

OPTIMAL CONTROL FOR STOCHASTIC HEAT EQUATION WITH MEMORY

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(Communicated by Igor Chueshov)

ABSTRACT. In this paper, we investigate the existence and uniqueness of solutions for a class of evolutionary integral equations perturbed by a noise arising in the theory of heat conduction. As a motivation of our results, we study an optimal control problem when the control enters the system together with the noise.

1. Introduction. Our main goal in this paper is to analyze a class of stochastic integro-differential equation arising in the theory of heat conduction for materials with memory and to present an application to an optimal control problem where the control enters the system together with the noise. Needless to say that many physical phenomena are better described if one considers in the equation of the model some terms which take into consideration the past history of the system. Further, it is sensible to assume that the model of certain phenomena from the real world are more realistic if some kind of uncertainty, for instance, some randomness or environmental noise, is also considered in the formulation.

We wish to mention that applications to optimal control problems naturally arise in the study of heating processes, for example in modeling heating with radiation boundary condition, simplified superconductivity, control of stationary flows, gluing in polymeric materials (for a thorough introduction to these problems we refer to the standard monograph by Lions [17] or Tröltzsch [24]).

Here we are concerned with the following semilinear heat equation

$$\partial_t v(t, x) = k_0 \Delta v(t, x) + \int_{-\infty}^t k_1(t-s) \Delta v(s) ds + g(t, x, v(t, x)) \quad (1)$$

in the bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with Dirichlet boundary condition

$$v|_{\partial\mathcal{O}}(t, x) = 0, \quad t \in \mathbb{R}, x \in \partial\mathcal{O} \quad (2)$$

2010 *Mathematics Subject Classification.* Primary: 45D05, 93E20; Secondary: 60H30.

Key words and phrases. Equations with memory, dynamical systems, semigroup theory, forward-backward systems, stochastic optimal control.

and initial condition given by

$$v(s, x) = v_0(s, x), \quad s \leq 0, x \in \mathcal{O}. \quad (3)$$

Notice that $v_0(\cdot)$ represents the *past history* of the system and should satisfy suitable smoothness properties (as we will see later on). Moreover, the function $k(t) = k_0 + \int_0^t k_1(s)ds$ is called the *convolution kernel* of the system and k_1 is assumed to be 3-monotone (see Hypothesis 1 for the precise definition of this term).

We are interested in the analysis of the system (1) when the function g :

- i.* is given by a Lipschitz continuous term f and an additive Gaussian noise W with covariance Q , i.e.

$$g(t, x, v) = f(t, v) + \sqrt{Q}\partial_t W(t);$$

- ii.* depends on a further parameter γ which introduces a control process in the system; this means that

$$g = g(t, x, v, \gamma) = f(t, v(t, x)) + \sqrt{Q}(r(t, v(t, x), \gamma(t, x)) + \partial_t W(t)), \quad (4)$$

where r is a function with appropriate regularity.

The main question arising around the first case (which we refer to as *uncontrolled problem*) is to determine existence and uniqueness of the solution. This problem can be handled by introducing an auxiliary variable which contains the information about the past history of the system. In this way we can reduce equation (1) to an abstract Cauchy problem on a suitable product space. Within this framework the system can be represented with the following evolutionary equation

$$\begin{cases} dX(t) = \mathbf{A}X(t)dt + \mathbf{F}(t, X(t))dt + \sqrt{\mathbf{Q}}dW(t) \\ X(0) = X_0, \end{cases} \quad (5)$$

where \mathbf{A} is an unbounded linear operator, \mathbf{F} is a Lipschitz continuous function in X , \mathbf{Q} a linear operator and \mathbf{W} a vector defined in term of the Wiener process $(W(t))_{t \geq 0}$. On this space, we can characterize the generation properties of the leading operator \mathbf{A} and prove that, in our setting, it is the generator of a C_0 -semigroup $e^{t\mathbf{A}}$.

There is a huge literature concerning integro-differential equations; see, for instance, the monograph [22]. Several semigroup approaches have been developed, among them we quote the history function approach by Miller [19] and Dafermos [11] and the more recent works by Bonaccorsi et al. [1, 2, 3] for stochastic models. In this paper we developed the same idea of Bonaccorsi et al. [3], which takes inspiration from the fundamental paper by Miller [19]. The main difficulty is to describe the semigroup $e^{t\mathbf{A}}$ and to study the stochastic convolution corresponding with the system. To this end we apply the resolvent method presented, for instance, in Clement and Da Prato [6], Monniaux and Prüss [20] and Prüss [22]. Anyway, differently from [3] (where k_1 is assumed completely monotone and a constraint is assumed between k_0 and the L^1 -norm of k_1), we are able to treat more general kernels.

We stress that our approach has the advantage that it naturally links the solution of a Volterra equation to a Markov process; this has important developments in view of the application to optimal control problems for Volterra equations.

In the case g contains a control parameter (which we refer to as *control problem*), the natural question is to determine a solution of the Volterra equation and a control

process γ , within a set of admissible controls, in such a way that they minimize a cost functional. In particular in this paper we consider a cost of the form:

$$\mathbb{J}(v_0, \gamma) = \mathbb{E} \int_0^T \int_{\bar{\mathcal{O}}} \ell(t, v(t, \xi), \gamma(t, \xi)) d\xi dt + \mathbb{E} \int_{\bar{\mathcal{O}}} \phi(v(T, \xi)) d\xi,$$

where ℓ and ϕ are given real functions.

In the same way as the uncontrolled problem, the model can be translated into an abstract setting. In particular, it can be rewritten in the form

$$\begin{cases} dX(t) = \mathbf{A}X(t)dt + \mathbf{F}(t, X(t))dt + \sqrt{\mathbf{Q}}(R(t, X(t), \gamma(t)))dt + dW(t) \\ X(0) = X_0, \end{cases}$$

where $\mathbf{A}, \mathbf{F}, \mathbf{Q}, \mathbf{W}, X_0$ are as above and R is given in terms of the function r introduced in (4). Notice here the special structure of the control term, which is clearly a restriction; however it arises from concrete models, such as gluing in polymeric materials (compare with subsection 6.2). Due to the special structure of the control term we are able to perform the synthesis of the optimal control, by solving in the weak sense the closed loop equation. The main object of investigation is the Hamilton-Jacobi-Bellman equation

$$\begin{cases} \frac{\partial}{\partial t} u(t, X) + \mathcal{L}_t[u(t, \cdot)](X) \\ \quad = \psi(t, X, u(t, X), \nabla u(t, X)\sqrt{\mathbf{Q}}), \quad t \in [0, T], X \in \mathcal{H}, \\ u(T, X) = \phi(X). \end{cases} \quad (\text{HJB})$$

where \mathcal{L} is the infinitesimal generator of the Markov semigroup corresponding to the process X :

$$\mathcal{L}_t[h](X) = \frac{1}{2} \text{Tr}(\nabla^2 h(X)\mathbf{Q}) + \langle \mathbf{A}X + \mathbf{F}(t, X), \nabla h(X) \rangle.$$

Moreover, ψ is the Hamiltonian function of the problem, defined in terms of ℓ and ϕ (see Section 6). It turns out that HJB admits a unique mild solution u which has a probabilistic representation. Namely, u can be characterized in terms of the solution $(Y(t), Z(t))$ of the following backward stochastic differential equation

$$\begin{aligned} dY(t) &= \psi(t, X(t), Z(t))d\sigma + Z(t)d\mathbf{W}(t) \quad t \in [0, T] \\ Y(T) &= \Phi(X(T)). \end{aligned}$$

Here X stands for the solution of (5) starting at time t from $X_0 \in \mathcal{H}$. It turns out that, setting $u(t, X_0) = Y(t)$, then u is the unique mild solution of (HJB).

Using the above probabilistic representation, we easily show existence of a feedback law. In the present paper, we assume that all the coefficients are Lipschitz functions, so that the problem can be handled by applying the methods developed in Fuhrman and Tessitore [13]. This is clearly a restriction. In forthcoming work, we shall treat more realistic situations and weaker regularity assumptions. Anyway, we emphasize that the paper, at our knowledge, is the first attempt to study optimal control problems for Volterra integral equations under hypothesis of the kernel which are typical of linear viscoelastic material behavior.

The paper is organized as follows: in the next subsection we give the physical motivation of our work. In Section 3 we introduce the main assumptions on the coefficients of the problem. In Section 4 we reformulate the uncontrolled problem into a semilinear abstract evolution equation and we study the properties of leading operator. In Section 5 we prove the first main result of the paper: we determine

existence and uniqueness of the solution of the Volterra equation (1) in the uncontrolled case. To this end, we study the so-called resolvent family (see Subsection 5.1) and the scalar resolvent family (see Subsection 5.2) associated with our problem. Moreover, we focus on the stochastic convolution of the rewritten equation (see Subsection 5.4). Finally, in Section 6 we perform the standard synthesis of the optimal control and we give an explicit example.

2. Motivation. Let us briefly explain one possible physical meaning of our model. Let \mathcal{O} be a 3-dimensional *homogeneous* and *isotropic* rigid body (see Prüss [22, p.125] for more details on the physical terminology) which is represented by an open set $\mathcal{O} \subset \mathbb{R}^d$ ($d = 1, 2, 3$) with boundary $\partial\mathcal{O}$ of class C^1 . Points in \mathcal{O} (i.e. material points) will be denoted by x, y, \dots . Suppose that the body \mathcal{O} is subject to temperature changes. We denote by $v = v(t, x)$ the temperature at time $t \in \mathbb{R}^+$, $q(t, x)$ the heat flux vector field, $e(t, x)$ the temperature and $f(t, x, v)$ the heat supply (possibly depending on the solution itself).

