# Thermalization and Quantum Information in Conformal Field Theory 

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Ashish Kakkar, Student<br>Dr. Anatoly Dymarsky, Major Professor<br>Dr. Christopher Crawford, Director of Graduate Studies

| DISSERTATION |
| :---: |
| A dissertation submitted in partial |
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| the degree of Doctor of Philosophy |
| in the College of Arts and Sciences |
| at the University of Kentucky |
| By |
| Ashish Kakkar |
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## ABSTRACT OF DISSERTATION

Thermalization and Quantum Information in Conformal Field Theory
The consequences of the constraints of conformal symmetry are far-reaching within theoretical physics. In this dissertation we address a series of questions in conformal field theory: 1) We calculate the spectrum of $q K d V$ charges in a large central charge expansion. 2) We determine the corrections to bulk information geometry from $1 / N$ contributions to holographic correlators. 3) We study the higher genus partitions functions of CFTs associated with classical and quantum error-correcting codes.

KEYWORDS: Conformal Field Theory

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Date: November 15, 2022

# Thermalization and Quantum Information in Conformal Field Theory 

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## TABLE OF CONTENTS

Acknowledgments ..... iii
Table of Contents ..... iv
List of Figures ..... vi
List of Tables ..... vii
Chapter 1 Introduction ..... 1
1.1 Integrability in 2d CFTs ..... 5
1.2 Generalized Eigenstate Thermalization ..... 8
1.3 Large c expansion in Holography ..... 9
1.4 Modular invariance of 2d CFTs ..... 10
Chapter 2 Spectrum of quantum KdV hierarchy in the semiclassical limit ..... 14
2.1 Introduction: Quantum KdV symmetry ..... 14
2.2 Calculation of $Q_{2 n-1}\left(h, I_{k}\right)$ ..... 18
2.3 "Energies" of primary states via ODE/IM correspondence ..... 28
2.4 Spectrum of quantum $Q_{2 k-1}$ ..... 30
2.5 Miscellaneous results ..... 35
2.6 Discussion ..... 38
Chapter 3 Information geometry and holographic correlators ..... 40
3.1 Introduction: Information geometry and holographic dictionary ..... 40
3.2 Information metric basics ..... 42
3.3 CFT correlators ..... 44
3.4 Four-point examples ..... 47
3.5 Transition amplitudes ..... 50
3.6 Future directions ..... 52
Chapter 4 Classical codes, Chiral CFTs and Lattices ..... 54
4.1 Introduction: Chiral CFTs and classical error correcting codes ..... 54
4.2 Review: Classical Codes and Chiral CFTs ..... 57
4.3 Chiral CFTs at Higher Genus ..... 63
4.4 Counting Higher Genus Enumerator Polynomials ..... 81
4.5 Discussion ..... 95
Chapter 5 Quantum codes, lattices and CFTs ..... 98
5.1 Introduction: Partition functions of Narain CFTs and quantum errorcorrecting codes . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 98
5.2 Quantum codes and Narain CFTs ..... 103
5.3 Narain lattices and code theories at higher genus ..... 120
5.4 Partition functions for code theories ..... 128
5.5 Beyond code theories ..... 136
5.6 Discussion ..... 144
Chapter 6 Conclusion ..... 147
Chapter A Appendices ..... 149
A. 1 One-zone potentials: details ..... 149
A. 2 Perturbative calculation for finite-zone potentials ..... 150
A. 3 Spectrum of $Q_{2 n-1}$ acting on primaries ..... 155
A. 4 Conventional Genus 2 Transformations ..... 157
A. 5 Siegel Modular Forms ..... 157
A. 6 Generators for invariant ring at genus 2 ..... 163
A. 7 Genus 2 partition functions in enumerator polynomial form ..... 165
Bibliography ..... 168

## LIST OF FIGURES

4.1 Factorization of a genus $g$ Riemann surface. ..... 71
4.2 Factorization of a genus 2 Riemann surface. The $a$ cycles are depicted in red, $b$ cycles in green. Under the factorization, the one-form $\omega_{2}$ dual to $a_{2}$vanishes on the left torus. Hence $\Omega_{12}=\oint_{b_{1}} \omega_{2}=0$, so that $r=e^{2 \pi i \Omega_{12}} \rightarrow 1.73$4.3 Allowed polytope for the genus 2 coefficients. Green dots represent poly-nomials which lie inside the bounds, and red outside. Plot (a) showsall 29 allowed genus 2 polynomials with integer coefficients. In (b), wezoomed into the bottom left, and allowed for half-integer coefficients in$P_{\text {inv }}^{1}$ allowing a larger number of solutions. Black stars represent lattices.
We can see the $k=4, k=5$, and $k=6$ lattices outside the allowed region. ..... 86
5.1 Graph defining the matrix in equation (5.2.72). ..... 119
5.2 Allowed regions for $n=3$. Plot (a): 2d region for genus 1. Dots represent polynomials with integer coefficients. Red dots, appearing at ( 0,1 ), (3, 1),$(1,2)$, and ( 0,4 ), represent factorization limits of genus 2 polynomials, aswell as physical code theories. Plot (b): rough 3d plot of genus 2 allowedregion. Remarkably, the factorizing, positive integer solutions lie on thevertices of this polytope!130
A. $1 Q_{1}^{[2]}$ (left), $Q_{2}^{[2]}$ (middle), and $Q_{3}^{[2]}$ (right). ..... 163
A. $2 Q_{4}^{[2]}$ ..... 163
A. $3 Q_{5}^{[2]}$ ..... 163
A. $4 Q_{6}^{[2]}$ ..... 163
A. $5 Q_{7}^{[2]}$ ..... 164
A. $6 Q_{8}^{[2]}$ ..... 164
A. $7 Q_{9}^{[2]}$ ..... 164
A. $8 Q_{10}^{2]}$ ..... 164
A. $9 Q_{11}^{[2]}$ ..... 164

## LIST OF TABLES

4.1 Dictionary relating error correcting codes and CFTs. ..... 57
4.2 Summary of the differences between genus 1 and higher genus. ..... 64
4.3 Summary of the differences between genus $g \leqslant 3$ and higher genus. ..... 77

4.4 Polynomials organized by number of spin 1 currents $N_{\text {currents }}=24 k$. Thenumber of currents may be computed from the genus 1 polynomial using| $N_{\text {currents }}$ | $=72+16 a_{4}$. | See | 106 |
| :--- | :--- | :--- | :--- | for the list of lattices organized by thenumber of currents. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 88

5.1 Number of linearly independent invariant polynomials, and number of generators of degree $n$ at genus $g=2$. . . . . . . . . . . . . . . . . . . . 133

## Chapter 1 Introduction

The course of development of understanding the fundamental laws of physics has been dictated by the study of symmetry. Uncovering symmetries of space-time and understanding their consequences has led to the development and understanding of quantum field theories (QFTs). QFTs are the most successful framework to describe the interactions between elementary particles as well as the statistical properties of collective excitations of particles. Studying the symmetry of systems allows one to severely constrain the allowed spectrum of particles and their interactions.

These interactions are organized based on the energy scales at which they are probed. The renormalization group describes how the interaction strengths or couplings change with the energy scale. The idea is to construct an effective description by coarse-graining the short distance degrees of freedom. This leads to a flow in the space of allowed theories [1.

An extremely important class of QFTs called Conformal Field Theories (CFTs) live at the fixed points of these flows. The symmetries of these field theories include space-time scale invariance, in addition to space-time translations and rotations which are present in any relativistic quantum field theory. These conformal symmetries are highly constraining and importantly emerge in the large distance description of a wide variety of physical systems such as those undergoing phase transition [2]. Working out the consequences of the presence of this symmetry has led to important insights in theoretical physics. This dissertation focuses on understanding the constraining power of conformal symmetry, primarily in 2 space-time dimensions.

The symmetries of CFTs are so constraining that they allow calculations to be done in regimes where interaction strengths are too strong to allow a perturbative description. In fact, this has led to the development of the conformal bootstrap [3], an ambitious approach of bypassing a Lagrangian description of field theories to constrain them using symmetry alone.

Quantum gravity also provides a compelling reason to develop an understanding of CFTs. According to our current understanding, one of the only known selfconsistent descriptions of quantum gravity in which we have a degree of qualitative control is via the holographic duality [4]. This is a conjectured duality between a certain gravitational theory living on $(d+1)$-dimensional asymptotically anti-de Sitter (AdS) space-time and a quantum field theory living on its $d$-dimensional boundary.

Crucially, it allows one to sharply formulate the black information paradox, first defined by Hawking [5]. The puzzle refers to the tension between the following two statements: i) as seen by an external observer, a black hole can be described in terms of a quantum system which evolves unitarily under time evolution, ii) a black hole has a temperature and is described as a thermal ensemble. Many studies attempt to use the AdS/CFT duality to describe the gravitational collapse of matter in AdS, in terms of the approach towards thermal equilibrium of the boundary CFT. The final formation of the black hole is expected to correspond to thermalization in the CFT [6].

So what is the mechanism that explains how a closed quantum system (satisfying certain conditions) is described as a thermal ensemble at late enough times, under unitary time evolution? An important modern tool to understand this process of thermalization in quantum mechanics is given by the Eigenstate Thermalization Hypothesis (ETH) [7]. ETH tells us that we only need to study the matrix elements of probe observable quantities in the energy eigenstates of a system in order to discern whether the system can be described by a canonical thermal ensemble at late enough times. The condition that the matrix elements need to satisfy is that the diagonal elements need to be smooth functions of the energy and the off-diagonal elements need to be exponentially suppressed. ETH has been extensively studied theoretically and it's predictions have found agreement with numerical simulations of lattice systems [8].

In two-dimensional CFTs, the conformal symmetry algebra is enhanced to the infinite-dimensional Lie algebra of holomorphic and anti-holomorphic local conformal transformations, the Witt algebra. Since we are in interested in quantum mechanical theories with such symmetries, we are led to study the centrally extended version of this symmetry algebra, which is called the Virasoro algebra. These symmetries allow all $2 d$ CFTs to admit an integrable sector with an infinite set of conserved charges [9]. Thus one naturally expects that in order to understand thermalization of an integrable $2 d$ CFT, one should consider thermalization to an ensemble defined by these non-trivial charges in addition to the Hamiltonian. Holographically, this point of view could enable us to describe the formation of a black hole in an $\mathrm{AdS}_{3}$ spacetime. Here, when quantum corrections are taken into account, conserved quantum KdV or qKdV charges would be present and understood to be soft hair of the black hole [10]. There are however a number of questions we have to answer before making progress on this challenging question, the first of them being how to calculate the spectrum of qKdV charges. In Chapter 2, we make initial steps in this direction
by understanding the spectrum of charges in the holographically relevant large central charge perturbative expansion. This chapter is based on my paper Spectrum of quantum KdV hierarchy in the semiclassical limit, written in collaboration with Anatoly Dymarsky, Sotaro Sugishita and Kiril Pavlenko [11].

Indeed, using the symmetries of a CFT is important for better understanding the holographic dictionary itself. Consider for example, a region $A$ in the boundary CFT. The dual region in the AdS bulk is referred to as the entanglement wedge and corresponds to a region enclosed by an extremal surface anchored on $A$ [12]. How does the bulk geometry of this entanglement wedge emerge from the CFT data on the boundary? This important question has attracted a lot of intense research and significant ideas from quantum information theory have been shown to be instrumental in understanding it.

In quantum information theory, a useful measure of distinguishability between two quantum states is the Bures distance. It is natural to expect this metric to emerge in the bulk. This is because the distinguishability of states in the bulk is expected to increase as the points where these states have support are geometrically separated in the boundary [13]. In Chapter 3, we explore perturbative corrections to the information metric associated with correlators in holographic CFTs. This chapter is based on my paper Information geometry and holographic correlators, written in collaboration with Allic Sivaramakrishnan and Hardik Bohra [14].

A recurring theme in both Chapters 2 and 3 is the constraining power of conformal symmetry. Interestingly, the symmetry constraints of a two-dimensional CFT can be formulated in terms of the consistency conditions that its correlation functions need to satisfy. The 4-point correlation functions admit a decomposition into conformal blocks. This decomposition then needs to satisfy unitarity and crossing symmetry. This technique is known as the conformal bootstrap and has been exploited in classifying all $2 d$ CFTs with central charge less than 1 [15]. For central charge greater than 1 , the expressions for conformal blocks are difficult to obtain and this regime has largely been unexplored.

These constraints are even more interesting when the CFT lives on higher genus Riemann surfaces, the simplest example being the torus. In these cases, in addition to the infinitesimal conformal transformations, there are also large or modular transformations which cannot be continuously connected to the identity. Invariance of CFT partition functions under such modular transformations, have been successfully applied to constrain CFTs and obtain universal bounds on the spectrum of $3 d$ AdS gravity 16 .

The celebrated Cardy formula is an important example of the physical consequences of studying this modular symmetry. It relates the density of states of a CFT at low and high temperatures, and via holography, has led to a microscopic description of the microstates of black holes in $\mathrm{AdS}_{3}$ (17.

The modular bootstrap program aims to chart the space of all allowed $2 d$ CFTs using the modular invariance and positivity of the torus partition function. In chiral CFTs, classical codes have been known to provide a natural mathematical framework which leads to solutions of the modular bootstrap equations [18]. More recently, this framework was extended to include a class of non-chiral CFTs [19]. These are perhaps the tip of the iceberg of interconnections between the bootstrap equations and areas of mathematics, the other well studied example being the sphere packing problem [20].

In [21], it was shown that and there are many modular-invariant "fake" partition functions that are not partition functions of any known theory. Many examples of isospectral yet physically distinct theories also exist. This leads one to ask the question: can the consideration of higher genus partition functions allow one to rule out unphysical theories? Perhaps the higher genus partition functions can tell apart physically distinct theories because they probe the spectrum as well as its interactions.

In order to approach these questions, in Chapter 4 we first extend the connection between classical codes and chiral CFTs to those that are defined on higher genus surfaces. We find that studying the higher genus partition function along with constraints arising from considering certain degeneration limits does allow one to constrain the space of allowed theories. The contents of this chapter are in my paper Classical codes and chiral CFTs at higher genus, written in collaboration with Brian McPeak and Johan Henriksson [22].

In Chapter 5, we extend to higher genus, the connection between quantum codes and partition functions of non-chiral CFTs defined by them. We find that in this class of theories, the constraints of higher genus modular transformations take the simple form of linear transformations on a certain set of variables. This allows one to solve these constraints. This chapter is based on my paper Narain CFTs and Quantum Codes at Higher Genus, written in collaboration with Brian McPeak and Johan Henriksson [23].

In the following sections of the introduction, I introduce in more detail various aspects of CFTs and other relevant background material that lead into the chapters that follow.

### 1.1 Integrability in 2d CFTs

The conserved current associated with conformal symmetry in 2d CFTs is the energymomentum tensor. Conservation in the conformal $z$ plane allows a split into a holomorphic component $T(z)=T_{z z}(z)$ and an anti-holomorphic component $\bar{T}(\bar{z})=$ $T_{\bar{z} \bar{z}}(\bar{z})$. These can be expanded in Laurent modes as follows:

$$
\begin{equation*}
T(z)=\sum_{n \in \mathbb{Z}} L_{n} z^{-n-2}, \bar{T}(\bar{z})=\sum_{n \in \mathbb{Z}} \bar{L}_{n} \bar{z}^{-n-2} . \tag{1.1.1}
\end{equation*}
$$

The modes $L_{n}$ satisfy the Virasoro algebra with central charge $c$ :

$$
\begin{align*}
{\left[L_{n}, L_{m}\right] } & =(n-m) L_{n+m}+\frac{c}{12}\left(n^{3}-n\right) \delta_{n+m, 0}, \\
{\left[\bar{L}_{n}, \bar{L}_{m}\right] } & =(n-m) \bar{L}_{n+m}+\frac{c}{12}\left(n^{3}-n\right) \delta_{n+m, 0}, \\
{\left[L_{n}, \bar{L}_{m}\right] } & =0 . \tag{1.1.2}
\end{align*}
$$

The Hilbert space of states is built by a collection of highest weight states (also referred to as primary states) $|h\rangle$ which are created by a primary field with weight $h$ inserted at the origin and satisfy $L_{-n}|h\rangle=0$ for $n>0$ and $L_{0}|h\rangle=h|h\rangle$.

In this section we will focus on the systems where there are an infinite number of conserved currents. At the classical level conservation of $T$ directly implies conservation of an infinite set of composite fields built from taking its powers and derivatives, $T^{2}, T^{3}, \cdots$.

In a seminal work, [9] it was shown that even at the quantum level, by taking powers of $T$ and its derivatives, one can construct an infinite set of currents $J_{2 k}$ such that their associated conserved charges (which will be refferred to as the qKdV charges) $Q_{2 k-1}=\int_{0}^{2 \pi} \frac{d x}{2 \pi} J_{2 k}$ form an infinite mutually commuting set,

$$
\begin{equation*}
\left[Q_{2 k-1}, Q_{2 l-1}\right]=0 . \tag{1.1.3}
\end{equation*}
$$

The first few currents have the following form:

$$
\begin{equation*}
J_{2}=T, J_{4}=: T^{2}:, J_{6}=: T^{3}:+\frac{c+2}{12}+(\partial T)^{2}, \tag{1.1.4}
\end{equation*}
$$

where $::$ is the normal ordering symbol and the derivative terms are added by construction to ensure 1.1 .3 holds. These charges have important physical interpretations. The first charge $Q_{1}$ is the generator of time tranlations ie. the Hamiltonian and the charge $Q_{3}$ generates a flow in phase space which is equivalent to the well-known KdV equation.

In order to study the classical spectrum of these charges in the classical field theory setting, we first look at how the problem of generating all integrals of motion is solved in finite-dimensional integrable systems. The relevant object to study is a pair of operators $L, M$ called a Lax pair, which are defined such that commutator of the pair is equivalent to equations of motion, ie.

$$
\begin{equation*}
\frac{d}{d t} L=[M, L] \tag{1.1.5}
\end{equation*}
$$

This is a useful reformulation of the equations of motion because they generate all the integrals of motion or action variables $I_{k}$ which can be calculated via the following trace over the phase space:

$$
\begin{equation*}
I_{k}=\operatorname{Tr} L^{k} \tag{1.1.6}
\end{equation*}
$$

The integrals of motion are now automatically conserved by cyclicity of trace $\frac{d}{d t} I_{k}=$ $\operatorname{Tr} L^{k-1}[M, L]=0$. So, the strategy to find the integrals of motion is to find such a pair of operators $\{L, M\}$.

In fact, such a pair of differential operators exists for the $k d V$ equation and is given by

$$
\begin{align*}
L & =-\frac{d^{2}}{d x^{2}}+u \\
M & =-4 \frac{d^{3}}{d x^{3}}-3\left(u \frac{d}{d x}+u \frac{d}{d x} u\right) \tag{1.1.7}
\end{align*}
$$

It is straightforward to check that $\dot{L}=[M, L]$ is equivalent to

$$
\begin{equation*}
\dot{u}=6 u u^{\prime}-4 u^{\prime \prime \prime} . \tag{1.1.8}
\end{equation*}
$$

We will solve the integrable field theory by means of the inverse scattering method [24]. This method transforms between the field on a time slice and a set of abstract scattering data. The object to study is a Schrödinger type equation, referred to as Hill's equation:

$$
\begin{equation*}
-\frac{d^{2}}{d x^{2}} \psi+u \psi=0 \tag{1.1.9}
\end{equation*}
$$

where $x \in S^{1}$ is the spatial circle in the CFT on a fixed time slice and $u(x)$ is a periodic potential $u(x+2 \pi)=u(x)$.

