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A Triple of Projective Billiards

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ABSTRACT

A projective billiard is a polygon in the real projective plane with a circumconic and an inconic. Similar to the classical billiards in conics, the intersection points between the extended sides of a projective billiard are located on a family of conics which form the associated Poncelet grid. We extend the projective billiard by the inner and outer billiard and disclose various relations between the associated grids and the diagonals, in particular other triples of projective billiards.

Key words: ellipse, billiard, caustic, Poncelet grid, billiard motion

MSC2020: 51N35

Trojka projektivnih biljara

SAŽETAK

Projektivni biljar je poligon u realnoj projektivnoj ravnini koji ima upisanu i opisanu koniku. Poput klasičnih biljara u konikama, sjecišta produljenih stranica projektivnog biljara se nalaze na familiji konika koje tvore pridruženu Ponceletovu mrežu. Proširujemo projektivni biljar unutarnjim i vanjskim biljarom i otkrivamo mnoštvo veza između pridruženih mreža i dijagonala, posebice drugih trojki projektivnih biljara.

Ključne riječi: elipsa, biljar, kaustika, Ponceletova mreža, biljarsko kretanje

1 Introduction

A *billiard* is the trajectory of a mass point in a domain called *billiard table* with ideal physical reflections in the boundary. Already for two centuries, billiards in ellipses (see Figures 1, 2, 8) and their projectively equivalent counterparts have attracted the attention of mathematicians, beginning with J.-V. Poncelet [4] and C.G.J. Jacobi [3] and continued, e.g., by S. Tabachnikov, who addresses in his book [10] a wide variety of themes around this topic. Computer animations carried out recently by D. Reznik [5] stimulated a new vivid interest on these well studied objects.

We focus on projective generalizations called *projective billiards*. This term stands for planar polygons $P_1P_2P_3\dots$ with a circumconic e and an inconic c called *caustic*. Not all projective billiards are projectively equivalent to Euclidean billiards (see, e.g., Figure 9), and not in all cases exist periodical polygons between the conics e and c. However, in all cases the intersection points between extended sides define a family of conics which form the associated *Poncelet grid*. The goal of this paper is to demonstrate that in a quite natural way any given projective billiard defines two more projective billiards with associated Poncelet grids. It will be demonstrated that not only the conics of these grids, but also configurations of related lines deserve our interest.

It needs to be pointed out, that the computation of the billiards' vertices can only be carried out either iteratively or with the help of Jacobian elliptic functions (see, e.g., [8]). Therefore, it is not straightforward to obtain results on vertices and their respective j-th followers for any given integer $j > 1$. Often such assertions are equivalent to identities in terms of elliptic functions (see, e.g., [9, Section 5]).

Structure of the article. In Section 2 we introduce the three Poncelet grids associated respectively with a projective billiard and its inner and outer polygons. Section 3 is devoted to the conics $e^{(j)}$, $c^{(j)}$, and $r^{(j)}$ of the three grids. In Section 4 we recall results on the envelopes of diagonals and determine the points of contact. Finally in Section 5, we study the configuration of the l-th diagonals of the projective billiards inscribed respectively in $e^{(j)}$, $c^{(j)}$, and $r^{(j)}$.

2 A triple of Poncelet grids

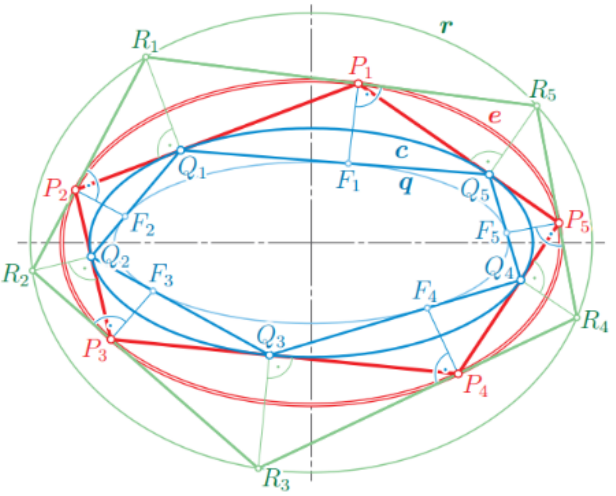

Figure 1: *Periodic billiard $P_1P_2\dots P_5$ inscribed in the ellipse e along with the polygon $Q_1Q_2\dots Q_5$ of contact points with an ellipse c as caustic, the polygon $F_1F_2\dots F_5$ of contact points with q, and the polygon $R_1R_2\dots \in r$ which is polar to $P_1P_2\dots P_5$ w.r.t. e.*

Let $P_1P_2P_3\dots$ be a polygon with circumconic e and inconic c in the real projective plane. Then there exists an associated Poncelet grid. We follow the notation in [7] and denote intersection points between extended sides[1] for $i,j = 1,2,\dots$ as

$$S_i^{(j)} := \begin{cases} [P_{i-k-1},P_{i-k}]\cap[P_{i+k},P_{i+k+1}] & \text{for } j=2k, \text{ and}\\ [P_{i-k},P_{i-k+1}]\cap[P_{i+k},P_{i+k+1}] & \text{for } j=2k-1\,. \end{cases} \tag{1}$$

For fixed j, the points $S_1^{(j)},S_2^{(j)},\dots$ are located on a conic $e^{(j)}$ which belongs to the dual pencil (*range*, in brief) spanned by e and c. This is due to a result of M. Chasles in 1843 (note, e.g., [7, Theorem 3.5]).

