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# RIORDAN MATRICES AND SUMS OF HARMONIC NUMBERS

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We obtain a general identity involving the row-sums of a Riordan matrix and the harmonic numbers. From this identity, we deduce several particular identities involving numbers of combinatorial interest, such as generalized Fibonacci and Lucas numbers, Catalan numbers, binomial and trinomial coefficients and Stirling numbers.

## 1. INTRODUCTION

In enumerative combinatorics, *Riordan matrices* [**14, 15, 16, 5, 7**] form an important class of combinatorial objects. They are infinite lower triangular matrices $R = [r_{n,k}]_{n,k\geqslant 0} = (g(x), f(x))$ whose columns have generating series $r_k(x) = g(x)f(x)^k$, where $g(x)$ and $f(x)$ are formal series with $g_0 = 1$, $f_0 = 0$ and $f_1 \neq 0$. In particular, a Riordan matrix $R = [r_{n,k}]_{n,k\geqslant 0} = (g(x), f(x))$ induces a transformation $\mathcal{T}_R$ on the set of formal series. Specifically, for any formal series $A(x) = \sum_{n\geqslant 0} a_n x^n$, $\mathcal{T}_R$ is defined by

$$(1) \qquad \mathcal{T}_R A(x) = g(x)A(f(x)) = \sum_{n\geqslant 0}\left(\sum_{k=0}^{n} r_{n,k}\, a_k\right) x^n .$$

Moreover, associated to a Riordan matrix $R = [r_{n,k}]_{n,k\geqslant 0} = (g(x), f(x))$ we also have the *row-sum sequence* $\{r_n\}_{n\in\mathbb{N}}$, where $r_n = \sum_{k=0}^{n} r_{n,k}$, having generating

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series

$$r(x) = \sum_{n \geqslant 0} r_n x^n = \frac{g(x)}{1 - f(x)} . \tag{2}$$

Riordan matrices provide a powerful tool for obtaining identities between combinatorial sequences, as shown, for instance, in [**17, 18**] (see also [**2**]). In this paper, we obtain a general identity involving the row-sums of a Riordan matrix and the *harmonic numbers* [**3**], defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

and having generating series

$$H(x) = \sum_{n \geqslant 0} H_n\, x^n = \frac{1}{1-x}\, \ln \frac{1}{1-x} . \tag{3}$$

From this identity we obtain several particular identities involving combinatorial sequences, such as generalized Fibonacci and Lucas numbers, Catalan numbers, binomial and trinomial coefficients, Stirling numbers.

## 2. THE MAIN IDENTITY

A *logarithmic series* is a formal series of the form

$$f(x) = \sum_{n \geqslant 1} f_n \frac{x^n}{n} .$$

To any ordinary series we can always associate a logarithmic series as follows:

$$f(x) = \sum_{n \geqslant 0} f_n x^n \qquad \rightsquigarrow \qquad F(x) = \sum_{n \geqslant 1} f_n \frac{x^n}{n} = \int_0^x \mathcal{R} f(t)\, \mathrm{d}t \tag{4}$$

where $\mathcal{R}$ denotes the (*incremental ratio*) operator defined by

$$\mathcal{R} f(x) = \frac{f(x) - f_0}{x} = \sum_{n \geqslant 0} f_{n+1}\, x^n .$$

Now, we can prove our main result.

**Theorem 1.** *Let* $R = [r_{n,k}]_{n,k \geqslant 0} = (g(x), f(x))$ *be a Riordan matrix with associated row-sum sequence* $\{r_n\}_{n \in \mathbb{N}}$. *Let* $\{s_n\}_{n \in \mathbb{N}}$ *be a sequence whose ordinary generating series* $s(x) = \sum_{n \geqslant 0} s_n x^n$ *satisfies the relation*

$$\mathcal{R} s(x) = \frac{f'(x)}{1 - f(x)} \qquad \text{or} \qquad s(x) = s_0 + \frac{x f'(x)}{1 - f(x)} . \tag{5}$$

*Then, the identity*

$$\sum_{k=1}^{n} \frac{1}{k}\, s_k r_{n-k} = \sum_{k=0}^{n} r_{n,k} H_k \tag{6}$$

*holds for every $n \in \mathbb{N}$.*

**Proof.** By applying transformation (4) to series $s(x)$ and by identity (5), we obtain the series

$$S(x) = \sum_{n\geqslant 1} s_n \frac{x^n}{n} = \int_0^x \mathcal{R}s(t)\mathrm{d}t = \int_0^x \frac{f'(t)}{1-f(t)}\,\mathrm{d}t = \ln\frac{1}{1-f(x)}\,.$$

Hence, the left-hand side of identity (6) has generating series

$$\begin{aligned}\sum_{n\geqslant 0}\left[\sum_{k=1}^{n}\frac{1}{k}\, s_k r_{n-k}\right] x^n &= \sum_{n\geqslant 0} r_n\, x^n \;\cdot\; \sum_{n\geqslant 1}\frac{s_n}{n}\, x^n \\ &= r(x)S(x) = \frac{g(x)}{1-f(x)}\ln\frac{1}{1-f(x)} = g(x)H(f(x)) = \mathcal{T}_R H(x)\,.\end{aligned}$$

This series is the Riordan transformation (1) of series (3), i.e. of the generating series of harmonic numbers. So, in conclusion, we have identity (6).

Remark. If the row-sums $r_n$ cannot be expressed in a closed form, identity (6) can be written as a double sum:

$$\sum_{k=1}^{n}\frac{s_k}{k}\sum_{i=0}^{n-k} r_{n-k,i} = \sum_{k=0}^{n} r_{n,k} H_k\,. \tag{7}$$

Riordan matrices form a group with respect to the matrix product [**14**], and this group admits several subgroups of combinatorial interest [**16**]. In particular, we have the *derivative subgroup*, consisting of the Riordan matrices $(g(x), f(x))$ such that $g(x) = f'(x)$. In this case, Theorem 1 yields the following result.

**Theorem 2.** *Let $R = [r_{n,k}]_{n,k\geqslant 0} = (f'(x), f(x))$ be a Riordan matrix belonging to the derivative subgroup, with associated row-sum sequence $\{r_n\}_{n\in\mathbb{N}}$. Then, the identity*

$$\sum_{k=1}^{n}\frac{1}{k}\, r_{k-1} r_{n-k} = \sum_{k=0}^{n} r_{n,k} H_k \tag{8}$$

*holds for every $n \in \mathbb{N}$.*

**Proof.** Since $g(x) = f'(x)$, from (2) and (5), we have $\mathcal{R}s(x) = r(x)$ and $s_n = r_{n-1}$ (for $n \geqslant 1$). So, identity (8) follows immediately from (6).

## 3. COMBINATORIAL SEQUENCES

From the general identity (6), we can obtain particular identities involving several other numerical sequences of combinatorial interest. In particular, we will consider the following numbers.

- The *Fibonacci numbers* $F_n$ and the *Lucas numbers* $L_n$, defined by the series

$$F(x) = \sum_{n\geqslant 0} F_n x^n = \frac{x}{1-x-x^2}$$
$$L(x) = \sum_{n\geqslant 0} L_n x^n = \frac{2-x}{1-x-x^2}\,.$$

For the Lucas numbers we also have the logarithmic generating series

$$\sum_{n\geqslant 1} L_n \frac{x^n}{n} = \ln \frac{1}{1-x-x^2}\,.$$

This result justifies the generalizations given here below. Moreover, we have the *Binet formulas*:

(9) $$F_n = \frac{\varphi^n - \widehat{\varphi}^n}{\sqrt{5}} \qquad \text{and} \qquad L_n = \varphi^n + \widehat{\varphi}^n$$

where $\varphi = (1+\sqrt{5})/2$ and $\widehat{\varphi} = (1-\sqrt{5})/2$.