We know that the vector fields that generate reparametrizations on a circle follow the Witt algebra. We would like to demand that Hill's equation continues being
satisfied under reparametrizations $x \longrightarrow \tilde{x}(x)$. This is ensured if $\psi(x)$ and $u(x)$ transform as

$$
\begin{gather*}
\tilde{\psi}(\tilde{x})=\psi(x)\left(\frac{d \tilde{x}}{d x}\right)^{1 / 2},  \tag{1.1.10}\\
\tilde{u}(\tilde{x})=\left(\frac{d \tilde{x}}{d x}\right)^{-2}(u(x)+2\{\tilde{x}, x\}) . \tag{1.1.11}
\end{gather*}
$$

Here $\{\tilde{x}, x\}$ is the conventional Schwarzian derivative, with the following definition

$$
\begin{equation*}
\{y, x\}=\frac{y^{\prime \prime \prime}}{y^{\prime}}-\frac{3}{2}\left(\frac{y^{\prime \prime}}{y^{\prime}}\right)^{2} \tag{1.1.12}
\end{equation*}
$$

The appearance of the Schwarzian derivative and correct scalings in the transformation law for $u(x)$ tells us that we should be thinking of $u(x)$ as the energy momentum tensor $T$ of the 2d CFT. $T$ is known to transform as a primary field of weight 2 with an extra Schwarzian term. To make this more precise, consider an infinitesimal transformation

$$
\begin{equation*}
\tilde{x}=c+\epsilon f(x) . \tag{1.1.13}
\end{equation*}
$$

To keep Hill's equation satisfied, the potential transforms as

$$
\begin{equation*}
u(\tilde{x})=u(x)+\epsilon \mathcal{D} f \tag{1.1.14}
\end{equation*}
$$

where

$$
\mathcal{D}:=\partial u+2 u \partial-2 \partial^{3} .
$$

This flow generated by $u(x)$ can be written in terms of a Poisson bracket:

$$
\begin{equation*}
\frac{c}{24}\left\{u\left(x_{1}\right), u\left(x_{2}\right)\right\}=-2 \pi \mathcal{D} \delta\left(x_{1}-x_{2}\right) . \tag{1.1.15}
\end{equation*}
$$

Fourier expanding $u(x)$

$$
\begin{equation*}
\frac{c}{24}(u(x)+1)=\sum_{k} l_{k} e^{i k x}, \tag{1.1.16}
\end{equation*}
$$

lets us identify the Virasoro algebra satisfied by its generators

$$
\begin{equation*}
\left\{l_{n}, l_{m}\right\}=(n-m) l_{n+m}+\frac{c}{12} k^{2}(k-1) \delta_{n+m} \tag{1.1.17}
\end{equation*}
$$

In the above analysis, we found Poisson bracket structure for $u$ by demanding that Hill's equation was kept invariant under reparametrization of coordinates (which are
secretly conformal transformations). It turns out that in order to generalize this to include the flow generated by the composite fields $Q_{2 k-1}^{c l}$ which are conserved classically, the auxiliary spectral problem to consider is

$$
\begin{equation*}
-\frac{d^{2}}{d x^{2}} \psi+\frac{u}{4} \psi=\lambda \psi . \tag{1.1.18}
\end{equation*}
$$

The eigenvalues $\lambda$ of periodic and anti-periodic wavefunctions $\psi$ are the spectral data. The crucial statement is that the iso-spectral deformations of $u$ are generated by the kdV generators $Q_{2 k-1}^{c l}$

$$
\begin{equation*}
\delta u=\frac{c}{24}\left\{Q_{2 k-1}^{c l}, u\right\} . \tag{1.1.19}
\end{equation*}
$$

An illustrative example of this is that the original kdV equation is just an example of the flow generated by $Q_{3}$

$$
\begin{equation*}
\dot{u}=\frac{c}{24}\left\{Q_{3}, u\right\}=6 u u^{\prime}-4 u^{\prime \prime \prime} . \tag{1.1.20}
\end{equation*}
$$

We will now clarify what we mean by thermalization to an ensemble described by this infinite set of integrable charges in 2d CFTs.

### 1.2 Generalized Eigenstate Thermalization

Understanding the set of conditions to determine if a system will thermalize, has important consequences in understanding the emergence of thermodynamics from the well known princeples of quantum mechanics. This question was posed sharply in [25, [7] as, what is the criterion to determine if the expectation values of few-body observable $O$ in an isolated quantum system prepared initially in a far from equilibrium state, will towards those of a thermal ensemble. Eigenstate Thermalization Hypothesis gives us one way of answering this question. The relevant object to study is the matrix elements of $O$ in energy eigenstates $\left|E_{i}\right\rangle$.
The condition is that the diagonal elements are smooth functions $f_{0}$ of the energy expectation $E_{i}$, ie.

$$
\begin{equation*}
\left\langle E_{i}\right| O\left|E_{i}\right\rangle=f_{O}\left(E_{i}\right) \tag{1.2.1}
\end{equation*}
$$

and the off-diagonal elements are exponentially suppressed by the entropy $S(E)$ as $e^{-\frac{S(E)}{2}}$.
If this condition holds, and we tune the temperature $\beta$ such that

$$
\begin{equation*}
\left\langle E_{i}\right| O\left|E_{i}\right\rangle=\operatorname{Tr} H e^{-\beta H} \tag{1.2.2}
\end{equation*}
$$

then expectations values of $O$ are given by the canonical ensemble at late times

$$
\begin{equation*}
\left\langle E_{i}\right| O\left|E_{i}\right\rangle=\operatorname{Tr} O e^{-\beta H} . \tag{1.2.3}
\end{equation*}
$$

ETH has not been proven from first principles but has passed numerous numerical and experimental checks. When the system is integrable and has many conserved charges $Q_{2 k-1}$ (where we label $H=Q_{1}$ ), it does not equilibrate in the usual sense. However, there is a natural generalization of the of ETH criterion because we expect it to thermalize to the generalized gibbs ensemble (GGE) decorated by the charges $Q_{2 k-1}$.
We define a set of mutual eigenstates $\left|E_{i}\right\rangle$ of $Q_{2 k-1}$ and examine the probe operator $O$ in this set of eigenstates. If the diagonal elements are smooth functions of

$$
\begin{equation*}
\left\langle E_{i}\right| O\left|E_{i}\right\rangle=f_{O} Q_{2 k-1}\left(E_{i}\right) \tag{1.2.4}
\end{equation*}
$$

with the off diagonal elements being suppressed as $e^{-\frac{S(E)}{2}}$ and we tune chemical potentials $\mu_{2 k-1}$ such that

$$
\begin{equation*}
\left\langle E_{i}\right| Q_{2 k-1}\left|E_{i}\right\rangle=\operatorname{Tr} Q_{2 k-1} e^{-\sum_{k} \mu_{2 k-1} Q_{2 k-1}} \tag{1.2.5}
\end{equation*}
$$

then expectations values of $O$ are given by the GGE

$$
\begin{equation*}
\left\langle E_{i}\right| O\left|E_{i}\right\rangle=\operatorname{Tr} O e^{-\sum_{k} \mu_{2 k-1} Q_{2 k-1}} \tag{1.2.6}
\end{equation*}
$$

### 1.3 Large c expansion in Holography

A generic 2d CFT is not expected to satisfy the holographic duality. The conditions that the CFT is assumed to follow for this to be the case are having a large c and a sparseness constraint on the low-lying spectrum. The matching of the algebra of asymptotic symmetries of the AdS3 with the Virasoro symmetry of the boundary CFT, allows one to identify $c=3 G_{N} / 2 l$ where $G_{N}$ is the Gravitational Newtons constant and $l$ is the AdS radius. In this sense, performing a $1 / c$ expansion is equivalent to a semi-classical $1 / G_{N}$ expansion in the bulk.

In fact, more generally, for large-N boundary field theories, the correlation functions factorize. The quantum effects in the correlation functions are suppressed by powers of N , which leads one to identify that $1 / N$ controls the effective theory expansion.

Chapter 2 In Chapter 2, we employ quasi-classical quantization to calculate spectrum of quantum KdV charges in a large $c$ expansion. The classical KdV charges can be expressed in terms of action variables $I_{i}$ as a power series expansion. Quantummechanically this series becomes the expansion in $1 / c$, while action variables become boson occupation numbers $n_{i}$. Crucially, classical expression, which is homogeneous in $I_{i}$, acquires quantum corrections which include terms of subleading powers. We conjecture that these "quantum" terms can be unambiguously fixed from the analytic form of $Q_{2 k-1}$ acting on the highest weight (primary) states. We confirm this hypothesis and find explicit expressions for the spectrum of $Q_{2 k-1}$ up to first three orders in $1 / c$ expansion. This knowledge of the spectrum of $Q_{2 n-1}$ can help understand the qKdV generalized Gibbs ensemble (GGE) [26, 27, 28, 29] and corresponding partition function and free energy.

Chapter 3 In Chapter 3, we explore perturbative corrections to quantum information geometry. In particular, we study a Bures information metric naturally associated with the correlation functions of a CFT. We compute the metric of holographic four-point functions and include corrections generated by tree Witten diagrams in the bulk. In this setting, we translate properties of correlators into the language of information geometry. Cross terms in the information metric encode non-identity operators in the OPE. We find that the information metric is asymptotically AdS. We also discuss an information metric for transition amplitudes.

### 1.4 Modular invariance of 2d CFTs

The partition function of 2 d CFTs on a torus is given by

$$
\begin{equation*}
Z=\sum_{h, \bar{h} \in \text { states }} q^{h-\frac{c}{24} \bar{q}} \overline{\bar{h}-\frac{\bar{c}}{24}} \tag{1.4.1}
\end{equation*}
$$

Here, we assume that there is a unique vacuum state with $h=\bar{h}=0$. The partition function on a Euclidean torus of modulus $\tau$ equals $Z(\tau, \bar{\tau})$ restricted to $\bar{\tau}=\tau^{*}$. Modular invariance of the partition function is the following statement:

$$
Z\left(\frac{a \tau+b}{c \tau+d}, \frac{a \bar{\tau}+b}{c \bar{\tau}+d}\right)=Z(\tau, \bar{\tau}), \quad\left(\begin{array}{ll}
a & b  \tag{1.4.2}\\
c & d
\end{array}\right) \in S L(2, \mathbb{Z})
$$

The states are organized by the symmetry of the theory so admit a decomposition into characters

$$
\begin{equation*}
Z(\tau, \bar{\tau})=\chi_{v a c}(\tau, \bar{\tau})+\chi_{h, \bar{h}}(\tau, \bar{\tau}) \tag{1.4.3}
\end{equation*}
$$

$\chi_{h, \bar{h}}$ gives the contribution to the sum given by the primary field of dimension $h, \bar{h}$ and all it's decendants. For the theory to be unitary, the coefficients in the expansion of characters must be positive, where $c_{1}, c_{2}$ are also positive.

$$
\begin{equation*}
\chi_{h, \bar{h}}=q^{h-\frac{c}{24}} \bar{q}^{\bar{h}-\frac{\bar{c}}{24}}\left(1+\text { positive coefficients } q^{c_{1}} \bar{q}^{c_{2}}+\ldots\right) \tag{1.4.4}
\end{equation*}
$$

We will now see how modular symmetry leaves the partition function of compact periodic scalar fields invariant by considering the example of the conformal field theory of a single periodic scalar field $X$ which satisfies $X \sim X+2 \pi R$.

Since all states are left invariant by translations by $R$, the states must have momentum quantized by $k=\frac{n}{R}, \quad n \in \mathbb{Z}$ Compactness has the important consequence that the field may wind around the compact dimension

$$
\begin{equation*}
X(x+2 \pi)=X(x)+2 \pi R w, \quad w \in \mathbb{Z} \tag{1.4.5}
\end{equation*}
$$

States are labelled by their momentum eigenvalue $n$ and their winding number $w$ which define the spectra of these theories. The momentum of the holomorphic (left) and anti-holomorphic (right) components of the field (labelled here by $p_{L}, p_{R}$ ) are written in terms of the quantum numbers $n, w$ as follows:

$$
\begin{align*}
& p_{L}=\frac{n}{R}+\frac{w R}{2}  \tag{1.4.6}\\
& p_{R}=\frac{n}{R}-\frac{w R}{2} \tag{1.4.7}
\end{align*}
$$

The set of points $p_{L}, p_{R}$ define an even self-dual lattice in $\Lambda=R^{(1,1)}$. By a standard construction [30], the fourier expansion of these fields allows one to relate $p_{L}, p_{R}$ to the Virasoro generators by

$$
\begin{align*}
& L_{0}=\frac{p_{L}^{2}}{2}+\sum_{n=1} \alpha_{-n} \alpha_{n}  \tag{1.4.8}\\
& \bar{L}_{0}=\frac{p_{R}^{2}}{2}+\sum_{n=1} \bar{\alpha}_{-n} \bar{\alpha}_{n} . \tag{1.4.9}
\end{align*}
$$

The partition function for a single boson can now be written as:

$$
\begin{align*}
Z(\tau, \bar{\tau}) & =(q \bar{q})^{1 / 24} \operatorname{Tr}\left(q^{L_{0}} \bar{q}^{L_{0}}\right)  \tag{1.4.10}\\
& =|\eta(\tau)|^{-2} \sum_{n, w} q^{p_{L}^{2} / 2} \bar{q}^{p_{R}^{2} / 2} \tag{1.4.11}
\end{align*}
$$

In this form, it can be seen that the partition function arises by summing over all possible vectors in the lattice $\Lambda$. Infact, in chapter 4 and 5, we consider multiple compact scalar fields, whose momentum and winding modes define Euclidean or Narian lattices.

Chapter 4 In chapter 4, we derive explicit expressions for the higher genus partition functions of a specific class of CFTs: code CFTs, which are constructed using classical error-correcting codes. An error correcting code can be used to define a lattice, simply by identifying the vectors in the code with the unit cell of the lattice. A key element of the relationship between codes and CFTs is that the CFT partition function is given by the code enumerator polynomial $W_{\mathcal{C}}$, which counts the degeneracy of codewords much like the partition function counts the states of the theory. The relation between the enumerator and the torus partition function is

$$
\begin{equation*}
Z(\tau)=\frac{W_{\mathcal{C}}\left(\vartheta_{0}, \vartheta_{1}\right)}{\Phi(\tau)} \tag{1.4.12}
\end{equation*}
$$

where $\vartheta_{i}=\vartheta_{i}(\tau)$ are Jacobi theta functions. In this chapter, we show that the relationship 1.4 .12 extends to higher genus partition functions, which are computed by the "higher-weight" enumerator polynomials. In this setting, the $S p(2 g, \mathcal{Z})$ modular transformations of genus g Riemann surfaces can be recast as a simple set of linear maps acting on $2 g$ polynomial variables. The CFT partition function is directly related to the enumerator polynomial, meaning that solutions of the linear constraints from modular invariance immediately give a set of seemingly consistent partition functions at a given genus. By requiring that the enumerator polynomial has positive integer coefficients, we can list every partition function which could possibly derive from an error-correcting code. Finally, we impose a further constraint, called "factorization limits:" when the higher genus Riemann surface degenerates into two lower-genus Riemann surfaces connected by an infinitely thin, long tube, then the higher genus partition function must factorize into the products of lower-genus partition functions. The higher genus constraints, plus consistency under degeneration limits of the Riemann surface, greatly reduces the number of possible code CFTs.

Chapter 5 In Chapter 5, we explore the connection between quantum error correcting codes and a certain class of Narain CFTs at higher genus. We prove that the higher-genus partition functions take the form of polynomials of higherweight theta functions, and that the higher-genus modular group acts as simple linear transformations on these polynomials. We explain how to solve the modular constraints
explicitly, which we do for genus 2 . The result is that modular invariance at genus 1 and genus 2 is much more constraining than genus 1 alone. This allows us to drastically reduce the space of possible code CFTs. We also consider a number of examples of "isospectral theories" - CFTs with the same genus 1 partition function - and we find that they have different genus 2 partition functions. Finally, we make connection to some 2d CFTs known from the modular bootstrap. The $\mathrm{n}=4$ theory conjectured to have the largest possible gap, the $S O(8)$ WZW model, is a code CFT, allowing us to give an expression for its genus 2 partition function.

## Chapter 2 Spectrum of quantum KdV hierarchy in the semiclassical limit

This chapter is essentially identical to: Spectrum of quantum KdV hierarchy in the semiclassical limit [11]

### 2.1 Introduction: Quantum KdV symmetry

Conformal invariance in two dimensions is a very powerful tool which gives rise to many non-pertubative relations constraining dynamics of 2d CFTs. Among them is universality of stress-energy tensor sector [2], namely any correlation function which includes only stress-energy tensor and its descendants depends only on central charge $c$ but not on any other details of the theory. An analytic form of all such correlators can in principle be found in a recursive form [31]. The stress-energy sector can be regarded as integrable, even if the whole theory is understood to be chaotic [32]. This can be justified formally by noting there is an infinite number of mutually commuting quantum KdV charges [9, 33, 34] - local charges $Q_{2 n-1}$ of the form

$$
\begin{equation*}
Q_{2 n-1}=\frac{1}{2 \pi} \int_{0}^{2 \pi} T_{2 n}(\varphi) d \varphi \tag{2.1.1}
\end{equation*}
$$

where the densities $T_{2 n}$ are appropriately regularized polynomials in stress-energy tensor $T(\varphi)$ and its derivatives. First charge

$$
\begin{equation*}
Q_{1}=L_{0}-\frac{c}{24}=\frac{1}{2 \pi} \int_{0}^{2 \pi} T d \varphi \tag{2.1.2}
\end{equation*}
$$

is the CFT Hamiltonian. (Here and below we consider 2d CFT on a cylinder. Because of standard factorization into left and right-moving sectors we restrict the discussion to one sector only.) Interest in integrable structure of 2d CFT stress-energy sector has been reignited recently in the context of Eigenstate Thermalization Hypothesis (ETH) [25]. Following original works [35, 36, 37, 38, 39, 40, 41, 27, 28] it has been conjectured and confirmed in [42] that 2d CFTs exhibit generalized ETH with the local equilibrium being described by qKdV Generalized Gibbs Ensemble (GGE). Schematically the role of qKdV charges is as follows. The CFT Hamiltonian 2.1.2 is highly degenerate with all CFT descendant states of the form

$$
\begin{equation*}
|E\rangle=L_{-m_{1}} \ldots L_{-m_{k}}|\Delta\rangle, \quad \sum_{i=1}^{k} m_{i}=m \tag{2.1.3}
\end{equation*}
$$

sharing the same energy $E=\Delta+m-c / 24$. Since all $Q_{2 n-1}$ commute, they can be simultaneously diagonalized giving rise to mathematically unique "integrable" basis of eigenstates. Unlike the energy eigenstates of the form (2.1.3), which fail the ETH, integrable eigenstates carry specific values of $Q_{2 k-1}$-charges and obey generalized ETH. This novel role of qKdV symmetries motivates the question of "solving" integrable structure, i.e. evaluating spectrum of qKdV charges and finding integrable eigenstates, which would allow detailed studies of generalized ETH and qKdV GGE thermodynamics.

In certain sense the question of finding qKdV spectra can be regarded as solved: there is not one but two distinct ways to write an algebraic Bethe-ansatz reducing the problem of finding spectra to a bunch of algebraic equations 43, 44. In practice complexity of these equations grows very rapidly with the level $m$ 2.1.3), making this approach useless in the context of ETH, at least so far. The ETH holds in thermodynamic limit, it may not and does not hold beyond that regime. Thermodynamic limit assumes the length of the spatial circle $L$ goes to infinity, with the energy density $E / L$ kept fixed. Using rescaling, one can always bring the circle to unit radius, the notations we use throughout the chapter. The energy $E$ then must go to infinity as $L^{2}$ with $L \rightarrow \infty$ being an auxiliary parameter keeping track of corrections to various ETH-related identities. For any given primary state $|\Delta\rangle$ this essentially means the descendant level $m$ must be taken to infinity, i.e. we arrive exactly at the limit where algebraic Bethe equations become most difficult.

A progress was achieved by taking an additional limit of large central charge. In this case $Q_{2 k-1}$-eigenstates, akin to (2.1.3), can be parametrized by a set of natural numbers, which can be conveniently combined into a Young tableau [28]. ${ }^{1}$ It is most convenient to use representation when $n_{k} \geq 0$ for $k=1,2, \ldots$ counts the number of rows of length $k$,

$$
\begin{equation*}
\left|n_{i}\right\rangle \equiv\left|n_{1}, \ldots\right\rangle, \quad \sum_{k=1}^{\infty} k n_{k}=m . \tag{2.1.4}
\end{equation*}
$$

We emphasize (2.1.4) are eigenstates of $Q_{2 n-1}$ and thus differ from 2.1.3). Corre-

[^0]sponding eigenvalues at leading order were conjectured in $\sqrt{\left.29\right|^{2}}$
\[

$$
\begin{align*}
Q_{2 n-1}\left|n_{i}\right\rangle & =\mathrm{Q}_{2 n-1}\left|n_{i}\right\rangle,  \tag{2.1.5}\\
\mathrm{Q}_{2 n-1} & =\tilde{\Delta}^{n}+\sum_{p=0}^{n-1} \xi_{n}^{p} \tilde{\Delta}^{n-1-p} \tilde{c}^{p}\left(\sum_{k=1}^{\infty} k^{2 p+1} n_{k}+\frac{\zeta(-2 p-1)}{2}\right)+O\left(\tilde{c}^{n-2}\right), \\
\xi_{n}^{p} & =\frac{(2 n-1) \sqrt{\pi} \Gamma(n+1)}{2 \Gamma(p+3 / 2) \Gamma(n-p)}, \quad \tilde{\Delta}=\Delta-\tilde{c}, \quad \tilde{c}=\frac{c-1}{24} . \tag{2.1.6}
\end{align*}
$$
\]

Here we assume the scaling when $c \rightarrow \infty$ while $\tilde{\Delta} / \tilde{c}=h$ is kept fixed. No thermodynamic limit is assumed. This is the limit of holographic correspondence, when CFT is dual to semiclassical gravity. The holographic picture provides an easy derivation for the leading $1 / c$ terms in (2.1.6) and provides interpretation for $n_{k}$ as the boson occupation numbers of the boundary gravitons, see Appendix ??. From the mathematical point of view simplicity of eigenstates parametrization with help of Young tableaux as well as relatively simple form of 2.1.6 can be readily understood from the semiclassical quantization of the co-adjoint orbit of Virasoro algebra. Indeed, as is explained in 45 in the large $c$ limit Virasoro algebra can be understood in quasi-classical terms, as quantization of the Kirillov-Kostant-Souriau symplectic form. Because of $U(1)$ symmetry semiclassical quantization of $Q_{1}$ is exact,

$$
\begin{equation*}
Q_{1}=\tilde{\Delta}+\left(\sum_{k=1}^{\infty} k n_{k}-\frac{1}{24}\right)=\Delta+m-\frac{c}{24}, \tag{2.1.7}
\end{equation*}
$$

but for all higher $Q_{2 n-1}, n>1$ it is not. It is a perturbation series in $1 / \tilde{c}$, which plays the effective role of Planck constant. In this chapter we develop a perturbative scheme to obtain the spectrum of $Q_{2 n-1}$ as a series in $1 / \tilde{c}$ expansion and calculate first two non-trivial terms. The result is summarized in 2.4.15).