If the polygon $P_1P_2\dots$ is N-periodic, then we can confine to $1\le j\le[\frac{N-3}{2}]$, since for even N the locus $e^{(j)}$ with $j=\frac{N-2}{2}$ is a line which has the same pole with respect to (w.r.t., for short) e and c. Under the *billiard motion* of $P_1P_2\dots$, i.e., the variation of the vertices along the circumconic e while c remains fixed, each conic $e^{(j)}$ of the Poncelet grid remains fixed as well (note [7, Theorem 3.6]).[2]

In the classical case of a Euclidean billiard $P_1P_2\dots$ in a conic e, the conics $e^{(j)}$ are confocal with e and the caustic c (Figure 2). If for a given ellipse e the caustic c is an ellipse, then the billiard is called *elliptic* and the conics e and c intersect in two pairs of complex conjugate points. Otherwise we obtain a *hyperbolic* billiard with a hyperbola as caustic (Figures 6 and 7). Then the two conics share four real points.

2.1 The outer polygon

The tangents $t_{P_1},t_{P_2},\dots$ to the circumconic e at the vertices $P_1,P_2,\dots$ of a projective billiard define a polygon $R_1R_2\dots$ called *outer polygon* in [5]. This polygon is polar to $P_1P_2\dots$ w.r.t. e and therefore inscribed in a conic r which is polar to c w.r.t. e (Figure 1). Similar to (1), the vertices $R_i^{(j)}$ of the associated Poncelet grid are points of intersection between tangents to e and denoted for $j=1,2,\dots$ as given below:

$$R_i^{(j)} := \begin{cases} t_{P_{i-k}}\cap t_{P_{i+k+1}} & \text{for } j=2k, \text{ and}\\ t_{P_{i-k}}\cap t_{P_{i+k}} & \text{for } j=2k-1, \end{cases}$$

$$= \begin{cases} [R_{i-k-1},R_{i-k}]\cap[R_{i+k},R_{i+k+1}] & \text{for } j=2k, \text{ and}\\ [R_{i-k-1},R_{i-k}]\cap[R_{i+k-1},P_{i+k}] & \text{for } j=2k-1, \end{cases} \tag{2}$$

hence $k=\left[\frac{j+1}{2}\right]$ (note Figure 2).

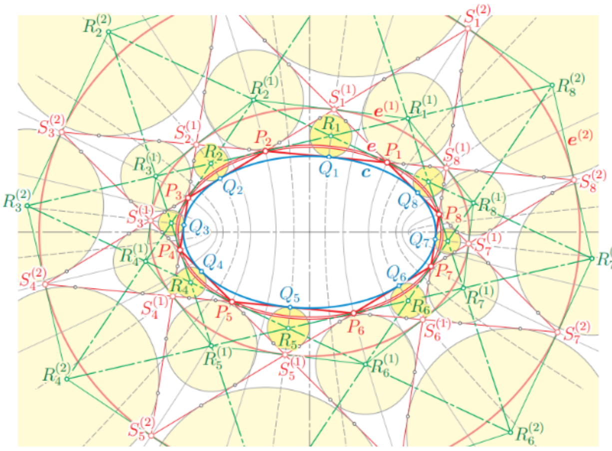

Figure 2: *Periodic billiard $P_1P_2\dots P_8$ in the ellipse e with the net of tangent lines to e at the vertices.*

2.2 The inner polygon

Beside the Poncelet grids associated with the pairs of conics (e,c) and (r,e), there is a third Poncelet grid. This time we focus on the polygon of contact points $Q_1,Q_2,\dots$ of the sides of $P_1P_2\dots$ with the caustic c. The polygon $Q_1Q_2\dots$

[1]Note that XY denotes the segment bounded by the points X and Y, while $[X,Y]$ denotes the connecting line.

[2]Beside the conics $e^{(j)}$, $j=1,2,\dots$, the Poncelet grid contains a second family of conics. In the case of classical billiards with ellipses e, c and $e^{(j)}$, the remaining conics are confocal hyperbolas (see, e.g., [7, Figures 5 or 6]) which vary under the billiard motion. However, here we focus only on $e^{(j)}$.

is called *inner polygon* in [5]. The vertices of the associated Poncelet grid are defined as

$$Q_i^{(j)} := \begin{cases} [Q_{i-k-1}, Q_{i-k}] \cap [Q_{i+k}, Q_{i+k+1}] & \text{for } j = 2k, \\ [Q_{i-k-1}, Q_{i-k}] \cap [Q_{i+k-1}, Q_{i+k}] & \text{for } j = 2k-1 \end{cases} \tag{3}$$

(note Figure 4).

The extended sides of the polygon $Q_1Q_2\ldots$ envelop a conic q which is polar to e w.r.t. c. The line $[Q_{i-1}, Q_i]$ contacts q at the c-pole F_i of the tangent t_{P_i} to e at P_i. Therefore, in the case of a Euclidean billiard it is the point of intersection between the chord $[Q_{i-1}, Q_i]$ and the normal to e at P_i (Figure 1). The latter is the locus of poles of the tangent t_{P_i} w.r.t. the conics of the confocal family.

