sum_{a \in \mathcal{N}_{2\ell+1}} f_a \cdot a.$$

REMARK 3.4. Conjecture 3.3 is true for $(1, 0, \dots, 0)$ -admissible colored partitions and any $\ell \geq 2$ (see [18]). In [19] Conjecture 3.3 is formulated for $(k, 0, \dots, 0)$ -admissible colored partitions with $\ell \geq 2$ and $k \geq 2$, and it was checked for some small values of ℓ, k and n (cf. [17]).

In [16] the product formula for the generating function for the number of (k_0, k_1) -admissible partitions on the array \mathcal{N}_3 is given; in the $\ell = 1$ case it is Lepowsky’s product formula for $A_1^{(1)}$ and the key ingredient—a construction of combinatorial bases of standard $A_1^{(1)}$ -modules—is independently obtained in [8] and [9].

The combinatorial identities for $(1, 0)$ -admissible partitions and for $(0, 1)$ -admissible partitions are equivalent² to the two Capparelli identities [6]. Moreover, it seems that these identities are related to the purely combinatorial approach in [1], based on Capparelli’s identity which was found using representation theory of $A_2^{(2)}$.

EXAMPLE 3.5. Conjecture 3.3 is based on a computer experiment. Here we present some results for admissible partitions for n up to 20.

We write

$$(1, 0, 1) \sim (r \equiv 1, 1, 3, 4, 4, 6, 6, 7, 9, 9 \pmod{10})$$

if the number of $(1, 0, 1)$ -admissible colored partitions of n is equal to the number of colored partitions of n with parts $r \equiv 1, 1, 3, 4, 4, 6, 6, 7, 9, 9 \pmod{10}$ for $n \leq 20$ (here $r \equiv 1, 1 \pmod{10}$ means that parts $r \equiv 1 \pmod{10}$ come in two colors). Since the parameters (k_0, k_1, k_2) and (k_2, k_1, k_0) give “isomorphic” colored partitions, we list below only mutually different conjectured identities.

²In [16] certain spanning sets $B(\Lambda_0) \subset L(\Lambda_0)$ and $B(\Lambda_1) \subset L(\Lambda_1)$ of the two fundamental $A_1^{(1)}$ -modules are constructed. The following three statements are equivalent: (i) $B(\Lambda_0)$ and $B(\Lambda_1)$ are linearly independent, (ii) the generating functions for $(1, 0)$ -admissible and $(0, 1)$ -admissible partitions are the principally specialized characters $\text{ch}_q L(\Lambda_0)$ and $\text{ch}_q L(\Lambda_1)$, and (iii) the two Capparelli identities hold.

For $w = 5$ we have:

$$\begin{aligned} (1, 0, 0) &\sim (r \text{ odd}; r \equiv 4 \pmod{8}), \\ (2, 0, 0) &\sim (r \text{ odd}; r \equiv 2, 4, 5, 6, 8 \pmod{10}), \\ (1, 1, 0) &\sim (r \text{ odd}; r \equiv 1, 3, 5, 7, 9 \pmod{10}), \\ (1, 0, 1) &\sim (r \equiv 1, 1, 3, 4, 4, 6, 6, 7, 9, 9 \pmod{10}), \\ (0, 2, 0) &\sim (r \equiv 1, 2, 2, 3, 3, 7, 7, 8, 8, 9 \pmod{10}), \\ (3, 0, 0) &\sim (r \text{ odd}; r \equiv 2, 3, 4, 5, 6, 7, 8, 9, 10 \pmod{12}), \\ (2, 1, 0) &\sim (r \text{ odd}; r \equiv 1, 2, 4, 5, 6, 7, 8, 10, 11 \pmod{12}), \\ (2, 0, 1) &\sim (r \text{ odd}; r \equiv 1, 2, 4, 5, 6, 7, 8, 10, 11 \pmod{12}), \\ (1, 2, 0) &\sim (r \text{ odd}; r \equiv 1, 2, 3, 4, 6, 8, 9, 10, 11 \pmod{12}), \\ (0, 3, 0) &\sim (r \text{ odd}; r \equiv 2, 3, 4, 5, 6, 7, 8, 9, 10 \pmod{12}), \end{aligned}$$

$$\begin{aligned} (3, 0, 1) &\sim (r \text{ odd}; r \equiv 1, 2, 3, 4, 6, 6, 7, 8, 8, 10, 11, 12, 13 \pmod{14}), \\ (2, 1, 1) &\sim (r, r \text{ odd}; r \equiv 1, 4, 6, 8, 10, 13 \pmod{14}), \\ (2, 0, 2) &\sim (r \equiv 1, 1, 2, 2, 3, 5, 5, 5, 6, 6, 8, 8, 9, 9, 9, 11, 12, 12, 13, 13 \pmod{14}), \\ (0, 4, 0) &\sim (r \equiv 1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 9, 9, 10, 10, 11, 11, 11, 12, 12, 13 \pmod{14}), \\ (3, 0, 2) &\sim (r, r \text{ odd}; r \equiv 2, 2, 6, 6, 8, 10, 10, 14, 14 \pmod{16}). \end{aligned}$$

For $w = 7$ we have:

$$\begin{aligned} (1, 0, 0, 1) &\sim (r \text{ odd}; r \equiv 1, 3, 4, 5, 7, 8, 9, 11 \pmod{12}), \\ (0, 1, 1, 0) &\sim (r \text{ odd}; r \equiv 1, 2, 3, 5, 7, 9, 10, 11 \pmod{12}), \\ (2, 0, 0, 1) &\sim (r \text{ odd}; r \equiv 1, 2, 3, 4, 5, 6, 6, 8, 8, 9, 10, 11, 12, 13 \pmod{14}), \\ (2, 0, 0, 2) &\sim (r, r \text{ odd}; r \equiv 2, 2, 4, 6, 6, 6, 10, 10, 10, 12, 14, 14 \pmod{16}). \end{aligned}$$