- The *generalized Fibonacci numbers of the first kind* $f_n^{[m]}$ (see [**6, 9**]) and the *generalized Lucas numbers of the first kind* $\ell_n^{[m]}$, with $m \geqslant 1$. They are defined by the generating series

(10) $$f^{[m]}(x) = \sum_{n\geqslant 0} f_n^{[m]} x^n = \frac{1}{1-x-x^2-\cdots-x^m}$$

(11) $$\ell^{[m]}(x) = \sum_{n\geqslant 1} \ell_n^{[m]} \frac{x^n}{n} = \ln \frac{1}{1-x-x^2-\cdots-x^m}\,.$$

Moreover, if $D$ denotes the operator of formal differentiation (with respect to $x$) and $\vartheta = xD$ (so that $\vartheta f(x) = xf'(x)$), then we have

(12) $$\sum_{n\geqslant 1} \ell_n^{[m]} x^n = \vartheta \ell^{[m]}(x) = \frac{x+2x^2+\cdots+mx^m}{1-x-x^2-\cdots-x^m}\,.$$

Combinatorially, $f_n^{[m]}$ is the number of all linear partitions of $\{1,2,\ldots,n\}$ consisting of blocks of size at most $m$ (see, for instance, [**10**] and [**11**]). In particular, for $m=2$, we have $f_n^{[2]} = F_{n+1}$ and $\ell_n^{[2]} = L_n$.

- The *generalized Fibonacci numbers of the second kind* $F_n^{[m]}$ and the *generalized Lucas numbers of the second kind* $L_n^{[m]}$, with $m \geqslant 1$. They are defined by the generating series

$$F^{[m]}(x) = \sum_{n \geqslant 0} F_n^{[m]} x^n = \frac{1 + x + x^2 + \cdots + x^{m-1}}{1 - x - x^m} \tag{13}$$

$$L^{[m]}(x) = \sum_{n \geqslant 1} L_n^{[m]} \frac{x^n}{n} = \ln \frac{1 + x + x^2 + \cdots + x^{m-1}}{1 - x - x^m}. \tag{14}$$

Moreover, we have

$$\sum_{n \geqslant 1} L_n^{[m]} x^n = \vartheta L^{[m]}(x) = \frac{x + (m-1)x^{m+1}}{(1 - x^m)(1 - x - x^m)} + \frac{x}{1 - x}. \tag{15}$$

Combinatorially, $F_n^{[m]}$ is the number of all subsets $X$ of $\{1, 2, \ldots, n\}$ such that $|x - y| \geqslant m$ for every $x, y \in X$, $x \neq y$ (see, for instance, [**12**]). In particular, for $m = 2$, we have $F_n^{[2]} = F_{n+2}$ and $L_n^{[2]} = L_n - (-1)^n$.

- The *polynomial coefficients* [**1**], defined by the identity

$$\sum_{k=0}^{(m-1)n} \binom{n; m}{k} x^k = (1 + x + x^2 + \cdots + x^{m-1})^n.$$

In particular, for $m = 2$ we have the *binomial coefficients*, and for $m = 3$ we have the *trinomial coefficients*.

- The *multiset coefficients* $\left(\!\binom{\alpha}{n}\!\right) = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{n!}$, such that

$$\sum_{n \geqslant 0} \left(\!\binom{\alpha}{n}\!\right) x^n = \frac{1}{(1 - x)^\alpha}.$$

- The *Catalan numbers* $C_n$, the *Motzkin numbers* $M_n$, the *central binomial coefficients* $\binom{2n}{n}$, and the *central trinomial coefficients* $T_n = \binom{n; 3}{n}$, defined by the series

$$C(x) = \sum_{n \geqslant 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}$$

$$M(x) = \sum_{n \geqslant 0} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$

$$B(x) = \sum_{n \geqslant 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1 - 4x}}$$

$$T(x) = \sum_{n \geqslant 0} T_n x^n = \frac{1}{\sqrt{1 - 2x - 3x^2}}.$$

- The *Stirling numbers of the first kind* and the *Stirling numbers of the second kind* [**3**]. For these numbers we have the identities

$$\sum_{n\geqslant k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!} = \frac{1}{k!}\left(\ln\frac{1}{1-x}\right)^k$$

$$\sum_{n\geqslant k} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{x^n}{n!} = \frac{1}{1-x}\,\frac{1}{k!}\left(\ln\frac{1}{1-x}\right)^k$$

$$\sum_{n\geqslant k} \begin{Bmatrix} n \\ k \end{Bmatrix} \frac{x^n}{n!} = \frac{(\mathrm{e}^x-1)^k}{k!}$$

$$\sum_{n\geqslant k} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \frac{x^n}{n!} = \mathrm{e}^x\frac{(\mathrm{e}^x-1)^k}{k!}\,.$$

- The *preferential arrangement numbers* (see, for instance, [**4**]), or *ordered Bell numbers*, $\mathcal{O}_n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} k!$ having generating exponential series

$$\mathcal{O}(x) = \sum_{n\geqslant 0} \mathcal{O}_n \frac{x^n}{n!} = \frac{1}{2-\mathrm{e}^x}\,.$$

Remark. As usual, we write $[x^n]f(x)$ for the coefficient of $x^n$ in the formal series $f(x)$. See, for instance, [**8**]. Moreover, the field of complex numbers will be denoted by $\mathbb{C}$. In what follows, for simplicity, the symbols $\alpha$, $\beta$ and $q$ are considered as complex parameters, but more generally they can be considered as indeterminates.

## 4. COMBINATORIAL IDENTITIES

All the following relations are obtained by applying identity (6) (or identity (8), when the Riordan matrix belongs to the derivative subgroup), and hold for every $n \in \mathbb{N}$.

**Proposition 3.** *For every $q \in \mathbb{C}$, $q \neq 1$, we have the identities*

$$\sum_{k=0}^{n} q^{n-k}\, H_k = \frac{q^{n+1}H_n(1/q) - H_n}{q-1} \qquad (q \neq 0) \tag{16}$$

$$\sum_{k=0}^{n} q^{k}\, H_k = \frac{H_n(q) - q^{n+1}H_n}{1-q} \tag{17}$$

*where $H_n(x) = \sum_{k=1}^{n} \frac{x^k}{k}$. In particular, we have the identities*

$$\sum_{k=0}^{n} (-1)^{n-k} H_k = \frac{H_n + (-1)^n H_n(-1)}{2} \tag{18}$$

$$\sum_{k=0}^{n} H_k = (n+1)H_n - n\,. \tag{19}$$

**Proof.** For the Riordan matrix

$$R = \left[q^{n-k}\right]_{n,k\geqslant 0} = \left(\frac{1}{1-qx}, x\right),$$

with $q \neq 1$, we have $r_n = 1 + q + \cdots + q^n = \dfrac{q^{n+1}-1}{q-1}$. Moreover, we have $\mathcal{R}s(x) = \dfrac{1}{1-x}$ and $s_n = 1$. If $q \neq 0$, the first member of (6) becomes

$$\sum_{k=1}^{n} \frac{1}{k}\frac{q^{n-k+1}-1}{q-1} = \frac{1}{q-1}\left(\sum_{k=1}^{n}\frac{q^{n-k+1}}{k} - \sum_{k=1}^{n}\frac{1}{k}\right) = \frac{q^{n+1}H_n(1/q) - H_n}{q-1}\,.$$

So, we have identity (16), which implies identity (17). Then, identity (18) can be obtained from identity (16) with $q = -1$. Finally, identity (19) can be obtained in a similar way from the same Riordan matrix in the case $q = 1$.