Lemma 1 *Referring to the previous notation, the circumconic r of the polygon $R_1R_2\ldots$ with sides tangent to e at P_i is polar to c w.r.t. e. The inconic q of the polygon $Q_1Q_2\ldots$ with circumconic c is polar to e w.r.t. c. In the billiard case (Figure 1), R_iQ_i is orthogonal to c at Q_i, and F_iP_i is orthogonal to e at P_i.*

Lemma 1 reveals that also the conics q and r are invariant under the billiard motion along e. Clearly, if the original projective billiard $P_1P_2\ldots$ is periodic, then $Q_1Q_2\ldots$ and $R_1R_2\ldots$ are periodic, too.

A polygon with circumconic e and inconic c can be periodic even when the two conics share two real and two complex conjugate points. An example is depicted in Figures 5 and 9 with the two conics as circles. Such polygons $P_1P_2\ldots$ are called *bicentric*. They were first treated in 1828 by Jacobi [3] in the case where c lies in the interior of e. In [6] various invariants of bicentric polygons are proved for the case that the circles e and c are either nested or disjoint.

3 More projective billiards in the three Poncelet grids

In the case of Euclidean billiards $P_1P_2\ldots$ in the plane or on the sphere (see [7, Fig. 7]), the tangents to e at P_i and those to $e^{(j)}$ at $S_i^{(j)}$ are angle bisectors of extended sides of $P_1P_2\ldots$. Therefore, the net of extended sides of $P_1P_2\ldots$ is *circular* with the points $R_i^{(j)}$ as centers of incircles of quadrilaterals (Figure 2). This result dates back to [1] in 2018. Below we present a generalization.

Theorem 1 *Given a projective billiard $P_1P_2\ldots$, then for each $j = 1, 2, \ldots$ the vertex $R_i^{(j)}$ of the Poncelet grid associated with the outer polygon $R_1R_2\ldots$ is located on the tangents to $e^{(j)}$ at $S_i^{(j)}$ and $S_{i+1}^{(j)}$. The points $R_1^{(j)}, R_2^{(j)}, \ldots$ belong to a conic $r^{(j)}$ which is contained in the range spanned by e and r. The polar conic of $r^{(j)}$ w.r.t. $e^{(j)}$ is the envelope of the extended sides of the polygon $S_1^{(j)}S_2^{(j)}\ldots$.*

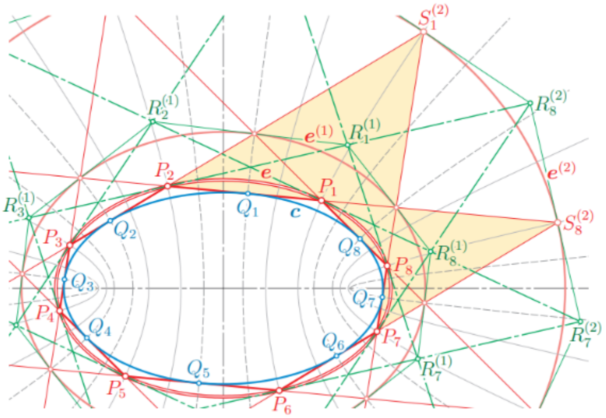

Figure 3: *N-periodic billiard with $N = 8$. In the proof of Theorem 1 we focus on the quadrilateral formed by the tangents from $S_1^{(2)}$ and $S_8^{(2)}$ to the caustic c.*

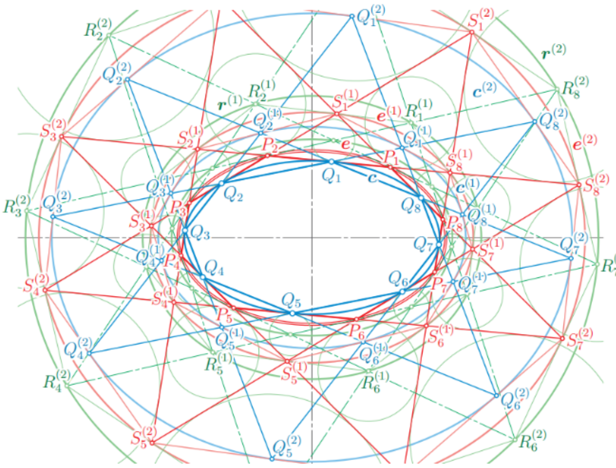

Figure 4: *The contact points of the sides of the polygon $S_i^{(j)}S_{i+1}^{(j)}\ldots$ with their envelope $c^{(j)}$ are the vertices $Q_i^{(j)}$ of the Poncelet grid associated with $Q_1Q_2\ldots$. In other words, the projective billiard $S_i^{(j)}S_{i+1}^{(j)}\ldots$ has $Q_1^{(j)}Q_2^{(j)}\ldots$ as its inner billiard.*

Proof. According to (1), the extended sides $[P_i, P_{i+1}]$ and $[P_{i+j+1}, P_{i+j+2}]$ through $S_{i+k}^{(j)}$ for $k = \left[\frac{j+1}{2}\right]$ and $[P_{i-1}, P_i]$ and $[P_{i+j}, P_{i+j+1}]$ through $S_{i+k-1}^{(j)}$ form a quadrilateral with $P_i, P_{i+j+1} \in e$ and $S_{i+k-1}^{(j)}, S_{i+k}^{(j)} \in e^{(j)}$ as pairs of opposite vertices (see the case $j = 2$, $N = 8$ and $i = 7$ in Figure 3). All four sides are tangents of the caustic c, while the conics and e, $e^{(j)}$ and c belong to a range. According to the mentioned result by Chasles and its extension in [7, Theorem 3.5]), the tangents to e at P_i and P_{i+j+1} and the tangents to $e^{(j)}$ at $S^{(j)}_{i+k-1}$ and $S^{(j)}_{i+k}$ are concurrent. By (2), their meeting point is $R^{(j)}_{i+k-1}$ (see Figure 2). After increasing all subscripts by 1, we obtain the analogue result for $R^{(j)}_{i+k}$.