In the strict $c \rightarrow \infty$ limit when the problem becomes classical, CFT stress-tensor $T$ can be substituted by an element of the co-adjoint orbit of Virasoro algebra $\frac{24}{c} u$, where $u$ is a potential of an auxiliary periodic Schrödinger equation. Then quantum KdV charges (2.1.1) reduce to conventional KdV Hamiltonians of the periodic problem

$$
\begin{equation*}
Q_{2 n-1}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(u^{n}+\ldots\right) d \varphi \tag{2.1.8}
\end{equation*}
$$

which we denote the same as the quantum ones, as it clear from the constant which, classical or quantum version, we had in mind. For the states with large but finite level $m$ number of non-zero $n_{k}$ will also be finite. At the classical level this

[^1]corresponds to finite-zone potentials $u$, which form a finite-dimensional symplectic manifold equipped with the structure of a completely integrable system. Hamiltonians $Q_{2 n-1}$ can be re-expressed in terms of the action variables $I_{k}$ and the orbit invariant $h$,
\[

$$
\begin{equation*}
Q_{2 n-1}=h^{n}+\sum_{k=1}^{\infty} \sum_{j=0}^{n-1} \xi_{n}^{j} h^{n-1-j} k^{2 j+1} I_{k}+O\left(I^{2}\right) \tag{2.1.9}
\end{equation*}
$$

\]

which at semiclassical level become integral quantum numbers $I_{k} \rightarrow n_{k} / \tilde{c}_{\text {. }}$. It is then easy to see that (2.1.9) becomes 2.1.6), up to an overall factor $\tilde{c}^{n}$ and certain corrections. At each power of $1 / \tilde{c}$ classical expression $Q_{2 n-1}\left(h, I_{k}\right)$ predicts only leading power of $n_{k}$ while all subleading powers are "quantum corrections" which must be fixed separately.

At leading $1 / \tilde{c}$ order quantum correction is just $n_{k}$-independent constant term proportional to $\zeta(-2 p-1) / 2$, see (2.1.6). It can be fixed trivially by introducing Maslov index $I_{k} \rightarrow\left(n_{k}+1 / 2\right) / \tilde{c}$, such that constant term can be formally rewritten as the vacuum energy of "quantum oscillators" with frequencies $\omega_{k}$ and occupation numbers $n_{k}$

$$
\mathrm{Q}_{2 n-1}=\tilde{\Delta}^{n}+\sum_{k=1}^{\infty}\left(n_{k}+1 / 2\right) \omega_{k}+O\left(\tilde{c}^{n-2}\right), \quad \omega_{k}=\sum_{j=0}^{n-1} \xi_{n}^{j} \tilde{\Delta}^{n-1-j} \tilde{c}^{j} k^{2 p+1} \cdot(2.1 \cdot 10)
$$

Unfortunately this simple trick fails beyond the leading order in $1 / \tilde{c}$. At $1 / \tilde{c}^{2}$ order one has to fix both constant and linear in $n_{k}$ terms, while simple $I_{k} \rightarrow\left(n_{k}+1 / 2\right) / \tilde{c}$ substitution leads to incorrect results.

We propose and verify up to $1 / \tilde{c}^{2}$ order that the subleading "quantum correction" terms can be unambiguously fixed starting from the analytic expression in terms of $1 / \tilde{c}$ pertubative series of the eigenvalues $Q_{2 n-1}^{0}$ of $Q_{2 n-1}$ acting on the primary state $|\Delta\rangle$. For leading $1 / \tilde{c}$ term this statement is trivial - taking all $n_{k}=0$ yields the constant term, which is simply leading $1 / \tilde{c}$ term in $\mathrm{Q}_{2 n-1}^{0}$. At the $1 / \tilde{c}^{2}$ order this statement is more nuanced: naively $Q_{2 n-1}^{0}$ only fixes the constant term with all $n_{k}=0$, but we show linear in $n_{k}$ terms can be also fixed starting from $\mathrm{Q}_{2 n-1}^{0}$. As a result we obtain spectrum of $Q_{2 n-1}$ up to first three orders in $1 / c$ expansion, including the leading $\Delta^{n}$ term. We then apply the obtained result to evaluate thermal expectation values of $Q_{2 n-1}$, free energy of the KdV Generalized Gibbs Ensemble at first few leading orders in $1 / c$.

The chapter is organized as follows. In section 2.2 we discuss classic completely integrable system associated with the finite zone potentials and evaluate $Q_{2 n-1}\left(h, I_{k}\right)$
as a perturbative series in $I_{k}$. In section 2.3 we discuss analytic form of $Q_{2 k-1}$ acting on primary states. These two pieces are combined in section 2.4 where we employ semiclassical quantization to obtain the spectrum of qKdV charges in the first three orders of $1 / \tilde{c}$ expansion. We also perform consistency checks, confirming our result. Section 2.5 is devoted to applications of the obtained result. In section 2.5 we calculate thermal expectation values of $Q_{2 n-1}$ and fix two leading orders in $1 / c$ of the associated differential operator $\mathcal{D}_{n}$

$$
\begin{equation*}
\operatorname{Tr}_{\Delta}\left(Q_{2 k-1} q^{Q_{1}}\right)=\mathcal{D}_{n} \chi_{\Delta}, \quad \chi_{\Delta} \equiv \operatorname{Tr}_{\Delta}\left(q^{Q_{1}}\right) \tag{2.1.11}
\end{equation*}
$$

We conclude with a discussion in section 2.6. Appendix A.1 provides technical details concerning Novikov's one-zone potentials. Appendix A.2 develops the technique of dealing with the multi-zone potentials in the limit of the infinitesimally small zones. Finally, appendix A. 3 provides the details of calculating the spectrum of $Q_{2 n-1}$ acting on primary states based on ODE/IM correspondence.

### 2.2 Calculation of $Q_{2 n-1}\left(h, I_{k}\right)$

In this section our goal is to find expression for $Q_{2 n-1}$ in terms of the orbit invariant $h$ and action variables $I_{k}$, by expanding pertubatively up to cubic order in $I_{k}$,

$$
\begin{align*}
Q_{2 n-1}=h^{n}+ & \sum_{k} f_{k}^{(n, 1)} I_{k}+f_{k}^{(n, 2)} I_{k}^{2}+f_{k}^{(n, 3)} I_{k}^{3}+\sum_{k<\ell} f_{k, \ell}^{(n, 2)} I_{k} I_{\ell}+  \tag{2.2.1}\\
& \sum_{k \neq \ell} f_{k, \ell}^{(n, 3)} I_{k}^{2} I_{\ell}+\sum_{k<\ell<p} f_{k, \ell, p}^{(n, 3)} I_{k} I_{\ell} I_{p}+\mathcal{O}\left(I^{4}\right)
\end{align*}
$$

Coefficients $f$ are $h$-dependent. First three $f_{k}^{(n, 1)}, f_{k}^{(n, 2)}, f_{k}^{(n, 3)}$ will be found using one-zone potentials in section 2.2. Using two-zone potentials we will find $f_{k, \ell}^{(n, 2)}$ and $f_{k, \ell}^{(n, 3)}$ in section 2.2, while coefficient $f_{k, \ell, p}^{(n, 3)}$ will be fixed using three-zone potentials in section 2.2. An alternative brute-force derivation of (2.2.1) up to quadratic order in $I_{k}$ is given in the appendix ??.

## Finite zone potentials: an introduction

The starting point is the "Schrödinger" equation

$$
\begin{equation*}
-\psi^{\prime \prime}+\frac{u}{4} \psi=\lambda \psi \tag{2.2.2}
\end{equation*}
$$

with the periodic real-valued potential $u(\varphi+2 \pi)=u(\varphi)$. For any real $\lambda$ there are two linearly-independent quasi-periodic solutions

$$
\begin{equation*}
\psi_{ \pm}(\varphi+2 \pi)=e^{ \pm 2 \pi i p(\lambda)} \psi_{ \pm}(\varphi) \tag{2.2.3}
\end{equation*}
$$

Here quasi-momentum $p(\lambda)$ could be either real or pure imaginary. Values of $\lambda \in \mathbb{R}$ for which $p(\lambda)$ is imaginary are called "forbidden zone." At the end of forbidden zones $p(\lambda)$ is integer or half-integer such that $\psi_{ \pm}$become periodic or antiperiodic and linearly dependent. Normally, for such $\lambda$, another linearly independent singular solution appears. Yet occasionally there are two linearly independent regular periodic or antiperiodic solutions for the same $\lambda$. In this case forbidden zone degenerates and disappears, with $p(\lambda)$ being real everyone in the vicinity of that point. We provide examples below.

A general potential $u$ would have an infinite number of forbidden zones, but there are special classes when only a finite number of forbidden zones are non-degenerate, Such $u$ are called finite zone potentials. They were introduced in a famous work 47] and often refereed to as Novikov potentials.

## Example: zero zone potential

Let us consider a constant potential $u=4 \lambda_{0}=Q_{1}$ with some real $Q_{1}$. A solution to (2.2.2) can be readily found

$$
\begin{equation*}
\psi_{ \pm}(\varphi)=e^{ \pm i p(\lambda) \varphi}, \quad p(\lambda)=\sqrt{\lambda-\lambda_{0}} . \tag{2.2.4}
\end{equation*}
$$

For any $\lambda>Q_{1} / 4$ quasi-potential is real, i.e. there are no forbidden zones, except for $\lambda \in\left(-\infty ; Q_{1} / 4\right)$. The solutions 2.2.4 are linearly independent, including $\lambda=$ $\left(Q_{1}+k^{2}\right) / 4$ for natural $k$, when $\psi_{ \pm}$are (anti)periodic. Values $\lambda=\left(Q_{1}+k^{2}\right) / 4$ mark the ends of degenerate forbidden zones.

## Example: "opening" a zone

Let us now consider the potential $u=Q_{1}+\epsilon \cos (k \varphi)+O\left(\epsilon^{2}\right)$ where $Q_{1}$ is a constant, $k$ is positive integer, and $\epsilon$ is some infinitesimal parameter. Using quantum mechanics perturbation theory we find at leading order that all eigenvalues of periodic and anti-periodic problems remain the same and double-degenerate, except for $\lambda_{k}$ which splits into

$$
\begin{equation*}
\lambda_{k}^{ \pm}=\frac{Q_{1}+k^{2}}{4} \pm \frac{\epsilon}{2} \tag{2.2.5}
\end{equation*}
$$

Hence now there are two forbidden zones, $\left(-\infty, Q_{1} / 4\right)$ and $\left(\lambda_{k}^{-}, \lambda_{k}^{+}\right)$.
Finite-zone potentials are characterized by the ends of non-degenerate zones $\lambda_{i}$. For the zero-zone potential above there is only one parameter $\lambda_{0}=Q_{1} / 4$. After one
zone is opened, there are three parameters: "energy" of the ground state $\lambda_{0}, \lambda_{1}=\lambda_{k}^{-}$ and $\lambda_{2}=\lambda_{k}^{+}$. In general an $m$-zone potential is characterized by

$$
\begin{equation*}
\lambda_{0}<\lambda_{1}<\cdots<\lambda_{2 m} \tag{2.2.6}
\end{equation*}
$$

with the forbidden zones $\left(-\infty, \lambda_{0}\right)$ and $\left(\lambda_{2 i-1}, \lambda_{2 i}\right), i=\overline{1, m}$. For each set $\left\{\lambda_{i}\right\}$ we can define a hyperelliptic curve

$$
\begin{equation*}
y^{2}=\prod_{i=0}^{2 m}\left(\lambda-\lambda_{i}\right) \tag{2.2.7}
\end{equation*}
$$

while the quasi-momentum $p$ being fixed in terms of its differential

$$
\begin{equation*}
d p=\frac{\lambda^{m}+r_{m-1} \lambda^{n-1}+\ldots r_{0}}{2 y} d \lambda, \quad p\left(\lambda_{0}\right)=0 \tag{2.2.8}
\end{equation*}
$$

The latter is defined in such a way that the integrals of $d p$ over $a$-cycles vanish

$$
\begin{equation*}
\oint_{a_{i}} d p=2 \int_{\lambda_{2 i-1}}^{\lambda_{2 i}} d p=0 \tag{2.2.9}
\end{equation*}
$$

This fixes $m$ coefficients $r_{0}, \ldots, r_{m-1}$. Furthermore for the potential associated with $\left\{\lambda_{i}\right\}$ to be $2 \pi$-periodic we must additionally require integrals over $b$-cycles

$$
\begin{equation*}
w_{i}=\oint_{b_{i}} d p=2 \int_{\lambda_{2 i-2}}^{\lambda_{2 i-1}} d p \tag{2.2.10}
\end{equation*}
$$

to be integer-valued

$$
\begin{equation*}
w_{i}=k_{i}-k_{i-1} . \tag{2.2.11}
\end{equation*}
$$

Here natural $k_{i}$ satisfying $k_{i+1}>k_{i}, k_{0} \equiv 0$, label opened zones. These are additional $m$ constrains, which reduce the total number of independent parameters $\lambda_{i}$ to $m+1$.

A given set $\left\{\lambda_{i}\right\}$ which satisfies 2.2.9 2.2.11), such that only $m+1$ parameters are independent, defines periodic potential $u(\varphi)$, but in a non-unique way. Individual potentials are labeled by points of the Jacobian of curve 2.2.7, with all of them sharing the same spectrum. In other words isospectral potentials form an $m$-dimensional torus, while full space of $m$-zone potentials is therefore $2 m+1$ dimensional.

At this point we would like to make a connection with the Virasoro algebra. Consider Hill's equation, which is "Schrodinger" equation (2.2.2) with $\lambda=0$,

$$
\begin{equation*}
-\psi^{\prime \prime}+\frac{u}{4} \psi=0 \tag{2.2.12}
\end{equation*}
$$

One can re-parametrize the circle going from $\varphi$ to $\tilde{\varphi}(\varphi)$ such that $\tilde{\varphi}(\varphi+2 \pi)=$ $\tilde{\varphi}(\varphi)+2 \pi$. Then wave-function and the potential also change accordingly

$$
\begin{align*}
\tilde{\psi}(\tilde{\varphi}) & =\psi(\varphi)\left(\frac{d \tilde{\varphi}}{d \varphi}\right)^{-1 / 2}  \tag{2.2.13}\\
\tilde{u}(\tilde{\varphi}) & =\left(\frac{d \tilde{\varphi}}{d \varphi}\right)^{-2}(u+2(S \tilde{\varphi})(\varphi)) \tag{2.2.14}
\end{align*}
$$

where Schwarzian derivative

$$
\begin{equation*}
(S \theta)(\varphi) \equiv \frac{\theta^{\prime \prime \prime}}{\theta^{\prime}}-\frac{3}{2}\left(\frac{\theta^{\prime \prime}}{\theta^{\prime}}\right)^{2} . \tag{2.2.15}
\end{equation*}
$$

From 2.2.14 it is clear that $u$ is an element from the co-adjoint orbit of Virasoro algebra with the Schwarzian derivative term appearing because of central extension [45]. All potentials $u(\varphi)$ related by circle reparametrizations, i.e. belonging to the same co-adjoint orbit share the same invariant - quasi-momentum at zero,

$$
\begin{equation*}
\psi(2 \pi) / \psi(0)=e^{2 \pi i p(0)} \tag{2.2.16}
\end{equation*}
$$

which is evident from (2.2.13). In other words

$$
\begin{equation*}
-4 p(0)^{2}=h \tag{2.2.17}
\end{equation*}
$$

is the invariant of $u$ characterizing the orbit itself. By choosing an appropriate $\tilde{\varphi}$ the potential always $3^{3}$ can be brought to a constant form, in which case

$$
\begin{equation*}
\tilde{u}(\tilde{\varphi})=h . \tag{2.2.18}
\end{equation*}
$$

The co-adjoint orbit is a symplectic space equipped with the Kirillov-KostantSouriau bracket

$$
\begin{equation*}
\frac{c}{24}\left\{u(\varphi), u\left(\varphi^{\prime}\right)\right\}=-2 \pi \mathcal{D} \delta\left(\varphi-\varphi^{\prime}\right), \quad \mathcal{D}=\partial u+2 u \partial-2 \partial^{3} \tag{2.2.19}
\end{equation*}
$$

Here, using linearity of symplectic form we introduce a formal parameter $c$, which later will be identified with the CFT central charge. Any Hamiltonian flow defined by (2.2.19) leaves $h$ invariant.

There is an infinite tower of the so-called KdV Hamiltonians $Q_{2 k-1}$, which can be defined recursively with help fo Gelfand-Dikii polynomials $R_{n}$,

$$
\begin{align*}
& Q_{2 n-1}=\frac{1}{2 \pi} \int_{0}^{2 \pi} R_{n} d \varphi \quad \partial R_{n+1}=\frac{n+1}{2 n+1} \mathcal{D} R_{n},  \tag{2.2.20}\\
& R_{0}=1, \quad R_{1}=u, \quad R_{2}=u^{2}-\frac{4}{3} \partial^{2} u, \quad R_{3}=u^{3}-4 u \partial^{2} u-2(\partial u)^{2}+\frac{8}{5} \partial^{4} u, \ldots
\end{align*}
$$

[^2]Their Hamiltonian flows generate isospectral deformations of $u$

$$
\begin{equation*}
\delta u=\frac{c}{24}\left\{Q_{2 n-1}, u\right\}=(2 n-1) \partial R_{n}, \tag{2.2.21}
\end{equation*}
$$

while they all remain in involution $\left\{Q_{2 n-1}, Q_{2 \ell-1}\right\}=0$.
We now consider a space of all $m$-zone potentials sharing the same $h$. This is a $2 m$-dimensional subspace within the orbit parametrized by $h$, which we will denote as $\mathcal{F}_{m}(h)$. The pullback of the symplectic form on this space is non-degenerate, hence it is also a symplectic manifold equipped with the Poisson bracket. Isospectral flows leave this manifold invariant. Upon restricting to $\mathcal{F}_{m}(h)$, only first $n \mathrm{KdV}$ Hamiltonians remain algebraically independent. The flows they generate move $u$ along the Jacobian of (2.2.7), which is the Liouvillian torus of a completely integrable dynamical system defined by $Q_{2 n-1}, n \leq m$. In other words the geometry of $\mathcal{F}_{m}(h)$ is a $m$-dimensional torus parametrized by angle variables fibered above a base parametrized by $m$ variables $Q_{2 n-1}$. Alternatively, one can introduce $m$ action variables $I_{k}$ parameterizing the base and forming canonical conjugate pairs with angle variables.

In terms of $d p(2.2 .8)$ values of KdV charges are given by an expansion at infinity

$$
\begin{equation*}
Q_{2 n-1}=\frac{2 \Gamma(n+1) \Gamma(1 / 2)}{\Gamma(n+1 / 2)} \frac{4^{n}}{2 \pi i} \oint_{\infty} d p \lambda^{n-1 / 2}, \tag{2.2.22}
\end{equation*}
$$

while the action variables are

$$
\begin{equation*}
I_{k}=\frac{i}{\pi} \oint_{a_{k}} p \frac{d \lambda}{\lambda}=\frac{1}{i \pi} \oint_{a_{k}} d p \ln \lambda \tag{2.2.23}
\end{equation*}
$$

Functional dependence of $Q_{2 n-1}$ for $n>m$ on the first $m$ ones readily follows from (2.2.22) and the form of $d p$ 2.2.8).

Our task is conceptually trivial: we want to learn an explicit change of variables on the base of $\mathcal{F}_{m}(h)$ from $Q_{2 n-1}$ to $I_{k}$. The expressions for $Q_{2 n-1}\left(h, I_{k}\right)$ is not available in the closed form, we therefore will find first few orders by expanding it in powers of $I_{k}$. There is one notable exception, using Riemann bilinear relation with two one-forms $d p$ and $p d \lambda / \lambda$ one can show in full generality

$$
\begin{equation*}
Q_{1}=h+\sum_{k} k I_{k} . \tag{2.2.24}
\end{equation*}
$$

Our main approach will be based on parameterizing both $Q_{2 n-1}$ and $I_{k}$ in terms of the spectral curve $i=\overline{0, m}$, with the infinitesimal $\lambda_{2 i}-\lambda_{2 i-1}$, and then reexpressing $Q_{2 n-1}$ in terms of $I_{k}$. There is an alternative straightforward approach,
to parametrize the potential $u(\varphi)$ in terms of its Fourier modes $u_{\ell}$, and then express both $Q_{2 n-1}$ and $I_{k}$ in terms of $u_{\ell}$. We develop this method in the appendix ?? and confirm the expansion (2.2.1) up to second order in $I_{k}$.