We denote by $v = v(t, x)$ the temperature at time $t \in \mathbb{R}^+$, $q(t, x)$ the heat flux vector field, $e(t, x)$ the temperature and $f(t, x)$ the external heat supply. Balance of energy then reads as:

$$\partial_t e(t, x) = -\operatorname{div} q(t, x) + f(t, x), \quad t \in \mathbb{R}, x \in \mathcal{O}, \quad (6)$$

with the boundary conditions basically either prescribed temperature or prescribed heat flux through the boundary. In particular, one (natural) choice is represented by Dirichlet boundary conditions:

$$v(t, x)|_{x \in \partial\mathcal{O}} = 0, \quad t \in \mathbb{R}.$$

For the relationship between e and v we shall use the following linear law (or, more formally, *constitutive law*):

$$e(t, x) = \int_0^\infty dm(r)v(t-r, x)dr + e_\infty, \quad t \in \mathbb{R}^+, x \in \mathcal{O};$$

where e_∞ is a suitable positive phenomenological constant. Analogously, for the constitutive law relating q and v we choose

$$q(t, x) = - \int_0^\infty dk(r)\nabla v(t-r, x), \quad t \in \mathbb{R}, x \in \mathcal{O},$$

where $m, k \in BV_{loc}(\mathbb{R}^+)$ are scalar functions.

Rearranging equation (6), we arrive at the following non autonomous heat-equation with memory

$$\begin{aligned} dm * \partial_t v(t, x) &= dk * \Delta v(t, x) + f(t, x), & t > 0, x \in \mathcal{O}; \\ v(t, x) &= 0, & t \in \mathbb{R}_+, x \in \partial\mathcal{O}, \end{aligned} \quad (7)$$

where $*$ denotes the symbol for the convolution product between two functions.

Remark 1. From the literature one can infer that m is a creep function, i.e. it is nonnegative, nondecreasing and concave which is also bounded. The natural form of this kind of functions is given by

$$m(t) = m_0 + m_\infty t + \int_0^t m_1(r)dr,$$

for $m_0 \geq 0$, $m_\infty \geq 0$ (in our case, $m_\infty = 0$) and $m_1 \in L^1(\mathbb{R}_+)$.

From a physical point of view, m_0 corresponds to the *instantaneous heat capacity*, i.e. the ratio of the change in heat energy of a unit mass of a substance to the change

in temperature of the substance. The function m_1 is called *energy-temperature relaxation function* while $\lim_{t \rightarrow \infty} m(t) = m_0 + \int_0^\infty m_1(s)ds$ is termed *equilibrium heat capacity*.

Concerning the function k , the literature is somewhat controversial. From Gurtin and Pipkin [16] and Nunziato [21] one can expect that k is a bounded creep function as well, in particular $k_\infty = 0$, $k_1 \in L^1(\mathbb{R}^+)$ and $\lim_{t \rightarrow \infty} k(t) = k_0 + \int_0^\infty k_1(s)ds > 0$. The constant k_0 is termed *instantaneous conductivity*, $k(\infty)$ is called *equilibrium conductivity* while k_1 is called *heat conduction relaxation function*. On the other hand, Clément and Nohel [8], Clément and Prüss [9] and Lunardi [18] and write $k(t) = k_0 - \int_0^t k_1(s)ds > 0$ with k_1 positive and nonincreasing; in this case k is 2-monotone (see Hypothesis 1 for the explanation of this term). Also Bonaccorsi et al. [3] consider k as above but they require k_1 completely monotone. In this theory the equilibrium conductivity $k(\infty) = k_0 - \int_0^\infty k_1(s)ds$ is smaller than the instantaneous conductivity, in contrast with Nunziato.

We notice that in this paper we are concerned with a *non-linear, homogeneous, controlled* version of (7).

In accordance with several works concerning with the same type of problems (see, for example, Clément and Nohel [8], Nunziato[21], Monniaux and Prüss [20], Grasselli and Pata [14, 15]), in this paper we are concerned, for simplicity, with $m(t) \equiv m_0 = 1$.

Moreover, we assume that k has the same form as in Gurtin and Pipkin [16] and Nunziato [21].

3. General assumptions. In equation (1), we are given the kernel $k : \mathbb{R} \rightarrow \mathbb{R}$, the non linear term $f : \mathbb{R} \rightarrow \mathbb{R}$ and the stochastic perturbation $(W(t))_{t \geq 0}$.

We assume the following.

Hypothesis 1. The kernel $\{k(t) : t \geq 0\}$ is a creep function, that is, $k(t) := k_0 + \int_0^t k_1(s)ds$, where $k_0 > 0$ and the function k_1 is **3-monotone**, that is, it satisfies the following conditions:

- h1) $k_1 \in L^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$;
- h2) k_1 is positive and nonincreasing;
- h3) $-k_1'$ is nonincreasing and convex;

Remark 2. We stress that the above assumption allows $k_1(t)$ to have a singularity at $t = 0$, whose order is less than 1, since $k(t)$ is a non-negative function in $L^1(\mathbb{R}^+)$. For instance, we are able to consider a weakly singular kernel of the following type

$$k_1(t) := \frac{e^{-\delta t}}{t^\gamma}, \quad 0 \leq \gamma < 1.$$

Remark 3. Following the terminology introduced by Prüss [22, Definition 4.4 pag. 94], the function k belongs to the class of *creep functions*. As stated in Section 1, the function k has a physical meaning within the theory of materials with memory.

Concerning the nonlinear part of the system we have:

Hypothesis 2. The function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following conditions:

1. f is continuous and differentiable on \mathbb{R} .

2. f is Lipschitz continuous with respect to x , uniformly on t , and has sublinear growth; this means that there exists a constant $L > 0$ such that

$$\begin{aligned} |f(t, x)| &\leq L(1 + |x|) & t \geq 0, x \in \mathbb{R} \\ |f(t, x) - f(t, y)| &\leq L|x - y| & t \geq 0, x, y \in \mathbb{R}. \end{aligned}$$

The conditions on the stochastic perturbation are given in the following.

Hypothesis 3.

1. The process $(W(t))_{t \geq 0}$ is a cylindrical Wiener process defined on a complete probability space with values in $L^2(\mathcal{O})$. In particular $W(t)$ is of the form

$$\langle W(t), x \rangle = \sum_{k=0}^{\infty} \langle e_k, x \rangle \beta_k(t), \quad t \geq 0, x \in L^2(\mathcal{O})$$

where $\{\beta_k\}_{k \in \mathbb{N}}$ is a sequence of real, standard, independent Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

2. Q is a linear bounded operator, symmetric and positive. With no loss of generality, we shall assume in the sequel that A and Q diagonalizes on the same basis of $L^2(\mathcal{O})$ (this is required only for simplicity);
3. If $\{\mu_j\}_{j \in \mathbb{N}}$ and $\{\lambda_j\}_{j \in \mathbb{N}}$ are respectively the eigenvalues of A and Q then we require

$$\mathrm{Tr}[(-\Delta)^{1-(1+\theta)/\delta} Q] = \sum_{j=1}^{\infty} \frac{\lambda_j}{\mu_j^{(1+\theta)/\delta - 1}} < \infty,$$

where δ is the quantity

$$\delta := 1 + \frac{2}{\pi} \sup\{|\arg \hat{k}(\lambda)| : \mathrm{Re} \lambda > 0\} \quad (8)$$

and θ is any real number in $(0, 1)$ such that $1 + \theta > \delta$.

Remark 4.

- i.* We notice that the quantity δ introduced in (8) depends only on the behavior of the Laplace transform of the kernel k . In Prüss and Monniaux [20] it is proved that, for the class of kernels considered by us (i.e. for 3-monotone kernels), the Laplace transform \hat{k} satisfies the following bound:

$$\sup\{|\arg \hat{k}(\lambda)| : \mathrm{Re} \lambda > 0\} = \theta < \frac{\pi}{2} \quad (9)$$

and, consequently, δ belongs to $(1, 2)$. Following the terminology in Prüss (see [22]) we say that the kernel k is θ -sectorial.

It can be proved that the sectoriality of the kernel plays a central role in the study of the Volterra equation (1). In particular, it allows to prove existence of the resolvent family corresponding with the problem, and consequently to investigate existence and uniqueness of the solution. For more details we refer to Section 5.1 and the monograph [22, Section 3].

- ii.* As has been observed, δ belongs to the interval $(1, 2)$; then condition 3 in Hypothesis 3 implies that $\mathrm{Tr}[QA^{-\epsilon}]$ for any $\epsilon \in [0, 2/\delta - 1)$. We stress that this condition is automatically satisfied if Q is of trace class.

4. Statement and reformulation of the uncontrolled equation.

4.1. **The abstract setting.** In this section we are concerned with the following (uncontrolled) class of integral Volterra equations perturbed by an additive Wiener noise

$$\begin{aligned} \partial_t v(t, x) &= k_0 \Delta v(t, x) + \int_0^\infty k_1(s) \Delta v(t-s, x) ds \\ &\quad + f(t, v(t, x)) + \sqrt{Q} \partial_t W(t, x) \quad t > 0 \\ v(-t, x) &= v_0(-t, x), \quad t \geq 0 \\ v(t, x) &= 0, \quad t \geq 0, x \in \partial\mathcal{O}. \end{aligned} \quad (10)$$

In the above equation v_0 represents the history of the system up to time $t = 0$. Our first purpose is to rewrite equation (10) as an evolution equation defined on a suitable Hilbert space.

To this end we denote with $L^2(\mathcal{O})$ the space of square integrable, real valued functions defined on \mathcal{O} with scalar product $\langle u, v \rangle_{L^2(\mathcal{O})} = \int_{\mathcal{O}} u(\xi)v(\xi)d\xi$, for any $u, v \in L^2(\mathcal{O})$. Sobolev spaces $H^1(\mathcal{O})$ and $H^2(\mathcal{O})$ are the spaces of functions whose first (resp. first and second) distributional derivatives are in $L^2(\mathcal{O})$. We set moreover $H_0^1(\mathcal{O})$ the subspace of $H^1(\mathcal{O})$ of functions which vanish (a.e.) on the boundary $\partial\mathcal{O}$.