**Proposition 4.** *For every $q \in \mathbb{C}$, we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\, H_k = (1+q)^n H_n - \sum_{k=1}^{n}\frac{q^k(1+q)^{n-k}}{k}\,. \tag{20}$$

*In particular, we have the identities*

$$\sum_{k=0}^{n}\binom{n}{k} H_k = 2^n H_n - \sum_{k=1}^{n}\frac{2^{n-k}}{k} \tag{21}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k H_k = -\frac{1}{n} \quad (n \geqslant 1) \tag{22}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k 2^{n-k} H_k = H_n - \sum_{k=1}^{n}\frac{2^k}{k} \tag{23}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} 2^k H_k = H_n - \sum_{k=1}^{n}\frac{(-1)^k}{k}\,. \tag{24}$$

*Moreover, we have the identities*

$$\sum_{k=0}^{n}\binom{n}{k} F_{n-k} H_k = F_{2n} H_n - \sum_{k=1}^{n}\frac{F_{2n-k}}{k} \tag{25}$$

$$\sum_{k=0}^{n}\binom{n}{k} L_{n-k} H_k = L_{2n} H_n - \sum_{k=1}^{n}\frac{L_{2n-k}}{k} \tag{26}$$

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} F_{n-k} H_k = F_{3n} H_n - \sum_{k=1}^{n} \frac{2^k F_{3n-2k}}{k} \tag{27}$$

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} L_{n-k} H_k = L_{3n} H_n - \sum_{k=1}^{n} \frac{2^k L_{3n-2k}}{k}\,. \tag{28}$$

**Proof.** For the Riordan matrix

$$R = \left[\binom{n}{k} q^{n-k}\right]_{n,k\geqslant 0} = \left(\frac{1}{1-qx}, \frac{x}{1-qx}\right)$$

we obtain the series

$$r(x) = \frac{1}{1-(1+q)x}, \quad \mathcal{R}s(x) = \frac{1}{(1-qx)(1-(1+q)x)} = \frac{1+q}{1-(1+q)x} - \frac{q}{1-qx}\,.$$

So, we have the coefficients $r_n = (1+q)^n$ and $s_n = (1+q)^n - q^n$. Now, by substituting in (6) and simplifying, we obtain identity (20).

Identities (21), (22), (23) and (24) can be obtained respectively for $q = 1$, $q = -1$, $q = -2$ and $q = -1/2$. Finally, to obtain identity (25), we substitute in identity (20) first $q = \varphi$ and then $q = \widehat{\varphi}$, simplify using the relations $\varphi^2 = \varphi + 1$ and $\widehat{\varphi}^2 = \widehat{\varphi} + 1$, take the difference of the two identities just obtained, divide both sides by $\sqrt{5}$, and finally simplify using Binet formulas (9). In a similar way, we can obtain also identity (26). Finally, to obtain identities (27) and (28), we can proceed in the same way starting with the substitutions $q = 2\varphi^2$ and then $q = 2\widehat{\varphi}^2$, and using the relations $\varphi^3 = 2\varphi + 1$ and $\widehat{\varphi}^3 = 2\widehat{\varphi} + 1$,

**Proposition 5.** *For every $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$, we have the identity*

$$\sum_{k=1}^{n} \frac{\alpha^k + \beta^k}{k} \, \frac{\alpha^{n-k+1} + \beta^{n-k+1}}{\alpha - \beta} = \sum_{k=\lceil n/2 \rceil}^{n} \binom{k}{n-k} (-\alpha\beta)^{n-k} (\alpha+\beta)^{2k-n} H_k\,. \tag{29}$$

*In particular, for every $m \in \mathbb{N}$, $m \geqslant 1$, we have the identities*

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k}\, L_{mk} \frac{F_{m(n-k+1)}}{F_m} &= \sum_{k=\lceil n/2 \rceil}^{n} \binom{k}{n-k} (-1)^{(m+1)(n-k)} L_m^{2k-n} H_k \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-1)^{(m+1)k} L_m^{n-2k} H_{n-k}\,. \end{aligned} \tag{30}$$

**Proof.** For the Riordan matrix

$$R = \left[\binom{k}{n-k} (-\alpha\beta)^{n-k} (\alpha+\beta)^{2k-n}\right]_{n,k\geqslant 0} = (1, (\alpha+\beta)x - \alpha\beta x^2)$$

we have the series

$$r(x) = \frac{1}{(1-\alpha x)(1-\beta x)} = \frac{\alpha}{\alpha-\beta}\frac{1}{1-\alpha x} + \frac{\beta}{\beta-\alpha}\frac{1}{1-\beta x},$$

$$\mathcal{R}s(x) = \frac{\alpha+\beta-2\alpha\beta x}{(1-\alpha x)(1-\beta x)} = \frac{\alpha}{1-\alpha x} + \frac{\beta}{1-\beta x}\,.$$

So, we have the coefficients $r_n = \dfrac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$ and $s_n = \alpha^n + \beta^n$.

Finally, to obtain identity (30), it is sufficient to set $\alpha = \varphi^m$ and $\beta = \widehat{\varphi}^m$, and to use Binet formulas (9).

**Proposition 6.** *For every $m \in \mathbb{N}$, $m \geqslant 2$, we have the identities*

$$\sum_{k=1}^{n} \frac{1}{k}\, \ell_k^{[m]} f_{n-k}^{[m]} = \sum_{k=\lceil n/m \rceil}^{n} \binom{k;\, m}{n-k} H_k = \sum_{k=0}^{\lfloor (m-1)n/m \rfloor} \binom{n-k;\, m}{k} H_{n-k}\,. \tag{31}$$

*In particular, for $m = 2$, we have the identities*

$$\sum_{k=1}^{n} \frac{1}{k}\, L_k F_{n-k+1} = \sum_{k=\lceil n/2 \rceil}^{n} \binom{k}{n-k} H_k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} H_{n-k}\,. \tag{32}$$

**Proof.** For the Riordan matrix

$$R = \left[\binom{k;\, m}{n-k}\right]_{n,k\geqslant 0} = (1, x + x^2 + \cdots + x^m)$$

we have the series

$$\begin{aligned} r(x) &= \frac{1}{1 - x - x^2 - \cdots - x^m} = f^{[m]}(x) \\ \mathcal{R}s(x) &= \frac{1 + 2x + \cdots + m x^{m-1}}{1 - x - x^2 - \cdots - x^m} = \mathcal{R}\vartheta \ell^{[m]}(x)\,. \end{aligned}$$

Hence, from identities (10) and (12), we have $r_n = f_n^{[m]}$ and $s_n = \ell_n^{[m]}$ (for $n \geqslant 1$). Finally, for $m = 2$, we have $f_n^{[2]} = F_{n+1}$ and $\ell_n^{[2]} = L_n$ (for $n \geqslant 1$).

Remark. Identity (32) is also a particular case of identity (30), for $m = 1$.

**Proposition 7.** *For every $m \in \mathbb{N}$, $m \geqslant 2$, we have the identity*

$$\sum_{k=1}^{n} \frac{1}{k}\, (L_k^{[m]} - 1) F_{n-k}^{[m]} = \sum_{k=0}^{n} \binom{\lfloor (n-k)/m \rfloor + k}{k} H_k\,. \tag{33}$$

*In particular, for $m = 2$, we have the identity*

$$\sum_{k=1}^{n} \frac{1}{k}\, (L_k - 1 - (-1)^k) F_{n-k+1} = \sum_{k=0}^{n} \binom{\lfloor (n+k)/2 \rfloor}{k} H_k\,. \tag{34}$$

**Proof.** For the Riordan matrix

$$R = \left[\binom{\lfloor (n-k)/m \rfloor + k}{k}\right]_{n,k\geqslant 0} = \left(\frac{1}{1-x}, \frac{x}{1-x^m}\right)$$

we have the series

$$r(x) = \frac{1 + x + x^2 + \cdots + x^{m-1}}{1 - x - x^m} = F^{[m]}(x)$$

and then the coefficient $r_n = F_n^{[m]}$. Moreover, we have

$$f'(x) = \frac{1 + (m-1)x^m}{(1 - x^m)^2} \qquad \text{and} \qquad \mathcal{R}s(x) = \frac{1 + (m-1)x^m}{(1 - x^m)(1 - x - x^m)}\,.$$

So, from (15), we obtain $s_n = L_n^{[m]} - 1$ (for $n \geqslant 1$).