## One-zone potentials

Before we consider one-zone potential in detail, we revisit the zero-zone potential $u=Q_{1} \equiv 4 \lambda_{0}$ and readily find differential

$$
\begin{equation*}
d p=\frac{d \lambda}{2 \sqrt{\lambda-\lambda_{0}}} \tag{2.2.25}
\end{equation*}
$$

to be defined on a Riemann sphere. This is the simplest possible case. In this case $p=\sqrt{\lambda-\lambda_{0}}, u(\varphi)=h=4 \lambda_{0}$ and the whole symplectic space $\mathcal{F}_{0}(h)$ shrinks to a point. All KdV Hamitonians are fixed by $h, Q_{2 n-1}=h^{n}$ with all action variables identically equal to zero.

Next, we consider the differential

$$
\begin{equation*}
d p=\frac{(\lambda-r) d \lambda}{2 \sqrt{\left(\lambda-\lambda_{0}\right)\left(\lambda-\lambda_{1}\right)\left(\lambda-\lambda_{2}\right)}} \tag{2.2.26}
\end{equation*}
$$

parameterized by $\lambda_{i}, r_{0}$. It is defined on a torus - a Riemann curve of genus one. We assume that $\left(\lambda_{2}, \lambda_{1}\right)$ correspond to $k$-th zone. After satisfying (2.2.9) and (2.2.11), which requires evaluating elliptic integrals, we find one-parametric family

$$
\begin{equation*}
\lambda_{2}=\lambda_{0}+\frac{k^{2}}{4} \theta_{3}(\tau)^{4}, \quad \lambda_{1}=\lambda_{0}+\frac{k^{2}}{4} \theta_{4}(\tau)^{4}, \quad r=\lambda_{0}+\frac{k^{2}}{4} \theta_{4}(\tau)^{4}\left(1+2 \frac{\partial \ln \theta_{3}^{2}(\tau)}{\partial \ln m}\right) \tag{2.2.27}
\end{equation*}
$$

where $m=\theta_{2}^{4}(\tau) / \theta_{3}^{4}(\tau)$ and $\tau=i \tau_{2}$ with positive $\tau_{2}$. In what follows we usf ${ }_{4}^{4} q=e^{i \pi \tau}$ such that $\theta_{2}=\sum_{n} q^{(n+1 / 2)^{2}}, \theta_{3}=\sum_{n} q^{n^{2}}, \theta_{4}=\sum_{n}(-1)^{n} q^{n^{2}}$.

To impose the orbit constraint $-4 p(0)^{2}=h$ it is more convenient to use the following trick. First we evaluate

$$
\begin{equation*}
Q_{1}=4\left(\lambda_{0}+\lambda_{1}+\lambda_{2}\right)-8 r, \tag{2.2.28}
\end{equation*}
$$

which expresses $\lambda_{0}$ in terms of $Q_{1}$ and $q$ expansion,

$$
\begin{equation*}
4 \lambda_{0}=Q_{1}-k^{2}\left(\theta_{2}^{4}-4 \theta_{4}^{4} \frac{\partial \ln \theta_{3}^{2}(\tau)}{\partial \ln m}\right)=Q_{1}-32 k^{2} q^{2}\left(1+2 q^{2}+4 q^{4}+4 q^{6}+\ldots\right) \tag{2.2.29}
\end{equation*}
$$

[^3]and then use 2.2 .23 to evaluate action variable perturbatively in $q$,
\[

$$
\begin{align*}
I_{k} & =\frac{2}{\pi} \int_{\lambda_{1}}^{\lambda_{2}} \frac{d \lambda(\lambda-r) \log \lambda}{\sqrt{\left(\lambda-\lambda_{0}\right)\left(\lambda-\lambda_{1}\right)\left(\lambda_{2}-\lambda\right)}}  \tag{2.2.30}\\
& =\sum_{n=1}^{\infty} \frac{2(-1)^{n}\left(\lambda_{1}-\lambda_{0}\right)^{n+1}}{n \sqrt{\lambda_{2}-\lambda_{0}} \lambda_{0}^{n}}\left[\frac{F\left(\frac{3}{2}, \frac{1}{2}, 1 ; m\right)}{F\left(\frac{1}{2}, \frac{1}{2}, 1 ; m\right)} F\left(n+\frac{1}{2}, \frac{1}{2}, 1 ; m\right)-F\left(n+\frac{3}{2}, \frac{1}{2}, 1 ; m\right)\right] . \tag{2.2.31}
\end{align*}
$$
\]

Here $F \equiv{ }_{2} F_{1}$ is the hypergeometric function such that $F\left(\frac{3}{2}, \frac{1}{2}, 1 ; m\right)=\theta_{3}^{2}$.
An infinite sum over $n$ above has to be evaluated individually for each term in $q$ expansion. This gives $I_{k}$ as a function of $\lambda_{0}$ and $q, I_{k}=\frac{32 k^{3} q^{2}}{k^{2}+4 \lambda_{0}}+O\left(q^{4}\right)$, which with help of 2.2.29 can be expressed as a function of $Q_{1}$ and $q$,

$$
\begin{align*}
I_{k}= & \frac{32 k^{3}}{k^{2}+Q_{1}} q^{2}+\frac{64 k^{3}\left(17 k^{4}+12 k^{2} Q_{1}+3 Q_{1}^{2}\right)}{\left(k^{2}+Q_{1}\right)^{3}} q^{4}  \tag{2.2.32}\\
& +\frac{128 k^{3}\left(5 k^{2}+Q_{1}\right)\left(77 k^{6}+69 k^{4} Q_{1}+27 k^{2} Q_{1}^{2}+3 Q_{1}^{3}\right)}{\left(k^{2}+Q_{1}\right)^{5}} q^{6}+\mathcal{O}\left(q^{8}\right)
\end{align*}
$$

At this point we use 2.2 .24 , which is exact, $Q_{1}=h+k I_{k}$. Using $I_{k}$ given as a $q$-series expansion with $Q_{1}$-dependent coefficients 2.2 .32 , with help of 2.2 we express $Q_{1}$ as a series in $q$ with $h$-dependent coefficients by iteratively substituting $Q_{1}$ written as an $h$-dependent series in $q$. Once we find $Q_{1}=Q_{1}(h, q), I_{k}$ can be deduced from (2.2),

$$
\begin{align*}
I_{k}= & \frac{32 k^{3}}{h+k^{2}} q^{2}+\frac{64\left(3 h^{2} k^{3}+12 h k^{5}+k^{7}\right)}{\left(h+k^{2}\right)^{3}} q^{4}  \tag{2.2.33}\\
& +\frac{128 k^{3}\left(3 h^{4}+42 h^{3} k^{2}+108 h^{2} k^{4}-58 h k^{6}+k^{8}\right)}{\left(h+k^{2}\right)^{5}} q^{6}+\mathcal{O}\left(q^{8}\right) \tag{2.2.34}
\end{align*}
$$

At this point it is straightforward to re-express $q$ as a $h$-dependent power series in $I_{k}, q^{2}=\frac{h+k^{2}}{32 k^{3}} I_{k}+O\left(I_{k}^{2}\right)$.

To obtain coefficients $f^{(n, i)}$ (2.2.1) we act as follows. From the definition 2.2.22) we can find $Q_{2 n-1}$ as a polynomial in $\lambda_{i}$ and $r$. Using expressions for $\lambda_{i}, r$ 2.2.27 and 2.2.29, where $Q_{1}$ is understood as a function of $h, q$ we write $Q_{2 n-1}$ as an $h$-dependent power series in $q$. After that it is straightforward to use $q^{2}=q^{2}\left(h, I_{k}\right)$ to re-express $Q_{2 n-1}$ as an $h$-dependent power series in $I_{k}$,

$$
\begin{equation*}
Q_{2 n-1}=h^{n}+f_{k}^{(n, 1)} I_{k}+f_{k}^{(n, 2)} I_{k}^{2}+f_{k}^{(n, 3)} I_{k}^{2}+\mathcal{O}\left(I_{k}^{3}\right), \tag{2.2.35}
\end{equation*}
$$

thus fixing $f^{(n, i)}$,

$$
\begin{align*}
f_{k}^{(n, 1)}= & \sum_{j=0}^{n-1} \frac{\sqrt{\pi}(2 n-1) \Gamma(n+1)}{2 \Gamma\left(j+\frac{3}{2}\right) \Gamma(n-j)} h^{n-1-j} k^{2 j+1}=\sum_{j=0}^{n-1} \xi_{n}^{j} h^{n-1-j} k^{2 j+1}  \tag{2.2.36}\\
f_{k}^{(n, 2)}= & \sum_{j=0}^{n-1} \frac{\sqrt{\pi}(2 n-1) \Gamma(n+1)(j(2 n+1)-2 n+2)}{16 \Gamma\left(j+\frac{3}{2}\right) \Gamma(n-j)} h^{n-1-j} k^{2 j}  \tag{2.2.37}\\
f_{k}^{(n, 3)}= & \frac{(2 n-1) n(n-1)}{64 k^{3}} h^{n}+  \tag{2.2.38}\\
& \sum_{j=0}^{n-1} \frac{\sqrt{\pi}(2 n-1) \Gamma(n+1) \mathrm{p}}{1536 \Gamma\left(j+\frac{5}{2}\right) \Gamma(n-j)} h^{n-1-j} k^{2 j-1}, \\
\mathrm{p}= & 4 j^{3}(2 n+1)(2 n+3)-2 j^{2}(2 n+1)(10 n-21)-3 j(2 n+3)(10 n-7) \\
& +36(n-1)(2 n-1) . \tag{2.2.39}
\end{align*}
$$

More technical details concering one-zone calculation can be found in appendix A.1.

## Two-zone potentials

In case of two zones the differential

$$
\begin{equation*}
d p=\frac{\left(\lambda-r_{1}\right)\left(\lambda-r_{2}\right) d \lambda}{2 \sqrt{\prod_{i=0}^{4}\left(\lambda-\lambda_{i}\right)}} \tag{2.2.40}
\end{equation*}
$$

depends on seven parameters subject to 4 constraints 2.2.9 and 2.2.11. Corresponding integrals can not be evaluated analytically. We therefore proceed by expanding perturbatively, assuming both zones, and hence corresponding action variables, are small. We introduce two infinitesimal variables $\epsilon_{1}, \epsilon_{2}$ of the same order, such that $\lambda_{2}-\lambda_{1}$ is of order $\epsilon_{1}$ and $\lambda_{4}-\lambda_{3}$ is of order $\epsilon_{2}$. Action variables are quadratic in $\epsilon_{i}, I_{k} \sim \epsilon_{1}^{2}, I_{\ell} \sim \epsilon_{2}^{2}$, where we assumed ( $\lambda_{1}, \lambda_{2}$ ) and ( $\lambda_{3}, \lambda_{4}$ ) correspond to $k$-th and $\ell$-zones respectively. Our goal is to find $Q_{2 n-1}$ up to third order in the pertubative expansion in $I_{k}, I_{\ell}$. Hence in what follows we must expand all quantities in $\epsilon_{i}$ up to sixth order. The details of this calculation can be found in in Appendix A.2.

After satisfying 2.2.9 and 2.2.11 we find $\lambda_{i}$ for $i \geq 1$ and $r_{i}$ in terms of $\lambda_{0}$ and $\epsilon_{1}, \epsilon_{2}$, as a perturbative expansion in $\epsilon_{i}$. Then, we evaluate $I_{k}, h$ and $Q_{2 n-1}$ also as function of $\lambda_{0}$ and $\epsilon_{1}, \epsilon_{2}$, similarly expanding in $\epsilon_{i}$ up to and including sixth order.

By matching both sides of (2.2.1) we find coefficients $f_{k, \ell}^{(m, n)}$, yielding

$$
\begin{align*}
f_{k, \ell}^{(n, 2)}= & \sum_{j=1}^{n-1} \frac{\sqrt{\pi}(2 n-1)^{2} \Gamma(n+1)}{4 \Gamma(n-j) \Gamma\left(j+\frac{3}{2}\right)} h^{n-1-j} \sum_{s=0}^{j-1} k^{2(j-s)-1} \ell^{2 s+1},  \tag{2.2.41}\\
f_{k, \ell}^{(n, 3)}= & \frac{\ell}{\left(k^{2}-\ell^{2}\right)^{2}}\left(-\frac{(2 n-1) n(n-1)}{4} h^{n}+\sum_{j=0}^{n-1} \frac{\sqrt{\pi}(2 n-1)^{2} \Gamma(n+1)}{64 \Gamma(n-j) \Gamma\left(j+\frac{5}{2}\right)} h^{n-}\left(2^{j} .24\right) 4 .\right) \\
\mathbf{q}= & -4(2 n+1) \frac{k^{2 j+4}-\ell^{2 j+4}}{k^{2}-\ell^{2}}+k^{2 j+2}(3+2 j)(j(2 n+1)-4 n+5) \\
& +\ell^{2 j+2} 2(3 j+n+5)+k^{2 j} \ell^{2}(3+2 j)(j(2 n+1)-2 n+2) \\
& +k^{2} \ell^{2 j}(3+2 j)(4 n-1) . \tag{2.2.43}
\end{align*}
$$

## Three-zone potentials

Extending calculations of the previous section using the technique of appendix A. 2 to the three-zone case we can fix

$$
\begin{equation*}
f_{k, \ell, p}^{(n, 3)}=\sum_{j=0}^{n-3} \frac{\sqrt{\pi}(2 n-1)^{3} \Gamma(n+1)(n-2-j)}{8 \Gamma(n-1-j) \Gamma\left(j+\frac{7}{2}\right)} h^{n-3-j} \sum_{s_{1}=0}^{j} \sum_{s_{2}=0}^{j-s_{1}} k^{2 j+1-2\left(s_{1}+s_{2}\right)} \ell^{2 s_{1}+1} p^{2 s_{2}+1} . \tag{2.2.44}
\end{equation*}
$$

## Consistency check

In case of an $m$-zone potential we can parametrize the differential $d p$ with help of $\lambda_{0}$ and $\epsilon_{i}, 1 \leq i \leq m$, cf. A.2.1 A.2.6),

$$
\begin{align*}
\lambda_{i} & =\lambda_{0}+\ldots, & & 1 \leq i \leq 2 m  \tag{2.2.45}\\
r_{i} & =\lambda_{0}+\ldots, & & 1 \leq i \leq m \tag{2.2.46}
\end{align*}
$$

where dots stand for $\epsilon_{i}$ but not $\lambda_{0}$-dependent terms. Similarly action variables $I_{k}$, charges $Q_{2 n-1}$ and the orbit parameter $h=-4 p(0)^{2}$ will be some functions of $\lambda_{0}$ and $\epsilon_{i}$. While dependence of $I_{k}$ and $h$ on $\lambda_{0}$ is non-trivial, since $Q_{2 n-1}$ are the coefficients of $1 / \lambda$ expansion of $p(\lambda)$ at infinity and $\lambda_{0}$ is simply the shift of the argument of $p(\lambda)$, we find

$$
\begin{equation*}
Q_{2 n-1}=\sum_{k=0}^{n} \frac{\Gamma(n+1)}{\Gamma(k+1) \Gamma(n-k+1)}\left(4 \lambda_{0}\right)^{n-k} Q_{2 k-1}^{0} \tag{2.2.47}
\end{equation*}
$$

Here $Q_{2 k-1}^{0}$ are the charges evaluated with help of 2.2.22 taking $\lambda_{0}=0$ in 2.2.45 2.2.46). Assuming we know $Q_{2 n-1}\left(h, I_{k}\right)$ where $h=h\left(\lambda_{0}, \epsilon_{i}\right)$ and $I_{k}=$ $I_{k}\left(\lambda_{0}, \epsilon_{i}\right)$, one can introduce $I_{k}^{0}=I_{k}\left(0, \epsilon_{i}\right)$ such that $Q_{2 k-1}^{0}=Q_{2 k-1}\left(0, I_{k}^{0}\right)$. Here first
argument is zero simply because $h\left(0, \epsilon_{i}\right)=0$. Then both sides of equation 2.2.47) become functions of $\lambda_{0}$ and $\epsilon_{i}$, providing a non-trivial check.

There is an alternative way to use 2.2 .47 to check the consistency of the perturbative expansion (2.2.1 with the coefficients found in the text. We can invert $h=h\left(\lambda_{0}, \epsilon_{i}\right)$ and $I_{k}=I_{k}\left(\lambda_{0}, \epsilon_{i}\right)$ to express both $\lambda_{0}$ and $I_{k}^{0}$ via $h$ and $I_{k}$,

$$
\begin{aligned}
\lambda_{0}= & h+\sum_{k}-\frac{h I_{k}}{k}+\frac{h\left(h+5 k^{2}\right) I_{k}^{2}}{8 k^{4}}-\frac{h\left(5 h^{2}+30 h k^{2}+41 k^{4}\right) I_{k}^{3}}{128 k^{7}}+\sum_{k<\ell} \frac{h I_{k} I_{\ell}}{k \ell} \\
& +\sum_{k \neq \ell} \frac{h I_{k}^{2} I_{\ell}\left(h^{2} \ell^{2}-h\left(k^{4}-4 k^{2} \ell^{2}+\ell^{4}\right)-5 k^{6}+11 k^{4} l^{2}-5 k^{2} \ell^{4}\right)}{8 k^{4} \ell(k-\ell)^{2}(k+\ell)^{2}}-\sum_{k<\ell<p} \frac{h I_{k} I_{\ell} I_{p}}{k \ell p} \\
& +\mathcal{O}\left(I^{4}\right), \\
I_{k}^{0}= & I_{k}+\frac{h I_{k}}{k^{2}}-\frac{h I_{k}^{2}\left(h+5 k^{2}\right)}{8 k^{5}}+\frac{h I_{k}^{3}\left(5 h^{2}+30 h k^{2}+41 k^{4}\right)}{128 k^{8}}-\sum_{\ell \neq k} \frac{h I_{k} I_{\ell}\left(h+k^{2}\right)}{k^{2} \ell\left(k^{2}-\ell^{2}\right)} \\
& +\frac{1}{8 k^{5} l\left(k^{2}-\ell^{2}\right)^{3}} \sum_{\ell \neq k}\left[h I _ { k } ^ { 2 } I _ { \ell } \left(2 h^{2}\left(-k^{4}+2 k^{2} \ell^{2}+\ell^{4}\right)+h k^{2}\left(7 k^{4}-14 k^{2} \ell^{2}+15 \ell^{4}\right)\right.\right. \\
& \left.\left.+k^{4}\left(5 k^{4}-10 k^{2} \ell^{2}+9 \ell^{4}\right)\right)\right]+\frac{1}{8 k^{2} \ell^{4}\left(k^{2}-\ell^{2}\right)^{3}} \sum_{\ell \neq k}\left[h I _ { k } I _ { \ell } ^ { 2 } \left(h^{2}\left(k^{4}+5 k^{2} \ell^{2}-2 \ell^{4}\right)\right.\right. \\
& \left.\left.+h\left(k^{6}+10 k^{4} \ell^{2}-9 k^{2} \ell^{4}+6 l^{6}\right)+k^{2} \ell^{2}\left(5 k^{4}-7 k^{2} \ell^{2}+6 l^{4}\right)\right)\right] \\
& +\sum_{p \neq \ell \neq k} \frac{h I_{k} I_{\ell} I_{p}\left(2 h^{2}+3 h k^{2}+k^{4}\right)}{k^{2} \ell p\left(k^{2}-\ell^{2}\right)\left(k^{2}-p^{2}\right)}+\mathcal{O}\left(I^{4}\right) .
\end{aligned}
$$

Now $Q_{2 n-1}^{0}\left(0, I_{k}^{0}\left(h, I_{k}\right)\right)$ is a function of $h, I_{k}$ and 2.2 .47 provides a non-trivial check for the coefficients in (2.2.1).

This check also ensures that $Q_{2 n-1}\left(h, I_{k}\right)$ satisfy another identity

$$
\begin{equation*}
\frac{1}{n+1} \frac{\partial Q_{2 n+1}}{\partial u_{0}}=Q_{2 n-1} \tag{2.2.48}
\end{equation*}
$$

which follows from the properties of Gelfand-Dikii polynomials (2.2.20). Here $Q_{2 n-1}[u(\varphi)]$ are understood as functionals of $u(\varphi)$ and the derivative is with respect the zero Fourier mode of $u(\varphi)$, while all other Fourier modes are kept fixed. The shift of $u_{0}$ with all other modes intact is equivalent to a shift of the spectrum by a constant, hence

$$
\begin{equation*}
\left(\frac{\partial}{\partial u_{0}}\right)_{u_{\ell}}=4\left(\frac{\partial}{\partial \lambda_{0}}\right)_{\epsilon_{i}} . \tag{2.2.49}
\end{equation*}
$$

Then (2.2.48) follows immediately from the right-hand-side of (2.2.47).
For an $m$-zone potential, all higher KdV charges $Q_{2 n-1}$ are some functions of first $m+1$ charges. Thus for one-zone potentials $Q_{5}, Q_{7}, \ldots$ are functions of $Q_{1}, Q_{3}$, see
e.g. section 2.4 of [10] for details. For the three-zone potentials higher $Q_{2 n-1}$ would depend on $Q_{1}, Q_{3}, Q_{5}, Q_{7}$ In principle this provides additional consistency check for 2.2.1). In practice the dependence is so complicated, it doesn't provide a useful check even for the one-zone case.