We let $H^{-1}(\mathcal{O})$ the topological dual of $H_0^1(\mathcal{O})$.

We recall that the operator Δ with Dirichlet boundary conditions (i.e. with domain $H_0^1(\mathcal{O})$) is the generator of a C_0 -semigroup of contractions; since Δ is self adjoint, the semigroup is analytic: see for instance [25, Theorem 1.5.7, Corollary 1.5.8].

In order to control the unbounded delay interval, we shall consider L^2 weighted spaces. Let

$$\rho(t) = \int_t^\infty k_1(s) ds. \quad (11)$$

Then we set $\mathcal{X} = L_\rho^2(\mathbb{R}_+; H_0^1(\mathcal{O}))$ be the space of functions $y : \mathbb{R}_+ \rightarrow D(A) = H_0^1(\mathcal{O})$ endowed with the inner product

$$\langle y_1, y_2 \rangle_{\mathcal{X}} = \int_0^\infty \rho(s) \langle \nabla y_1(s), \nabla y_2(s) \rangle ds$$

and $\|y\|_{\mathcal{X}}$ the corresponding norm. On this space, we introduce the delay operator K with domain $D(K) = \mathcal{X}$ by setting

$$K\eta = \int_0^\infty k_1(s) \Delta \eta(s) ds$$

Finally, we define the Hilbert space $\mathcal{H} = L^2(\mathcal{O}) \times L_\rho^2(\mathbb{R}_+; H_0^1(\mathcal{O}))$ endowed with the energy norm $\|x\|_{\mathcal{H}}^2 = \|v\|_{L^2(\mathcal{O})}^2 + \|\eta\|_{\mathcal{X}}^2$, $x = \begin{pmatrix} v \\ \eta \end{pmatrix}$.

Our aim is to reduce problem (10) to an abstract Cauchy problem on the product space \mathcal{H} in such a way that the first component gives the evolution of the system while the second contains all the information concerning the whole history of the solution. The state variable in the Hilbert space \mathcal{H} will be denoted by $X(t)$. Thus $(X(t))_{t \geq 0}$ is a process in \mathcal{H} and the initial condition is assumed to belong to \mathcal{H} and satisfies suitable properties to be precised.

We introduce the linear operator \mathbf{A} defined as:

$$\begin{aligned}\mathbf{A} \begin{pmatrix} v \\ \eta \end{pmatrix} &= \begin{pmatrix} \Delta & K \\ 0 & -\partial_s \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix} \\ &= \begin{pmatrix} k_0 \Delta v + \int_0^\infty k_1(s) \Delta \eta(s) ds \\ -\partial_s \eta \end{pmatrix}\end{aligned}$$

with domain $D(\mathbf{A}) \subset \mathcal{H}$

$$D(\mathbf{A}) := \left\{ \begin{pmatrix} v \\ \eta(\cdot) \end{pmatrix} \in H_0^1(\mathcal{O}) \times W_\rho^{1,2}(\mathbb{R}_+; H_0^1(\mathcal{O})) : \eta(0) = v, \Delta v + K\eta \in \mathcal{X} \right\}.$$

In order to handle the contribution of temperature values taken in the past, we introduce the new variable

$$\eta^t(s) = v(t-s), \quad s \geq 0.$$

Moreover, we introduce the non linear operator $\mathbf{F} : [0, T] \times \mathcal{H} \rightarrow \mathcal{H}$

$$\mathbf{F} \left(t, \begin{pmatrix} v \\ \eta(\cdot) \end{pmatrix} \right) := \begin{pmatrix} f(t, v) \\ 0 \end{pmatrix}$$

where f is the non linear term in Equation (10). Finally we introduce the linear operator \mathbf{Q} and the stochastic perturbation \mathbf{W} on \mathcal{H} as

$$\mathbf{Q} := \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \quad \mathbf{W}(t) = \begin{pmatrix} W(t) \\ 0 \end{pmatrix}.$$

With the above notation, problem (10) can be rewritten in the form

$$\begin{cases} dX(t) = (\mathbf{A}X(t) + \mathbf{F}(X(t)))dt + \sqrt{\mathbf{Q}}d\mathbf{W}(t), & t \in [0, T], \\ X(0) = X_0 \in \mathcal{H}, \end{cases} \quad (12)$$

where $X(t)$ stands for the pair $\begin{pmatrix} v(t) \\ \eta^t(\cdot) \end{pmatrix}$ and $X_0 := \begin{pmatrix} \bar{v} \\ \bar{\eta}(\cdot) \end{pmatrix}$ is the initial condition.

In the following (see Section 4.2) we will see that the dynamics of the system is described in terms of the transition semigroup $e^{t\mathbf{A}}$ generated by the linear operator \mathbf{A} . As a consequence we will read the solution of the original Volterra equation in the first component of X .

Before proceeding, let us recall the definition of mild solution for the stochastic Cauchy problem (12).

Definition 4.1. Given an \mathcal{F}_t -adapted cylindrical Wiener process on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, a process $(X(t))_{t \geq 0}$ is a mild solution of (12) if it belongs to $L^2(0, T; L^2(\Omega; \mathcal{H}))$ and satisfies \mathbb{P} -a.s. the following integral equation

$$X(t) = e^{t\mathbf{A}}X_0 + \int_0^t e^{(t-s)\mathbf{A}}\mathbf{F}(X(s))ds + \int_0^t e^{(t-s)\mathbf{A}}\sqrt{\mathbf{Q}}dW(s), \quad t \geq 0. \quad (13)$$

Condition (13) implies that the integrals on the right-member are well defined. In particular, the second integral, which we shall refer to as *stochastic convolution*, is a mean-square continuous Gaussian process with values in \mathcal{H} . For the analysis of the stochastic convolution and its properties, we refer Section 5.

4.2. Generation properties. In this section we are dealing with the generation properties of the leading (matrix) operator and prove that, in our setting, the operator is quasi- m -dissipative (see inequality (14) below) and that the range of $\mu - \mathbf{A}$ is dense in \mathcal{H} for some (and all) $\mu > 0$. In this way we will be able to apply the Lumer-Phillips theorem to conclude that $\mathbf{A} - \mu$, and hence \mathbf{A} , generates a C_0 -semigroup.

We start by proving the dissipativity properties.

Theorem 4.2. *The operator $(\mathbf{A}, D(\mathbf{A}))$ is quasi- m -dissipative: for any $\phi = (u, \eta)^t \in D(\mathbf{A})$ there exists $\lambda_0 > 0$ such that, for any $\lambda \geq \lambda_0$*

$$\langle \mathbf{A}\phi, \phi \rangle_{\mathcal{H}} \leq \lambda \|\phi\|_{\mathcal{H}}. \quad (14)$$

Proof. We proceed in the same spirit as Bonaccorsi et al. [3, Theorem 3.1], but we include the proof for completeness. The difference, here, is that we have no conditions linking the constant k_0 and the function k_1 . In contrast with [3], this point does not allow to prove the pure dissipativity of \mathbf{A} , but only quasi- m -dissipativity.

We compute the scalar product

$$\begin{aligned} \langle \mathbf{A}\phi, \phi \rangle_{\mathcal{H}} &= k_0 \langle \Delta u, u \rangle_{L^2(\mathcal{O})} \\ &\quad + \int_{\mathbb{R}^+} k_1(r) \langle u, \Delta \eta(r) \rangle_{L^2(\mathcal{O})} dr - \int_{\mathbb{R}^+} \rho(r) \langle \nabla \eta(r), \nabla \eta'(r) \rangle dr \end{aligned}$$

and we get

$$\begin{aligned} &\langle \mathbf{A}\phi, \phi \rangle_{\mathcal{H}} \\ &= -k_0 \|x\|_{H_0^1(\mathcal{O})}^2 - \int_{\mathbb{R}^+} k_1(r) \langle x, \eta(r) \rangle_{H_0^1(\mathcal{O})} dr - \int_{\mathbb{R}^+} \rho(r) \frac{1}{2} \frac{d}{dr} \|\eta(r)\|_{H_0^1(\mathcal{O})}^2 dr \\ &\leq -k_0 \|x\|_{H_0^1(\mathcal{O})}^2 + \int_{\mathbb{R}^+} k_1(r) \|x\|_{H_0^1(\mathcal{O})} \|\eta(r)\|_{H_0^1(\mathcal{O})} dr - \frac{1}{2} \rho(r) \|\eta(r)\|_{H_0^1(\mathcal{O})}^2 \Big|_0^{+\infty} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^+} \rho'(r) \|\eta(r)\|_{H_0^1(\mathcal{O})}^2 dr. \end{aligned}$$

Now recall that $\rho \geq 0$ while $\rho' = -k_1 \leq 0$; choose some $\varepsilon > 0$ and use the bound $ab - \frac{1}{2}(1-\varepsilon)b^2 \leq \frac{1}{2(1-\varepsilon)}a^2$ with $a = \|x\|_{H_0^1(\mathcal{O})}$ and $b = \|\eta(r)\|_{H_0^1(\mathcal{O})}$ to get

$$\begin{aligned} \langle \mathbf{A}\phi, \phi \rangle_{\mathcal{H}} &\leq \left(-k_0 + \frac{1-\varepsilon/2}{1-\varepsilon} \rho(0) \right) \|x\|_{H_0^1(\mathcal{O})}^2 + \frac{\varepsilon}{2} \int_{\mathbb{R}^+} \rho'(r) \|\eta(r)\|_{H_0^1(\mathcal{O})}^2 dr \\ &\leq \lambda \|\phi\|_{\mathcal{H}}^2, \end{aligned}$$

for any $\lambda > \lambda_0 := \min \left\{ 0, -k_0 + \frac{1-\varepsilon/2}{1-\varepsilon} \rho(0) \right\}$. \square

Next, we consider the properties of the resolvent $R(\mu, \mathbf{A})$.

Theorem 4.3. *For every $\mu > 0$ the equation*

$$(\mu - \mathbf{A})\phi = \psi, \quad \psi \in \mathcal{H}, \quad (15)$$

has a unique solution $\phi \in D(\mathbf{A})$.