Finally, for $m = 2$, we have the coefficients $\binom{\lfloor (n-k)/2 \rfloor + k}{k} = \binom{\lfloor (n+k)/2 \rfloor}{k}$ (related to the *Terquem problem* [**13**, p. 17]). Moreover, we have $r_n = F_n^{[2]} = F_{n+2}$ and $s_n = L_n^{[2]} - 1 = L_n - 1 - (-1)^n$.

**Proposition 8.** *We have the identities*

$$\sum_{k=1}^{n} \frac{1}{k}\,(L_{2k} - 2)F_{2n-2k+2} = \sum_{k=0}^{n} \binom{n+k+1}{n-k} H_k \tag{35}$$

$$\sum_{k=1}^{n} \frac{1}{k}\,(L_{2k} - 2)L_{2n-2k+1} = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{2n+1}{2k+1}\, H_k\,. \tag{36}$$

**Proof.** For the Riordan matrix

$$R = \left[\binom{n+k+1}{n-k}\right]_{n,k \geqslant 0} = \left(\frac{1}{(1-x)^2}, \frac{x}{(1-x)^2}\right)$$

we have the series

$$r(x) = \frac{1}{1 - 3x + x^2} \qquad \text{and} \qquad s(x) = 1 - \frac{2}{1-x} + \frac{2 - 3x}{1 - 3x + x^2}\,.$$

So $r_n = F_{2n+2}$ and $s_n = L_{2n} - 2$ (for $n \geqslant 1$). This proves identity (35). Similarly, for the Riordan matrix

$$R = \left[\binom{n+k}{n-k}\frac{2n+1}{2k+1}\right]_{n,k \geqslant 0} = \left(\frac{1+x}{(1-x)^2}, \frac{x}{(1-x)^2}\right)$$

we have the series

$$r(x) = \frac{1+x}{1 - 3x + x^2} \qquad \text{and} \qquad \mathcal{R}s(x) = \frac{3 - 2x}{1 - 3x + x^2} - \frac{2}{1-x}\,.$$

So $r_n = L_{2n+1}$ and $s_n = L_{2n} - 2$ (for $n \geqslant 1$). This proves identity (36).

**Proposition 9.** *We have the identity*

$$\sum_{k=1}^{n} \binom{2k-1}{k-1} \binom{2n-2k+1}{n-k+1} \frac{1}{k} = \sum_{k=0}^{n} \binom{2n-k}{n-k}\, H_k\,. \tag{37}$$

**Proof.** The Riordan matrix

$$R = \left[\binom{2n-k}{n-k}\right]_{n,k\geqslant 0} = (B(x), xC(x))$$

belongs to the derivative subgroup. So, in this case, we have

$$r(x) = \mathcal{R}s(x) = B(x)C(x) = \frac{1-\sqrt{1-4x}}{2x\sqrt{1-4x}}\,.$$

Hence $r_n = \binom{2n+1}{n+1}$ and $s_n = r_{n-1} = \binom{2n-1}{n-1}$ (for $n \geqslant 1$).

**Proposition 10.** *For every $m \in \mathbb{N}$, we have the identity*

$$\begin{aligned}(38)\qquad &\sum_{k=1}^{n} \binom{2k-1}{k-1}\frac{1}{k}\binom{m+2n-2k}{n-k}\frac{m+1}{m+n-k+1}\\ &= \sum_{k=0}^{n}\binom{m+2n-k}{n-k}\frac{m+k}{m+2n-k}\,H_k\,.\end{aligned}$$

*In particular, for $m = 0, 1, 2$, we have the identities*

$$(39)\qquad \sum_{k=1}^{n}\binom{2k-1}{k-1}\frac{1}{k}\,C_{n-k} = \sum_{k=0}^{n}\binom{2n-k}{n-k}\frac{k}{2n-k}\,H_k$$

$$(40)\qquad \sum_{k=1}^{n}\binom{2k-1}{k-1}\frac{1}{k}\,C_{n-k+1} = \sum_{k=0}^{n}\binom{2n-k}{n-k}\frac{k+1}{n+1}\,H_k$$

$$(41)\qquad \sum_{k=1}^{n}\binom{2k-1}{k-1}\frac{3(n-k+1)}{k(n-k+3)}\,C_{n-k+1} = \sum_{k=0}^{n}\binom{2n-k+1}{n-k}\frac{k+2}{n+2}\,H_k\,.$$

**Proof.** Consider the Riordan matrix

$$R = \left[\binom{m+2n-k}{n-k}\frac{m+k}{m+2n-k}\right]_{n,k\geqslant 0} = (C(x)^m, xC(x))\ .$$

Then, we have

$$r(x) = \frac{C(x)^m}{1-xC(x)} = C(x)^{m+1} \qquad \text{and} \qquad r_n = \binom{m+2n}{n}\frac{m+1}{m+n+1}\,.$$

Moreover, we have $f'(x) = \frac{1}{\sqrt{1-4x}}$, and hence

$$s(x) = \frac{1}{2}\left(1+\frac{1}{\sqrt{1-4x}}\right) \quad \text{and} \quad s_n = \frac{1}{2}\binom{2n}{n} = \binom{2n-1}{n-1} \quad (\text{for } n \geqslant 1)\,.$$

In particular, for $m = 0$, we have $r(x) = C(x)$ and hence $r_n = C_n$. Similarly, for $m = 1$, we have $r(x) = C(x)^2 = \mathcal{R}C(x)$ and hence $r_n = C_{n+1}$. Finally, for

$m = 2$, we have $r(x) = C(x)^3 = C(x)\mathcal{R}C(x) = \mathcal{R}^2 C(x) - \mathcal{R}C(x)$. So, we have the coefficients

$$r_n = C_{n+2} - C_{n+1} = \frac{3(n+1)}{n+3}\, C_{n+1}\,.$$

In this way, we have identities (39), (40) and (41).

**Proposition 11.** *For every $m \in \mathbb{N}$, we have the identities*

$$\sum_{k=1}^{n} \binom{2k-1}{k-1}\binom{m+2n-2k}{n-k} \frac{(m+3)(n-k)}{k(m+n-k+1)(m+n-k+2)} = \sum_{k=0}^{n} \binom{m+2n-k}{n-k} \frac{(m+k+2)(n-k)}{(m+n+1)(m+2n-k)}\, H_k \tag{42}$$

$$\sum_{k=1}^{n} \binom{2k-1}{k-1}\binom{m+2n-2k}{n-k} \frac{2(m+2)(m+1)+(3m+5)(n-k)}{k(m+n-k+1)(m+n-k+2)} = \sum_{k=0}^{n} \binom{m+2n-k}{n-k} \frac{2m^2+3mn+mk+3nk+2m+2n-k^2}{(m+n+1)(m+2n-k)}\, H_k\,. \tag{43}$$

*In particular, for $m = 0$, we have the identities*

$$\sum_{k=1}^{n} \binom{2k-1}{k-1}\binom{2n-2k}{n-k} \frac{3(n-k)}{k(n-k+1)(n-k+2)} = \sum_{k=0}^{n} \binom{2n-k}{n-k} \frac{(k+2)(n-k)}{(n+1)(2n-k)}\, H_k \tag{44}$$

$$\sum_{k=1}^{n} \binom{2k-1}{k-1}\binom{2n-2k}{n-k} \frac{5n-5k+4}{k(n-k+1)(n-k+2)} = \sum_{k=0}^{n} \binom{2n-k}{n-k} \frac{3nk+2n-k^2}{(n+1)(2n-k)}\, H_k\,. \tag{45}$$

**Proof.** To obtain identities (42) and (43), it is sufficient to substitute $m$ with $m+1$ in identity (39), and then to take the difference and the sum between this identity and identity (39).