## 2.3 "Energies" of primary states via ODE/IM correspondence

In the previous section we found classical expression for $Q_{2 n-1}$ in term of action variables $I_{k}$ and the orbit invariant $h$. Following the standard rules of semiclassical quantization $I_{k}$ should be promoted to an integer quantum number, while $h$ will become the dimension of the highest weight (primary) state $\Delta$, marking representation of the Virasoro algebra. It is easy to see, this naive receipt fails already for the values of $Q_{2 n-1}$ on a primary state $|\Delta\rangle$. Indeed, taking all $I_{k}$ to zero, we readily find $Q_{2 n-1}=h^{k}$, which upon the naive quantization yields $\mathrm{Q}_{2 n-1}^{0}=\Delta^{n}$ where

$$
\begin{equation*}
Q_{2 n-1}|\Delta\rangle=\mathrm{Q}_{2 n-1}^{0}|\Delta\rangle \tag{2.3.1}
\end{equation*}
$$

This answer is missing $c$-dependent terms. Explicit values of $\mathrm{Q}_{2 n-1}^{0}$ for $n \leq 8$ were calculated in 48 via brute-force approach, using explicit expressions for $Q_{2 n-1}$ in terms of free field representation. The pattern is clear, while $\Delta^{n}$ is indeed the leading term, full expression is a polynomial in both $\Delta$ and $c$ of order $n$.

There is no known receipt to obtain exact $Q_{2 n-1}^{0}$ from the semiclassical quantization, hence our strategy will be the following. We will combine exact expression for $Q_{2 n-1}^{0}$ in the large $c$ limit, which will be obtained in this section by a different method, with the classical result of section 2.2 , to find spectrum of excited states in the large $c$ limit in next section.

To find $\mathrm{Q}_{2 n-1}^{0}$ we use ODE/IM correspondence, initiated in [49, 50 and more recently developed in [51] (also see [52]), which relates qKdV spectrum to solutions of an auxiliary Schrödinger equation

$$
\begin{equation*}
\partial_{x}^{2} \Psi(x)+\left(E-x^{2 \alpha}-\frac{l(l+1)}{x^{2}}\right) \Psi(x)=0 \tag{2.3.2}
\end{equation*}
$$

where

$$
\begin{equation*}
(l+1 / 2)^{2}=4(\alpha+1) \tilde{\Delta}, \quad \tilde{c}=-\frac{\alpha^{2}}{4(\alpha+1)} \tag{2.3.3}
\end{equation*}
$$

Equation 2.3.2 can be solved using WKB approximation by systematically expanding in a small parameter. This leads to a quadratic ODE which can be solved
iteratively. We delegate all details to Appendix A.3 and only write down iterative relation which defines coefficients $c_{k}^{(n)}$ for $n \geq 1, n \geq k \geq 0$,

$$
\begin{equation*}
\sum_{j=0}^{n} \sum_{p=0}^{j} \sum_{q=0}^{n-j} \delta_{p+q, k} c_{p}^{(j)} c_{q}^{(n-j)}-2\left[n-k-u-\frac{n-2}{2 \alpha}\right] c_{k-1}^{(n-1)}+(2 k-3 n+4) c_{k}^{(n-1)}=0 \tag{2.3.4}
\end{equation*}
$$

and we formally assumed $c_{-1}^{(n)}=c_{n+1}^{(n)}=0, u^{2}=-\tilde{\Delta} / \tilde{c}$, and the starting values are

$$
\begin{equation*}
c_{0}^{(0)}=-\frac{1}{\alpha}, \quad c_{0}^{(1)}=-\frac{1}{2}, \quad c_{1}^{(1)}=\frac{1}{2 \alpha}-u . \tag{2.3.5}
\end{equation*}
$$

Coefficients $c_{k}^{(n)}$ determine values of $Q_{2 n-1}$ acting on primaries 51,

$$
\begin{equation*}
\mathrm{Q}_{2 n-1}^{0}=\frac{(2 n-1) \Gamma(n+1)}{\sqrt{\pi} \Gamma\left(1-\frac{2 n-1}{2 \alpha}\right) 4^{n}(\alpha+1)^{n}} \sum_{k=0}^{2 n} c_{k}^{(2 n)} \Gamma\left(k+\frac{3}{2}-3 n\right) \Gamma\left(2 n-k-\frac{2 n-1}{2 \alpha}\right) . \tag{2.3.6}
\end{equation*}
$$

Although this is not obvious, $\mathrm{Q}_{2 n-1}^{0}$ given by (2.3.6) is a polynomial in terms of $\tilde{\Delta}$ and $\tilde{c}$. After some algebra we find leading order expansion

$$
\begin{equation*}
\mathrm{Q}_{2 n-1}^{0}=\tilde{\Delta}^{n}+\sum_{j=0}^{n-1} \tilde{R}_{n, j}^{(1)} \tilde{\Delta}^{n-j-1} \tilde{c}^{j}+\sum_{j=0}^{n-2} \tilde{R}_{n, j}^{(2)} \tilde{\Delta}^{n-j-2} \tilde{c}^{j}+\sum_{j=0}^{n-3} \tilde{R}_{n, j}^{(3)} \tilde{\Delta}^{n-j-3} \tilde{c}^{j}+\mathcal{O}\left(\tilde{c}^{n-3}\right) \tag{2.3.7}
\end{equation*}
$$

where

$$
\begin{align*}
& \tilde{R}_{n, j}^{(1)}=\frac{(2 n-1) \sqrt{\pi} \Gamma(n+1)}{4 \Gamma\left(j+\frac{3}{2}\right) \Gamma(n-j)} \zeta(-2 j-1)=\xi_{n}^{j} \frac{\zeta(-2 j-1)}{2},  \tag{2.3.8}\\
& \tilde{R}_{n, j}^{(2)}=\frac{(2 n-1) \sqrt{\pi} \Gamma(n+1)}{24 \times 4 \Gamma\left(j+\frac{5}{2}\right) \Gamma(n-j-1)} \times  \tag{2.3.9}\\
&\left\{-6 \zeta(-2 j-3)\left(2 j+3-(2 n-1) y_{1}(j+1)\right)+3(2 n-1) \zeta_{2}(j)\right\}, \\
& \tilde{R}_{n, j}^{(3)}=\frac{(2 n-1) \sqrt{\pi} \Gamma(n+1)}{24^{2} \times 4 \Gamma\left(j+\frac{7}{2}\right) \Gamma(n-j-2)}\left\{6^{2} \zeta(-2 j-5)\left(2 j^{2}+7 j+5\right)-(2 n-1) r_{n, j}\right\}, \\
& r_{n, j}=12 \zeta_{3}(j)+36 \zeta_{2}(j+1)\left(y_{1}(j+2)+j+2\right)+3\left(4 j^{2}+18 j+23\right) \zeta(-2 j-3)+ \\
& \quad 36 \zeta(-2 j-5)\left(y_{1}^{2}(j+2)+2(j+2) y_{1}(j+2)+y_{2}(j+2)\right)+(2 n+1) p_{j} . \tag{2.3.10}
\end{align*}
$$

Functions $\zeta_{2}, \zeta_{3}, y_{1}, y_{3}$ are defined in the Appendix A.3, where we also give values of $p_{j}$ for $0 \leq j \leq 17$.

### 2.4 Spectrum of quantum $Q_{2 k-1}$

At this point we are ready to combine classical pertubative expression for $Q_{2 n-1}\left(h, I_{k}\right)$ 2.2.1 with the "energies" of primary state (2.3.7) to obtain $Q_{2 n-1}$ up to first two non-trivial orders in $1 / \tilde{c}$ expansion.

The naive semi-classical quantization would map the co-adjoint orbit invariant $h$ and the actions variables $I_{k}$ on the classical side to dimension of the primary state $\Delta$ and the excited state quantum numbers $n_{k}$ correspondingly,

$$
\begin{equation*}
h \rightarrow \frac{24 \Delta}{c}, \quad I_{k} \rightarrow \frac{24 n_{k}}{c} . \tag{2.4.1}
\end{equation*}
$$

Also classical charge $Q_{2 n-1}$ should be rescaled by $(c / 24)^{n}$. Starting from (2.2.1) this correctly reproduces full quantum spectrum of $Q_{1}$ and the leading $\Delta^{n}$ term in $Q_{2 n-1}$. But it falls short of reproducing sub-leading terms even for the primary state (2.3.7). The relation between classical and quantum quantities 2.4 .1 is only correct at the leading $c$ order. In 29$]$ we observed that using $c-1$ as an expansion parameter leads to more elegant expressions. This is confirmed by (2.3.7), which looks most naturally if written in terms of $\tilde{\Delta}$ and $\tilde{c}$. We therefore propose the following quantization map, which agrees with the naive one at leading order,

$$
\begin{equation*}
h \rightarrow \frac{\tilde{\Delta}}{\tilde{c}}, \quad I_{k} \rightarrow \frac{n_{k}}{\tilde{c}}, \quad \tilde{\Delta}=\Delta-\tilde{c}, \quad \tilde{c}=\frac{c-1}{24} . \tag{2.4.2}
\end{equation*}
$$

This does not solve the problem of reproducing subleadig terms in $Q_{2 n-1}^{0}$, but this can be fixed, at least at first subleading order, by introducing the Maslov index, $n_{k} \rightarrow \tilde{n}_{k}=n_{k}+1 / 2$. We thus arrive at the following map,

$$
\begin{equation*}
Q_{2 n-1}\left(h, I_{k}\right) \rightarrow \mathrm{Q}_{2 n-1}=\tilde{c}^{n} Q_{2 n-1}\left(\tilde{\Delta} / \tilde{c},\left(n_{k}+1 / 2\right) / \tilde{c}\right) \tag{2.4.3}
\end{equation*}
$$

Infinite sums due to Maslov index contributing to "vacuum energy" should be regularized using zeta-function regularization. It is now straightforward to see that we immediately reproduce the leading $1 / \tilde{c}$ term (2.3.8),

$$
\begin{align*}
& Q_{2 n-1}=h^{n}+\sum_{k} f_{k}^{(n, 1)}(h) I_{k}+O\left(I^{2}\right) \\
& \rightarrow Q_{2 n-1}=\tilde{\Delta}^{n}+\tilde{c}^{n-1} \sum_{k} f_{k}^{(n, 1)}(\tilde{\Delta} / \tilde{c}) \tilde{n}_{k}+O\left(\tilde{c}^{n-2}\right) \\
& =\tilde{\Delta}^{n}+\sum_{k} \sum_{j=0}^{n-1} \xi_{n}^{j} \tilde{\Delta}^{n-1-j} \tilde{c}^{j} k^{2 j+1}\left(n_{k}+1 / 2\right)+O\left(\tilde{c}^{n-2}\right) \\
& =\tilde{\Delta}^{n}+\sum_{j=0}^{n-1} \xi_{n}^{j} \tilde{\Delta}^{n-1-j} \tilde{c}^{j}\left(\sum_{k} k^{2 j+1} n_{k}+\frac{\zeta(-2 j-1)}{2}\right)+O\left(\tilde{c}^{n-2}\right) \tag{2.4.4}
\end{align*}
$$

In other words, at first sub-leading order $\tilde{c}^{n-1}$ the quantization prescription 2.4 .3 ) leads to 2.1.6 which passes all available tests: matches the spectrum of $Q_{1}, Q_{3}, Q_{5}$ (see section 2.4 below) and thermal expectation values for $Q_{9}, \ldots, Q_{13}$ (see section 2.5 below) at the order $\tilde{c}^{n-1}$.

There is another way to write 2.4 .4 . We can express $Q_{2 n-1}$ as $Q_{2 n-1}^{0}$ plus the terms from the classical $Q_{2 n-1}$ 2.2.1 which non-trivially depend on $I_{k}$ using the substitution 2.4.2, i.e. without the Maslov index,

$$
\begin{equation*}
\mathrm{Q}_{2 n-1}=\mathrm{Q}_{2 n-1}^{0}+\tilde{c}^{n-1} \sum_{k} f_{k}^{(n, 1)}(\tilde{\Delta} / \tilde{c}) n_{k}+O\left(\tilde{c}^{n-2}\right) \tag{2.4.5}
\end{equation*}
$$

At $\tilde{c}^{n-1}$ order it is the same as 2.4.4.
To obtain the quantum spectrum at next order $\tilde{c}^{n-2}$, we could try the prescription (2.4.3), apply the zeta-function regularization and notice that many but not all terms from 2.3.9 are reproduced. Thus, we see that the quantization (2.4.3) is exact only at leading $1 / \tilde{c}$ order, at higher orders the expression obtained from the classical $Q_{2 n-1}$ has to be modified as well. Indeed, starting from the classical 2.2.1 and using substitution (2.4.2 we would find that terms contributing at the order $\tilde{c}^{n-p}$ are homogeneous polynomials in $n_{k}$ of order $p$. This is very restrictive and obviously incorrect. We already saw that even at the first sub-leading order $\tilde{c}^{n-1}$ the homogeneous (linear) in $n_{k}$ terms have to be amended by a constant, i.e. $\left(n_{k}\right)^{0}$ term. This suggest the following "quantization rules": to obtain the quantum spectrum $\mathrm{Q}_{2 n-1}$ in $1 / \tilde{c}$ expansion, one starts with the classical perturbation expression 2.2.1 and make the substitution (2.4.2), together with the overall rescaling by $\tilde{c}^{n}$. As the order $\tilde{c}^{n-p}$ this fixes leading, homogeneous in $n_{k}$ terms of order $p$. These terms should be amended by the sub-leading terms of order $p-1, p-2, \ldots, 0$ in $n_{k}$. These terms should be regarded as quantum corrections and should be determined separately, they do not follow from the classical answer in any simple way. More explicitly,

$$
\begin{align*}
& \mathrm{Q}_{2 n-1}=\tilde{\Delta}^{n}+\tilde{c}^{n-1}\left(\sum_{k} g_{k}^{(1)} n_{k}+g^{(1)}\right)+\tilde{c}^{n-2}\left(\sum_{k_{1}, k_{2}} g_{k_{1}, k_{2}}^{(2)} n_{k_{1}} n_{k_{2}}+\sum_{k} g_{k}^{(2)} n_{k}+g^{(2)}\right) \\
& +\tilde{c}^{n-3}\left(\sum_{k_{1}, k_{2}, k_{3}} g_{k_{1}, k_{2}, k_{3}}^{(3)} n_{k_{1}} n_{k_{2}} n_{k_{3}}+\sum_{k_{1}, k_{2}} g_{k_{1}, k_{2}}^{(3)} n_{k_{1}} n_{k_{2}}+\sum_{k} g_{k}^{(3)} n_{k}+g^{(3)}\right)+\ldots \tag{2.4.6}
\end{align*}
$$

Here $g^{(p)}$ with different number of indexes denote different quantities. The leading
terms $g_{k_{1}, \ldots, k_{p}}^{(p)}$ are given by classical expressions (2.2.1) upon the substitution 2.4.2

$$
\begin{align*}
g_{k}^{(1)} & =f_{k}^{(n, 1)}(\tilde{\Delta} / \tilde{c})  \tag{2.4.7}\\
g_{k \ell}^{(2)} & =\frac{1}{2} f_{k \ell}^{(n, 2)}, \quad g_{k k}^{(2)}=f_{k}^{(n, 2)}  \tag{2.4.8}\\
g_{k \ell m}^{(3)} & =\frac{1}{6} f_{k l m}^{(n, 3)}, \quad g_{k k \ell}^{(3)}=\frac{1}{3} f_{k \ell}^{(n, 3)}, \quad g_{k k k}^{(3)}=f_{k}^{(n, 3)}, \tag{2.4.9}
\end{align*}
$$

for $k \neq \ell \neq m$ and $g^{(p)}$ are given by 2.3.8 2.3.9 2.3.10. This is essentially the generalization of (2.4.5) to higher orders in $1 / \tilde{c}$. Coefficients $g_{k}^{(2)}, g_{k \ell}^{(3)}, g_{k}^{(3)}$, etc. are quantum corrections and a priory not known.

To fix $g_{k}^{(2)}$ we employ the following strategy, we will try to "salvage" the Maslov index quantization 2.4.3 by adding minimal possible terms subleading in powers of $n_{k}$,
$\mathrm{Q}_{2 n-1}=\tilde{\Delta}^{n}+\tilde{c}^{n-1} \sum_{k} g_{k}^{(1)} \tilde{n}_{k}+\tilde{c}^{n-2}\left(\sum_{k_{1}, k_{2}} g_{k_{1}, k_{2}}^{(2)} \tilde{n}_{k_{1}} \tilde{n}_{k_{2}}+\sum_{k} \tilde{g}_{k}^{(2)} \tilde{n}_{k}+\tilde{g}^{(2)}\right)+\ldots$

This expression is understood in terms of the zeta-function regularization and $\tilde{g}_{k}^{(2)}, \tilde{g}^{(2)}$ are different from $g_{k}^{(2)}, g^{(2)}$. Our goal is to reproduce "vacuum energy" $\mathrm{Q}_{2 n-1}^{0}$. There is infinitely many ways to do that, for example by taking $g_{k}^{(2)}=0$, $\tilde{g}^{(2)}=g^{(2)}$, but we will additionally require that the zeta-functions from 2.3.9 will become the sums of the form $\sum_{k} k^{p}$ in 2.4.10). This leads to

$$
\begin{equation*}
\tilde{g}_{k}^{(2)}=\sum_{j=0}^{n-1} \frac{1}{4} \xi_{n}^{j}\left((2 n-1) y_{1}(j)-2 j-1\right) \tilde{\Delta}^{n-1-j} \tilde{c}^{j} k^{2 j+1}, \tag{2.4.11}
\end{equation*}
$$

and very simple

$$
\begin{equation*}
\tilde{g}^{(2)}=-\frac{n(n-1)(2 n-1) \tilde{\Delta}^{n-1}}{96 \tilde{c}} . \tag{2.4.12}
\end{equation*}
$$

This term is necessary to subtract $n_{k}$-independent $\tilde{\Delta}^{n-1} \tilde{c}^{-1}$ term coming from $\sum_{k} \tilde{g}_{k}^{(2)} \tilde{n}_{k}$ to match $\mathrm{Q}_{2 n-1}^{0}$ 2.3.7 which has no terms with the negative powers of $c$.

For convenience we give the full expression 2.4.10 explicitly

$$
\begin{align*}
\mathrm{Q}_{2 n-1}= & \tilde{\Delta}^{n}+\sum_{k} \sum_{j=0}^{n-1} \frac{(2 n-1) \sqrt{\pi} \Gamma(n+1)}{2 \Gamma\left(j+\frac{3}{2}\right) \Gamma(n-j)} \tilde{\Delta}^{n-1-j} \tilde{c}^{j} k^{2 j+1} \tilde{n}_{k}  \tag{2.4.13}\\
& -\frac{n(n-1)(2 n-1) \tilde{\Delta}^{n-1}}{96 \tilde{c}} \\
& +\sum_{k} \sum_{j=0}^{n-1} \frac{(2 n-1) \sqrt{\pi} \Gamma(n+1)}{8 \Gamma\left(j+\frac{3}{2}\right) \Gamma(n-j)}\left((2 n-1) y_{1}(j)-2 j-1\right) \tilde{\Delta}^{n-1-j} \tilde{c}^{j-1} k^{2 j+1} \tilde{n}_{k} \\
& -\sum_{k} \sum_{j=0}^{n-1} \frac{(2 n-1) \sqrt{\pi} \Gamma(n+1)(2 n j+2 n-3 j-2)}{16 \Gamma\left(j+\frac{3}{2}\right) \Gamma(n-j)} \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1} k^{2 j} \tilde{n}_{k}^{2} \\
& +\frac{1}{2} \sum_{k, \ell} \sum_{j=1}^{n-1} \frac{(2 n-1)^{2} \sqrt{\pi} \Gamma(n+1)}{4 \Gamma\left(j+\frac{3}{2}\right) \Gamma(n-j)} \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1} \sum_{s=0}^{j-1} k^{2(j-s)-1} \ell^{2 s+1} \tilde{n}_{k} \tilde{n}_{\ell}+\mathcal{O}\left(c^{n-3}\right) .
\end{align*}
$$

We conjecture this is the full quantum spectrum of $Q_{2 n-1}$ up to $\tilde{c}^{n-2}$ order and verify that it passes all available checks.