Proof. We give only a sketch of the proof, since it essentially repeats the arguments of Bonaccorsi et al. [3, Proposition 3.2]. Let $\phi = \begin{pmatrix} u \\ \eta \end{pmatrix}$ and $\psi = \begin{pmatrix} v \\ \xi \end{pmatrix}$. Then equation

(15) is equivalent to

$$\begin{aligned}\mu u - k_0 \Delta u - K \eta &= v \\ \mu \eta(s) + \partial_s \eta(s) &= \xi(s).\end{aligned}$$

From the variation of constant formula we get

$$\eta(s) = e^{-\mu s} \eta(0) + \int_0^s e^{-\mu(s-r)} \xi(r) dr;$$

moreover, using the monotonicity property of ρ it is possible to prove that $\eta \in W_\rho^{1,2}(\mathbb{R}_+; H_0^1(\mathcal{O}))$. Straightforward calculation gives

$$u = \frac{1}{c_{k,\mu}} R\left(\frac{\mu}{c_{k,\mu}}, \Delta\right) \tilde{v},$$

where $c_{k,\mu}$ is a positive constant depending only on k and μ while \tilde{v} is the function

$$\tilde{v} := v + \int_0^{+\infty} k_1(r) \Delta \int_0^s e^{-\mu(s-r)} \eta(r) ds dr.$$

Obviously $u \in D(\Delta)$. Finally, we notice that

$$k_0 \Delta u + K \eta = \mu u - v \in L^2(\mathcal{O});$$

hence, it turns out that $\phi = \begin{pmatrix} u \\ \eta \end{pmatrix} \in D(\mathbf{A})$. \square

Taking into account the above results, we can deduce the generation properties for the operator \mathbf{A} . Precisely, we have

Proposition 1. *Under Hypothesis 1, 2 and 3 the operator $(\mathbf{A}, D(\mathbf{A}))$ generates a strongly continuous semigroup.*

Proof. The result follows by a direct application of a perturbation method and the Lumer-Phillips Theorem. \square

5. The stochastic uncontrolled equation. In this section we aim to prove existence and uniqueness of the solution for the uncontrolled equation

$$\begin{aligned}\partial_t v(t, x) &= k_0 \Delta v(t, x) + \int_0^\infty k_1(s) \Delta v(t-s, x) ds \\ &\quad + f(v(t, x)) + \sqrt{Q} \partial_t W(t, x), \quad t > 0 \\ v(s, x) &= v_0(s, x), \quad s \leq 0, \\ v(t, x) &= 0, \quad t \geq 0, \quad x \in \partial \mathcal{O}.\end{aligned}\tag{16}$$

where the coefficients k_0, k_1, f, Q satisfy the assumptions made in Section 3.

To this end we first focus the stochastic convolution corresponding with our problem, that is the process

$$W_{\mathbf{A}}(t) := \int_0^t e^{(t-s)\mathbf{A}} \sqrt{Q} dW(s), \quad t \geq 0.$$

In particular, our purpose is to prove that $(W_{\mathbf{A}}(t))_{t \geq 0}$ is a well-defined mean-square continuous Gaussian process with values in \mathcal{H} . Following the approach of Da Prato and Clement [6], Bonaccorsi et al. [3], we can give a meaning to the stochastic convolution through the study of the so-called *resolvent family* associated with an abstract homogeneous linear Volterra equation of type

$$v(t) = k * \Delta v(t) \tag{17}$$

where k is a kernel satisfying Hypothesis 1 and where Δ denotes the Laplace operator on $\bar{\mathcal{O}}$ with homogeneous Dirichlet boundary conditions. The concept of the resolvent plays a central role for the theory of linear Volterra equations and can be applied to inhomogeneous problem to derive a variation of parameters formula. The main tools for the resolvent are described in detail in the monograph [22]. In the next subsection we recall a few basic concepts and results.

5.1. The resolvent family. Following [22, Section 1], we define the resolvent family for the equation (17) as

Definition 5.1. A family $(S(t))_{t \geq 0}$ of bounded linear operators in X is called a resolvent for equation (17) if the following conditions are satisfied:

- (S1) $S(0) = I$ and, for all $x \in X$, $t \mapsto S(t)x$ is continuous on \mathbb{R}^+ ;
- (S2) $S(t)$ commutes with Δ , that is for a.e. $t \geq 0$, $S(t)D(\Delta) \subset D(\Delta)$ and

$$\Delta S(t)\bar{v} = S(t)\Delta\bar{v}, \quad v \in D(\Delta);$$

- (S3) for any $\bar{v} \in D(\Delta)$, $t \mapsto S(t)\bar{v}$ is a strong solution of (10) on $[0, T]$, for any $T > 0$.

It turns out that if the kernel k satisfies Hypothesis 1 (or, more generally, if it is θ -sectorial for $\theta < \pi$), then equation (17) admits a resolvent $(S(t))_{t \geq 0}$ which is uniformly bounded in $L^2(\mathcal{O})$ (see [22, Corollary 3.3]). Consequently (see [22, Proposition 1.1]), problem (17) is well-posed and its strong solution is given by the function $v(t) = S(t)\bar{v}$. Besides, since k belongs to $BV_{loc}(\mathbb{R}_+)$, $S(t)$ turns out to be differentiable and consequently (by differentiation of equation (17)) the function v is the mild solution of the homogeneous Cauchy problem

$$\begin{cases} v'(t) = dk * \Delta v(t) \\ v(0) = \bar{v} \in H_0^1(\mathcal{O}). \end{cases} \quad (18)$$

Here the term $dk * \Delta v(t)$ denotes the function

$$\int_0^t k_0 \Delta v(t-s) \delta_0(s) + \int_0^t k_1(t-s) \Delta v(s) ds.$$

Analogously, it can be proved that if g is a function belonging to $L^1(0, T; X)$, then the Cauchy problem

$$\begin{cases} v' = dk * \Delta v + g \\ v(0) = \bar{v} \in H_0^1(\mathcal{O}) \end{cases} \quad (19)$$

is well-posed too and its (unique) mild solution can be represented through the variation of parameter formula as

$$v(t) = S(t)\bar{v} + \int_0^t S(t-\tau)g(\tau)d\tau, \quad t \geq 0.$$

For a full discussion about the notion of well-posedness for equation (17), of mild solution for problems of type (18), (19) and their relationship between the resolvent family we refer to [22, Section 1].

Here we want to emphasize that the above arguments can be applied to the inhomogeneous Volterra equation (16) to obtain existence and uniqueness of a mild

solution and its representation in terms of the resolvent family corresponding with the kernel $k = 1 * dk$. In fact, equation (16) is equivalent to

$$\begin{aligned} v_t(t, x) &= k_0 \Delta v(t, x) + \int_0^t k_1(s) \Delta v(s, x) ds \\ &+ \int_t^\infty k(s) \Delta v(t-s, x) ds + f(v(t, x)) + \sqrt{Q} \partial_t W(t, x), \quad t > 0, \end{aligned} \quad (20)$$

with boundary and initial conditions given by:

$$\begin{aligned} v(-t, x) &= v_0(-t, x), \quad t \geq 0, \\ v(t, x) &= 0, \quad t \geq 0, x \in \partial \mathcal{O}. \end{aligned}$$

In abstract form, we have

$$\begin{aligned} v'(t) &= dk * \Delta v(t) + \int_t^\infty k(s) \Delta v_0(t-s) ds + f(v(t)) + \sqrt{Q} dW(t), \quad t > 0 \\ v(0) &= v_0(0) \in H_0^1(\mathcal{O}). \end{aligned}$$

Therefore, integrating (20) over $[0, t]$ we obtain

$$\begin{aligned} v(t) &= v_0(0) + \int_0^t (dk * \Delta v)(s) ds + \int_0^t ds \left(\int_0^\infty k(s+r) \Delta v_0(-r) dr \right) \\ &+ \int_0^t f(v(s)) ds + \sqrt{Q} W(t). \end{aligned} \quad (21)$$

Now, by the associativity property of the convolution product, the second term in the right member of (21) gives

$$\begin{aligned} \int_0^t (dk * \Delta v)(s) ds &= 1 * (dk * \Delta v) = (1 * dk) * \Delta v \\ &= k * \Delta v = \int_0^t k(s) \Delta v(t-s) ds. \end{aligned}$$

Hence equation (20) can be rewritten as follows:

$$v(t) = v_0(0) + \int_0^t k(t-s) \Delta v(s) ds + h(t) \quad (22)$$

where the function h is given by

$$h(t) = \int_0^t ds \left(\int_0^\infty k(s+r) \Delta v_0(r) dr \right) + \int_0^t f(v(s)) ds + \sqrt{Q} W(t).$$

Now the variation of parameters formula implies that the function

$$v(t) = S(t)v_0(0) + \int_0^t S(t-s)h(s) ds$$

is a mild solution of the Volterra equation (20), provided that $h \in L^1(0, T; X)$. We notice that the condition $v_0 \in L_\rho^2(\mathbb{R}^+; H_0^1(\mathcal{O}))$ assures the requested regularity for the function h .

5.2. The scalar resolvent family. Suppose that $(S(t))_{t \geq 0}$ is the resolvent family for equation (17) and let $\{\mu_j\}_{j \in \mathbb{N}}$ be the set of eigenvalues of Δ with respect to the basis $\{e_j\}_{j \in \mathbb{N}}$. For any $j \in \mathbb{N}$, we introduce the following one-dimensional Volterra equation

$$s_j(t) + \mu_j(k * s_j)(t) = 1. \quad (23)$$

Then (see [22, Section 1.3]) a unique solution to (23) exists and it satisfies

$$S(t)e_j = s_j(t)e_j, \quad t \geq 0.$$

In particular, the resolvent family $S(t)$ admits a decomposition in the basis $\{e_j\}$ of $L^2(\mathcal{O})$ in terms of the solutions s_j to (23).