For $w = 9$ we have:

$$\begin{aligned} (1, 0, 0, 0, 1) &\sim (r \equiv 1, 1, 3, 3, 4, 4, 5, 6, 6, 8, 8, 9, 10, 10, 11, 11, 13, 13 \pmod{14}), \\ (0, 1, 1, 0, 0) &\sim (r \text{ odd}; r \equiv 1, 2, 3, 4, 5, 7, 9, 10, 11, 12, 13 \pmod{14}), \\ (0, 1, 1, 0, 1) &\sim (r \text{ odd}; r \equiv 1, 1, 2, 3, 4, 4, 6, 6, 7, 8, \\ &\quad 9, 10, 10, 12, 12, 13, 14, 15 \pmod{16}), \\ (2, 1, 0, 0, 1) &\sim (r, r \text{ odd}; r \equiv 1, 2, 4, 4, 5, 6, 7, 8, 8, \\ &\quad 10, 10, 11, 12, 13, 14, 14, 16, 17 \pmod{18}). \end{aligned}$$

EXAMPLE 3.6. In the following identities, the right hand sides can be interpreted as the number of colored partitions satisfying congruence conditions, with the extra requirement that in one color the parts have to be distinct.

Associated to the module $L(0, 1, 0)$ we have that, for $n \leq 20$, the number of $(0, 1, 0)$ -admissible colored partitions of n equals

$$\text{coeff}_{q^n} \prod_{r \equiv 2 \pmod 4} (1 + q^r) / \prod_{r \equiv 1, 3, 5, 7 \pmod 8} (1 - q^r).$$

Associated to the module $L(1, 1, 1)$ we have that, for $n \leq 20$, the number of $(1, 1, 1)$ -admissible colored partitions of n equals

$$\text{coeff}_{q^n} \prod_{r \text{ odd}} (1 + q^r) / \prod_{r \text{ odd}} (1 - q^r)^2.$$

4. ARRAYS WITH EVEN WIDTH $w \geq 4$

Let $\mathcal{N}^e = \mathcal{N}_4$ be the array of natural numbers

$$(4.1) \quad \begin{array}{cccccc} 1 & 3 & 5 & 7 & 9 & \\ & 2 & 4 & 6 & 8 & 10 \\ 1 & 3 & 5 & 7 & 9 & \cdots \\ & 2 & 4 & 6 & 8 & 10 \end{array}$$

This array consists of two copies of the set of natural numbers \mathbb{N} arranged in 4 rows, with diagonals of width $w = 4$. We consider colored partitions

$$(4.2) \quad n = \sum_{a \in \mathcal{N}^e} f_a \cdot a,$$

where f_a is the frequency of the part $a \in \mathcal{N}^e$ in the colored partition (4.2) of n . Let $k_0, k_1, k_2 \in \mathbb{N}_0$, $k = k_0 + k_1 + k_2 > 0$. We say that an array of frequencies \mathcal{F}^e

$$(4.3) \quad \begin{array}{cccccc} f_1 & f_3 & f_5 & f_7 & & \\ & f_2 & f_4 & f_6 & f_8 & \cdots \\ f_1 & f_3 & f_5 & f_7 & & \\ & f_2 & f_4 & f_6 & f_8 & \end{array}$$

is $(k_0, k_1, k_2)^e$ -admissible if the extended array of frequencies $\mathcal{F}^{e(k_0, k_1, k_2)}$

$$(4.4) \quad \begin{array}{cccccc} k_2 & f_1 & f_3 & f_5 & f_7 & \\ & 0 & f_2 & f_4 & f_6 & f_8 \cdots \\ k_1 & f_1 & f_3 & f_5 & f_7 & \\ & k_0 & f_2 & f_4 & f_6 & f_8 \end{array}$$

satisfies the *difference condition*

$$(4.5) \quad \sum_{m \in \mathcal{Z}} m \leq k$$

for all downward paths \mathcal{Z} in $\mathcal{F}^{e(k_0, k_1, k_2)}$. So, for example, f_1 in the first row must be $\leq k_2$ because of (4.5) for downward path $\mathcal{Z} = \{f_1, 0, k_1, k_0\}$. We say that colored partitions (4.2) with $(k_0, k_1, k_2)^e$ -admissible arrays of frequencies (4.3) are $(k_0, k_1, k_2)^e$ -admissible colored partitions.

We extend these notions for $\ell \geq 1$ by starting with the array $\mathcal{N}^e = \mathcal{N}_{2\ell}$ of ℓ copies of the set of natural numbers \mathbb{N} arranged in 2ℓ rows and diagonals of width $w = 2\ell$; with numbers in \mathcal{N}^e increasing by one going to the right on any diagonal. For example, for $\ell = 3$ ($w = 6$) instead of (4.4) we have the extended array of frequencies $\mathcal{F}^{e(k_0, k_1, k_2, k_3)}$

$$(4.6) \quad \begin{array}{cccccc} k_3 & f_1 & f_3 & f_5 & f_7 & \\ & 0 & f_2 & f_4 & f_6 & f_8 \\ k_2 & f_1 & f_3 & f_5 & f_7 & \\ & 0 & f_2 & f_4 & f_6 & f_8 & \cdots \\ k_1 & f_1 & f_3 & f_5 & f_7 & \\ & k_0 & f_2 & f_4 & f_6 & f_8 \end{array}$$

and the corresponding notion of $(k_0, k_1, k_2, k_3)^e$ -admissible colored partitions on the array \mathcal{N}_6 .