The Poncelet grid associated with $R_1R_2\dots$ contains conics $r^{(j)}$ passing through the vertices $R^{(j)}_1, R^{(j)}_2, \dots$. All conics $r^{(j)}$ belong to the range spanned by e and r and are motion invariant, too. Since the polar line of $R^{(j)}_i \in r^{(j)}$ w.r.t. $e^{(j)}$ is the line $[S^{(j)}_i, S^{(j)}_{i+1}]$, the polar conic $c^{(j)}$ of $r^{(j)}$ w.r.t. $e^{(j)}$ envelops the polygon $S^{(j)}_1S^{(j)}_2\dots$. $\square$

Theorem 2 *Referring to the previous notation, the sides of the polygon $S^{(j)}_1S^{(j)}_2\dots$ contact the enveloping conic $c^{(j)}$ at the vertices $Q^{(j)}_1, Q^{(j)}_2, \dots$. Hence, the envelope $c^{(j)}$ coincides with the conic of the Poncelet grid associated with $Q_1Q_2\dots$ (Figure 4).*

Proof. We replace the polygon $P_1P_2\dots$ inscribed in e and circumscribed to c by the polygon $R_1R_2\dots$ inscribed in r and circumscribed to e. Then by virtue of Theorem 1, the side $[R^{(j)}_i, R^{(j)}_{i+1}]$ contacts the envelope $e^{(j)}$ at the point $S^{(j)}_i$. This implies for our original polygon $P_1P_2\dots$ that $[S^{(j)}_i, S^{(j)}_{i+1}]$ contacts the envelope $c^{(j)}$ at the vertex $Q^{(j)}_i$ of the Poncelet grid associated with the j-th diagonals of $Q_1Q_2\dots$. $\square$

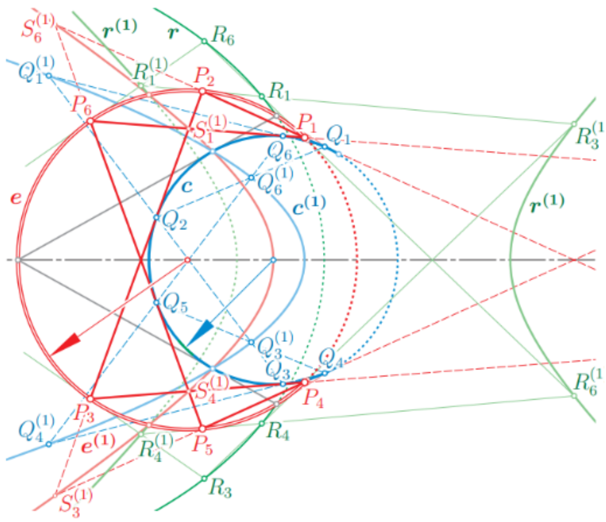

Figure 5: *Periodic projective billiard $P_1P_2\dots P_6$ in the bicentric case with the hyperbolas $e^{(1)}$ (red), $r^{(1)}$ (green), $c^{(1)}$ (blue), and the ellipse r (green).*

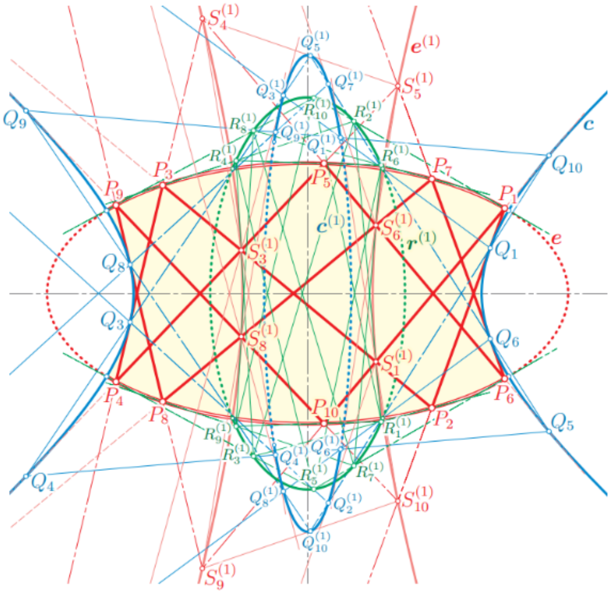

Figure 6: *A periodic hyperbolic billiard $P_1P_2\dots P_{10}$ along with the polygons $S^{(1)}_1S^{(1)}_2\dots S^{(1)}_{10}$ (red), $Q^{(1)}_1Q^{(1)}_2\dots Q^{(1)}_{10}$ (blue), $R^{(1)}_1R^{(1)}_2\dots R^{(1)}_{10}$ (green), and the respective circumconics $e^{(1)}$, $c^{(1)}$ and $r^{(1)}$.*