In the sequel we state and prove some useful estimates on the scalar resolvent functions s_j . They are crucial to study the stochastic convolution and descend immediately from the assumption on the kernel k .

Lemma 5.2. *Let k satisfy Hypothesis 1. Then, there exist suitable positive constants M and C such that for any $j \in \mathbb{N}$, equation (23) admits a solution $s_j(t)$ such that the following properties hold:*

1. $|s_j(t)| \leq M$ for all $t > 0$;
2. $\int_0^\infty |s'_j(t)| dt \leq C$;
3. $\int_0^\infty t |s'_j(t)| dt \leq C \mu_j^{-1}$;
4. $\int_0^{+\infty} |s_j(t)| dt \leq C \mu_j^{-1}$.

Proof. Assertion 1 follows from [22, Corollary 3.3], while assertions 2 and 3 are contained in Monniaux and Pruss [20, Proposition 6] (observe the relation $s'_j(t) = -\mu_j r_j(t)$ to connect the notations). To prove 4, we notice that

$$s_j(t) = s_j(R) - \int_t^R s'_j(\tau) d\tau.$$

Hence assertion 2 implies that the limit of $s_j(R)$ for $R \rightarrow \infty$ exists; moreover, we have

$$\lim_{R \rightarrow \infty} s_j(R) = s_j(0) + \lim_{R \rightarrow \infty} \int_0^R s'_j(\tau) d\tau = 1 + \int_0^\infty s'_j(\tau) d\tau.$$

We observe that the last term in the above equality can be rewritten as

$$\begin{aligned} \lim_{\lambda \rightarrow 0^+} \int_0^\infty e^{-\lambda \tau} s'_j(\tau) d\tau &= \lim_{\lambda \rightarrow 0^+} s_j(\tau) e^{-\lambda \tau} \Big|_0^\infty + \lim_{\lambda \rightarrow 0^+} \lambda \hat{s}_j(\lambda) \\ &= -1 + \lim_{\lambda \rightarrow 0^+} \lambda \hat{s}_j(\lambda). \end{aligned}$$

Further, since s_j satisfies equation (23), we get

$$\lambda \hat{s}_j(\lambda) = \frac{1}{\lambda + \mu_j \hat{k}(\lambda)} = \frac{\lambda}{\lambda^2 + k_0 + \hat{k}_1(\lambda)}; \quad (24)$$

in fact, we have

$$\hat{s}_j(\lambda) + \mu_j \hat{k}(\lambda) \hat{s}_j(\lambda) = \frac{1}{\lambda},$$

and

$$\hat{k}(\lambda) = \frac{k_0}{\lambda} + \frac{\hat{k}_1(\lambda)}{\lambda}.$$

We notice that, since k_1 belongs to $L^1(\mathbb{R}^+)$, for any $\lambda \geq 0$ it holds

$$|\hat{k}_1(\lambda)| = \left| \int_0^\infty e^{-\lambda t} k_1(t) dt \right| \leq \int_0^\infty k_1(t) dt < \infty.$$

Taking into account the last inequality and equality (24) we see that the limit of $s_j(R)$ for $R \rightarrow \infty$ satisfies

$$\lim_{R \rightarrow \infty} s_j(R) = \lim_{\lambda \rightarrow 0^+} \frac{\lambda}{\lambda^2 + k_0 + \hat{k}_1(\lambda)} = 0.$$

Therefore

$$s_j(t) = - \int_t^\infty s'_j(\tau) d\tau$$

yields

$$\int_0^\infty |s_j(\tau)| d\tau \leq \int_0^\infty \int_t^\infty |s'_j(\tau)| d\tau dt = \int_0^\infty \tau |s'_j(\tau)| d\tau \leq C\mu^{-1/\delta},$$

by assertion 3. □

For further use, we conclude this subsection with an estimate concerning the norm of s_j in $L^2(\mathbb{R}^+)$.

Lemma 5.3. *Suppose that the kernel k is subject to Hypothesis 1. Then for each $\theta \in (0, 1)$ and for any $T > 0$, there exists a constant $C_{\theta, T} > 0$ (depending only on θ and T) such that*

$$\int_0^T d\tau \int_0^\tau s_j^2(\sigma) d\sigma \leq C_{\theta, T} \mu_j^{-(\theta+1)/\delta}.$$

Proof. From assertion 1 of Lemma 5.2 we obtain

$$\int_0^\tau s_j^2(\sigma) d\sigma \leq M \int_0^\tau |s_j(\sigma)| d\sigma < M\mu_j^{-1/\delta},$$

as well as

$$\int_0^\tau s_j^2(\sigma) d\sigma \leq M^2\tau;$$

hence employing point 4 of Lemma 5.2

$$\int_0^\tau s_j^2(\sigma) d\sigma \leq M^{-2\theta} \tau^{-\theta} \left(\int_0^\tau |s_j(\sigma)| d\sigma \right)^{1+\theta} \leq C_{\theta} \mu_j^{-(1+\theta)/\delta} \tau^{-\theta}.$$

Now integrating both members of the previous inequality we obtain the thesis. □

5.3. The representation of the semigroup. In the following we show that the semigroup corresponding with the linear operator $e^{t\mathbf{A}}$ can be computed explicitly in terms of the *resolvent family* $(S(t))_{t \geq 0}$.

We recall that since the linear operator \mathbf{A} generates a C_0 -semigroup, there exists a unique mild solution $(X(t))_{t \geq 0}$ for the deterministic equation

$$\begin{cases} X'(t) = \mathbf{A}X(t), \\ \begin{pmatrix} v(0) \\ \eta^0(\cdot) \end{pmatrix} = \begin{pmatrix} \bar{v} \\ \bar{\eta}(\cdot) \end{pmatrix} \in D(\mathbf{A}). \end{cases} \quad (25)$$

The variation of parameter formula for abstract evolution equations applies to equation (25) and we can write:

$$X(t) = e^{t\mathbf{A}}X_0.$$

If we set

$$e^{t\mathbf{A}} := \begin{pmatrix} e_{11}^{t\mathbf{A}} & e_{12}^{t\mathbf{A}} \\ e_{21}^{t\mathbf{A}} & e_{22}^{t\mathbf{A}} \end{pmatrix},$$

then, for any $t > 0$, we have

$$\begin{aligned} v(t) &= e_{11}^{t\mathbf{A}}(t)\bar{v} + e_{12}^{t\mathbf{A}}(t)\bar{\eta}; \\ \eta^t(\cdot) &= e_{21}^{t\mathbf{A}}(t)\bar{v} + e_{22}^{t\mathbf{A}}(t)\bar{\eta}. \end{aligned} \quad (26)$$

By construction, the first component of X satisfies the inhomogenous Volterra equation

$$v(t) = v_0(0) + \int_0^t k(t-s)\Delta v(s)ds + \int_0^t ds \left(\int_0^\infty k(s+r)\Delta v_0(-r)dr \right) \quad (27)$$

and the variation of parameters formula for Volterra equations applied to (27) (see Subsection 5.1) yields

$$\begin{aligned} v(t) &= S(t)\bar{v} + \int_0^t S(t-\tau)h(\tau)d\tau \\ \eta^t(s) &= \begin{cases} v(t-s) = S(t-s)\bar{v} + \int_0^{t-s} S(t-s-\tau)h(\tau)d\tau, & 0 < s \leq t \\ \bar{\eta}(t-s), & s > t, \end{cases} \end{aligned} \quad (28)$$

where

$$\begin{aligned} h(t) &= \int_{\mathbb{R}_+} k(t+\sigma)\Delta v_0(-\sigma)d\sigma \\ &= \int_{\mathbb{R}_+} k(t+\sigma)\Delta\bar{\eta}(\sigma)d\sigma. \end{aligned}$$

Comparing the first terms in equalities (26) and (28), we obtain

$$e_{11}^{t\mathbf{A}}\bar{v} = S(t)\bar{v} \quad e_{12}^{t\mathbf{A}}\bar{\eta} = \int_0^t S(t-\tau)h(\tau)d\tau. \quad (29)$$

Moreover, from the second part of (26) and (28), we have for $s \geq 0$

$$\begin{aligned} (e_{21}^{t\mathbf{A}}\bar{v})(s) &= S(t-s)\bar{v} \mathbf{1}_{[0,t]}(s) \\ (e_{22}^{t\mathbf{A}}\bar{\eta})(s) &= \begin{cases} \int_0^{t-s} S(t-s-\tau)h(\tau)d\tau, & 0 < s \leq t \\ \bar{\eta}(t-s), & s > t. \end{cases} \end{aligned} \quad (30)$$

Thus the semigroup $e^{t\mathbf{A}}$ is completely described in terms of the resolvent family. As we will see in the next subsection, the above characterization allows to study the stochastic convolution process.