CONJECTURE 4.1. *Let $k_0, k_1, \dots, k_\ell \in \mathbb{N}_0, k = k_0 + k_1 + \dots + k_\ell > 0$. Then the generating function for the number of $(k_0, k_1, \dots, k_\ell)^e$ -admissible colored partitions*

$$n = \sum_{a \in \mathcal{N}_{2\ell}} f_a \cdot a$$

is the infinite periodic product

$$(4.7) \quad \frac{\prod_{a \in \{0\}^\ell; b \in \Delta(k_1+1, \dots, k_\ell+1); j \equiv a, \pm b \pmod{2\ell+2k+1}} (1 - q^j)}{\prod_{j \in \mathbb{N}} (1 - q^j)^\ell}.$$

EXAMPLE 4.2. By using Example 2.1 for the congruence triangle $\Delta(2, 1, 1, 2)$, we see that the conjectured product (4.7) for $(2, 1, 0, 0, 1)^e$ -admissible colored partitions is

$$\prod_{j \equiv 1, 1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10, 11, 11, 12, 12, 12, 13, 13, 14, 14, 15, 16, 16 \pmod{17}} (1 - q^j)^{-1}.$$

REMARK 4.3. Conjecture 4.1 is true for $\ell = 1$: the extended array of frequencies $\mathcal{F}^{e(k_0, k_1)}$ is

$$(4.8) \quad \begin{array}{cccccc} k_1 & f_1 & f_3 & f_5 & f_7 & \\ & k_0 & f_2 & f_4 & f_6 & f_8 & \cdots \end{array},$$

and $(k_0, k_1)^e$ -admissible colored partitions are classical partitions

$$n = \sum_{a \in \mathbb{N}} f_a \cdot a$$

satisfying difference and initial conditions

$$f_a + f_{a+1} \leq k, \quad f_1 \leq k_1.$$

So the generating functions for the number of $(1, 0)^e$ -admissible and $(0, 1)^e$ -admissible partitions are the product sides of two Rogers-Ramanujan identities, and for $k = k_0 + k_1 > 1$ we have the product sides of Gordon identities (cf. [2], [3], [11]). These combinatorial identities have a Lie-theoretic interpretation (see [15]), but for $2\ell > 2$ there is no obvious connection of $(k_0, k_1, \dots, k_\ell)^e$ -admissible colored partitions (4.2) with representation theory of affine Lie algebras.

EXAMPLE 4.4. In this example we write

$$(0, 0, 1)^e \sim (r \equiv 1, 3, 4, 6 \pmod{7})$$

if the number of $(0, 0, 1)^e$ -admissible colored partitions of n is equal to the number of colored partitions of n with parts $r \equiv 1, 3, 4, 6 \pmod{7}$ for $n \leq 20$.

For $w = 2$ we have Rogers-Ramanujan and Gordon identities:

$$\begin{aligned} (1, 0)^e &\sim (r \equiv 2, 3 \pmod{5}), \\ (0, 1)^e &\sim (r \equiv 1, 4 \pmod{5}), \\ (2, 0)^e &\sim (r \equiv 2, 3, 4, 5 \pmod{7}), \\ (1, 1)^e &\sim (r \equiv 1, 3, 4, 6 \pmod{7}), \\ (0, 2)^e &\sim (r \equiv 1, 2, 5, 6 \pmod{7}), \\ (3, 0)^e &\sim (r \equiv 2, 3, 4, 5, 6, 7 \pmod{9}), \\ (2, 1)^e &\sim (r \equiv 1, 3, 4, 5, 6, 8 \pmod{9}), \\ (1, 2)^e &\sim (r \equiv 1, 2, 4, 5, 7, 8 \pmod{9}), \\ (0, 3)^e &\sim (r \equiv 1, 2, 3, 6, 7, 8 \pmod{9}). \end{aligned}$$

For $w = 4$ we have:

$$\begin{aligned} (1, 0, 0)^e &\sim (r \equiv 2, 3, 4, 5 \pmod{7}), \\ (0, 1, 0)^e &\sim (r \equiv 1, 2, 5, 6 \pmod{7}), \\ (0, 0, 1)^e &\sim (r \equiv 1, 3, 4, 6 \pmod{7}), \\ (2, 0, 0)^e &\sim (r \equiv 2, 3, 4, 4, 5, 5, 6, 7 \pmod{9}), \\ (1, 1, 0)^e &\sim (r \equiv 1, 2, 3, 4, 5, 6, 7, 8 \pmod{9}), \\ (1, 0, 1)^e &\sim (r \equiv 1, 2, 3, 4, 5, 6, 7, 8 \pmod{9}), \\ (0, 2, 0)^e &\sim (r \equiv 1, 2, 2, 3, 6, 7, 7, 8 \pmod{9}), \\ (0, 1, 1)^e &\sim (r \equiv 1, 1, 3, 4, 5, 6, 8, 8 \pmod{9}), \\ (0, 0, 2)^e &\sim (r \equiv 1, 2, 3, 4, 5, 6, 7, 8 \pmod{9}), \\ (3, 0, 0)^e &\sim (r \equiv 2, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 9 \pmod{11}), \\ (2, 1, 0)^e &\sim (r \equiv 1, 2, 3, 4, 5, 5, 6, 6, 7, 8, 9, 10 \pmod{11}), \\ (2, 0, 1)^e &\sim (r \equiv 1, 2, 3, 4, 4, 5, 6, 7, 7, 8, 9, 10 \pmod{11}), \end{aligned}$$

$$\begin{aligned}
(1, 2, 0)^e &\sim (r \equiv 1, 2, 2, 3, 4, 5, 6, 7, 8, 9, 9, 10 \pmod{11}), \\
(1, 1, 1)^e &\sim (r \equiv 1, 1, 3, 3, 4, 5, 6, 7, 8, 8, 10, 10 \pmod{11}), \\
(1, 0, 2)^e &\sim (r \equiv 1, 2, 2, 3, 5, 5, 6, 6, 8, 9, 9, 10 \pmod{11}), \\
(0, 3, 0)^e &\sim (r \equiv 1, 2, 2, 3, 3, 4, 7, 8, 8, 9, 9, 10 \pmod{11}), \\
(0, 2, 1)^e &\sim (r \equiv 1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10 \pmod{11}), \\
(0, 1, 2)^e &\sim (r \equiv 1, 1, 2, 4, 4, 5, 6, 7, 7, 9, 10, 10 \pmod{11}), \\
(0, 0, 3)^e &\sim (r \equiv 1, 2, 3, 3, 4, 5, 6, 7, 8, 8, 9, 10 \pmod{11}).
\end{aligned}$$