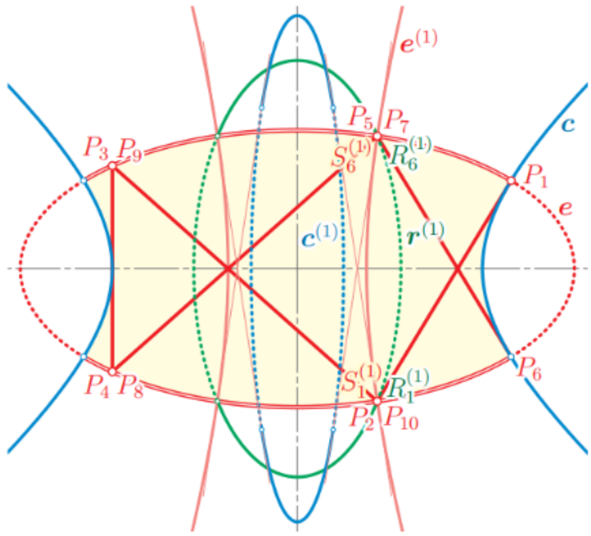

Figure 7: *Twofold pose of a periodic hyperbolic billiard $P_1P_2\dots P_{10}$ with $c^{(1)}$, $e^{(1)}$, and $r^{(1)}$.*

Corollary 1 *Let $P_1P_2\dots$ be a projective billiard with $R_1R_2\dots$ and $Q_1Q_2\dots$ as respective outer and inner polygon. Then for fixed $j \in \{1,2,\dots\}$, the vertices $S^{(j)}_1, S^{(j)}_2, \dots$ on the conic $e^{(j)}$ of the Poncelet grid associated with $P_1P_2\dots$ form another projective billiard with the polygons $R^{(j)}_1R^{(j)}_2\dots$ as outer billiard with circumconic $r^{(j)}$ and*

$Q_1^{(j)}Q_2^{(j)}\dots$ as inner billiard with the inconic $c^{(j)}$, which is polar to $r^{(j)}$ w.r.t. $e^{(j)}$.

The Figures 5–7 illustrate that the triples $(c^{(j)},e^{(j)},r^{(j)})$ can look quite different in comparison with $(c^{(1)},e^{(1)},r^{(1)})$ or $(c^{(2)},e^{(2)},r^{(2)})$ in Figure 4.

As shown at the hyperbolic billiard in Figure 6, the conic $r^{(1)}$ passes through the intersection points of the hyperbola $e^{(1)}$ with e. This follows from particular poses with a twofold covered billiard (see Figure 7): When $P_1\in e$ is specified at an intersection point[3] with the caustic c, then P_2 coincides with P_{10} as well as with $S_1^{(1)}$ and $R_1^{(1)}$. There is a general statement in the background:

Theorem 3 *Referring to the previous notation, for each $j=1,2,\dots$ the conics $r^{(j)}$, $e^{(j)}$ and e belong to a pencil. The same is true for the three conics $e^{(j)}$, $c^{(j)}$ and c (Figure 5).*

Proof. We argue with help of the complex extension of the real projective plane. Whenever the point $R_{i+k}^{(j)}=t_{P_i}\cap t_{P_{i+j}}$ for $k=[\frac{j+1}{2}]$ is located on e, then follows $R_{i+k}^{(j)}=P_i=P_{i+j}$ and consequently $S_{j+k}=[P_{i-1},P_i]\cap[P_{i+j},P_{i+j+1}]=R_{i+k}^{(j)}$. This means that each point of intersection between e and $r^{(j)}$ belongs also to $e^{(j)}$. Therefore, if e and $r^{(j)}$ share four mutually different points, then $e^{(j)}$ belongs to the pencil spanned by $r^{(j)}$ and e.

The remaining cases with intersection points of higher order between $r^{(j)}$ and e can be seen respectively as a limit where some of the four intersection points tend to coincidence. It cannot happen that in the limit the symmetric coefficient matrices of the three conics become linearly independent when everywhere else in the neighborhood they are linearly dependent.

The second statement follows just by replacing the triple $(r^{(j)},e^{(j)},e)$ by $(e^{(j)},c^{(j)},c)$. □

4 Diagonals

In view of the envelopes of the j-th diagonals $[P_i,P_{i+j+1}]$ of our polygon $P_1P_2P_3\dots$, we recall from [9] a result which was first stated in 1822 by V.-P. Poncelet [4] and reproved in 1828 by C.G.J. Jacobi for the case of nested circles e and c. Moreover, we recall from [9] how to find the enveloping points. However, the proofs of the Theorems 1 and 2 in [9] cover only the cases of elliptic and hyperbolic billiards, where affine scalings are available between involved conics. The following theorem addresses the general case.

Theorem 4 *Let $P_1P_2P_3\dots$ be a polygon inscribed in the conic e and circumscribed to the conic c with contact points $Q_1,Q_2,Q_3,\dots$. Then for fixed $j=1,2,\dots$, the envelope of the j-diagonals $[P_i,P_{i+j+1}]$ is a conic $h_{e|j}$ included in the pencil spanned by e and c, provided that in the particular case of N-periodic billiards with even N holds $j\le[\frac{N-3}{2}]$.*
The diagonal $[P_i,P_{i+j+1}]$ contacts $h_{e|j}$ at the intersection with the adjacent j-th diagonals $[Q_{i-1},Q_{i+j}]$ and $[Q_i,Q_{i+j+1}]$ of the inner billiard $Q_1Q_2Q_3\dots$ (Figures 8 or 9).