**Proposition 12.** *For every $m \in \mathbb{N}$, we have the identity*

$$\sum_{k=1}^{n} \left(2^{2k-1} - \binom{2k-1}{k-1}\right) \binom{m+2n-2k}{n-k} \frac{1}{k} = \sum_{k=0}^{n} \binom{m+2n}{n-k} \frac{m+2k+1}{m+n+k+1}\, H_k\,. \tag{46}$$

*In particular, for $m = 0$, we have the identity*

$$\sum_{k=1}^{n} \left(2^{2k-1} - \binom{2k-1}{k-1}\right) \binom{2n-2k}{n-k} \frac{1}{k} = \sum_{k=0}^{n} \binom{2n}{n-k} \frac{2k+1}{n+k+1}\, H_k\,. \tag{47}$$

**Proof.** For the Riordan matrix

$$R = \left[\binom{m+2n}{n-k} \frac{m+2k+1}{m+n+k+1}\right]_{n,k\geqslant 0} = \big(C(x)^{m+1}, C(x)-1\big)$$

we have the series

$$\begin{aligned} r(x) &= \frac{1}{\sqrt{1-4x}} \left(\frac{1-\sqrt{1-4x}}{2x}\right)^m \\ \mathcal{R}s(x) &= \frac{1-\sqrt{1-4x}}{2x(1-4x)} = \frac{1}{2}\left(\mathcal{R}\frac{1}{1-4x} + \mathcal{R}\frac{1}{\sqrt{1-4x}}\right), \end{aligned}$$

and consequently the coefficients (for $n \geqslant 1$)

$$r_n = \binom{m+2n}{n}, \quad s_n = \frac{1}{2}\left(4^n - \binom{2n}{n}\right) = \left(2^{2n-1} - \binom{2n-1}{n-1}\right).$$

**Proposition 13.** *For every $m \in \mathbb{N}$, we have the identity*

$$\begin{aligned} &\sum_{k=1}^{n} \left(2^{2k-1} - \binom{2k-1}{k-1}\right) \binom{m+2n-2k}{n-k} \frac{n-k}{k(m+n-k+1)} \\ &= \sum_{k=0}^{n} \binom{m+2n}{n-k} \frac{(n-k)(m+2k+3)}{(m+n+k+1)(m+n+k+2)}\, H_k\,. \end{aligned} \tag{48}$$

**Proof.** To obtain identity (48), it is sufficient to take the difference between identity (49) with $m+1$ substituted to $m$ and identity (49) itself.

**Proposition 14.** *We have the identity*

$$\sum_{k=1}^{n} \binom{2k-1}{k-1} \binom{2n-2k}{n-k} \frac{1}{k} = \sum_{k=0}^{n} \left(\!\!\binom{n}{n-k}\!\!\right) H_k\,. \tag{49}$$

**Proof.** For the Riordan matrix

$$R = \left[\left(\!\!\binom{n}{n-k}\!\!\right)\right]_{n,k\geqslant 0} = \left(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}, \frac{1-\sqrt{1-4x}}{2}\right)$$

we have the series

$$r(x) = \frac{1}{\sqrt{1-4x}} \qquad \text{and} \qquad \mathcal{R}s(x) = \frac{1}{2}\,\mathcal{R}\frac{1}{\sqrt{1-4x}}\,.$$

So we have the coefficients

$$r_n = \binom{2n}{n} \qquad \text{and} \qquad s_n = \frac{1}{2}\binom{2n}{n} = \binom{2n-1}{n-1}.$$

**Proposition 15.** *We have the identity*

$$\sum_{k=1}^{n} \frac{(3^k - T_k)(3^{n-k} + T_{n-k})}{4k} = \sum_{k=0}^{n} \binom{n;\,3}{n-k} H_k \,. \tag{50}$$

**Proof.** For the Riordan matrix

$$R = \left[\binom{n;\,3}{n-k}\right]_{n,k\geqslant 0} = \left[\binom{n;\,3}{n+k}\right]_{n,k\geqslant 0} = (T(x), xM(x))$$

generated by the trinomial coefficients, we have the series

$$r(x) = \frac{1}{2}\left(\frac{1}{1-3x} + \frac{1}{\sqrt{1-2x-3x^2}}\right)$$
$$s(x) = \frac{1}{2}\left(3 + \frac{3x}{1-3x} - \frac{1}{\sqrt{1-2x-3x^2}}\right),$$

and the coefficients $r_n = (3^n + T_n)/2$ and $s_n = (3^n - T_n)/2$ (for $n \geqslant 1$).

**Proposition 16.** *For every $\alpha, q \in \mathbb{C}$, we have the identity*

$$\sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{n-k} \binom{\alpha+i}{i} q^i = \sum_{k=0}^{n} \binom{\alpha+n-k}{n-k} q^{n-k} H_k \,. \tag{51}$$

*In particular, for every $m, r \in \mathbb{N}$, we have the identities*

$$\sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{n-k} \binom{\alpha+i}{i} F_{mi+r} = \sum_{k=0}^{n} \binom{\alpha+n-k}{n-k} F_{m(n-k)+r} H_k \tag{52}$$

$$\sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{n-k} \binom{\alpha+i}{i} L_{mi+r} = \sum_{k=0}^{n} \binom{\alpha+n-k}{n-k} L_{m(n-k)+r} H_k \,. \tag{53}$$

**Proof.** For the Riordan matrix

$$R = \left[\binom{\alpha+n-k}{n-k}\right]_{n,k\geqslant 0} = \left(\frac{1}{(1-qx)^{\alpha+1}}, x\right),$$

we have the series

$$r(x) = \frac{1}{(1-x)(1-qx)^{\alpha}} \qquad \text{and} \qquad \mathcal{R}s(x) = \frac{1}{1-x}$$

and consequently the coefficients

$$r_n = \sum_{k=0}^{n} \binom{\alpha+k}{k} q^k \qquad \text{and} \qquad s_n = 1\,.$$

Finally, substitute $q = \varphi^m$ in identity (51) and multiply both sides by $\varphi^r$, Repeat the same for $q = \widehat{\varphi}^m$. To obtain identity (52), take the difference of these two identities and divide by $\sqrt{5}$. To obtain identity (53), take the sum of these two identities. In both cases, use Binet formulas.

**Proposition 17.** *For every $\alpha, q \in \mathbb{C}$, we have the identities*

$$\sum_{k=1}^{n} \frac{(1+q)^k - q^k}{k} \sum_{i=0}^{n-k} \binom{\alpha+n-k}{i} q^i = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} q^{n-k} H_k . \tag{54}$$

$$\sum_{k=1}^{n} \frac{(1+q)^k - q^k}{k} \sum_{i=0}^{n-k} \left(\!\!\binom{\alpha}{i}\!\!\right) (1+q)^{n-k-i} = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} q^{n-k} H_k . \tag{55}$$

**Proof.** For the Riordan matrix

$$R = \left[ \binom{\alpha+n}{n-k} q^{n-k} \right]_{n,k \geqslant 0} = \left( \frac{1}{(1-qx)^{\alpha+1}}, \frac{x}{1-qx} \right),$$

we have the series

$$\begin{aligned} r(x) &= \frac{1}{(1-x)^{\alpha}(1-(1+q)x)} \\ \mathcal{R}s(x) &= \frac{1}{(1-x)(1-(1+q)x)} = \frac{1+q}{1-(1+q)x} - \frac{q}{1-qx} \end{aligned}$$

and consequently the coefficients

$$r_n = \sum_{k=0}^{n} \binom{\alpha+n}{k} q^k = \sum_{k=0}^{n} \left(\!\!\binom{\alpha}{k}\!\!\right) (1+q)^{n-k} \qquad \text{and} \qquad s_n = (1+q)^n - q^n . \quad \square$$

The following result generalizes Propositions 3, 4, 16 and 17.