For $w = 6$ we have:

$$\begin{aligned}
(1, 0, 0, 0)^e &\sim (r \equiv 2, 3, 4, 5, 6, 7 \pmod{9}), \\
(0, 1, 0, 0)^e &\sim (r \equiv 1, 2, 4, 5, 7, 8 \pmod{9}), \\
(0, 0, 1, 0)^e &\sim (r \equiv 1, 2, 3, 6, 7, 8 \pmod{9}), \\
(0, 0, 0, 1)^e &\sim (r \equiv 1, 3, 4, 5, 6, 8 \pmod{9}), \\
(2, 0, 0, 0)^e &\sim (r \equiv 2, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 9 \pmod{11}), \\
(1, 1, 0, 0)^e &\sim (r \equiv 1, 2, 3, 4, 4, 5, 6, 7, 7, 8, 9, 10 \pmod{11}).
\end{aligned}$$

For $w = 8$ we have:

$$\begin{aligned}
(0, 1, 1, 0, 1)^e &\sim (r \equiv 1, 1, 1, 2, 3, 4, 4, 4, 6, 6, 6, 7, \\
&\quad 8, 9, 9, 9, 11, 11, 11, 12, 13, 14, 14, 14 \pmod{15}), \\
(2, 1, 0, 0, 1)^e &\sim (r \equiv 1, 1, 2, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7, 7, 8, 8, \\
&\quad 9, 9, 10, 10, 11, 11, 12, 12, 12, 13, 13, 14, 14, 15, 16, 16 \pmod{17}), \\
(0, 1, 1, 1, 1)^e &\sim (r \equiv 1, 1, 1, 1, 3, 3, 3, 4, 5, 5, 5, 6, 7, 7, 8, 8, \\
&\quad 9, 9, 10, 10, 11, 12, 12, 12, 13, 14, 14, 14, 16, 16, 16, 16 \pmod{17}).
\end{aligned}$$

REMARK 4.5. It seems that for pairs (level k , rank ℓ) there is some sort of duality

$$(k, \ell) \longleftrightarrow (\ell, k).$$

In particular, the Rogers-Ramanujan case $k = 1$, $w = 2$ is self-dual and $k = 2$, $w = 2$ is dual to $k = 1$, $w = 4$. In the self-dual case $k = 2$, $w = 4$ we see that $(1, 1, 0)^e$ -admissible, $(1, 0, 1)^e$ -admissible and $(0, 0, 2)^e$ -admissible partitions have the same product formula, but already for $n = 1$ and $n = 2$ we see that three types of colored partitions on \mathcal{N}_4 are mutually different.

5. AN ALGORITHM FOR CONSTRUCTING ADMISSIBLE ARRAYS OF FREQUENCIES

In this section we describe an algorithm for constructing $(k_0, k_1, \dots, k_\ell)$ -admissible and $(k_0, k_1, \dots, k_\ell)^e$ -admissible arrays of frequencies. First we consider the simplest case of $(k, 0, 0)$ -admissible arrays of frequencies \mathcal{F} . The

difference condition (3.5) forces the extended array of frequencies $\mathcal{F}^{(k,0,0)}$ to look like

$$(5.1) \quad \begin{array}{cccccccc} & 0 & 0 & 0 & f_{5_1} & f_{7_1} & & \\ & & 0 & 0 & f_{4_1} & f_{6_1} & f_{8_1} & \\ 0 & 0 & 0 & f_{3_2} & f_{5_2} & f_{7_2} & \dots & \\ & 0 & f_{2_2} & f_{4_2} & f_{6_2} & f_{8_2} & & \\ k & f_{1_3} & f_{3_3} & f_{5_3} & f_{7_3} & & & \end{array}$$

In order to avoid any confusion and facilitate the exposition, here we have colored our arrays as in Example 3.1.

At the beginning we let $\mathcal{F} = 0$, i.e. all the frequencies in \mathcal{F} are zero. Then we construct a nontrivial $(k, 0, 0)$ -admissible array of frequencies by changing \mathcal{F} in steps. We start our construction from the top 5_1 till the bottom 1_3 of the first diagonal $\{5_1, 4_1, 3_2, 2_2, 1_3\} \subset \mathcal{N}$ of length 5, then from the top till the bottom of the second diagonal $\{7_1, 6_1, 5_2, 4_2, 3_3\}$ of length 5, and so on. At each step for $a \in \mathcal{N}$ we choose a frequency f_a and determine the corresponding maximum

$$(5.2) \quad m_a = \max \left\{ \sum_{c \in \mathcal{Z}} f_c \mid \mathcal{Z} \text{ is a downward path which ends in } a \right\}.$$

So we start with $a = 5_1$ and we choose any value $f_{5_1} \in \{0, 1, \dots, k\}$. For the first point 5_1 there is only one downward path $\mathcal{Z} = \{5_1\}$ which ends in 5_1 , so

$$m_{5_1} = f_{5_1}.$$

For the second point 4_1 we should choose f_{4_1} . There are two downward paths which end in 4_1 : $\mathcal{Z}_1 = \{5_1, 4_1\}$ and $\mathcal{Z}_2 = \{3_1, 4_1\}$. Since $f_{3_1} = 0$, we have

$$m_{4_1} = f_{5_1} + f_{4_1}$$

and the level k difference condition $m_{4_1} \leq k$ forces us to choose

$$f_{4_1} \leq k - f_{5_1}.$$

At each step we choose f_a and determine m_a , so assume we completed a list of frequencies f_a and maxima m_a in the first two diagonals and in the top two places $9_1, 8_1$ in the third diagonal. Next we should choose f_b for $b = 7_2$. Denote the points 6_1 and 8_1 —the points above and adjacent to b —as x and y :

$$(5.3) \quad \begin{array}{ccccccc} & & & \bullet & \bullet & \bullet & \\ & & & & x & y & \\ & & & \bullet & \bullet & b & \\ & & \bullet & \bullet & & & \\ & \bullet & & & & & \\ \bullet & & \bullet & & & & \end{array}$$

Then choose a frequency f_b so that $f_b + \max\{m_x, m_y\} \leq k$. Since every downward path \mathcal{Z} comes to b via x or via y , it is clear that

$$(5.4) \quad m_b = f_b + \max\{m_x, m_y\} \leq k.$$

In this sequence of steps we have constructed the frequencies f_c for c denoted as x, y, b or \bullet in (5.3), and all the other frequencies in \mathcal{F} are (still) 0. By construction, i.e. by the recursive condition (5.4) for each constructed f_c , the difference condition (3.5) holds for all downward paths \mathcal{Z} in $\mathcal{F}^{(k,0,0)}$. Hence the (so far) constructed sequence of frequencies is $(k, 0, 0)$ -admissible. In this way we proceed till the end of a finite (chosen in advance) number of diagonals.