Proof. (i) According to (1), the extended sides $[P_i,P_{i+1}]$ and $[P_{i+j+1},P_{i+j+2}]$ intersect at the point $S_{i+k+1}^{(j)}$, $k:=[\frac{j}{2}]$, on the conic $e^{(j)}$, which belongs to the range spanned by e and c. The restriction on j in the periodic case as mentioned in Theorem 4 excludes the case where $e^{(j)}$ is a line.

The polarity in the caustic c transforms this into the following statement: The connecting lines $[Q_i,Q_{i+j+1}]$ envelop a conic $h_{c|j}$ which belongs to the pencil spanned by c and the polar conic q of e w.r.t. c (Figures 1 and 8). In order to obtain the first part of our statement, it is sufficient to replace the polygon $Q_1Q_2Q_3\dots$ inscribed in c and circumscribed to q by the original polygon $P_1P_2P_3\dots$ with the circumconic e and the inconic c.

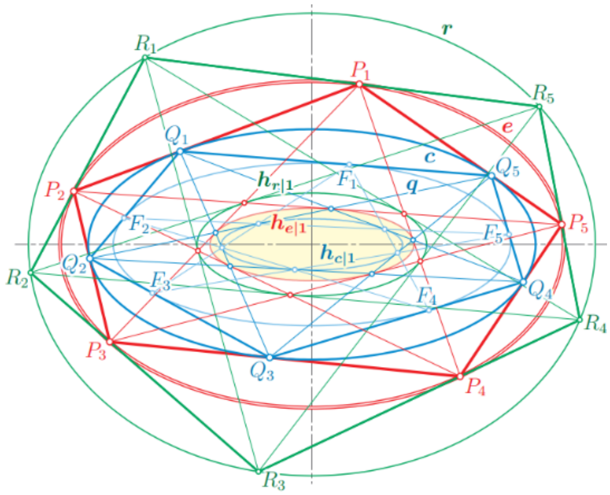

Figure 8: *Envelopes $h_{e|1}$, $h_{c|1}$ and $h_{r|1}$ of the diagonals of the periodic elliptic billiard $P_1P_2\dots P_5$ and of its inner and outer polygons $Q_1Q_2\dots$ and $R_1R_2\dots$. Triples of these diagonals together with that of $F_1F_2\dots$ meet at 15 points in the interior of $P_1P_2\dots$.*

[3]Twofold covered poses of projective billiards arise when one vertex is specified either as a point of intersection between the circumconic e and the inconic c or as the contact point with a common tangent between e and c (note the gray pose in Figure 5).

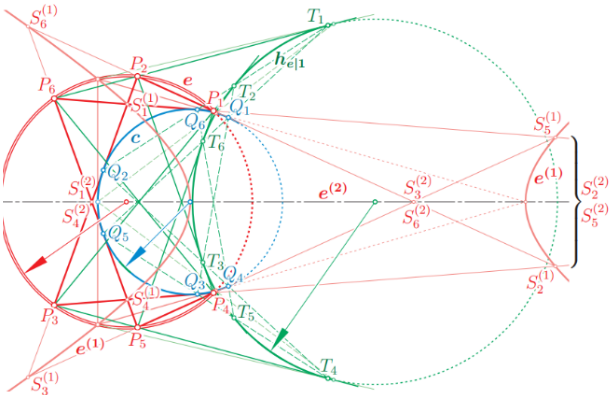

Figure 9: *In the bicentric case with circumcircle e and intersecting incircle c (blue) the envelope of the first diagonals (green solid) of the periodic polygon $P_1P_2\dots P_6$ is the circle $h_{e|1}$ (green) with contact points $T_1, T_2, \dots$. The hyperbola $e^{(1)}$ (pink) and the diameter $e^{(2)}$ belong to the associated Poncelet grid.*

(ii) The point of contact between $[Q_i, Q_{i+j+1}]$ and the envelope $h_{c|j}$ is the c-pole of the tangent to $e^{(j)}$ at $S^{(j)}_{i+k+1}$. By virtue of Theorem 1, this tangent passes through $R^{(j)}_{i+k}$ and $R^{(j)}_{i+k+1}$. Hence, the requested point of contact is the meeting point of the polar lines of $S^{(j)}_{i+k+1}$, $R^{(j)}_{i+k}$ and $R^{(j)}_{i+k+1}$ w.r.t. c.
The c-polar line of $S^{(j)}_{i+k+1}$ is the diagonal $[Q_i, Q_{i+j+1}]$. Since by (2) the point $R^{(j)}_{i+k}$ is the intersection of the tangents to e at P_i and P_{i+j+1}, the c-polar of $R^{(j)}_{i+k}$ connects the contact points F_i and F_{i+j+1} of respective sides of the polygon $Q_1Q_2\dots$ with its envelope q. After increasing all subscripts by 1, we obtain $[F_{i+1}, F_{i+j+2}]$ as the c-polar of $R^{(j)}_{i+k+1}$.
In order to prove the second claim, it is sufficient to replace the polygon $Q_1Q_2\dots$ with the inconic q by the polygon $P_1P_2\dots$ with the inconic c and the contact point F_{i+1} of the side $[Q_i, Q_{i+1}]$ by the contact point Q_i of the side $[P_i, P_{i+1}]$. □

In Figure 8, the particular case $j = 1$ is depicted along with the configuration of the j-th diagonals of $R_1R_2\dots$, $P_1P_2\dots$, $Q_1Q_2\dots$, and $F_1F_2\dots$ with triples of concurrent lines. The depicted enveloping conics $h_{r|1}$, $h_{e|1}$ and $h_{c|1}$ of the j-th diagonals of $R_1R_2\dots$, $P_1P_2\dots$ and $Q_1Q_2\dots$ in Figure 8 reveal that we obtain a sequence of triples of conics like (r, e, c). This reminds on sequences of billiards as presented in [2].