**Proposition 18.** *For every $\alpha, \beta, q \in \mathbb{C}$, we have the identity*

$$\begin{aligned} &\sum_{k=1}^{n} \left[ \sum_{i=1}^{k} \left(\!\!\binom{i\beta}{k-i}\!\!\right) \frac{q^{k-i}}{i} \right] \left[ \sum_{i=0}^{n-k} \binom{\alpha+(n-k-i)\beta+i}{i} q^i \right] \\ &= \sum_{k=0}^{n} \binom{\alpha+k\beta+n-k}{n-k} q^{n-k} H_k . \end{aligned} \tag{56}$$

**Proof.** Consider the Riordan matrix

$$R = \left[ \binom{\alpha+k\beta+n-k}{n-k} q^{n-k} \right]_{n,k \geqslant 0} = \left( \frac{1}{(1-qx)^{\alpha+1}}, \frac{x}{(1-qx)^{\beta}} \right).$$

In this case, we only have

$$r_n = \sum_{k=0}^{n} r_{n,k} = \sum_{k=0}^{n} \binom{\alpha+k\beta+n-k}{n-k} q^{n-k} = \sum_{k=0}^{n} \binom{\alpha+(n-k)\beta+k}{k} q^k .$$

Moreover, we have the series

$$\begin{aligned}\mathcal{R}s(x) &= \frac{1+(\beta-1)qx}{(1-qx)((1-qx)^{\beta}-x)} \\ &= \frac{1}{(1-qx)^{\beta}}\,\frac{1}{1-\dfrac{x}{(1-qx)^{\beta}}} + \frac{1}{(1-qx)^{\beta+1}}\,\frac{\beta qx}{1-\frac{x}{(1-qx)^{\beta}}} \\ &= \sum_{k\geqslant 0}\frac{x^k}{(1-qx)^{(k+1)\beta}} + q\beta\sum_{k\geqslant 0}\frac{x^{k+1}}{(1-qx)^{(k+1)\beta+1}} \\ &= \sum_{k\geqslant 0}\sum_{n\geqslant k}\left(\!\!\binom{(k+1)\beta}{n-k}\!\!\right)q^{n-k}x^n + q\beta\sum_{k\geqslant 0}\sum_{n\geqslant k}\left(\!\!\binom{(k+1)\beta+1}{n-k-1}\!\!\right)q^{n-k-1}x^n \\ &= \sum_{n\geqslant 0}\left(\sum_{k=0}^{n}\left[\left(\!\!\binom{(k+1)\beta}{n-k}\!\!\right)+\beta\left(\!\!\binom{(k+1)\beta+1}{n-k-1}\!\!\right)\right]q^{n-k}\right)x^n\,.\end{aligned}$$

Now, using the properties of the multiset coefficients, we have

$$\begin{aligned}&\left(\!\!\binom{(k+1)\beta}{n-k}\!\!\right)+\beta\left(\!\!\binom{(k+1)\beta+1}{n-k-1}\!\!\right) \\ &= \left(\!\!\binom{(k+1)\beta}{n-k}\!\!\right)+\beta\left(\!\!\binom{(k+1)\beta}{n-k}\!\!\right)\frac{n-k}{(k+1)\beta} = \left(\!\!\binom{(k+1)\beta}{n-k}\!\!\right)\frac{n+1}{k+1}\,.\end{aligned}$$

So, we have

$$\mathcal{R}s(x) = \sum_{n\geqslant 0}\left(\sum_{k=0}^{n}\left(\!\!\binom{(k+1)\beta}{n-k}\!\!\right)\frac{n+1}{k+1}\,q^{n-k}\right)x^n$$

and consequently

$$s_n = \sum_{k=0}^{n-1}\left(\!\!\binom{(k+1)\beta}{n-k-1}\!\!\right)\frac{n}{k+1}\,q^{n-k-1} = \sum_{k=1}^{n}\left(\!\!\binom{k\beta}{n-k}\!\!\right)\frac{n}{k}\,q^{n-k}\,.$$

**Proposition 19.** *For every $m \in \mathbb{N}$, we have the identity*

$$\begin{aligned}\sum_{k=1}^{n}\frac{5^k-4^k}{k}\binom{2m+2n-2k}{m+n-k}\sum_{i=0}^{n-k}\binom{m+n-k}{m-i}\binom{2m+2i}{m+i}^{-1} \\ = \binom{2m+2n}{m+n}\sum_{k=0}^{n}\binom{m+n}{m+k}\binom{2m+2k}{m+k}^{-1}H_k\,.\end{aligned} \tag{57}$$

**Proof.** For the Riordan matrix

$$R = \left[\binom{m+n}{m+k}\binom{2m+2n}{m+n}\binom{2m+2k}{m+k}^{-1}\right]_{n,k\geqslant 0} = \left(\frac{1}{(1-4x)^m\sqrt{1-4x}},\frac{x}{1-4x}\right),$$

we have the series

$$\mathcal{R}s(x) = \frac{1}{(1-4x)(1-5x)} = \frac{5}{1-5x} - \frac{4}{1-4x}$$

and consequently the coefficients $s_n = 5^n - 4^n$. For the row-sums we have no a particular closed form, and so

$$r_n = \binom{2m+2n}{m+n} \sum_{k=0}^{n} \binom{m+n}{m+k} \binom{2m+2k}{m+k}^{-1}.$$

**Proposition 20.** *We have the identities*

$$\sum_{k=1}^{n} \binom{n}{k} \mathcal{O}_{k-1}\mathcal{O}_{n-k} - \frac{n}{2}\,\mathcal{O}_{n-1} = \frac{1}{2}\sum_{k=0}^{n} \left\{ n \atop k \right\} H_k k! \tag{58}$$

$$\sum_{k=1}^{n} \binom{n}{k} \mathcal{O}_{k-1}\mathcal{O}_{n-k} - \frac{n+1}{2}\,\mathcal{O}_{n-1} + \frac{1}{4}\,\delta_{n,1} = \frac{1}{4}\sum_{k=0}^{n} \left\{ n+1 \atop k+1 \right\} H_k k!\,. \tag{59}$$

**Proof.** For the Riordan matrix

$$R = \left[ \left\{ n \atop k \right\} \frac{k!}{n!} \right]_{n,k \geqslant 0} = (1, \mathrm{e}^x - 1)\,,$$

we have the series

$$r(x) = \frac{1}{2-\mathrm{e}^x} = \mathcal{O}(x), \quad \mathcal{R}s(x) = \frac{\mathrm{e}^x}{2-\mathrm{e}^x} = \frac{2}{2-\mathrm{e}^x} - 1 = 2\mathcal{O}(x) - 1$$

and consequently the coefficients $r_n = \dfrac{\mathcal{O}_n}{n!}$ and $s_n = \dfrac{2\mathcal{O}_{n-1}}{(n-1)!} - \delta_{n,1}$ (for $n \geqslant 1$). So, identity (6) becomes

$$\sum_{k=1}^{n} \frac{2\mathcal{O}_{k-1}}{k\,(k-1)!} \frac{\mathcal{O}_{n-k}}{(n-k)!} - \frac{\mathcal{O}_{n-1}}{(n-1)!} = \sum_{k=0}^{n} \left\{ n \atop k \right\} \frac{k!}{n!}\, H_k\,.$$

Now, by multiplying both sides for $n!/2$, we obtain identity (58).