REMARK 5.1. Assume we want to construct and count all colored partitions

$$n = \sum_{a \in \mathcal{N}_5} f_a \cdot a \quad \text{for } n \leq 15,$$

with $(k, 0, 0)$ -admissible frequencies f_a . Since the biggest part in a partition of $n \leq 15$ is at most 15, it is enough to consider only the first eight nontrivial diagonals of frequency array (5.1). We can write compactly the first eight full-length diagonals in the array (3.1), the constructed part of the frequency array (5.3), and the constructed part of the maxima array as³

$$(5.5) \quad \begin{array}{cccccccccccccccc} 1_3 & 2_2 & 3_2 & 4_1 & 5_1 & f_{1_3} & f_{2_2} & f_{3_2} & f_{4_1} & f_{5_1} & m_{1_3} & m_{2_2} & m_{3_2} & m_{4_1} & m_{5_1} \\ 3_3 & 4_2 & 5_2 & 6_1 & 7_1 & f_{3_3} & f_{4_2} & f_{5_2} & f_{6_1} & f_{7_1} & m_{3_3} & m_{4_2} & m_{5_2} & m_{6_1} & m_{7_1} \\ 5_3 & 6_2 & 7_2 & 8_1 & 9_1 & \cdot & \cdot & \cdot & f_{8_1} & f_{9_1} & \cdot & \cdot & \cdot & m_{8_1} & m_{9_1} \\ 7_3 & 8_2 & 9_2 & 10_1 & 11_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 9_3 & 10_2 & 11_2 & 12_1 & 13_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 11_3 & 12_2 & 13_2 & 14_1 & 15_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 13_3 & 14_2 & 15_2 & 16_1 & 17_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 15_3 & 16_2 & 17_2 & 18_1 & 19_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Then we can choose a frequency $0 \leq f_{7_2} \leq k$ for 7_2 so that the recursive condition (5.4) holds, i.e.

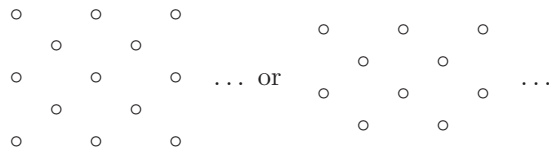
$$(5.6) \quad m_{7_2} = f_{7_2} + \max\{m_{6_1}, m_{8_1}\} \leq k.$$

In this way we proceed till the end of the eighth row.

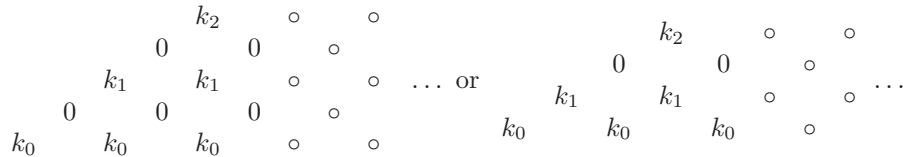
Now we consider, again for $\ell = 2$, a construction of (k_0, k_1, k_2) -admissible or $(k_0, k_1, k_2)^e$ -admissible arrays of frequencies \mathcal{F} or \mathcal{F}^e with non-zero frequencies in a finite (chosen in advance) number of diagonals. We denote with \circ the places of the array of not yet constructed frequencies in the chosen finite

³The diagram (5.5) is obtained from (5.1) by a (clockwise) 45 degree rotation, followed by the shear linear transformation $(x, y) \mapsto (x + y, y)$. In particular, the rows in (5.5) are obtained from diagonals in (5.1).

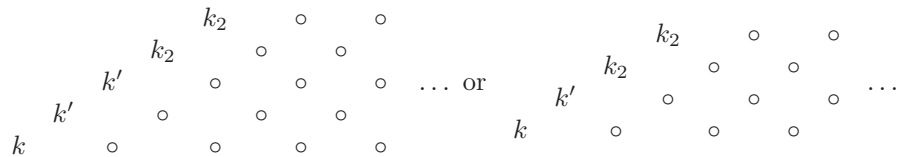
number of diagonals as



and we extend them on the left with the prescribed fixed frequencies

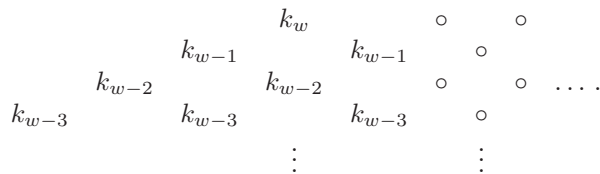


Note that added fixed frequencies satisfy the difference conditions (3.5) or (4.5). Now the first diagonal is fixed and “constructed”, and the corresponding diagonal of the maxima is



with $k' = k_1 + k_2$ and $k = k_0 + k_1 + k_2$. So we pass to the first point in the second diagonal, choose a frequency from the set $\{0, 1, \dots, k\}$ and determine the corresponding maximum. After that we pass to the second point and so on, just like before, except that in the second diagonal four or three frequencies are already fixed and if the condition (3.5) or (4.5) does not hold, we should return to the beginning of the diagonal and give another try with the frequency of the first point. In the third diagonal we have to construct frequency for three points—the other two or one are already given and fixed, and so on. It is clear that the constructed sequence of frequencies is (k_0, k_1, k_2) -admissible.