Corollary 2 *Let $P_1P_2\dots$ be a projective billiard with $R_1R_2\dots$ and $Q_1Q_2\dots$ as outer and inner polygon, while $F_1, F_2, \dots$ are the contact points of the inner polygon with its inconic q. Then the j-th diagonals of $Q_1Q_2\dots$ are the sides of another projective billiard, where the j-th diagonals of $P_1P_2\dots$ are the sides of the outer polygon and that of $F_1F_2\dots$ sides of the inner polygon (Figure 8).*

For later use we record a consequence of the Theorems 2 and 4:

Lemma 2 *Referring to the previous notation, the conic $h_{e|j}$ is polar to $c^{(j)}$ w.r.t. the caustic c. The enveloping point of $[S^{(j)}_i, S^{(j)}_{i+1}]$ is the c-pole*

$$\begin{cases} Q^{(j)}_i \ \text{ of } \ d := [P_{i-k}, P_{i+k+1}] & \text{for } \ j = 2k, \text{ and} \\ Q^{(j)}_{i+1} \ \text{ of } \ d := [P_{i-k+1}, P_{i+k+1}] & \text{for } \ j = 2k-1\,. \end{cases}$$

The line d is a j-th diagonal of $P_1P_2P_3\dots$ and a diagonal of the quadrilateral consisting of the tangents drawn from $S^{(j)}_i$ and $S^{(j)}_{i+1}$ to the caustic c.

The composition of the polarities in c and e is a collinear transformation κ. It takes Q_i to R_i and by (3) and (2) $Q^{(j)}_i$ to $R^{(j)}_i$ for all i. Moreover, it sends c to r and $c^{(j)}$ via $h_{e|j}$ to $r^{(j)}$ and the envelope of the j-th diagonals of $Q_1Q_2\dots$ to the envelope of j-th diagonals of $R_1R_2\dots$ (Figure 8). Lines with equal poles w.r.t. e and c remain fixed under κ as for example the axes of symmetry of e in the case of classical billiards.

5 Configurations of lines related to the Poncelet grids

The term 'Poncelet grid' usually stands for a configuration of conics, which are confocal in the particular case of Euclidean billiards. Below we demonstrate that a Poncelet grid is also combined with a configuration of lines.

The following theorem deals with the l-th diagonals of the polygon $S^{(j)}_1 S^{(j)}_2\dots$ inscribed in the conic $e^{(j)}$ of the Poncelet grid associated with $P_1P_2\dots$ and circumscribed to the conic $c^{(j)}$. Note that in the case $l = j$ we obtain extensions of the sides of the original billiard $P_1P_2\dots$.

Theorem 5 *The l-th diagonal $[S^{(j)}_i, S^{(j)}_{i+l+1}]$ of the polygon $S^{(j)}_1 S^{(j)}_2\dots$ inscribed in $e^{(j)}$ contains three meeting points of at least five l-th diagonals of other polygons of the three involved grids (Figure 10):*
(i) *The contact point with the envelope of the l-th diagonals of $S^{(j)}_1 S^{(j)}_2\dots$ is common to $[Q^{(j)}_{i-1}, Q^{(j)}_{i+l}]$, $[Q^{(j)}_i, Q^{(j)}_{i+l+1}]$ as well as for $j = 2k$ to $[Q_{i-k-1}, Q_{i-k+l}]$ and $[Q_{i+k}, Q_{i+k+l+1}]$ and for $j = 2k-1$ to $[Q_{i-k}, Q_{i-k+l+1}]$ and $[Q_{i+k}, Q_{i+k+l+1}]$.*
(ii) *The intersection point with the preceding diagonal*

$[S^{(j)}_{i-1}, S^{(j)}_{i+l}]$ belongs also to $[R^{(j)}_{i-1}, R^{(j)}_{i+l}]$, as well as for even j to $[P_{i-k-1}, P_{i-k+l}]$ and $[P_{i+k}, P_{i+k+l+1}]$ and in the odd case to $[P_{i-k-1}, P_{i-k+l}]$ and $[P_{i+k-1}, P_{i+k+l}]$. A similar result holds for the follower $[S^{(j)}_{i+1}, S^{(j)}_{i+l+2}]$.

Proof. (i) The first statement is a direct consequence of Theorem 4, applied to the projective billiard $S^{(j)}_1 S^{(j)}_2 \dots$ with the circumconic $e^{(j)}$ and the inconic $c^{(j)}$.