Finally, identity (59) can be obtained in a similar way, starting from the Riordan matrix

$$R = \left[ \left\{ n+1 \atop k+1 \right\} \frac{k!}{n!} \right]_{n,k \geqslant 0} = (\mathrm{e}^x, \mathrm{e}^x - 1)\,.$$

Remark. The preferential arrangement numbers can be expressed in terms of harmonic numbers as follows

$$\mathcal{O}_n = \sum_{k=0}^{n} \left\{ n+1 \atop k+1 \right\} H_{k+1}(k+1)! - \frac{1}{2}\sum_{k=0}^{n} \left\{ n+2 \atop k+2 \right\} H_{k+1}(k+1)! + \frac{1}{2}\,\delta_{n,0}\,. \tag{60}$$

Indeed, it is sufficient to subtract identity (59) to identity (58), to substitute $n$ with $n+1$, and to divide by 2.

**Proposition 21.** *We have the identities*

$$\sum_{k=1}^{n} \binom{n}{k} a_{k-1} a_{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} H_k k! \tag{61}$$

$$\sum_{k=1}^{n} \binom{n}{k} a_{k-1} b_{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} H_k k! \tag{62}$$

*where*

$$a_n = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} k! \qquad \text{and} \qquad b_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} k!\,.$$

**Proof.** The Riordan matrix

$$R = \left[ \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{k!}{n!} \right]_{n,k \geqslant 0} = \left( \frac{1}{1-x}, \ln \frac{1}{1-x} \right)$$

belongs to the derivative subgroup. Moreover, we have

$$r_n = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \frac{k!}{n!} = \frac{a_n}{n!}\,.$$

So, by substituting in identity (8) and by multiplying by $n!$, we obtain identity (61). Similarly, for the Riordan matrix

$$R = \left[ \begin{bmatrix} n \\ k \end{bmatrix} \frac{k!}{n!} \right]_{n,k \geqslant 0} = \left( 1, \ln \frac{1}{1-x} \right)$$

we have

$$r_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{k!}{n!} = \frac{b_n}{n!}$$

and $s_n = a_{n-1}$ as before. Again, by substituting in identity (8) and by multiplying for $n!$, we obtain identity (62).

## 5. OTHER COMBINATORIAL IDENTITIES

From identity (20) we can obtain another general identity and several other particular identities involving combinatorial sequences.

**Theorem 22.** *For a Riordan matrix $R = [r_{n,k}]_{n,k \geqslant 0} = (g(x), f(x))$, the identity*

$$\sum_{k=0}^{n} \binom{n}{k} r_{m,n-k} H_k = f_{m,n} H_n - \sum_{k=1}^{n} \frac{1}{k} F_{m,n,k} \tag{63}$$

*holds for every $m, n \in \mathbb{N}$, where*

$$f_{m,n} = [x^m] g(x)(1+f(x))^n = \sum_{k=0}^{n} \binom{n}{k} r_{m,k} \tag{64}$$

$$F_{m,n,k} = [x^m] g(x) f(x)^k (1+f(x))^{n-k} = \sum_{i=0}^{n-k} \binom{n-k}{i} r_{m,i+k} \,. \tag{65}$$

**Proof.** By substituting $q = f(x)$ in identity (20), and then by multiplying both sides by $g(x)$, we obtain the relation

$$\sum_{k=0}^{n} \binom{n}{k} g(x) f(x)^{n-k} \, H_k = g(x)(1+f(x))^n H_n - \sum_{k=1}^{n} \frac{g(x) f(x)^k (1+f(x))^{n-k}}{k} \,.$$

Now, by taking the coefficient of $x^m$, we obtain identity (63).

**Proposition 23.** *For every $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{m+\alpha}{m-n+k} H_k = \binom{m+n+\alpha}{m} H_n - \sum_{k=1}^{m} \binom{m+n-k+\alpha}{m-k} \frac{1}{k} \,. \tag{66}$$

*In particular, for $m = n$ and $\alpha = 0$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k}^2 H_k = \binom{2n}{n} H_n - \sum_{k=1}^{n} \binom{2n-k}{n-k} \frac{1}{k} \tag{67}$$

*and, for $m = n = \alpha$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2n}{k} H_k = \binom{3n}{n} H_n - \sum_{k=1}^{n} \binom{3n-k}{n-k} \frac{1}{k} \,. \tag{68}$$

**Proof.** Consider the Riordan matrix

$$R = \left[ \binom{n+\alpha}{n-k} \right]_{n,k \geqslant 0} = \left( \frac{1}{(1-x)^{\alpha+1}}, \frac{x}{1-x} \right) .$$

Then, $1 + f(x) = 1/(1-x)$ and so we have

$$f_{m,n} = [x^m] \frac{1}{(1-x)^{n+\alpha+1}} = \binom{m+n+\alpha}{m}$$

$$F_{m,n,k} = [x^{m-k}] \frac{1}{(1-x)^{n+\alpha+1}} = \binom{m-k+n+\alpha}{m-k} .$$

**Proposition 24.** *For every $\alpha \in \mathbb{C}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{\alpha}{n-k} H_k = \binom{\alpha+n}{n} H_n - \sum_{k=1}^{n} \binom{\alpha+n-k}{n} \frac{1}{k} \,. \tag{69}$$

**Proof.** Immediate consequence of identity (66), where $\alpha = 0$ and $m$ is substituted with a new symbol $\alpha$ (by applying the identity principle for polynomials).

**Proposition 25.** *For every $m, r \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2m+r}{m-n+k} H_k = \binom{2m+n+r}{m} H_n - \sum_{k=1}^{n} \binom{2m+n-k+r}{m-k} \frac{1}{k}.$$

*In particular, for $m = n$ and $r = 1$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2n+1}{2n-k+1} H_k = \binom{3n+1}{2n+1} H_n - \sum_{k=1}^{n} \binom{3n-k+1}{2n+1} \frac{1}{k} \tag{70}$$

*and, for $m = n = r$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{3n}{k} H_k = \binom{4n}{n} H_n - \sum_{k=1}^{n} \binom{4n-k}{n-k} \frac{1}{k}. \tag{71}$$

**Proof.** Consider the Riordan matrix

$$R = \left[ \binom{2n+r}{n-k} \right]_{n,k \geqslant 0} = \left( \frac{1}{\sqrt{1-4x}} \left( \frac{1-\sqrt{1-4x}}{2x} \right)^r, \frac{1-2x-\sqrt{1-4x}}{2x} \right).$$

In this case, we have

$$1 + f(x) = \frac{1-\sqrt{1-4x}}{2x} \qquad \text{and} \qquad f(x) = x(1+f(x))^2. \tag{72}$$

So, we have

$$f_{m,n} = [x^m] \frac{1}{\sqrt{1-4x}} \left( \frac{1-\sqrt{1-4x}}{2x} \right)^{n+r} = \binom{2m+n+r}{m}.$$

Moreover, we have

$$f(x)^k (1+f(x))^{n-k} = x^k (1+f(x))^{n+k}. \tag{73}$$

So, we have

$$F_{m,n,k} = [x^{m-k}] \frac{1}{\sqrt{1-4x}} \left( \frac{1-\sqrt{1-4x}}{2x} \right)^{n+k+r} = \binom{2m+n-k+r}{m-k}. \qquad \square$$

Notice that the identity obtained in Proposition 25 can also be obtained from identity (66) for $\alpha = m + r$.