From here it is now straightforward to find $\mathrm{Q}_{2 n-1}$ in the representation (2.4.6). Coefficient

$$
\begin{align*}
g_{k}^{(2)}= & \sum_{j=0}^{n-1} \frac{(2 n-1) \sqrt{\pi} \Gamma(n+1)}{8 \Gamma\left(j+\frac{3}{2}\right) \Gamma(n-j)} v(n, j, k) \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1}  \tag{2.4.14}\\
v(n, j, k)= & (2 n-1) \sum_{s=0}^{j-1} \zeta(2(s-j)+1) k^{2 s+1}+\left((2 n-1) y_{1}(j)-2 j-1\right) k^{2 j+1} \\
& -\frac{1}{2}(2 n j+2 n-3 j-2) k^{2 j}
\end{align*}
$$

is significantly more bulky than (2.4.11), while the full expression is

$$
\begin{align*}
\mathrm{Q}_{2 n-1} & =\mathrm{Q}_{2 n-1}^{0}+\sum_{k} \sum_{j=0}^{n-1} \xi_{n}^{j} \tilde{\Delta}^{n-j-1} \tilde{c}^{j} k^{2 j+1} n_{k}  \tag{2.4.15}\\
& +\sum_{k, \ell} \sum_{j=1}^{n-1} \xi_{n}^{j} \frac{(2 n-1)}{4} \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1} \sum_{s=0}^{j-1} k^{2(j-s)-1} \ell^{2 s+1} n_{k} n_{\ell} \\
& -\sum_{k} \sum_{j=0}^{n-1} \xi_{n}^{j} \frac{(2 n j+2 n-3 j-2)}{8} \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1} k^{2 j} n_{k}^{2} \\
& +\sum_{k} \sum_{j=0}^{n-1} \xi_{n}^{j} \frac{v(n, j, k)}{4} \tilde{\Delta}^{n-j-1} \tilde{c}^{j-1} n_{k}+\mathcal{O}\left(c^{n-3}\right)
\end{align*}
$$

To summarize, we have found the (conjectured) spectrum of all qKdV charges at first two sub-leading orders in $1 / c$ expansion 2.4 .13 2.4.15) and observed certain
patterns which may help fix the spectrum at higher orders. Let us spell the step to find the next $1 / \tilde{c}^{3}$ order, i.e. fix the terms of order $\tilde{c}^{n-3}$ in 2.4.6). The classical result for $Q_{2 n-1}$ in terms of action variables $I_{k}$ was calculated up to cubic order in 2.2 .382 .2 .42 2.2.44). "Energies" of primary states $Q_{2 n-1}^{0}$ were also calculated to this order, see eq. 2.3.10. Thus $g_{k_{1} k_{2} k_{3}}^{(3)}$ and $g^{(3)}$ are known, and to find the spectrum one would only need to fix $g_{k_{1} k_{2}}^{(3)}$ and $g_{k}^{(3)}$. To do that one would need to find $\tilde{g}_{k_{1} k_{2}}^{(3)}$ and $\tilde{g}_{k}^{(3)}$ from the expansion 2.4 .10 to reproduce 2.3.10 via zeta-function regularization and minimal possible $\tilde{g}^{(3)}$, which presumably will only include terms with negative powers of $\tilde{c}$. "Restoring" $\tilde{g}_{k_{1} k_{2}}^{(3)}$ and $\tilde{g}_{k}^{(3)}$ from $\tilde{R}_{n, j}^{(3)}$ is not a mathematically well-posed problem. We expect that all zeta-functions $\zeta(-2 j-1)$ in $\tilde{R}_{n, j}^{(3)}$ to lead to the sums $\sum_{k} k^{2 j+1} \tilde{n}_{k}$ - the rule which successfully worked at second $1 / \tilde{c}$ order. At third order this rule should be amended by others, as suggested by a non-polynomial dependence on $k$ in (2.2.42). In practice, restoring $\tilde{g}_{k}^{(3)}$ from $\tilde{R}_{n, j}^{(3)}$ may require establishing the analytic form of coefficients $p_{j}$ in (2.3.10) and then reverse-engineering corresponding $k_{1}, k_{2}, k_{3}$-dependent sums. Once hypothetical $\tilde{g}_{k_{1} k_{2}}^{(3)}$ and $\tilde{g}_{k}^{(3)}$, and accordingly $g_{k_{1} k_{2}}^{(3)}$ and $g_{k}^{(3)}$ are fixed, a non-trivial set of checks is provided by the spectrum of $Q_{3}, Q_{5}$ generated by computer algebra, as well as the requirement that thermal expectation values $\left\langle Q_{2 n-1}\right\rangle_{q}$ discussed in section 2.5 must have certain modular properties.

## Computer algebra check

For $n=1$ the expansion 2.4.15 reduces to 2.1.7 which is a simple check. A more sophisticated check is provided by $Q_{3}$ and $Q_{5}$ which are known explicitly in terms of the Virasoro algebra generators [9]

$$
\begin{align*}
Q_{3} & =\left(L_{0}^{2}-\frac{c+2}{12} L_{0}+\frac{c(5 c+22)}{2990}\right)+\tilde{Q}_{3},  \tag{2.4.16}\\
\tilde{Q}_{3} & =2 \sum_{k=1}^{\infty} L_{-k} L_{k}, \\
Q_{5} & \left.=\left(L_{0}^{3}-\frac{c+4}{8} L_{0}^{2}+\frac{(c+2)(3 c+20)}{576} L_{0}-\frac{c(3 c+14)(7 c+68)}{290304}\right)+\tilde{Q}_{2}, 4.17\right) \\
\tilde{Q}_{5} & =\sum_{k, l=0}^{\infty} L_{-k-l} L_{k} L_{l}+2 \sum_{k=1, l=0}^{\infty} L_{-k} L_{k-l} L_{l}+\sum_{k, l=1}^{\infty} L_{-k} L_{-l} L_{k+l}+ \\
& +\sum_{n=1}^{\infty}\left(\frac{c+2}{6} m^{2}-\frac{c}{4}-1\right) L_{-n} L_{n}-L_{0}^{3}, \tag{2.4.18}
\end{align*}
$$

and 53

Using computer algebra spectrum of $Q_{3}, Q_{5}$ for all descendants at a small levels $m$ can be evaluated explicitly, as an expansion in powers of $1 / c$. The resulting expressions can be compared with the spectrum following from 2.4.15, which we will write in terms of quantum numbers $n_{k}$ packaged as follows

$$
\begin{gather*}
m_{p, r} \equiv \sum_{k} k^{p} n_{k}^{r}, \quad m_{p} \equiv m_{p, 1}, \quad m \equiv m_{1}, \quad h=\tilde{\Delta} / \tilde{c}, \quad  \tag{2.4.19}\\
\mathrm{Q}_{3}=\quad \tilde{\Delta}^{2}+\tilde{\Delta}\left(6 m_{1}-\frac{1}{4}\right)+\tilde{c}\left(4 m_{3}+\frac{1}{60}\right)+  \tag{2.4.20}\\
\left(m_{3}-\frac{3}{2} m_{2}-\frac{1}{4} m_{1}\right)-\frac{3}{2} m_{2,2}+3 m_{1}^{2}+\frac{3}{2} h\left(2 m_{1}-m_{0}-m_{0,2}\right)+\frac{3}{320}+\mathcal{O}(1 / \tilde{c}),
\end{gather*}
$$

and similarly

$$
\begin{align*}
\mathrm{Q}_{5}= & \tilde{\Delta}^{3}+\left(15 m_{1}-\frac{5}{8}\right) \tilde{\Delta}^{2}+\tilde{\Delta} \tilde{c}\left(20 m_{3}+\frac{1}{12}\right)+\tilde{c}^{2}\left(8 m_{5}-\frac{1}{63}\right)+ \\
& \tilde{\Delta}\left(\frac{5}{12}\left(-5 m_{1}-42 m_{2}+44 m_{3}\right)-\frac{35}{2} m_{2,2}+25 m_{1}^{2}+\frac{15}{2} h\left(2 m_{1}-m_{0}-m_{0,2}\right)+\frac{23}{192}\right)+ \\
& \tilde{c}\left(\frac{1}{12}\left(m_{1}-10 m_{3}-120 m_{4}+64 m_{5}\right)-10 m_{4,2}+20 m_{1} m_{3}-\frac{85}{6048}\right)+\mathcal{O}\left(\tilde{c}^{0}\right), \tag{2.4.21}
\end{align*}
$$

and
We checked, these expressions are in agreement with the computer algebra generated spectrum for $m \leq 12$, which serves as a non-trivial consistency check of (2.4.15).

### 2.5 Miscellaneous results

Explicit expression for the spectrum of quantum $Q_{2 n-1}$ in large $c$ limit opens the opportunity to make progress in a number of adjacent directions. In this section we discuss several applications of our results.

## Thermal expectation values of $Q_{2 n-1}$

Our first application is toward thermal exaction value of $Q_{2 n-1}$, i.e. averaged over the CFT Gibbs ensemble $\left\langle Q_{2 n-1}\right\rangle_{q} \equiv \operatorname{Tr}\left(q^{L_{0}-c / 24} Q_{2 n-1}\right)$. This question appears naturally, though in a more complicated form, to calculate the averaged value of $Q_{2 n-1}$ over the KdV Generalized Gibbs Ensemble (see section ?? below), if one wants to match the GGE chemical potentials to describe equilibration endpoint of
some initial state. The expectation value $\left\langle Q_{2 n-1}\right\rangle_{q}$, which is essentially the onepoint function of $T_{2 n}$ (2.1.1) on the torus, exhibits modular properties and can be represented as a covariant differential operator acting on the CFT torus partition function [54]. In fact, one can average $Q_{2 n-1}$ over a particular Verma module, $\left\langle Q_{2 n-1}\right\rangle_{\Delta} \equiv \operatorname{Tr}_{\Delta}\left(q^{L_{0}-c / 24} Q_{2 n-1}\right)$, where sum goes over all Virasoro descendants of the primary state $|\Delta\rangle$. This sum too is a modular object and can be evaluated with help of the same differential operator

$$
\begin{align*}
\left\langle Q_{2 n-1}\right\rangle_{\Delta} & =\mathcal{D}_{n} \chi_{\Delta}, \quad \chi_{\Delta} \equiv \operatorname{Tr}_{\Delta}\left(q^{L_{0}-c / 24}\right)=q^{\tilde{\Delta}-\frac{1}{24}} / \eta  \tag{2.5.1}\\
\mathcal{D}_{n} & =D^{n}+\sum_{j=1}^{n-1} P_{n}^{j}(c, q) D^{n-j-1}, \quad D^{n}=D_{2(n-1)} \ldots D_{2} D_{0} \tag{2.5.2}
\end{align*}
$$

and $D_{r}=q \partial_{q}-\frac{r}{12} E_{2}$ is Serre derivative. Each $P_{n}^{j}$ is a degree $j$ polynomial in $c$ with each coefficient being a modular form of weight $2 j+2$,

$$
\begin{equation*}
P_{n}^{j}(c, q)=\sum_{k=1}^{j+1} P_{n, j}^{(k)} \tilde{c}^{j-k+1} E_{2 j+2}^{(n, k)}(q) . \tag{2.5.3}
\end{equation*}
$$

Here $P_{n, j}^{(k)}$ are numerical coefficients and $E_{2 j+2}^{(n, k)}$ is some modular form, which is a linear combination of $E_{4}^{a} E_{6}^{b}$ with $4 a+6 b=2 j+2$ for non-negative integer $a, b$, normalized such that $E_{2 j+2}^{(n, j)}=1+O(q)$. For $j=1,2,3,4,6$ there is a unique modular form of the weight $2(j+1)$ and therefore for these $j$, independently of $n$ and $k, E_{2 j+2}^{(n, k)}=E_{2 j+2}$ where

$$
\begin{equation*}
E_{2 n}=1+\frac{2}{\zeta(1-2 n)} \sigma_{2 n-1}, \quad \sigma_{p}=\sum_{k=1}^{\infty} \frac{k^{p} q^{k}}{1-q^{k}} \tag{2.5.4}
\end{equation*}
$$

For instance, in the simplest case of $Q_{3}$ the operator $\mathcal{D}_{2}$ is given by

$$
\begin{equation*}
\left\langle Q_{3}\right\rangle_{\Delta}=\mathcal{D}_{2} \chi_{\Delta}=\left[D^{2}+\frac{c}{1440} E_{4}\right] \chi_{\Delta} . \tag{2.5.5}
\end{equation*}
$$

In this case $P_{2,1}^{(1)}=1 / 60$ and $P_{2,1}^{(2)}=1 / 1440$. Explicit expressions for $\mathcal{D}_{n}$ for $n \leq 7$ were found in 54. For higher $n$ the modular form $E_{2 i+2}^{(n, j)}$ and coefficients $P_{n, j}^{(k)}$ are not known.

Strictly speaking (2.5.1 2.5.2 is an unproven ansatz proposed in 54. We find it to be consistent with the large $c$ spectrum of $Q_{2 n-1} 2.4 .15$ and fix two leading in $c$ terms in $P_{n}^{j}$. To compare with 2.5.1), we need to calculate $\left\langle Q_{2 n-1}\right\rangle_{\Delta}$ starting from (2.4.15). Here the following straightforward identities will be helpful

$$
\begin{align*}
& \left\langle\sum_{k=1}^{\infty} n_{k} k^{p}\right\rangle_{\Delta}=\sigma_{p} \chi_{\Delta}, \quad\left\langle\sum_{k=1}^{\infty} n_{k}^{2} k^{p}\right\rangle_{\Delta}=\left(2 q \partial_{q} \sigma_{p-1}-\sigma_{p}\right) \chi_{\Delta},  \tag{2.5.6}\\
& \left\langle\sum_{k=1}^{\infty} n_{k} k^{p} \sum_{\ell=1}^{\infty} n_{\ell} \ell^{p^{\prime}}\right\rangle_{\Delta}=\left(q \partial_{q} \sigma_{p+p^{\prime}-1}+\sigma_{p} \sigma_{p^{\prime}}\right) \chi_{\Delta}, \tag{2.5.7}
\end{align*}
$$

where by $n_{k}$ we mean the quantum numbers (2.1.4). Then (2.1.6) immediately yields

$$
\begin{equation*}
\left\langle Q_{2 n-1}\right\rangle_{\Delta}=\tilde{\Delta}^{n} \chi_{\Delta}+\sum_{j=0}^{n-1} \tilde{\Delta}^{n-p-1} \tilde{c}^{p} \xi_{n}^{p}\left(\sigma_{2 p+1}+\frac{\zeta(-2 p-1)}{2}\right) \chi_{\Delta}+O\left(\tilde{c}^{n-2}\right), \tag{2.5.8}
\end{equation*}
$$

where we assumed the usual limit, $h=\tilde{\Delta} / \tilde{c}$ is kept fixed while $\tilde{c} \rightarrow \infty$. Comparing this with 2.5.1), we immediately see that the leading $\tilde{\Delta}^{n}$ term is coming from (we drop $\chi_{\Delta}$ for simplicity)

$$
\begin{equation*}
D^{n} \rightarrow\left(q \partial_{q}\right)^{n} \rightarrow \tilde{\Delta}^{n} . \tag{2.5.9}
\end{equation*}
$$

Similarly we can trace origin of all $\tilde{c}^{n-1}$ terms,

$$
\begin{aligned}
& D^{n} \rightarrow\left(q \partial_{q}\right)^{n}-\frac{n(n-1)}{12} E_{2}\left(q \partial_{q}\right)^{n-1} \rightarrow \tilde{\Delta}^{n-1} n\left(\sigma_{1}-\frac{1}{24}\right)-\tilde{\Delta}^{n-1} \frac{n(n-1)}{12} E_{2} \\
& =-\frac{n(2 n-1)}{24} E_{2}
\end{aligned}
$$

which agrees with (2.5.8), and

$$
\begin{equation*}
P_{n, j}^{(1)} \tilde{c}^{j} E_{2 j+2}^{(n, 1)} D^{n-j-1} \rightarrow P_{n, j}^{(1)} \tilde{c}^{j} E_{2 j+2}^{(n, 1)}\left(q \partial_{q}\right)^{n-j-1} \rightarrow P_{n, j}^{(1)} \tilde{\Delta}^{n-j-1} \tilde{c}^{j} E_{2 j+2}^{(n, 1)}, \tag{2.5.10}
\end{equation*}
$$

for $n-1 \geq j>0$. From here immediately follows

$$
\begin{equation*}
P_{n, j}^{(1)}=\tilde{R}_{n, j}^{(1)}, \quad E_{2 j+2}^{(n, 1)}=E_{2 j+2}, \quad n-1 \geq j \geq 1 . \tag{2.5.11}
\end{equation*}
$$

To fix $P_{n, j}^{(2)}$ it is convenient to take $q \rightarrow 0$ limit and compare $\left\langle Q_{2 n-1}\right\rangle_{\Delta}$ with 2.3.7), yielding

$$
\begin{aligned}
& P_{n, 1}^{(2)}=\tilde{R}_{n, 0}^{(2)}-\frac{n(n-1)\left(12 n^{2}-16 n-1\right)}{3456}=\frac{n(n-1)\left(12 n^{2}-38 n+31\right)}{8640} \\
& P_{n, j}^{(2)}=\tilde{R}_{n, j-1}^{(2)}+\frac{(n-j)(2(n-j)-1)}{24} P_{n, j-1}^{(1)}, \quad n-1 \geq j \geq 2 .
\end{aligned}
$$

Evaluation of $E_{2 j+2}^{(n, 2)}$ is a more challanging task and requires first using 2.5.6 2.5.7 and then combining pieces into modular forms to match (2.5.1 2.5.2). We note, there are terms in 2.4.15 proportional to $\tilde{\Delta}^{n-1} \tilde{c}^{-1}$, but 2.5.1 has no negative powers of c. Hence these terms must vanish after averaging, which follows from the identity $q \partial_{q} \sigma_{-1}-\sigma_{1}=0$ and serves as a consistency check. The final expression reads

$$
\begin{align*}
& P_{n, j}^{(2)} E_{2 j+2}^{(n, 2)}=\frac{(2 n-1) \sqrt{\pi} \Gamma(n+1)}{8 \Gamma(j+3 / 2) \Gamma(n-j)}\left(\left((2 n-1) y_{1}(j)-2 j-1\right) \frac{\zeta(-2 j-1)}{2} E_{2 j+2}-\right. \\
& \left.(n-1-j) \zeta(-2 j+1) D_{2 j} E_{2 j}+\frac{(2 n-1)}{4} \sum_{s=1}^{j-2} \zeta(-2 s-1) \zeta(-2(j-s)+1) E_{2 s+2} E_{2(j-s)}\right) . \tag{2.5.12}
\end{align*}
$$

It is valid for $n-1 \geq j \geq 2$. For $j=1$, there is a unique modular form $E_{2 j+2}^{(n, 2)}=E_{4}$. Also, as was mentioned above $E_{2 j+2}^{(n, 2)}=E_{2 j+2}$ for $j=2,3,4,6$, which can be checked straightforwardly. Because of the identities between modular forms there are other ways to write 2.5.12.

Explicit form of $\mathrm{Q}_{2 n-1}^{0}$ up to $\tilde{c}^{n-3}$ order allows us, in principle, to calculate $P_{n, j}^{(3)}$, although calculation of $E_{2 j+2}^{(n, 3)}$ would require first extending 2.4.15 to the next $1 / c$ order. Given involved form of $P_{n, j}^{(2)}$ and $E_{n, j}^{(2)}$ we do not expect the answer to be simple.

### 2.6 Discussion

In this chapter we obtained spectrum of quantum KdV charges $Q_{2 n-1}$ in first two non-trivial orders in $1 / c$ expansion. Our result (2.4.13) and (2.4.15) is valid in the semiclassical limit of large central charge $c \rightarrow \infty$ with the ratio of $\Delta / c$ kept fixed. This limit is inspired by holographic correspondence, when CFT is dual to weakly coupled gravity. Accordingly, dynamics of stress-energy sector becomes semiclassical, with the leading (classical) contribution governed by integrable dynamics on the coadjoint orbit of the Virasoro algebra. Under semiclassical quantization classical action variables $I_{k}$ are promoted to integer quantum numbers $n_{k}$, and the spectrum of $Q_{2 n-1}$ looks most elegant in terms of variables $\tilde{\Delta}$ and $\tilde{c}$ 2.4.2. At each order in $1 / \tilde{c}$ the quantum answer is a polynomial in $n_{k}$. Classical calculation fixes the leading term with the highest power of $n_{k}$, while all other terms should be regarded as "quantum corrections." We have seen that semiclassical quantization, combined with the values of qKdV charges $Q_{2 n-1}$ acting on primary states, is sufficient to completely fix these quantum corrections and obtain the spectrum of excited states at least in first two orders in $1 / c$. We conjecture this quantization scheme can be extended to higher orders in $1 / c$. We laid the groundwork for the next order $1 / c^{3}$ by calculating classical $Q_{2 n-1}\left(h, I_{k}\right)$ as well as "energies" on primary states $\mathrm{Q}_{2 n-1}^{0}$, albeit in the latter case not all terms are known analytically. To complete the job one would need to find analytic expressions for $Q_{2 n-1}^{0}$ and develop a dictionary that maps each term to an infinite sum, yielding this term back via zeta-function regularization.

The obtained spectrum has several immediate applications. First, in section 2.5 we calculated two leading terms in large $c$ expansion of the "thermal expectation values" $\left\langle Q_{2 n-1}\right\rangle_{\Delta} \equiv \operatorname{Tr}_{\Delta}\left(q^{L_{0}-c / 24} Q_{2 n-1}\right)$, where sum goes over a particular Verma module, and compared them with the predictions of [54]. Covariance under modular transformation of $\left\langle Q_{2 n-1}\right\rangle_{\Delta}$ in each order in $1 / c$ serves as a non-trivial check of our
main result (2.4.15). We also fixed two leading terms in the differential operator $\mathcal{D}_{n}$ yielding thermal expectation values via $\left\langle Q_{2 n-1}\right\rangle_{\Delta}=\mathcal{D}_{n} \operatorname{Tr}_{\Delta}\left(q^{L_{0}-c / 24}\right)$, see 2.5.11) and (2.5.12).