In the Python code below we consider the w odd and w even cases simultaneously with the general initial conditions for parameters $[k_1, k_2, \dots, k_{w-1}, k_w]$: for a frequency array \mathcal{F} for \mathcal{N}_w we have the extended frequency array $\mathcal{F}^{[k_1, k_2, \dots, k_{w-1}, k_w]}$



and the corresponding notion of $[k_1, k_2, \dots, k_{w-1}, k_w]$ -admissible colored partitions on the array \mathcal{N}_w . Note that

- $(k_0, k_1, \dots, k_\ell)$ -admissible = $[k_0, 0, k_1, 0, \dots, 0, k_\ell]$ -admissible for $w = 2\ell + 1$,
- $(k_0, k_1, \dots, k_\ell)^e$ -admissible = $[k_0, k_1, 0, \dots, 0, k_\ell]$ -admissible for $w = 2\ell$.

We conjecture product formulas for generating functions for the number of the corresponding admissible colored partitions, and it seems that for any other $[k_1, \dots, k_w]$ there is no infinite periodic product formula for the generating function for the number of the corresponding $[k_1, \dots, k_w]$ -admissible colored partitions.

REMARK 5.2. In higher ranks it may be convenient to write the extended $[k_1, \dots, k_w]$ -admissible frequency arrays compactly as infinite matrices

$$(5.7) \quad F = (f_{ij})_{i=0,1,\dots; j=w,\dots,1}$$

with fixed prescribed frequencies in the upper left corner of the matrix, depending on odd $w = 2\ell + 1$ or even $w = 2\ell$. Here $j = w$ denotes the left column, and $j = 1$ the right column. For $\ell = 4$ we have matrices of the form

$$\begin{array}{cccccccccccccccc}
 k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & k_9 & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 \\
 k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & k_8 & \bullet & k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & \bullet \\
 k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & \bullet & \bullet & \bullet & k_1 & k_2 & k_3 & k_4 & k_5 & \bullet & \bullet & \bullet \\
 k_1 & k_2 & k_3 & k_4 & \bullet & \bullet & \bullet & \bullet & \bullet & k_1 & k_2 & k_3 & \bullet & \bullet & \bullet & \bullet & \bullet \\
 k_1 & k_2 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & k_1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \text{OR} & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
 \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
 \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ & & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
 & & & & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & \vdots \\
 & & & & & & & & & & & & & & & & & \vdots
 \end{array}$$

For a frequency matrix (5.7) we assume that there are finitely many non-zero elements f_{ij} and we say that F has a finite support. We can write the matrix F as an infinite sequence of rows

$$(5.8) \quad F = (\phi_i)_{i=0,1,\dots}, \quad \phi_i = (f_{ij})_{j=w,\dots,1}.$$

Like in (5.5), for a frequency matrix F we have the associated *maxima matrix*

$$(5.9) \quad M = (m_{ij})_{i=0,1,\dots; j=w,\dots,1},$$

defined for $i = 0$ as

$$m_{0j} = k_w + \dots + k_{w-j+1}, \quad j = 1, \dots, w,$$

and for $i = 1, 2, \dots$ recursively like in (5.6),

$$(5.10) \quad m_{i1} = f_{i1}, \quad m_{ij} = f_{ij} + \max\{m_{i-1,j-1}, m_{i,j-1}\}, \quad j = 2, \dots, w.$$

Our argument above shows that a frequency matrix F is $[k_1, \dots, k_w]$ -admissible if and only if

$$(5.11) \quad m_{ij} \leq k \quad \text{for all } i = 1, 2, \dots, \quad j = 1, \dots, w.$$

We can also write our arrays of natural numbers $\mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5, \dots$ as infinite matrices N_w^0 , $w = 2, 3, 4, 5, \dots$ of natural numbers, extended with fixed zeros in the upper left corner of the matrix,

$$N_w^0 = (n_{ij})_{i=0,1,\dots; j=w,\dots,1}, \quad n_{ij} = \max\{0, 2i - j\},$$

i.e. as infinite matrices $N_2^0, N_3^0, N_4^0, N_5^0, \dots$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	1	0	0	1	0	0	0	1	0	0	0	0	1	
2	3	1	2	3	0	1	2	3	0	0	1	2	3	
4	5	3	4	5	2	3	4	5	1	2	3	4	5	
6	7	5	6	7	4	5	6	7	3	4	5	6	7	...
8	9	7	8	9	6	7	8	9	5	6	7	8	9	
10	11	9	10	11	8	9	10	11	7	8	9	10	11	
⋮		⋮		⋮		⋮		⋮		⋮		⋮		

Then for each $[k_1, \dots, k_w]$ -admissible frequency matrix F we have a $[k_1, \dots, k_w]$ -admissible colored partition

$$n = \sum_{i=0}^{\infty} \sum_{j=1}^w f_{ij} n_{ij}.$$

EXAMPLE 5.3. Here we describe a construction of all $(0, 1, 0)$ -admissible frequencies with the support in the first 5 diagonals in the array \mathcal{N}_5 . By using the notation in Remark 5.2, we want to construct all $[0, 0, 1, 0, 0]$ -admissible frequency matrices F , together with the associated maxima matrices M ,

$$(5.12) \quad F = (\phi_0, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5), \quad M = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5),$$

with $\phi_0 = (0, 0, 1, 0, 0)$, $\phi_1 = (0, 0, 1, 0, \bullet)$, $\phi_2 = (0, 0, \bullet, \bullet, \bullet)$, $\phi_3 = (\circ, \circ, \circ, \circ, \circ)$, \dots , and μ_i is i -th row in M . Denote by $A(i)$ the set of all possible quintuples of frequencies in the i -th row, i.e. the set of quintuples of frequencies with the total sum $\leq k = 1$. $A(0)$ is completely determined, and for the other we obviously have:

- $A(0) = \{(0, 0, 1, 0, 0)\},$
- $A(1) = \{(0, 0, 1, 0, 0)\},$
- $A(2) = \{(0, 0, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\},$
- $A(3) = \{(0, 0, 0, 0, 0), (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\},$
- $A(4) = \{(0, 0, 0, 0, 0), (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\},$
- $A(5) = \{(0, 0, 0, 0, 0), (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}.$

If we use printouts from the Python code 21AAIC.py in the Appendix, for $N = 6$ and ‘highest weight’ = $[0, 0, 1, 0, 0]$, and if we activate “print(‘i =’, i, ‘all fs =’, all fs)” on line 76, we get $A(1), \dots, A(5)$ as above (with mixed-up left and right). We obtain the first six rows of N_5^0 if we activate “print(‘i =’, i, ‘row1 =’, row1)” on line 73.