**Proposition 26.** *For every $m, r \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2m+r}{m-n+k} \frac{2n-2k+r+1}{m+n-k+r+1} H_k = \binom{2m+n+r}{m} \frac{n+r+1}{m+n+r+1} H_n - \sum_{k=1}^{m} \frac{1}{k} \binom{2m+n-k+r}{m-k} \frac{n+k+r+1}{m+n+r+1} .$$

*In particular, for $m = n$ and $r = 0$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2n}{k} \frac{2n-2k+1}{2n-k+1} H_k = \binom{3n}{n} \frac{n+1}{2n+1} H_n - \sum_{k=1}^{n} \frac{1}{k} \binom{3n-k}{n-k} \frac{n+k+1}{2n+1} .$$

*and, for $m = n = r$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{3n}{k} \frac{3n-2k+1}{3n-k+1} H_k = \binom{4n}{n} \frac{2n+1}{3n+1} H_n - \sum_{k=1}^{n} \frac{1}{k} \binom{4n-k}{n-k} \frac{2n+k+1}{3n+1} .$$

**Proof.** Consider the Riordan matrix

$$\left[ \binom{2n+r}{n-k} \frac{2k+r+1}{n+k+r+1} \right]_{n,k \geqslant 0} = \left( \left( \frac{1-\sqrt{1-4x}}{2x} \right)^{r+1}, \frac{1-2x-\sqrt{1-4x}}{2x} \right).$$

Also in this case, we have (72) and (73). So, we obtain

$$f_{m,n} = [x^m] \left( \frac{1-\sqrt{1-4x}}{2x} \right)^{n+r+1} = \binom{2m+n+r+1}{m} \frac{n+r+1}{2m+n+r+1}$$

and

$$F_{m,n,k} = [x^{m-k}] \left( \frac{1-\sqrt{1-4x}}{2x} \right)^{n+k+r+1} = \binom{2m+n-k+r+1}{m-k} \frac{n+k+r+1}{2m+n-k+r+1} .$$

**Proposition 27.** *For every $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, we have the identity*

$$(74) \qquad \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+n-k;\, s}{m-n+k} H_k = H_n \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+k;\, s}{m-k} - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{m-k} \binom{n-k}{i} \binom{\alpha+i+k;\, s}{m-i-k} .$$

*In particular, for $\alpha = 0$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n-k;\, s}{m-n+k} H_k = \binom{n;\, s+1}{m} H_n - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} \binom{i+k;\, s}{m-i-k}$$

*and for $\alpha = 0$ and $m = n$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n-k;\, s}{k} H_k = \binom{n;\, s+1}{n} H_n - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} \binom{i+k;\, s}{n-i-k}.$$

*Moreover, for $\alpha = m = n$, we have*

$$\begin{aligned}&\sum_{k=0}^{n} \binom{n}{k} \binom{2n-k;\, s}{k} H_k \\ &= H_n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k;\, s}{n-k} - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} \binom{n+i+k;\, s}{n-i-k}.\end{aligned}$$

**Proof.** Consider the Riordan matrix

$$R = \left[ \binom{\alpha+k;\, s}{n-k} \right]_{n,k \geqslant 0} = \left( (1+x+x^2+\cdots+x^{s-1})^{\alpha}, x+x^2+\cdots+x^s \right).$$

Then, for $\alpha = 0$ we have

$$f_{m,n} = [x^m](1+x+x^2+\cdots+x^s)^n = \binom{n;\, s+1}{m}.$$

More generally, for (64) and (65), we have

$$f_{m,n} = \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+k;\, s}{m-k} \quad \text{and} \quad F_{m,n,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} \binom{\alpha+i+k;\, s}{m-i-k}.$$

**Proposition 28.** *For every $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, we have the identity*

$$\begin{aligned}&\sum_{k=0}^{n} \binom{n}{k} H_k \sum_{i=0}^{m-n+k} \binom{\alpha}{i} \binom{n-k+i;\, s}{m-n+k-i} \\ &= \binom{\alpha+n;\, s+1}{m} H_n - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{m-k} \binom{\alpha+n-k}{i} \binom{i+k;\, s}{m-i-k}.\end{aligned} \tag{75}$$

*In particular, for $\alpha = m = n$, we have*

$$\begin{aligned}&\sum_{k=0}^{n} \binom{n}{k} H_k \sum_{i=0}^{k} \binom{n}{i} \binom{n-k+i;\, s}{k-i} \\ &= \binom{2n;\, s+1}{n} H_n - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{n-k} \binom{2n-k}{i} \binom{i+k;\, s}{n-i-k}.\end{aligned} \tag{76}$$

**Proof.** Consider the Riordan matrix

$$\left[\sum_{i=0}^{n-k}\binom{\alpha}{i}\binom{i+k;\,s}{n-k-i}\right]_{n,k\geqslant 0}=\big((1+x+x^2+\cdots+x^s)^\alpha, x+x^2+\cdots+x^s\big)\,.$$

Then, we have

$$f_{m,n}=[x^m](1+x+x^2+\cdots+x^s)^{\alpha+n}=\binom{\alpha+n;\,s+1}{m}.$$

Moreover, we have

$$\begin{aligned}F_{m,n,k}&=[x^m](1+x+x^2+\cdots+x^s)^{\alpha+n-k}(x+x^2+\cdots+x^s)^k\\&=\sum_{i=0}^{m-k}\binom{\alpha+n-k}{i}\binom{i+k;\,s}{m-k-i}.\end{aligned}$$

**Proposition 29.** *For every $m\in\mathbb{N}$, we have the identities*

$$\sum_{k=0}^{n}\binom{n}{k}\left\{{m \atop n-k}\right\}(n-k)!H_k=n^mH_n-\sum_{k=1}^{n}\frac{1}{k}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(n-k+i)^m \tag{77}$$

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\left\{{m+1 \atop n-k+1}\right\}(n-k)!H_k\\&=(n+1)^mH_n-\sum_{k=1}^{n}\frac{1}{k}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}(n-k+i+1)^m.\end{aligned} \tag{78}$$

**Proof.** Consider the Riordan matrix

$$R=\left[\left\{{n \atop k}\right\}\frac{k!}{n!}\right]_{n,k\geqslant 0}=(1,\mathrm{e}^x-1)\,.$$

Then, we have $f_{m,n}=[x^m]\mathrm{e}^{nx}=n^m/m!$ and

$$\begin{aligned}F_{m,n,k}&=[x^m](\mathrm{e}^x-1)^k\mathrm{e}^{(n-k)x}\\&=[x^m]\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\mathrm{e}^{(n-k+1)x}=\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\frac{(n-k+i)^m}{m!}.\end{aligned}$$

Multiplying by $m!$, we obtain identity (77). Finally, identity (78) can be obtained in a similar way starting from the Riordan matrix

$$R=\left[\left\{{n+1 \atop k+1}\right\}\frac{k!}{n!}\right]_{n,k\geqslant 0}=(\mathrm{e}^x,\mathrm{e}^x-1)\,.$$

**Proposition 30.** *For every $m \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha+n-k)^{m-n+k}}{(m-n+k)!} H_k = H_n \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha+k)^{m-k}}{(m-k)!} - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(\alpha+i+k)^{m-i-k}}{(m-i-k)!} . \tag{79}$$

*In particular, for $\alpha = 0$ and $m = n$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)^k}{k!} H_k = H_n \sum_{k=0}^{n} \binom{n}{k} \frac{k^{n-k}}{(n-k)!} - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(i+k)^{n-i-k}}{(n-i-k)!} .$$

**Proof.** Consider the Riordan matrix

$$R = \left[ \frac{(\alpha+k)^{n-k}}{(n-k)!} \right]_{n,k\geqslant 0} = (\mathrm{e}^{\alpha x}, x\mathrm{e}^{x}) .$$

Then, for (64) and (65), we have

$$f_{m,n} = \sum_{k=0}^{n} \binom{n}{k} \frac{(\alpha+k)^{m-k}}{(m-k)!} \quad \text{and} \quad F_{m,n,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{(\alpha+i+k)^{m-i-k}}{(m-i-k)!} .$$

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