There are several potential applications of our results, which we hope to address in the future. The obtained spectrum of $Q_{2 n-1}$ will be helpful to study generalized Eigenstate Thermalization Hypothesis of 2d CFTs [42] at the subleading order in $1 / c$. We also expect the semiclassical quantization approach developed in this chapter could be helpful in the context of Intermediate Long Wave hierachry, which is closely related to qKdV problem. More generally, it would be interesting to bridge the gap between the semiclassical approach of this work with the Bethe anzatz approach of 44 by taking "holographic" limit $c \rightarrow \infty$ with fixed $h=\tilde{\Delta} / \tilde{c}$ of the appropriate Bethe anzatz equations.

## Chapter 3 Information geometry and holographic correlators

This chapter is essentially identical to:
Information geometry and holographic correlators (14]

### 3.1 Introduction: Information geometry and holographic dictionary

How does quantum information encode effective field theory? This question is relevant in holography, where the quantum extremal surface proposal for quantum corrections implies novel features of black hole evaporation [12, 55, 56]. Despite recent progress, effective field theory remains far less understood in terms of quantum information than in the language of Lagrangians, correlation functions, and the S-matrix. Developing this subject may prove useful. We may learn more about effective field theory via constraints coming from quantum information. We may also identify new perturbative structures in a quantum information description of gravity. Even a better technical understanding of quantum corrections at first order may have far-reaching implications for our understanding of black holes.

While computations of entanglement entropy in AdS/CFT have been illuminating, the mechanics of effective field theory can be studied in another setting as well. In certain cases, quantum information quantities can be related directly to correlation functions or the S-matrix, or to their ingredients. For recent results in this direction, see for example $[57,58,59,60,61,62,63])$. This approach exposes the role of effective field theory, allowing direct study of its interplay with quantum information.

In this work we study the Bures information metric, which is a measure of the distinguishability of nearby states. We explore perturbative corrections and focus in particular on the information metric associated with correlators in holographic conformal field theories (CFTs). To summarize the setup, we consider the Bures distance between pure states $D_{B}\left(\psi\left(x_{1}, x_{2}\right), \psi\left(x_{3}, x_{4}\right)\right)^{2}$ near $x_{1}=x_{3}, x_{2}=x_{4}$ using states,

$$
\begin{equation*}
\left|\psi\left(x_{1}, x_{2}\right)\right\rangle=\frac{\mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{1}\left(x_{1}\right)|0\rangle}{\sqrt{\left\langle\mathcal{O}_{1}\left(x_{1}^{*}\right) \mathcal{O}_{2}\left(x_{2}^{*}\right) \mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{1}\left(x_{1}\right)\right\rangle}}, \tag{3.1.1}
\end{equation*}
$$

where we use the notation $(\mathcal{O}(x))^{\dagger}=\mathcal{O}\left(x^{*}\right)$. Up to a normalization factor, the Bures distance is a four-point function,

$$
\begin{equation*}
D_{B}\left(\psi\left(x_{1}, x_{2}\right), \psi\left(x_{3}, x_{4}\right)\right)^{2} \sim\left\langle\mathcal{O}_{1}\left(x_{1}^{*}\right) \mathcal{O}_{2}\left(x_{2}^{*}\right) \mathcal{O}_{1}\left(x_{3}\right) \mathcal{O}_{2}\left(x_{4}\right)\right\rangle \tag{3.1.2}
\end{equation*}
$$

In a holographic CFTs, the Bures information metric of this two-operator state is

$$
\begin{equation*}
g_{x_{1}^{\mu} x_{2}^{\nu}} \equiv \frac{d^{2}}{d x_{1}^{\mu} d x_{2}^{\nu}} D_{B}\left(\psi\left(x_{1}, x_{2}\right), \psi\left(x_{3}, x_{4}\right)\right)^{2} \approx g_{x_{1}^{\mu} x_{2}^{\nu}}^{(0)}+\frac{1}{N^{2}} g_{x_{1}^{\mu} x_{2}^{\nu}}^{(2)}+\ldots \tag{3.1.3}
\end{equation*}
$$

and encodes features of four-point correlators in a simple way. ${ }^{1}$ Taking a similar approach, we also discuss transition amplitudes induced by a unitary $U=e^{-i \lambda H}$ and work perturbatively in $\lambda$. Our aim here is to take initial steps in describing the information geometry of $2 n$-point processes in quantum field theory, but it is straightforward to explore this story more fully using standard methods.

As four-point functions appear explicitly, the connection to $1 / N$ perturbation theory is direct. The $1 / N$ corrections are computed by four-point Witten diagrams in the bulk, which have been studied extensively at tree level and more recently at one loop [64, 65, 66, 67, 68, 69, 70, 71. By using these known results as input, computing the information metric itself is relatively simple (3.1.3). This approach applies equally well in all dimensions. By comparison, it is more challenging to probe $1 / N$ corrections by taking the partial trace of density matrix [72, 13]. The reduced density matrix approach probes entanglement wedge structure and has been explored to order $\mathcal{O}\left(N^{0}\right)$ in $\mathrm{CFT}_{2}$. However, we expect that computing $1 / N$ corrections on a replica manifold will be more challenging than in the original theory, and far less tractable in general dimensions. In short, the pure state and reduced density matrix approaches probe different and complementary features of the information metric, and may be useful for different purposes.

Here, we give an outline of this chapter. In Section 2, we review basics of information geometry and then discuss perturbative corrections to the information metric. We show that when the fidelity factorizes, the information metric also factorizes. In Section 3, we review the two-point function metric derived in [72, 13], and then study the metric of four-point functions in $\mathrm{CFT}_{d}$. We show that correlators with a weak-coupling expansion have an information metric that also has a weak-coupling expansion. In Section 4, we compute the information metric in explicit four-point examples. Correlators in mean field theory (MFT) with pairwise-identical operators factorize and have a factorized information metric. MFT correlators with identical scalars do not factorize, or equivalently have operators besides the identity exchanged in every channel. The resulting information metric does not factorize. We then com-

[^4]pute an $\mathcal{O}\left(1 / N^{2}\right)$ correction dual to a tree Witten diagram in bulk $\phi_{1}^{2} \phi_{2}^{2}$ theory. While the MFT contribution factorizes, the tree-level information metric does not, and the tree diagram has operator exchanges besides the identity in all channels. In all four-point examples, we find that the information metric is asymptotically AdS. In Section 5, we address similar questions for transition amplitudes of qubits, a simple model for the S-matrix. In this context, we find an information metric for transition amplitudes with identical in and out states. The metric takes the form $\left\langle H^{2}\right\rangle-\langle H\rangle^{2}$. In Section 6, we discuss future directions.

### 3.2 Information metric basics

## Review

We review the Bures distance $D_{B}^{2}$ and the associated metric, which we will refer to as the information metric. We follow the approach in [13, 72, to which we refer the reader for further details and discussion of other distance measures. The Bures distance between density matrices $\rho_{1}, \rho_{2}$ is

$$
\begin{equation*}
D_{B}\left(\rho_{1}, \rho_{2}\right)^{2}=2\left(1-\sqrt{F\left(\rho_{1}, \rho_{2}\right)}\right) \tag{3.2.1}
\end{equation*}
$$

where $F$ is the fidelity,

$$
\begin{equation*}
F\left(\rho_{1}, \rho_{2}\right)=\left(\operatorname{tr}\left(\sqrt{\sqrt{\rho_{1}} \rho_{2} \sqrt{\rho_{1}}}\right)\right)^{2} . \tag{3.2.2}
\end{equation*}
$$

Though not manifest above, fidelity is symmetric. We will study pure states, for which $D_{B}^{2}$ takes a simple form. When $\rho_{i}=\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$ with $\left|\psi_{i}\right\rangle$ normalized, $F\left(\rho_{1}, \rho_{2}\right)=$ $\left|\left\langle\psi_{1} \mid \psi_{2}\right\rangle\right|^{2}$ and

$$
\begin{equation*}
D_{B}\left(\rho_{1}, \rho_{2}\right)^{2}=2\left(1-\left|\left\langle\psi_{1} \mid \psi_{2}\right\rangle\right|\right) . \tag{3.2.3}
\end{equation*}
$$

In other words, the Bures distance between pure states is simply the magnitude of the inner product.

Consider a family of density matrices $\rho\left(\lambda_{i}\right)$ that depend smoothly on parameters $\lambda_{i}$. The Bures distance of nearby $\rho$ 's can be described by a metric as

$$
\begin{equation*}
\left.D_{B}\left(\rho\left(\lambda_{i}\right), \rho\left(\lambda_{i}+d \lambda_{i}\right)\right)^{2} \approx \sum_{i} d \lambda_{i} \frac{d}{d \lambda_{i}^{\prime}}\right|_{\lambda_{i}^{\prime}=\lambda_{i}} D_{B}\left(\rho\left(\lambda_{i}\right), \rho\left(\lambda_{i}^{\prime}\right)\right)^{2}+\sum_{i, j} g_{i j}\left(\lambda_{i}\right) d \lambda_{i} d \lambda_{j}, \tag{3.2.4}
\end{equation*}
$$

where

$$
\begin{equation*}
\left.g_{i j} \equiv \frac{d^{2}}{d \lambda_{i}^{\prime} d \lambda_{j}^{\prime}}\right|_{\substack{\lambda_{i}^{\prime}=\lambda_{i} \\ \lambda_{j}^{\prime}=\lambda_{j}}} \sqrt{F\left(\rho\left(\lambda_{i}\right), \rho\left(\lambda_{i}^{\prime}\right)\right)} \tag{3.2.5}
\end{equation*}
$$

is the Bures metric. Assuming $D_{B}\left(\rho\left(\lambda_{i}\right), \rho\left(\lambda_{i}^{\prime}\right)\right)^{2}$ is analytic in a neighborhood of $\lambda_{i}^{\prime}=\lambda_{i}$, then it has a minimum at $\lambda_{i}^{\prime}=\lambda_{i}$ and so

$$
\begin{equation*}
D_{B}\left(\rho\left(\lambda_{i}\right), \rho\left(\lambda_{i}+d \lambda_{i}\right)\right)^{2} \approx \sum_{i, j} g_{i j}\left(\lambda_{i}\right) d \lambda_{i} d \lambda_{j} \tag{3.2.6}
\end{equation*}
$$

The information metric therefore captures the distinguishability of nearby density matrices. Following the quantum Cramer-Rao bound, the inverse metric $g_{i j}^{-1}$ bounds the error in estimating values of $\lambda_{i}$ through measurement.

## Perturbative corrections

We now study the information metric in the context of perturbation theory. For a family of density matrices $\rho$ parametrized by $\lambda_{1}, \lambda_{2}$,

$$
\begin{equation*}
D_{B}\left(\rho\left(\lambda_{1}, \lambda_{2}\right), \rho\left(\lambda_{1}, \lambda_{2}+d \lambda_{2}\right)\right)^{2} \approx g_{22}\left(\lambda_{1}, \lambda_{2}\right)\left(d \lambda_{2}\right)^{2} \tag{3.2.7}
\end{equation*}
$$

Suppose $\rho\left(\lambda_{1}, \lambda_{2}\right)$ has an expansion in $\lambda_{1}$ about for example $\lambda_{1}=\lambda$. It follows that $g_{22}\left(\lambda_{1}, \lambda_{2}\right)$ can also be expanded in $\lambda_{1}$,

$$
\begin{equation*}
g_{22}\left(\lambda_{1}, \lambda_{2}\right)=\sum_{n=0} g_{22}^{(n)}\left(\lambda, \lambda_{2}\right)\left(\lambda_{1}-\lambda\right)^{n} . \tag{3.2.8}
\end{equation*}
$$

This statement is intuitive when the $g_{22}^{(n)}\left(\lambda_{1}, \lambda\right)$ are computed from objects within the same Hilbert space, which is natural in quantum mechanics. In weakly coupled quantum field theory, expanding an interacting quantity in a coupling $\lambda_{1}$ gives $g_{22}^{(n)}\left(\lambda, \lambda_{2}\right)$ computed from elements of the Hilbert space of the free theory. $\lambda_{2}$ parametrizes the state in the exact theory. Concretely, when $\lambda_{1}$ is a coupling constant, $\lambda_{2}$ can be the position or momentum that specifies the state.

Finally, we show that factorization of the fidelity into the fidelities of subsystems implies factorization of the information metric. Suppose that

$$
\begin{equation*}
D_{B}\left(\rho\left(\lambda_{1}, \lambda_{2}\right), \rho\left(\lambda_{3}, \lambda_{4}\right)\right)^{2}=2\left(1-\sqrt{F_{1}\left(\lambda_{1}, \lambda_{3}\right) F_{2}\left(\lambda_{2}, \lambda_{4}\right)}\right), \tag{3.2.9}
\end{equation*}
$$

where $F_{1}, F_{2}$ are themselves fidelities,

$$
\begin{align*}
& D_{B}\left(\rho_{1}\left(\lambda_{1}\right), \rho_{1}\left(\lambda_{3}\right)\right)^{2}=2\left(1-\sqrt{F_{1}\left(\lambda_{1}, \lambda_{3}\right)}\right) \\
& D_{B}\left(\rho_{2}\left(\lambda_{2}\right), \rho_{2}\left(\lambda_{4}\right)\right)^{2}=2\left(1-\sqrt{F_{2}\left(\lambda_{2}, \lambda_{4}\right)}\right) \tag{3.2.10}
\end{align*}
$$

for families of density matrices $\rho_{1}\left(\lambda_{1}\right), \rho_{2}\left(\lambda_{2}\right)$ that admit information metrics $g_{11} d \lambda_{1}^{2}$ and $g_{22} d \lambda_{2}^{2}$ respectively. Expanding $D_{B}\left(\rho\left(\lambda_{1}, \lambda_{2}\right), \rho\left(\lambda_{3}, \lambda_{4}\right)\right)^{2}$ using $\lambda_{3}=\lambda_{1}+d \lambda_{1}$ and $\lambda_{4}=\lambda_{2}+d \lambda_{2}$ therefore gives what we refer to as a factorized metric,

$$
\begin{equation*}
D_{B}\left(\rho\left(\lambda_{1}, \lambda_{2}\right), \rho\left(\lambda_{1}+d \lambda_{1}, \lambda_{2}+d \lambda_{2}\right)\right)^{2} \approx g_{\lambda_{1} \lambda_{1}} d \lambda_{1}^{2}+g_{\lambda_{2} \lambda_{2}} d \lambda_{2}^{2} \tag{3.2.11}
\end{equation*}
$$

The cross term

$$
\begin{align*}
& \left.\frac{d^{2}}{d \lambda_{3} d \lambda_{4}}\right|_{\substack{\lambda_{3}=\lambda_{1} \\
\lambda_{4}=\lambda_{2}}} D_{B}\left(\rho\left(\lambda_{1}, \lambda_{2}\right), \rho\left(\lambda_{3}, \lambda_{4}\right)\right)^{2} \\
& \quad=-2\left(\left.\frac{d}{d \lambda_{3}}\right|_{\lambda_{3}=\lambda_{1}} \sqrt{F\left(\rho_{1}\left(\lambda_{1}\right), \rho_{1}\left(\lambda_{3}\right)\right)}\right)\left(\left.\frac{d}{d \lambda_{4}}\right|_{\lambda_{4}=\lambda_{2}} \sqrt{F\left(\rho_{2}\left(\lambda_{2}\right), \rho_{2}\left(\lambda_{4}\right)\right)}\right)=0 \tag{3.2.12}
\end{align*}
$$

because each factor equals the first order term in $D_{B}\left(\rho_{1}\left(\lambda_{1}\right), \rho_{1}\left(\lambda_{1}+d \lambda_{1}\right)\right)^{2}$ and $D_{B}\left(\rho_{2}\left(\lambda_{2}\right) \rho_{2}\left(\lambda_{2}+d \lambda_{2}\right)\right)^{2}$ respectively. As these Bures distances admit information metrics by assumption, the first order terms are zero. An immediate corollary is that the presence of cross terms in the metric implies the failure of factorization of fidelity into sub-fidelities ${ }^{2}$ As we will see shortly, this notion of factorization will be related to factorization in correlators.

### 3.3 CFT correlators

Our main focus will be four-point correlators in holographic $\mathrm{CFT}_{d}$. Some of the explicit expressions we give will be for $\mathrm{CFT}_{2}$ for simplicity. Nevertheless, we expect many of our conclusions apply more generally.

## Review: two-point function

Following [13, 72], we review the information metric for the Euclidean two-point function of scalar primaries. We work with real operators, which obey $(\mathcal{O}(x, \tau))^{\dagger}=$ $\mathcal{O}(x,-\tau)$ [73]. We begin with density matrix

$$
\begin{equation*}
\rho(x)=\frac{\mathcal{O}(x)|0\rangle\langle 0| \mathcal{O}\left(x^{*}\right)}{\left\langle\mathcal{O}(x) \mathcal{O}\left(x^{*}\right)\right\rangle} \tag{3.3.1}
\end{equation*}
$$

where $\frac{\mathcal{O}(x)|0\rangle}{\sqrt{\left\langle\mathcal{O}\left(x^{*}\right) \mathcal{O}(x)\right\rangle}}$ has unit norm. We use notation $x^{\mu}=\left(x^{i}, \tau\right)$ with $\left(x^{\mu}\right)^{*} \equiv\left(x^{i},-\tau\right)$ and suppress the indices in the arguments of $\mathcal{O}$ for compactness. Raised indices run over spatial coordinates while subscripts label external points. Expectation values are taken in the vacuum. The information distance is

$$
\begin{equation*}
D_{B}\left(\rho\left(x_{1}\right), \rho\left(x_{2}\right)\right)^{2}=2\left(1-\frac{\left|\left\langle\mathcal{O}\left(x_{1}^{*}\right) \mathcal{O}\left(x_{2}\right)\right\rangle\right|}{\sqrt{\left\langle\mathcal{O}\left(x_{1}^{*}\right) \mathcal{O}\left(x_{1}\right)\right\rangle\left\langle\mathcal{O}\left(x_{2}^{*}\right) \mathcal{O}\left(x_{2}\right)\right\rangle}}\right) . \tag{3.3.2}
\end{equation*}
$$

[^5]The CFT two-point function is fixed by conformal symmetry to be $\langle\mathcal{O}(x) \mathcal{O}(y)\rangle=$ $(x-y)^{-2 \Delta}$, where $\Delta$ is the scaling dimension of $\mathcal{O}$.

$$
\begin{equation*}
D_{B}\left(\rho\left(x_{1}\right), \rho\left(x_{2}\right)\right)^{2}=2\left(1-\frac{\left(4 \tau_{1} \tau_{2}\right)^{\Delta}}{\left(\left(x_{1}^{i}-x_{2}^{i}\right)^{2}+\left(\tau_{1}+\tau_{2}\right)^{2}\right)^{\Delta}}\right) . \tag{3.3.3}
\end{equation*}
$$

An information metric is obtained from the expansion

$$
\begin{equation*}
x_{2}^{\mu}=x_{1}^{\mu}+d x_{1}^{\mu} . \tag{3.3.4}
\end{equation*}
$$

The resulting metric describes the distinguishability of states created by inserting operators at nearby locations. The information metric in $\mathrm{CFT}_{2}$ is [13, 72

$$
\begin{equation*}
d s^{2}=\frac{\Delta}{2 \tau_{1}^{2}}\left(d x_{1}^{2}+d \tau_{1}^{2}\right) \tag{3.3.5}
\end{equation*}
$$

which is proportional to the metric of Poincare $\mathrm{AdS}_{3}$. See 72 for additional examples of this equivalence. The general dimension case is similar to the two-dimensional case. For relating $\mathrm{CFT}_{2}$ expressions to those in $\mathrm{CFT}_{d}$, it is useful to note that $\frac{d}{d x_{1}^{i}} \frac{d}{d x_{1}^{j}}\left(\left(x_{1}^{i}-x_{2}^{i}\right)^{2}+\left(\tau_{1}+\tau_{2}\right)^{2}\right)=2 \frac{d}{d x_{1}^{j}}\left(x_{1}-x_{2}\right)^{i}=2 \delta^{i j}$. This implies $g_{x^{i} x^{j}} \sim \delta^{i j}$ for the two-point function metric. As we expand about $x_{2}^{i}=x_{1}^{i}$, we also have $g_{x^{i} \tau}=g_{\tau x^{i}}=0$. The information metric in $d$-dimensions is therefore

$$
\begin{equation*}
d s^{2}=\frac{\Delta}{2 \tau_{1}^{2}}\left(\sum_{i}\left(d x_{1}^{i}\right)^{2}+d \tau_{1}^{2}\right) \tag{3.3.6}
\end{equation*}
$$

which is proportional to the Euclidean Poincare $\operatorname{AdS}_{d}$ metric. As the two-point function is determined by conformal symmetry, this metric is the same for all $\mathrm{CFT}_{d}$. The reduced density matrix obtained by tracing out a spatial subregion does probe theory-dependent information [13, 72, but we take a different approach here.