We may construct F in steps $i = 0, \dots, 4$, and by using (5.10) at each step we determine the rows of the associated maxima matrix

(5.13)

$$\phi_0, \phi_1, \dots, \phi_i; \mu_0, \mu_1, \dots, \mu_i \rightsquigarrow \phi_0, \phi_1, \dots, \phi_i, \phi_{i+1}; \mu_0, \mu_1, \dots, \mu_i, \mu_{i+1},$$

$\phi_r \in A(r)$. Since we are using (5.10) to determine μ_{i+1} , we have a map

$$(\phi_{i+1}, \mu_i) \mapsto \mu_{i+1},$$

and if μ_{i+1} does not satisfy the criteria (5.11), we discard the newly constructed matrix from the further procedure. In Python code 21AAIC.py this function is defined on line 29 as “filter frequencies(fs, ms1, *ks)”.

Since in our construction we need only the last maxima row μ_i to determine whether the newly constructed frequency matrix in (5.13) is $[0, 0, 1, 0, 0]$ -admissible, we could have discarded already used up $\mu_0, \mu_1, \dots, \mu_{i-1}$. On the other hand, if we are interested in the corresponding $[0, 0, 1, 0, 0]$ -admissible partitions, we can keep track of the total contribution τ_i of rows ϕ_r , $r \leq i$, to the corresponding $[0, 0, 1, 0, 0]$ -admissible partition,

$$\tau_i = \sum_{r=1}^i \sum_{j=1}^w f_{rj} n_{rj}.$$

Hence we should record our steps in the construction with data

$$(5.14) \quad \phi_0, \phi_1, \dots, \phi_i; \tau_i; \mu_i \rightsquigarrow \phi_0, \phi_1, \dots, \phi_i, \phi_{i+1}; \tau_{i+1}; \mu_{i+1}.$$

Finally, if we are interested only in the number of constructed partitions, and do not need the constructed $[0, 0, 1, 0, 0]$ -admissible frequency matrices $\phi_0, \phi_1, \dots, \phi_i$, we should record our steps in the construction only with data

$$(5.15) \quad \tau_i; \mu_i \rightsquigarrow \phi_{i+1}; \tau_{i+1}; \mu_{i+1},$$

i.e. we should regard the step (5.15) as the map

$$(5.16) \quad ((\tau_i, \mu_i), \phi_{i+1}) \mapsto (\tau_{i+1}, \mu_{i+1}),$$

and this is what the Python code 21AAIC.py does in two loops “for total0, ms0 in all total ms0:” and “for fs1 in all fs:” starting at lines 77 and 78. If in 21AAIC.py we change the “While loop” and the “result” into

```
while True:
    all_total_ms1 = []
    row1 = get_row(i, w)
    all_fs = list(all_frequencies(i, *highest_weight))
    for total0, ms0 in all_total_ms0:
        for fs1 in all_fs:
```

```

ms = filter_frequencies(fs1, ms0, *highest_weight)
if ms is None:
    continue
total1 = row_fs_value(row1, fs1) + total0
all_total_ms1.append((total1, ms))
if i == N:
    #if i == w//2:
        break
    i += 1
    all_total_ms0 = all_total_ms1

result = []
for total1, ms1 in all_total_ms1:
    if total1 <= 6:
        result.append(total1)
print(result)
#print(len(all_total_ms1))

```

then the activated “`print(len(all total ms1))`” on line 92 shows that there is altogether 164 final pairs (τ_5, μ_5) in our construction of $(0, 1, 0)$ -admissible partitions with the support in the first $N = 5$ rows. If we print only the list of $\tau_5 \leq 6$ which appear in the final step of (5.15) we get

```

highest_weight = [0, 0, 1, 0, 0]
k = 1  w = 5
[0, 5, 6, 3, 4, 5, 6, 2, 6, 3, 4, 5, 1, 6, 5, 6, 4,
 5, 6, 2, 6, 3]

```

From this list we see that $\tau = 6$ appears 7 times, i.e. there is 7 $(0, 1, 0)$ -admissible partitions of 6. The Python code 21AAIC.py is just a bit faster and more polished version of this code and gives for the number of all $(0, 1, 0)$ -admissible partitions of $n \leq N = 6$ the list

```

highest_weight = [0, 0, 1, 0, 0]
k = 1  w = 5
[[1, 1], [2, 2], [3, 3], [4, 3], [5, 5], [6, 7]]

```

Of course, this example would have been much shorter if we simply wrote “by hand” all $(0, 1, 0)$ -admissible partitions of $n \leq 6$, as we did for all $(2, 0, 0)$ -admissible partitions of $n \leq 8$ in Example 3.1.

REMARK 5.4. A basis of the finite dimensional representation $L_{C_\ell}(k_1, \dots, k_\ell)$ is parametrized in [10] in terms of symplectic Dyck paths—this result is closely related to our Conjecture 3.3. For $\ell = 4$ Feigin-Fourier-Littelmann’s theorem can be rephrased in terms of $(0, k_1, k_2, k_3, k_4)$ -admissible 5×9 matrices⁴

$$\begin{array}{cccccccccc} k_0 & 0 & k_1 & 0 & k_2 & 0 & k_3 & 0 & k_4 & \\ k_0 & 0 & k_1 & 0 & k_2 & 0 & k_3 & 0 & \bullet & \\ k_0 & 0 & k_1 & 0 & k_2 & 0 & \bullet & \bullet & \bullet & ; \\ k_0 & 0 & k_1 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \\ k_0 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \end{array}$$

in [10] these matrices are written in the form

$$\begin{array}{cccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet & & \\ & & \bullet & \bullet & \bullet & & & \\ & & & \bullet & & & & \\ & & & & \bullet & & & \\ & & & & & \bullet & & \\ & & & & & & \bullet & \end{array}$$