In order to prove the second statement of (i), we apply Lemma 2 to the polygon $S^{(j)}_i S^{(j)}_{i+l+1} S^{(j)}_{i+2(l+1)} \dots \in e^{(j)}$, which is formed by l-th diagonals of $S^{(j)}_1 S^{(j)}_2 \dots$, but also by diagonals of a certain type in the polygon (or the union of polygons) with the caustic c and the side lines $[S^{(j)}_i, S^{(j)}_{i+j+1}]$. Hence, the contact point of $[S^{(j)}_i, S^{(j)}_{i+l+1}]$ with the envelope of the l-th diagonals is the c-pole of a diagonal in the quadrilateral formed by the tangents drawn from $S^{(j)}_i$ and $S^{(j)}_{i+l+1}$ to c. According to (1), these tangents contact c respectively

$$\begin{cases} \text{for } \; j = 2k & \text{at } \; Q_{i-k-1}, Q_{i+k} \; \text{ and } \; Q_{i-k+l}, Q_{i+k+l+1}, \\ \text{for } \; j = 2k-1 & \text{at } \; Q_{i-k}, Q_{i+k} \; \text{ and } \; Q_{i-k+l+1}, Q_{i+k+l+1}. \end{cases}$$

Due to the rules of the polarity w.r.t. c, the requested pole is the intersection of the connections of respective contact points, i.e., $[Q_{i-k-1}, Q_{i-k+l}] \cap [Q_{i+k}, Q_{i+k+l+1}]$ for even j and $[Q_{i-k}, Q_{i-k+l+1}] \cap [Q_{i+k}, Q_{i+k+l+1}]$ for odd j.

(ii) From Theorem 4 applied to $r^{(j)}$ and $e^{(j)}$ follows that the contact point of $[R^{(j)}_{i-1}, R^{(j)}_{i+l}]$ with the envelope of the l-th diagonals of $R^{(j)}_1 R^{(j)}_2 \dots$ is common to $[S^{(j)}_{i-1}, S^{(j)}_{i+l}]$ and $[S^{(j)}_i, S^{(j)}_{i+l+1}]$.

In order to prove the second statement, we replace in Lemma 2 the pair of conics $(c, e^{(j)})$ by $(e, r^{(j)})$ and apply this result to the polygons $R^{(j)}_i R^{(j)}_{i+l+1} R^{(j)}_{i+2(l+1)} \dots$ formed by l-th diagonals of $R^{(j)}_1 R^{(j)}_2 \dots$. Hence, the contact point of $[R^{(j)}_{i-1}, R^{(j)}_{i+l}]$ with the envelope of these l-th diagonals is the e-pole of a diagonal d in the quadrilateral formed by the tangents drawn from $R^{(j)}_{i-1}$ and $R^{(j)}_{i+l}$ to e. According to (2), the requested diagonal d of the quadrilateral connects the points

$$t_{P_{i-k-1}} \cap t_{P_{i-k+l}} \quad \text{and} \quad \begin{cases} t_{P_{i+k}} \cap t_{P_{i+k+l+1}} & \text{for } \; j = 2k, \\ t_{P_{i+k-1}} \cap t_{P_{i+k+l}} & \text{for } \; j = 2k-1, \end{cases}$$

The e-pole of d is the intersection of the connections of respective contact points with e, which confirms the claim. □

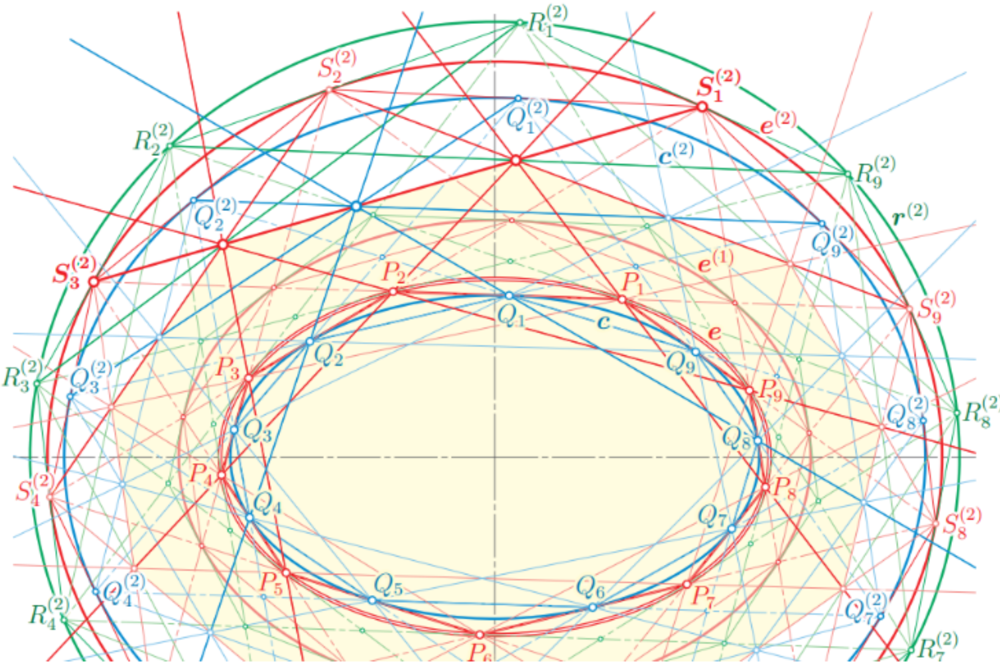

Figure 10: *Each l-th diagonal $[S^{(j)}_i, S^{(j)}_{i+l+1}]$ of the projective billiard $S^{(j)}_1 S^{(j)}_2 \dots$ in $e^{(j)}$ contains three meeting points with at least four other l-th diagonals of involved polygons (Theorem 5). Here the case $j = 2$ and $l = 1$ of a periodic elliptic billiard $P_1 P_2 \dots P_9$ is depicted; note the diagonal $S^{(2)}_1 S^{(2)}_3$ (red).*

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