5.4. The stochastic convolution. We are now in the position to prove the main result of this section. We recall that $(W(t))_{t \geq 0}$ is a cylindrical Wiener process of the form

$$\langle W(t), x \rangle = \sum_{k=0}^{\infty} \langle e_k, x \rangle \beta_k(t), \quad t \geq 0, x \in L^2(\mathcal{O})$$

where $\{\beta_k\}_{k \in \mathbb{N}}$ is a sequence of real, standard, independent Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. We have:

Lemma 5.4. *Under Hypothesis 3, for all $T > 0$ the process $(W_{\mathbf{A}}(t))_{0 \leq t \leq T}$ defined as*

$$\mathbf{W}_{\mathbf{A}}(t) := \int_0^t e^{(t-s)\mathbf{A}} d\mathbf{W}(s), \quad (31)$$

is a Gaussian random variable with mean 0 and covariance operator

$$\mathbf{Q}_t := \int_0^t e^{s\mathbf{A}} \mathbf{Q} e^{s\mathbf{A}^*} ds.$$

Proof. It is well-known that the thesis follows provided that

$$\int_0^T \|e^{\tau\mathbf{A}} \mathbf{Q}\|_{HS}^2 d\tau < C_T,$$

where C_T is a positive constant depending only on $T > 0$. Recalling the representation of $e^{t\mathbf{A}}$ given in (29) and (30), we have that

$$\begin{aligned} \int_0^t \|e^{\tau\mathbf{A}} \mathbf{Q}\|_{HS}^2 d\tau &= \sum_{j=1}^{\infty} \int_0^t \left| e^{\tau\mathbf{A}} \begin{pmatrix} \sqrt{Q} \\ 0 \end{pmatrix} \right|_{\mathcal{H}}^2 d\tau \\ &= \sum_{j=1}^{\infty} \int_0^t \left| \begin{pmatrix} S(\tau) \sqrt{\lambda_j} e_j \\ S(\tau - \cdot) \sqrt{\lambda_j} e_j \mathbf{1}_{[0, \tau]}(\cdot) \end{pmatrix} \right|_{\mathcal{H}}^2 d\tau \\ &= \sum_{j=1}^{\infty} \lambda_j \int_0^t \|S(\tau) e_j\|_{L^2(\mathcal{O})}^2 d\tau + \\ &\quad \sum_{j=1}^{\infty} \lambda_j \int_0^t \int_0^{\infty} \rho(\sigma) \|S(\tau - \sigma) e_j\|_{H_0^1(\mathcal{O})}^2 \mathbf{1}_{[0, \tau]} d\sigma d\tau. \end{aligned} \quad (32)$$

We consider separately the two series in the previous formula. We recall that $S(t)e_j = s_j(t)$ for any $j \in \mathbb{N}$ (see Subsection 5.2); hence we get

$$\sum_{j=1}^{\infty} \lambda_j \int_0^t \|S(\tau) e_j\|_{L^2(\mathcal{O})}^2 d\tau = \sum_{j=1}^{\infty} \lambda_j \int_0^t |s_j(\tau)|^2 d\tau \leq \sum_{j=1}^{\infty} \lambda_j \int_0^t |s_j(\tau)| d\tau,$$

where the last inequality follows from Lemma 5.2, point 1. Moreover, since it holds also that $\int_0^{\infty} |s_j(\tau)| d\tau < (\mu_j k_0)^{-1}$ (see Lemma 5.2, point 4), it follows that

$$\sum_{j=1}^{\infty} \lambda_j \int_0^t \|S(\tau) e_j\|_{L^2(\mathcal{O})}^2 d\tau \leq \frac{1}{k_0} \sum_{j=1}^{\infty} \frac{\lambda_j}{\mu_j}.$$

Concerning the second series in (32), applying Fubini's theorem, we get

$$\sum_{j=1}^{\infty} \lambda_j \mu_j \int_0^t \int_0^{\tau} \rho(\sigma) |s_j(\tau - \sigma)|^2 d\sigma d\tau \leq \sum_{j=1}^{\infty} \lambda_j \mu_j \int_0^t d\sigma \rho(\sigma) \int_{\sigma}^t d\tau |s_j(\tau - \sigma)|^2$$

and, taking into account Lemma 5.3 and the definition of the function ρ (see (11)),

$$\begin{aligned} \sum_{j=1}^{\infty} \lambda_j \mu_j \int_0^t \int_0^\tau \rho(\sigma) |s_j(\tau - \sigma)|^2 d\sigma d\tau &\leq \sum_{j=1}^{\infty} \lambda_j \mu_j \int_0^t \rho(0) C_\theta \mu_j^{-(1+\theta)/\rho} (\tau - \sigma)^{-\theta} \\ &\leq C_\theta T^{1-\theta} \rho(0) \sum_{j=1}^{\infty} \frac{\lambda_j}{\mu_j^{(1+\theta)/\delta-1}}. \end{aligned}$$

By the above estimates and condition 3 in Hypothesis 3, we conclude that, for any $\theta \in (0, 1)$ such that $1 + \theta > \delta$,

$$\int_0^T \|e^{\tau \mathbf{A}} \mathbf{Q}\|_{HS}^2 d\tau \leq C_T,$$

where $C_T := C_\theta T^{1-\theta} \rho(0) \text{Tr}[Q(-\Delta)^{(1+\theta)/\delta-1}]$. \square

5.5. Existence and uniqueness. Now we turn on the existence and uniqueness of the solution for the uncontrolled equation (10).

Recalling what has been showed in the previous section, the above equation can be rewritten as an abstract equation on the space $\mathcal{H} := L^2(\mathcal{O}) \times L_\rho^2(\mathbb{R}_+; H_0^1(\mathcal{O}))$

$$\begin{cases} dX(t) = \mathbf{A}X(t)dt + \mathbf{F}(X(t))dt + \sqrt{\mathbf{Q}}dW(t) & t > 0, \\ X(0) = X_0. \end{cases} \quad (33)$$

We recall that, from Proposition 1 \mathbf{A} is the generator of a C_0 -semigroup, while from the assumption on the function f we get that $\mathbf{F} : \mathcal{H} \rightarrow \mathcal{H}$ is Lipschitz continuous. Moreover, \mathbf{Q} is a linear operator on \mathcal{H} involving the covariance operator Q , $X_0 = (v_0(0, \cdot), (v_0(-s, x))_{s \geq 0})^t$ and the stochastic convolution $W_{\mathbf{A}}(t)$ introduced in (31) is a well-defined Gaussian process (see Lemma 5.4).

Existence and uniqueness of mild solution for the abstract evolution equation (33) is a classical result within the theory of stochastic equation in infinite dimension. The proof follows from a fixed point argument and can be found in [12, Theorem 7.4]

Theorem 5.5. *For arbitrary $T > 0$, and any $X_0 \in \mathcal{H}$ there exists a unique mild solution $(X(t))_{t \geq 0}$ of equation (33) which belongs to the space $L^p(\Omega; C([0, T]; \mathcal{H}))$ for any $p \geq 1$.*

An immediate consequence of the above result is that also the original stochastic Volterra equation (16) admits a unique mild solution. The definition of mild solution involves the resolvent family introduced in Subsection 5.1 and reads as follows

Definition 5.6. A $L^2(\mathcal{O})$ -valued process $(v(t))_{t \geq 0}$ is a mild solution of the stochastic Volterra equation of (16) if $v \in L^2(0, T; L^2(\Omega; L^2(\mathcal{O})))$ and satisfies

$$\begin{aligned} v(t) &= S(t)v_0(0) + \int_0^t S(t-s)f(s, v(s))ds \\ &\quad + \int_0^t S(t-s) \int_{\mathbb{R}_+} k(s+r)\Delta v_0(-r)dr + \int_0^t S(t-s)\sqrt{Q}dW(s) \end{aligned}$$

Theorem 5.7. *For arbitrary $T > 0$ and any $v_0 \in L_\rho^2(\mathbb{R}_+; H_0^1(\mathcal{O}))$ there exists a unique mild solution $v = v(t)$, $t \geq 0$ of equation (16) which belongs to the space $C([0, T]; L^2(\mathcal{O}))$ which is \mathcal{F}_t -adapted for any $t \geq 0$.*

Proof. The proof follows directly from Theorem 5.5. In fact, the mild solution of (16) is represented by the first component of the process $(X(t))_{t \geq 0}$. \square

6. Synthesis of the optimal control.

6.1. **The main result.** In this section we proceed with the study of the optimal control problem associated with the stochastic Volterra equation

$$\begin{aligned} \partial_t v(t, x) = & k_0 \Delta v(t, x) + \int_{-\infty}^t k_1(t-s) \Delta v(s, x) ds \\ & + f(t, v(t, x)) + \sqrt{Q}(r(t, v(t, x), \gamma(t, x)) + \partial_t W(t, x)), \end{aligned} \quad (34)$$

in the bounded domain $\mathcal{O} \subset \mathbb{R}^d$, with Dirichlet boundary condition $v(t, x) = 0, t \in [0, T], x \in \partial\mathcal{O}$ and initial condition $v(t, x) = v_0(t, x), t \leq 0, x \in \mathcal{O}$. Here f is the nonlinear function introduced in Hypothesis 2 and $\gamma = \gamma(\omega, t, x)$ is the control variable, which is assumed to be a predictable real-valued process \mathcal{F}_t -adapted. The optimal control that we wish to treat consists in minimizing over all admissible controls a cost functional of the form

$$\mathbb{J}(v_0, \gamma) := \mathbb{E} \int_0^T \int_{\bar{\mathcal{O}}} \ell(t, v(t, \xi), \gamma(t, \xi)) d\xi dt + \mathbb{E} \int_{\bar{\mathcal{O}}} \phi(v(T, \xi)) d\xi,$$

where ℓ and ϕ are given real-valued functions.

We will work under the following general assumptions. Concerning the function r, ℓ, ϕ we require:

Hypothesis 4.

1. $r : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\ell : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions and there exist $m \in \mathbb{N}$ and $C \geq 0$ such that for a.e. $t \in [0, T]$ and for θ_1, θ_2, y in \mathbb{R} ,

$$\begin{aligned} |r(t, x_1, y) - r(t, x_2, y)| + |\ell(t, \theta_1, y) - \ell(t, \theta_2, y)| \\ \leq C(1 + |\theta_1| + |\theta_2|)^m |\theta_1 - \theta_2|, \\ |r(t, \theta_1, y)| + |\ell(t, \theta_1, y)| \leq C. \end{aligned}$$

2. $\phi \in C^1(\mathbb{R})$ and there exist $L > 0$ and $k \in \mathbb{N}$ such that for every $\theta \in \mathbb{R}$

$$|\phi'(\theta)| \leq L(1 + |\theta|)^k.$$

In order to characterize the optimal control through a feedback law, we impose the following additional condition on the nonlinear term f :

Hypothesis 5. The function $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable, for every $t \in [0, T]$ the function $f(t, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable and there exists a constant C_f such that

$$\left| \frac{\partial}{\partial x} f(t, \xi) \right| \leq C_f, \quad t \in [0, T], \xi \in \mathbb{R}.$$

To handle the control problem, we first restate equation (34) in an evolution setting and we provide the synthesis of the optimal control by using the forward-backward system approach.