6. APPENDIX: PYTHON CODE FOR COUNTING ADMISSIBLE COLORED PARTITIONS

Here we give the Python code which is in part explained in Example 5.3 where diagonals in the array (3.3) became the rows in the matrix (5.12).

```
"""The program 21Aaic counts the number P(n) of [k1,k2,...,kw]
-admissible colored partitions of n <= N on the array Nw of w
rows of natural numbers. We input by hand N and k1,k2,...,kw as
the list 'highest_weight' on line 57. The result is a list of
pairs [n,P(n)]."""
```

```
def get_row(i, w):
    return [max(0, x) for x in range(i*2 - 1, i*2 - w - 1, -1)]
```

```
def all_subfrequencies(c, r):
    for g0 in range(r + 1):
        if c == 1:
            yield [g0]
        else:
            for gs in all_subfrequencies(c - 1, max(0, r - g0)):
                yield [g0] + gs
```

```
def all_frequencies(i, *ks):
    rest = []
    for x in reversed(ks):
        rest.append(x)
```

⁴Here $k_0 = 0$, but it works for any k_0 since $L_{C_\ell}(k_1, \dots, k_\ell) \subset L_{C_\ell^{(1)}}(k_0, k_1, \dots, k_\ell)$.

```

if i > w//2:
    for gs in all_subfrequencies(w, k):
        yield gs
else:
    for gs in all_subfrequencies(i*2-1, sum(ks[-i*2+1:])):
        yield gs + rest[2 * i-1:]

def filter_frequencies(fs, ms1, *ks):
    k = sum(ks)
    ms = []
    for j, f in enumerate(fs):
        if j:
            m = ms1[j - 1]
            m0 = ms[-1]
            if m0 > m:
                m = m0
            m += f
            if m > k:
                return None
            ms.append(m)
        else:
            ms.append(f)
    return ms

def row_fs_value(row, fs):
    s = 0
    for v, f in zip(row, fs):
        s += v * f
    return s

if __name__ == '__main__':
    """One should put by hand N and the 'highest wight'.
    For w=2n+1 for (k0,k1,...,kn) put [k0,0,k1,0,...,0,kn].
    For w=2n for (k0,k1,...,kn)e put [k0,k1,0,...,0,kn]."""
    N = 6
    highest_weight = [0, 0, 1, 0, 0]
    print("highest_weight =", highest_weight)
    w = len(highest_weight)
    k = sum(highest_weight)
    print('k =', k, ' w =', w)
    i = 1
    frequencies = {}
    result = []
    ms0 = []
    for j in range(0, len(highest_weight)):
        ms0.append(sum(highest_weight[-j-1:]))
    all_total_ms0 = [(0, ms0)]

```

```

while True:
    all_total_ms1 = []
    row1 = get_row(i, w)
    #print( 'i =', i, 'row1 =', row1)
    min_next_row = get_row(i + 1, w)[-1]
    all_fs = list(all_frequencies(i, *highest_weight))
    #print( 'i =', i, 'all_fs =', all_fs)
    for total0, ms0 in all_total_ms0:
        for fs1 in all_fs:
            ms = filter_frequencies(fs1, ms0, *highest_weight)
            if ms is None:
                continue
            total1 = row_fs_value(row1, fs1) + total0
            if total1 <= N:
                if total1 > total0:
                    frequencies[total1] = 1 + frequencies.get(total1, 0)
                if total1 <= N - min_next_row:
                    all_total_ms1.append((total1, ms))
    if row1[-2] > 0:
        result.append([row1[-1], frequencies.get(row1[-1], 0)])
        result.append([row1[-2], frequencies.get(row1[-2], 0)])
    if max(row1[-2:]) >= N:
        break
    i += 1
    all_total_ms0 = all_total_ms1

if w%2 == 0:
    result.remove([0,0])
print(result)

```

EXAMPLE 6.1. In the Rogers-Ramanujan case, i.e. for $w = 2$ and $k = 1$ we get:

```
highest_weight = [0, 1]
```

```
[1, 1], [2, 1], [3, 1], [4, 2], [5, 2], [6, 3], [7, 3], [8, 4],
[9, 5], [10, 6], [11, 7], [12, 9], [13, 10], [14, 12],
[15, 14], [16, 17], [17, 19], [18, 23], [19, 26], [20, 31];
```

```
highest_weight = [1, 0]
```

```
[1, 0], [2, 1], [3, 1], [4, 1], [5, 1], [6, 2], [7, 2], [8, 3],
[9, 3], [10, 4], [11, 4], [12, 6], [13, 6], [14, 8], [15, 9],
[16, 11], [17, 12], [18, 15], [19, 16], [20, 20].
```

EXAMPLE 6.2. For $(2, 1, 0, 0, 1)^e$ -admissible partitions in Example 4.4 we get:

```
highest_weight = [2, 1, 0, 0, 0, 0, 0, 1]
```

```
[1, 2], [2, 4], [3, 8], [4, 15], [5, 27], [6, 47], [7, 78],
[8, 128], [9, 205], [10, 323], [11, 499], [12, 763],
[13, 1148], [14, 1709], [15, 2516], [16, 3669],
[17, 5297], [18, 7589], [19, 10779], [20, 15204].
```

EXAMPLE 6.3. Related to Remark 5.4, we can modify slightly the above Python code for calculating dimensions of representations $L_{C_\ell}(k_1, \dots, k_\ell)$ (one way to do it is to use the change described in Example 5.3, activate “if i == w//2:” on line 82 and “print(len(all total ms1))” on line 92, deactivate “#if i == N:” on line 81 and set highest weight = [0,0,1,0,1,0,2,0,2] for dim L [1, 1, 2, 2]). For example, for $\ell = 4$ we have:

```
dim L [1, 1, 2, 2] = 3459456,
dim L [2, 1, 2, 2] = 9848916,
dim L [0, 2, 2, 2] = 4321512,
dim L [1, 2, 2, 2] = 16358760,
dim L [2, 2, 2, 2] = 43046721.
```

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