Arguing as in Section 4, given a control process γ and any $t \in [0, T]$, $v_0 \in L^2_{\rho}(\mathbb{R}_+; H^1_0(\mathcal{O}))$ we rewrite the problem (34) in the following abstract form

$$\begin{cases} dX(t) = \mathbf{A}X(t)dt + \mathbf{F}(t, X(t))dt + \sqrt{Q}(R(t, X(t), \gamma(t)))dt + dW(t) \\ X(0) = X_0, \end{cases} \quad (35)$$

where $X_0 = (v_0(0), v_0(\cdot))^t$ and $R : [0, T] \times \mathcal{H} \times \mathcal{X} \rightarrow \mathcal{H}$ is the mapping defined by

$$R\left(t, \begin{pmatrix} v \\ \eta \end{pmatrix}, \gamma\right) = \begin{pmatrix} r(t, v, \gamma) \\ 0 \end{pmatrix}, \quad t \in [0, T], \quad \begin{pmatrix} v \\ \eta \end{pmatrix} \in \mathcal{H}, \gamma \in \mathcal{X}.$$

In this setting the cost functional will depend on X_0 and γ and is given by

$$\mathbb{J}(X_0, \gamma) = \mathbb{E} \int_0^T L(t, X(t), \gamma(t)) dt + \mathbb{E}[\Phi(X(T))] \quad (36)$$

where $L : [0, T] \times \mathcal{H} \times \mathcal{X} \rightarrow \mathbb{R}$ is given by

$$L\left(t, \begin{pmatrix} v \\ \eta \end{pmatrix}, \gamma\right) = \int_{\mathcal{O}} \ell(t, v(\xi), \gamma(\xi)) d\xi$$

for any $t > 0$, $\begin{pmatrix} v \\ \eta \end{pmatrix} \in \mathcal{H}$, $\gamma \in \mathcal{X}$ and $\Phi : \mathcal{H} \rightarrow \mathbb{R}$ is defined as

$$\Phi\left(\begin{pmatrix} v \\ \eta \end{pmatrix}\right) = \int_{\mathcal{O}} \phi(v(\xi)) d\xi, \quad \begin{pmatrix} v \\ \eta \end{pmatrix} \in \mathcal{H}.$$

There are different ways to give a precise meaning to the above problem; one of them is the so called *weak formulation* and will be specified below.

In the weak formulation the class of *admissible control systems* (a.c.s.) is given by the set $\mathbb{U} := (\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \geq 0}, \hat{\mathbb{P}}, \hat{W}, \hat{\gamma})$, where $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ is a complete probability space; the filtration $(\hat{\mathcal{F}}_t)_{t \geq 0}$ verifies the usual conditions, the process \hat{W} is a Wiener process with respect to the filtration $(\hat{\mathcal{F}}_t)_{t \geq 0}$ and the control $\hat{\gamma}$ is an \mathcal{F}_t -predictable process taking value in some subset \mathcal{U} of \mathcal{X} with respect to the filtration $(\hat{\mathcal{F}}_t)_{t \geq 0}$.

With an abuse of notation, for given $X_0 \in \mathcal{H}$, we associate to every a.c.s. a cost functional $\mathbb{J}(x, \mathbb{U})$ given by the right side of (36). Although formally the same, it is important to note that now the cost is a functional of the a.c.s. and not a functional of $\hat{\gamma}$ alone. Any a.c.s. which minimizes $\mathbb{J}(x, \cdot)$, if it exists, is called optimal for the control problem starting from X_0 at time t in the weak formulation. The minimal value of the cost is then called the optimal cost. Finally we introduce the value function $V : [0, T] \times \mathcal{H} \rightarrow \mathbb{R}$ of the problem as:

$$V(X_0) = \inf_{\gamma \in \mathcal{U}} \mathbb{J}(X_0, \gamma), \quad X_0 \in \mathcal{H},$$

where the infimum is taken over all a.c.s. \mathbb{U} .

At this moment it is convenient to list the relevant properties of the objects introduced so far in this section. Therefore we formulate the following proposition.

Proposition 2. *Under Hypothesis 1, 2, 3, 4 and 5, the following properties hold:*

1. *The functions R and L are Borel measurable and there exist constants $C, m, k \in \mathbb{N}$ such that for any $t > 0, X_1, X_2 \in \mathcal{H}$ and $\gamma \in \mathcal{U}$*

$$\begin{aligned} |R(t, X_1, \gamma) - R(t, X_2, \gamma)| + |L(t, X_1, \gamma) - L(t, X_2, \gamma)| \\ \leq C(1 + |X_1| + |X_2|)^m |X_1 - X_2|, \\ |R(t, X_1, \gamma)| + |L(t, 0, \gamma)| \leq C. \end{aligned}$$

2. *Φ is Gâteaux differentiable and there exists $C_\Phi > 0$ such that for every $X_1, X_2 \in \mathcal{H}$*

$$|\Phi(X_1) - \Phi(X_2)| \leq C_\Phi |X_1 - X_2|$$

3. $\mathbf{F} : [0, T] \times \mathcal{H} \rightarrow \mathcal{H}$ is a measurable function and there exists a constant $C_{\mathbf{F}}$ such that

$$|\mathbf{F}(t, 0)| \leq C_{\mathbf{F}}, \quad |\mathbf{F}(t, X_1) - \mathbf{F}(t, X_2)| \leq C_{\mathbf{F}}|X_1 - X_2|,$$

for every $t \in [0, T]$, $X_1, X_2 \in \mathcal{H}$. Moreover, for every $t \in [0, T]$, $\mathbf{F}(t, \cdot)$ has a Gâteaux derivative $\nabla \mathbf{F}(t, X)$ at every point $X \in \mathcal{H}$. Finally, the function $(X, H) \mapsto \nabla \mathbf{F}(t, X)[H]$ is continuous as a map $\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$.

Optimal control problems associated with equation (35) and the cost functional (36) when the coefficients has the properties listed in Proposition 2 has been exhaustively studied by Fuhrman and Tessitore in [13, Theorem 7.2]. Within their approach the existence of an optimal control is related to the existence of the solution of a suitable forward backward system (FBSDE) that is a system in which the coefficients of the backward equation depend on the solution of the forward equation. Moreover, the optimal control can be selected using a feedback law given in terms of the solution to the corresponding FBSDE.

We introduce the Hamiltonian function $\psi : [0, T] \times \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$ setting

$$\psi(t, X, Z) = \inf_{\gamma \in \mathcal{U}} \{L(t, X, \gamma) + \langle Z, R(t, X, \gamma) \rangle\}, \quad t \in [0, T], \quad X \in \mathcal{H}, \quad Z \in \mathcal{H},$$

and we define the following set

$$\Gamma(t, X, Z) = \{\gamma \in \mathcal{U} : L(t, X, \gamma) + \langle Z, R(t, X, \gamma) \rangle = \psi(t, X, Z)\}, \quad (37)$$

$$t \in [0, T], \quad X \in \mathcal{H}, \quad Z \in \mathcal{H}.$$

For further use we require some additional properties of the function ψ :

Hypothesis 6.

1. For all $t \in [0, T]$, for all $X, Z \in \mathcal{H}$ there exists a unique $\Gamma(t, X, Z)$ that realizes the minimum in (37). Namely:

$$\psi(t, X, Z) = L(t, X, \Gamma(t, X, Z)) + \langle Z, r(t, X, \Gamma(t, X, Z)) \rangle$$

with $\Gamma \in C([0, T] \times \mathcal{H} \times \mathcal{H}; \mathcal{U})$.

2. For almost every $s \in [0, T]$ the map $\psi(s, \cdot, \cdot)$ is Gâteaux differentiable on $\mathcal{H} \times \mathcal{H}$ and the maps $(X, H, Z) \mapsto \nabla_X \psi(s, X, Z)[H]$ and $(X, Z, K) \mapsto \nabla_Z \psi(s, X, Z)[K]$ are continuous on $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ and $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ respectively.

Remark 5. It is easy to prove that combining the previous assumption with Proposition 2 we can deduce the following properties of ψ :

1. ψ is a measurable mapping and there exists a constant C such that

$$|\psi(t, X_1, Z) - \psi(t, X_2, Z)| \leq C(1 + |X_1| + |X_2|)|X_2 - X_1|$$

for all $X_1, X_2, Z \in \mathcal{H}$ and $t \in [0, T]$.

2. Setting $C_{\mathcal{U}} := \sup \{|\gamma| : \gamma \in \mathcal{U}\}$ we have

$$|\psi(s, X, Z_1) - \psi(s, X, Z_2)| \leq C_{\mathcal{U}}|Z_1 - Z_2|,$$

for every $s \in [0, T]$, $X, Z_1, Z_2 \in \mathcal{H}$.

Finally, $\sup_{s \in [0, T]} |\psi(s, 0, 0)| \leq C$.

Now, let us consider an arbitrary set-up $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{W})$ and

$$\tilde{X}(t) = e^{t\mathbf{A}} \tilde{X}_0 + \int_0^t e^{(t-\sigma)\mathbf{A}} \mathbf{F}(\sigma, \tilde{X}_\sigma) d\sigma + \int_0^t e^{(t-\sigma)\mathbf{A}} \sqrt{\tilde{\mathbf{Q}}} d\tilde{\mathbf{W}}(\sigma), \quad t \in [0, T], \quad (38)$$

where $\tilde{\mathbf{W}}(t) = (\tilde{W}(t), 0)^t$. By Theorem 5.5 stated in Subsection 5.5, equation (38) is well-posed and the solution $(\tilde{X}(t))_{t \geq 0}$ is a continuous process in \mathcal{H} , adapted to the filtration $(\tilde{\mathcal{F}}_t)_{t \geq 0}$. Moreover, the law of $(\tilde{\mathbf{W}}, \tilde{X})$ is uniquely determined by X_0 , \mathbf{A} , \mathbf{F} and $\sqrt{\mathbf{Q}}$. We define the process

$$\mathbf{W}^{\mathbb{U}}(t) = \tilde{\mathbf{W}}(t) - \int_0^t R(s, \tilde{X}(s), \hat{\gamma}(s)) ds, \quad t \in [0, T],$$

and we note that, since R is bounded, by the Girsanov theorem there exists a probability measure \mathbb{P} on (Ω, \mathcal{F}) such that $\mathbf{W}^{\mathbb{U}}$ is a Wiener process under \mathbb{P} . Rewriting equation (38) in terms of $\mathbf{W}^{\mathbb{U}}$ we get that \tilde{X} solves the controlled state equation (in weak sense)

$$\begin{aligned} \tilde{X}(t) = \tilde{X}_0 + \int_0^t e^{(t-\sigma)\mathbf{A}} \mathbf{F}(\sigma, \tilde{X}_\sigma) d\sigma + \\ \int_0^t e^{(t-\sigma)\mathbf{A}} \sqrt{\mathbf{Q}} d\mathbf{W}^{\mathbb{U}}(\sigma) + \int_0^t e^{(t-\sigma)\mathbf{A}} R(s, \tilde{X}(s), \hat{\gamma}(s)) ds. \end{aligned} \quad (39)$$

Next we consider the backward stochastic differential equation

$$\tilde{Y}(t) + \int_t^T \tilde{Z} d\tilde{W}(\sigma) = \Phi(\tilde{X}(T)) + \int_t^T \psi(\sigma, \tilde{X}(\sigma), \tilde{Z}(\sigma)) d\sigma, \quad t \in [0, T], \quad (40)$$

where ψ is the Hamiltonian function and Φ is the function defining the final cost. Under our assumptions, we can apply [13, Proposition 3.2 and Theorem 4.8] and state that there exists a solution $(\tilde{X}, \tilde{Y}, \tilde{Z})$ of the forward-backward system (38)-(40) on the interval $[0, T]$, where \tilde{Y} is unique up to indistinguishability and \tilde{Z} is unique up to modification. Moreover from the proof of [13, Theorem 4.8] it follows that the law of (\tilde{Y}, \tilde{Z}) is uniquely determined by the law of $(\tilde{\mathbf{W}}, \tilde{X})$ and by Φ and Ψ . We note that $\tilde{Y}(t)$, being measurable with respect to the degenerate σ -algebra $\tilde{\mathcal{F}}_0$, is deterministic; in particular $\tilde{Y}(t) = \mathbb{E}(\tilde{Y}(t))$ only depends on the law of \tilde{Y} , and thus it is a functional of X_0 , \mathbf{A} , \mathbf{F} , $\sqrt{\mathbf{Q}}$, Φ , Ψ . To stress dependence on the initial datum X_0 , we will denote the solution of (38) and (40) by $\{(\tilde{X}^{X_0}(t), \tilde{Y}^{X_0}(t), \tilde{Z}^{X_0}(t)), t \in [0, T]\}$.

We recall from [9, Theorem 6.2] that, Proposition 2 and Hypothesis 6, imply existence and uniqueness of a (mild) solution $u \in C^{0,1}([0, T] \times \mathcal{H}; \mathbb{R})$ of the Hamilton Jacobi Bellman equation corresponding with our control problem:

$$\begin{cases} \frac{\partial}{\partial t} u(t, X) + \mathcal{L}_t[u(t, \cdot)](X) \\ \quad = \psi(t, X, u(t, X), \nabla u(t, X) \sqrt{\mathbf{Q}}), & t \in [0, T], X \in \mathcal{H}, \\ u(T, X) = \phi(X). \end{cases}$$

Here \mathcal{L} is the infinitesimal generator of the Markov semigroup corresponding to the process X :

$$\mathcal{L}_t[h](X) = \frac{1}{2} \text{Tr}(\nabla^2 h(X) \mathbf{Q}) + \langle \mathbf{A}X + \mathbf{F}(t, X), \nabla h(X) \rangle.$$

Moreover, \mathbb{P} -a.s. for a.e. $t \in [0, T]$, we have

$$\tilde{Y}^{X_0}(t) = u(t, \tilde{X}^{X_0}(t)), \quad \tilde{Z}^{X_0}(t) = \nabla u(t, \tilde{X}^{X_0}(t)) \sqrt{\mathbf{Q}}.$$

The relevance of the solution of the Hamilton-Jacobi-Bellman equation to our control problem is explained in the following proposition.

Proposition 3. *Assume that Hypotheses 1, 2, 3, 4, 5 and 6 hold. For every $t \in [0, T]$ and $X_0 \in \mathcal{H}$, and for every a.c.s. \mathbb{U} we have $u(0, X_0) \leq \mathbb{J}(X_0, \mathbb{U})$ and equality holds if and only if the following feedback law is verified, \mathbb{P} -a.s. for almost every $t \in [0, T]$:*

$$\hat{\gamma}(t) = \Gamma(t, \tilde{X}(t), \nabla u(t, \tilde{X}(t)))\sqrt{\mathbf{Q}}. \quad (41)$$

Finally, there exists at least an a.c.s. \mathbb{U} verifying (41). In such a system, the closed loop equation admits a solution

$$\begin{cases} d\bar{X}(t) = \mathbf{A}\bar{X}(t)dt + \mathbf{F}(t, \bar{X}(t))dt + \\ \quad \sqrt{\mathbf{Q}} \left(R(t, \bar{X}(t), \Gamma(t, \bar{X}(t), \nabla u(t, \bar{X}(t)))\sqrt{\mathbf{Q}})dt + d\mathbf{W}(t) \right), \quad t \in [s, T] \\ \bar{X}(s) = X_0 \in \mathcal{H}, \end{cases} \quad (42)$$

and if $\bar{\gamma}(t) = \Gamma(t, \bar{X}(t), \nabla u(t, \bar{X}(t)))\sqrt{\mathbf{Q}}$ then the couple $(\bar{\gamma}, \bar{X})$ is optimal for the control problem.

Proof. The result follows immediately from the paper of Fuhrman and Tessitore [13, Theorem 7.2]. \square

6.2. Example: Gluing in polymeric materials. In manufacturing polymeric materials there arises frequently the problem of gluing. For this purpose the surface of the pieces first have to be heated, for example by means of radiation (infrared or microwave radiation). In a very simplified model this leads to a control problem for a one-dimensional heat equation for materials with memory. It is in particular assumed that the absorption of the radiation in the material follows an exponential distribution $\alpha e^{-\alpha x}$, where x denotes the distance from the surface and $\alpha > 0$ is a constant. If the constitutive laws for isotropic homogeneous materials with memory are employed (see Subsection 2) with $m \equiv 1$ and a stochastic perturbation effect the radiation which controls the temperature, this leads to the problem

$$\begin{aligned} \partial_t v(t, x) &= dk * \partial_{xx}^2 v(t, x) + i(t)\alpha e^{-\alpha x} + \sigma(x)\partial_t W(t, x), \quad t, x > 0, \\ v(0, x) &= 0, \quad v(t, 0) = v(t, 1) = 0 \quad t, x \in [0, 1], \\ v(s, x) &= v_0(s, x), \quad s \leq 0, x \in [0, 1]. \end{aligned}$$

Here $v(t, x)$ denotes the temperature, $W(t, x)$ is the so-called space-time white noise on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and bounded function such that, for all $\xi \in [0, 1]$, $\sigma(\xi) \neq 0$. The process $i(t)$ is the intensity of the radiation which serves as the control variable. It is subjected to the constrain

$$0 \leq i(t) \leq \gamma_0, \quad \mathbb{P} - a.s., \quad a.a. \quad t > 0, \quad (43)$$

where γ_0 denotes the maximal available intensity. A possible control problem consists in minimizing the total amount of energy displaced by the system over an interval $[0, T]$. In other words we can treat the cost functional

$$\mathbb{J}(v_0, \gamma) = \mathbb{E} \int_0^T \frac{1}{2} \int_0^1 [\alpha^2 i^2(t) e^{-2\alpha \xi} + v^2(t, \xi)] d\xi dt + \frac{1}{2} \mathbb{E} \int_0^1 v^2(T, \xi) d\xi.$$

that we wish to minimize over all controls $i \in L^2([0, T] \times \Omega; \mathbb{P})$ satisfying the constraint (43).

The above problem falls under the scope of the general result proved in the previous subsection letting $\mathcal{H} := L^2(0, 1) \times L^2_{\rho}(\mathbb{R}_+; H^1_0(0, 1))$,

$$\mathcal{U} = \{ \gamma \in L^2(0, 1) : \gamma(\xi) = i e^{-\alpha \xi}, i \leq \gamma_0 \},$$

$\mathbf{W}(t) = (W(t), 0)$. Moreover we define

$$(\mathbf{Q}X)(\xi) = \begin{pmatrix} \sigma^2(\xi)v(\xi) & 0 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} v \\ \eta \end{pmatrix} \in \mathcal{H};$$

$$R(t, X, \gamma)(\xi) = \begin{pmatrix} \gamma(\xi) \\ \sigma(\xi) \\ 0 \end{pmatrix}, \quad t > 0, X \in \mathcal{H}, \gamma \in \mathcal{U};$$

$$L(t, X, \gamma) = |\gamma|_{L^2(0,1)}^2 + |v|_{L^2(0,1)}^2 = \int_0^1 (\gamma^2(\xi) + v^2(\xi))d\xi, \quad X = \begin{pmatrix} v \\ \eta \end{pmatrix} \in \mathcal{H};$$

$$\Phi(X) = |v|_{L^2(0,1)}^2, \quad X = \begin{pmatrix} v \\ \eta \end{pmatrix} \in \mathcal{H}.$$

Notice that all the functions above satisfy the regularity required in the previous subsection. Then Proposition 3 can be applied and we obtain a characterization of the optimal control by a feedback law.

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Received December 2012; revised July 2013.

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