## The four-point function

In order to obtain a theory-specific information metric, we now turn to two-operator states,

$$
\begin{equation*}
\rho\left(x_{1}, x_{2}\right)=\frac{\mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{1}\left(x_{1}\right)|0\rangle\langle 0| \mathcal{O}_{1}\left(x_{1}^{*}\right) \mathcal{O}_{2}\left(x_{2}^{*}\right)}{\left\langle\mathcal{O}_{1}\left(x_{1}^{*}\right) \mathcal{O}_{2}\left(x_{2}^{*}\right) \mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{1}\left(x_{1}\right)\right\rangle} \tag{3.3.7}
\end{equation*}
$$

The Bures distance is

$$
\begin{align*}
& D_{B}\left(\rho\left(x_{1}, x_{2}\right), \rho\left(x_{3}, x_{4}\right)\right)^{2} \\
& \quad=2\left(1-\frac{\left|\left\langle\mathcal{O}_{1}\left(x_{3}^{*}\right) \mathcal{O}_{2}\left(x_{4}^{*}\right) \mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{1}\left(x_{1}\right)\right\rangle\right|}{\sqrt{\left\langle\mathcal{O}_{1}\left(x_{1}^{*}\right) \mathcal{O}_{2}\left(x_{2}^{*}\right) \mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{1}\left(x_{1}\right)\right\rangle\left\langle\mathcal{O}_{1}\left(x_{3}^{*}\right) \mathcal{O}_{2}\left(x_{4}^{*}\right) \mathcal{O}_{2}\left(x_{4}\right) \mathcal{O}_{1}\left(x_{3}\right)\right\rangle}}\right) \tag{3.3.8}
\end{align*}
$$

With $\tau_{1}<\tau_{2}<0<-\tau_{4}<-\tau_{3}$, the correlators above are time-ordered in Euclidean. ${ }^{3}$ This expression is a valid Bures distance for all $\tau_{i}<0$.

Various limits of $D_{B}^{2}$ are determined by familiar properties of the four-point function. $D_{B}^{2}$ is finite in the OPE limits $x_{12}^{2} \rightarrow 0, x_{34}^{2} \rightarrow 0$ and determined by the $\mathcal{O}_{1} \mathcal{O}_{2}$ OPE. In the limit $\tau_{i} \rightarrow 0$, the normalization factor diverges and gives $D_{B}^{2} \rightarrow 0$. As is standard, $\tau$ acts as a UV regulator for a state formed by inserting local operators, which would otherwise contain arbitrarily high energy excitations. Cluster decomposition implies that when we translate $x_{3}, x_{4}$ by a large distance,

$$
\begin{align*}
& D_{B}\left(\rho\left(x_{1}, x_{2}\right), \rho\left(x_{3}, x_{4}\right)\right)^{2} \\
& \quad \approx 2\left(1-\frac{\left|\left\langle\mathcal{O}_{1}\left(x_{1}^{*}\right) \mathcal{O}_{2}\left(x_{2}^{*}\right)\right\rangle\left\langle\mathcal{O}_{1}\left(x_{3}\right) \mathcal{O}_{2}\left(x_{4}\right)\right\rangle\right|}{\sqrt{\left\langle\mathcal{O}_{1}\left(x_{1}^{*}\right) \mathcal{O}_{2}\left(x_{2}^{*}\right) \mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{1}\left(x_{1}\right)\right\rangle\left\langle\mathcal{O}_{2}\left(x_{4}^{*}\right) \mathcal{O}_{1}\left(x_{3}^{*}\right) \mathcal{O}_{1}\left(x_{3}\right) \mathcal{O}_{2}\left(x_{4}\right)\right\rangle}}\right) \tag{3.3.9}
\end{align*}
$$

We consider the information metric obtained from the expansion

$$
\begin{equation*}
x_{3}^{\mu}=x_{1}^{\mu}+d x_{1}^{\mu}, \quad x_{4}^{\mu}=x_{2}^{\mu}+d x_{2}^{\mu} . \tag{3.3.10}
\end{equation*}
$$

One can check that the first-order terms are automatically zero,

$$
\begin{equation*}
\left.\frac{d}{d x_{3}^{\mu}}\right|_{\substack{x_{3}=x_{1} \\ x_{4}=x_{2}}} D_{B}\left(\rho\left(x_{1}, x_{2}\right), \rho\left(x_{3}, x_{4}\right)\right)^{2}=0,\left.\quad \frac{d}{d x_{4}^{\mu}}\right|_{\substack{x_{3}=x_{1} \\ x_{4}=x_{2}}} D_{B}\left(\rho\left(x_{1}, x_{2}\right), \rho\left(x_{3}, x_{4}\right)\right)^{2}=0 . \tag{3.3.11}
\end{equation*}
$$

In the next section, we find that the small- $\tau$ limit gives Euclidean Poincare AdS,

$$
\begin{equation*}
\tau_{k} \rightarrow 0: \quad g_{\mu \nu} d x^{\mu} d x^{\nu} \approx \frac{\Delta_{k}}{2} \frac{\sum_{i}\left(d x_{k}^{i}\right)^{2}+d \tau_{k}^{2}}{\tau_{k}^{2}} \tag{3.3.12}
\end{equation*}
$$

Specifically, we will show that the identity contribution to the OPE in the $13 \rightarrow$ 24 channel gives an asymptotically-AdS information metric. We expect this is the leading contribution to the information metric for general correlators, including at higher points.

As the four-point function is theory-dependent, we can study perturbative corrections. Suppose the states have a perturbative expansion in $\lambda_{1}$ about $\lambda$. The correlators and $D_{B}^{2}$ can also be expanded in $\lambda_{1}$. To every order in $\lambda_{1}-\lambda$, the Bures distance is 0 for $x_{3}=x_{1}, x_{2}=x_{4}$ because the states are identical at these locations by construction for all $\lambda_{1}, x_{3}=x_{1}, x_{2}=x_{4}$ is therefore a minimum of $D_{B}^{2}$ for any $\lambda_{1}$, and so the information metric that arises from expanding about this point is still

[^6]the leading correction to $D_{B}^{2}$ at all orders in $\lambda_{1}-\lambda$. (3.2.8) now follows, which is that the metric has an expansion to all orders in $\lambda_{1}-\lambda$ :
\[

$$
\begin{equation*}
g_{\mu \nu}=\left.\frac{d^{2}}{d x_{3}^{\mu} d x_{4}^{\nu}}\right|_{\substack{x_{3}=x_{1} \\ x_{4}=x_{2}}} D_{B}\left(\rho\left(x_{1}, x_{2}, \lambda_{1}\right), \rho\left(x_{3}, x_{4}, \lambda_{1}\right)\right)^{2}=\sum_{n=0} g_{\mu \nu}^{(n)}(\lambda)\left(\lambda_{1}-\lambda\right)^{n} . \tag{3.3.13}
\end{equation*}
$$

\]

The same argument applies to states created by $n$ operator insertions.

### 3.4 Four-point examples

We now demonstrate the statements in the previous section. We first study the MFT correlator, which captures the contribution of the identity operator to any CFT correlator. We find the MFT information metric is asymptotically AdS. We then consider a $1 / N^{2}$ correction in holographic CFTs computed by a tree Witten diagram in the bulk. We find that the tree-level contribution preserves the asymptotically AdS behavior of the information metric, consistent with the fact that the block decomposition of tree diagrams does not contain the identity exchange. We will often present $\mathrm{CFT}_{2}$ expressions for notational simplicity.

## Mean Field Theory

The MFT correlator is computed by taking Wick contractions as in free field theory. The MFT four-point function is

$$
\begin{equation*}
\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{3}\left(x_{3}\right) \mathcal{O}_{4}\left(x_{4}\right)\right\rangle_{M F T}=\frac{\delta_{\mathcal{O}_{1}, \mathcal{O}_{2}} \delta_{\mathcal{O}_{3}, \mathcal{O}_{4}}}{x_{12}^{2 \Delta_{1}} x_{34}^{2 \Delta_{3}}}+\frac{\delta_{\mathcal{O}_{1}, \mathcal{O}_{3}} \delta_{\mathcal{O}_{2}, \mathcal{O}_{4}}}{x_{13}^{2 \Delta_{1}} x_{24}^{2 \Delta_{4}}}+\frac{\delta_{\mathcal{O}_{1}, \mathcal{O}_{4}} \delta_{\mathcal{O}_{3}, \mathcal{O}_{2}}}{x_{14}^{2 \Delta_{1}} x_{32}^{2 \Delta_{2}}} . \tag{3.4.1}
\end{equation*}
$$

Choosing $\mathcal{O}_{3}=\mathcal{O}_{1}, \mathcal{O}_{4}=\mathcal{O}_{2}$ with $\mathcal{O}_{1} \neq \mathcal{O}_{2}$,

$$
\begin{equation*}
\left\langle\mathcal{O}_{1}\left(x_{1}\right) \mathcal{O}_{2}\left(x_{2}\right) \mathcal{O}_{1}\left(x_{3}\right) \mathcal{O}_{2}\left(x_{4}\right)\right\rangle_{M F T}=x_{13}^{-2 \Delta_{1}} x_{24}^{-2 \Delta_{2}} \tag{3.4.2}
\end{equation*}
$$

The Bures distance is
$D_{B}\left(\rho\left(x_{1}, x_{2}\right), \rho\left(x_{3}, x_{4}\right)\right)^{2}=2\left(1-\frac{\left(2 \tau_{1}\right)^{\Delta_{1}}\left(2 \tau_{2}\right)^{\Delta_{2}}\left(2 \tau_{3}\right)^{\Delta_{1}}\left(2 \tau_{4}\right)^{\Delta_{2}}}{\left(\left(x_{1}-x_{3}\right)^{2}+\left(\tau_{1}+\tau_{3}\right)^{2}\right)^{\Delta_{1}}\left(\left(x_{2}-x_{4}\right)^{2}+\left(\tau_{2}+\tau_{4}\right)^{2}\right)^{\Delta_{2}}}\right)$.
Using the expansion

$$
\begin{equation*}
x_{3}^{\mu}=x_{1}^{\mu}+d x_{1}^{\mu}, \quad x_{4}^{\mu}=x_{2}^{\mu}+d x_{2}^{\mu} \tag{3.4.4}
\end{equation*}
$$

the information metric for $\mathrm{CFT}_{d}$ is

$$
\begin{equation*}
d s^{2}=\frac{\Delta_{1}}{2 \tau_{1}^{2}}\left(\sum_{i}\left(d x_{1}^{i}\right)^{2}+d \tau_{1}^{2}\right)+\frac{\Delta_{2}}{2 \tau_{2}^{2}}\left(\sum_{i}\left(d x_{2}^{i}\right)^{2}+d \tau_{2}^{2}\right) . \tag{3.4.5}
\end{equation*}
$$

The lack of cross terms in the metric above is consistent with (3.2.11), as this MFT correlator factorized into products of lower-point correlators. In CFT language, this factorization is the statement that only the identity operator is exchanged in the $13 \rightarrow 24$ channel. Operators with the dimensions of double trace operators $\left[\mathcal{O}_{1} \mathcal{O}_{2}\right]$ are exchanged in other channels.

Next, we consider a MFT correlator with four identical operators,

$$
\begin{equation*}
\left\langle\mathcal{O}\left(x_{1}\right) \mathcal{O}\left(x_{2}\right) \mathcal{O}\left(x_{3}\right) \mathcal{O}\left(x_{4}\right)\right\rangle_{M F T}=\frac{1}{x_{12}^{2 \Delta} x_{34}^{2 \Delta}}+\frac{1}{x_{13}^{2 \Delta} x_{24}^{2 \Delta}}+\frac{1}{x_{14}^{2 \Delta} x_{32}^{2 \Delta}} \tag{3.4.6}
\end{equation*}
$$

Unlike the pairwise-identical case, this correlator does not factorize. The expression for the full metric is large but straightforward to obtain, so we only give some explicit expressions at specific values of $\Delta$. We have also checked that the first order terms vanish, confirming the information metric is the leading contribution to $D_{B}^{2}$. Even with $\Delta=1$ the full metric is a large expression, but it simplifies for $x_{1}=x_{2}$,

$$
\begin{align*}
d s^{2}= & \frac{\left(\tau_{1}-\tau_{2}\right)^{2}}{2 \tau_{1}^{2} \tau_{2}^{2}\left(\tau_{1}+\tau_{2}\right)^{2}\left(\tau_{1}^{4}+14 \tau_{2}^{2} \tau_{1}^{2}+\tau_{2}^{4}\right)^{2}} \\
& \left(\left(d \tau_{1}^{2} \tau_{2}^{2}+d \tau_{2}^{2} \tau_{1}^{2}\right)\left(\tau_{1}^{4}+62 \tau_{2}^{2} \tau_{1}^{2}+\tau_{2}^{4}\right)\left(\tau_{1}+\tau_{2}\right)^{4}+\left(\tau_{2}^{2} d x_{1}^{2}+\tau_{1}^{2} d x_{2}^{2}\right)\left(\tau_{1}+\tau_{2}\right)^{4}\left(\tau_{1}^{2}-\tau_{2}^{2}\right)^{2}\right. \\
& \left.+64\left(d x_{1} d x_{2} \tau_{1}^{4} \tau_{2}^{4}\left(\tau_{1}-\tau_{2}\right)^{2}-d \tau_{1} d \tau_{2} \tau_{1}^{3} \tau_{2}^{3}\left(\tau_{1}^{4}+3 \tau_{2} \tau_{1}^{3}+8 \tau_{2}^{2} \tau_{1}^{2}+3 \tau_{2}^{3} \tau_{1}+\tau_{2}^{4}\right)\right)\right) \tag{3.4.7}
\end{align*}
$$

The full metric (with $x_{1} \neq x_{2}$ ) has the limiting behavior

$$
\begin{equation*}
\tau_{1} \rightarrow 0: \quad d s^{2} \approx \frac{1}{2 \tau_{1}^{2}}\left(d x_{1}^{2}+d \tau_{1}^{2}\right) \tag{3.4.8}
\end{equation*}
$$

and similarly for $\tau_{2} \rightarrow 0$ due to symmetry in $\tau_{1}, \tau_{2}$. For general $\Delta$, we find

$$
\begin{equation*}
\tau_{i} \rightarrow 0: \quad d s^{2} \approx \frac{\Delta}{2 \tau_{i}^{2}}\left(d x_{i}^{2}+d \tau_{i}^{2}\right) \tag{3.4.9}
\end{equation*}
$$

which we verified explicitly in $d=2$ up to $\Delta=7$ for integer values of $\Delta$. As $x_{i}$ does not appear in the $\mathrm{CFT}_{2}$ expression above, we expect the same asymptotic behavior in $\mathrm{CFT}_{d}$.

The information metric in the identical operator case did not factorize. In the correlator, operators above the identity are exchanged in the $13 \rightarrow 24$ channel. All cross terms in the metric associated with an OPE channel are therefore proportional to the OPE coefficients of some operator exchanged in that channel.

## Holographic correction: tree level

Now we specialize to a holographic CFT. At each order in the $1 / N$ expansion, correlators of light single trace operators are computed by Witten diagrams in AdS.$^{4}$ The $\mathcal{O}\left(N^{0}\right)$ contribution is dual to free propagation in the bulk and is computed by MFT correlators. The next correction to four-point functions occurs at $\mathcal{O}\left(1 / N^{2}\right)$, and is computed by tree Witten diagrams. We consider pairwise identical operators $\mathcal{O}_{1}=\mathcal{O}_{3}, \mathcal{O}_{2}=\mathcal{O}_{4}$. We assume the bulk theory has a $\left(\phi_{1} \phi_{2}\right)^{2}$ vertex, where $\phi_{i}$ are dual to $\mathcal{O}_{i}$. The tree-level contribution is therefore the contact diagram

$$
\begin{equation*}
\mathcal{A}^{\phi_{1}^{2} \phi_{2}^{2}}\left(x_{i}\right)=\int_{A d S} d^{d+1} y \sqrt{-g} \prod_{i}^{4} K_{\Delta_{i}}\left(x_{i}, y\right) \equiv D_{\Delta_{1} \Delta_{2} \Delta_{1} \Delta_{2}}, \tag{3.4.10}
\end{equation*}
$$

where $K_{\Delta_{i}}(x, y)$ is the bulk to boundary propagator for the bulk field with boundary dual $\mathcal{O}_{i}$. For particular scaling dimensions, $D$-functions are known in closed form. For instance,

$$
\begin{equation*}
\frac{2 x_{13}^{2} x_{24}^{2}}{\Gamma\left(2-\frac{d}{2}\right)} D_{1111}\left(x_{i}\right)=\frac{1}{z-\bar{z}}\left(2 \operatorname{Li}_{2}(z)-2 \operatorname{Li}_{2}(\bar{z})+\log (z \bar{z}) \log \frac{1-z}{1-\bar{z}}\right) \tag{3.4.11}
\end{equation*}
$$

The contact diagram for other integer scaling dimensions can be found using $D$ function identities. We choose $\mathcal{O}_{1}, \mathcal{O}_{2}$ to be distinct scalars with equal dimension, $\Delta_{1}=\Delta_{2}=1$. We will once again work in $d=2$, in which $\Delta=1$ is above the unitarity bound. The information metric is found by expanding $D_{B}^{2}$ in the small parameter $1 / N^{2}$. We have checked explicitly that the first order terms, $\frac{1}{N^{2}} d x_{i}^{\mu}$, are zero. The leading contribution to $D_{B}^{2}$ therefore comes from the information metric. The metric is

$$
\begin{equation*}
d s^{2}=\frac{d x_{1}^{2}+d \tau_{1}^{2}}{2 \tau_{1}^{2}}+\frac{d x_{2}^{2}+d \tau_{2}^{2}}{2 \tau_{2}^{2}}+\sum_{i, j} \frac{1}{N^{2}} g_{i j}^{(2)} d x^{i} d x^{j} \tag{3.4.12}
\end{equation*}
$$

where the leading term is the metric of pairwise identical MFT correlator studied earlier. The leading term factorizes but the $1 / N^{2}$ correction does not. As in the MFT case, the explicit form of the $1 / N^{2}$ contribution to the metric is lengthy. With $x_{1}=x_{2}$, the metric takes a simpler form,

$$
\begin{aligned}
& d s^{2}=\frac{\pi^{d / 2} \Gamma\left(2-\frac{d}{2}\right)}{\left(\tau_{1}-\tau_{2}\right)^{4}\left(\tau_{1}+\tau_{2}\right)^{4}} \\
& \left(-8\left(\tau_{1}+\tau_{2}\right)^{2}\left(\tau_{1}-\tau_{2}\right)^{2}\left(\tau_{1}^{2}\left(d x_{2}^{2}+d \tau_{2}^{2}\right)-\tau_{2} \tau_{1}\left(d \tau_{1} d \tau_{2}+d x_{1} d x_{2}\right)+\tau_{2}^{2}\left(d \tau_{1}^{2}+d x_{1}^{2}\right)\right)\right. \\
& \left.+\left(\tau_{1}-\tau_{2}\right)^{4} \log \left(\frac{\left(\tau_{1}-\tau_{2}\right)^{4}}{16 \tau_{1}^{2} \tau_{2}^{2}}\right) X_{-}\left(\tau_{1}, \tau_{2}\right)-2\left(\tau_{1}+\tau_{2}\right)^{4} \log \left(\frac{\left(\tau_{1}+\tau_{2}\right)^{2}}{4 \tau_{1} \tau_{2}}\right) X_{+}\left(\tau_{1}, \tau_{2}\right)\right)
\end{aligned}
$$

[^7]where
\[

$$
\begin{align*}
& X_{-}\left(\tau_{1}, \tau_{2}\right)=4\left(\tau_{1}^{2}\left(d x_{2}^{2}+d \tau_{2}^{2}\right)+\tau_{2}^{2}\left(d x_{1}^{2}+d \tau_{1}^{2}\right)\right)+\left(d x_{1} d x_{2}+d \tau_{1} d \tau_{2}\right)\left(\tau_{1}^{2}-6 \tau_{2} \tau_{1}+\tau_{2}^{2}\right) \\
& X_{+}\left(\tau_{1}, \tau_{2}\right)=-4\left(\tau_{1}^{2}\left(d x_{2}^{2}+d \tau_{2}^{2}\right)+\tau_{2}^{2}\left(d x_{1}^{2}+d \tau_{1}^{2}\right)\right)+\left(d x_{1} d x_{2}+d \tau_{1} d \tau_{2}\right)\left(\tau_{1}+\tau_{2}\right)^{2} \tag{3.4.13}
\end{align*}
$$
\]

We have checked numerically that the full metric $\left(x_{1} \neq x_{2}\right)$ obeys

$$
\begin{equation*}
\lim _{\tau_{i} \rightarrow 0} \tau_{i} g_{\mu \nu}^{(2)}\left(\tau_{1}, \tau_{2}, x_{1}, x_{2}\right)=0 \tag{3.4.14}
\end{equation*}
$$

In other words, $g_{\mu \nu}^{(2)}\left(\tau_{1}, \tau_{2}, x_{1}, x_{2}\right)$ does not change the $1 / \tau_{i}^{2}$ divergence we found coming from the MFT contribution. The metric therefore remains asymptotically AdS up to order $1 / N^{2}$.

### 3.5 Transition amplitudes

We now discuss transition amplitudes and find somewhat different structure from the correlation function case. Nevertheless, we find that transition amplitudes admit an information metric in a certain sense. We study a quantum-mechanical setup that describes relevant features of transition amplitudes in quantum field theory. Consider a transition between states $\left|\psi_{f}\right\rangle,\left|\psi_{i}\right\rangle$ induced by unitary $U$. The transition amplitude is $\left\langle\psi_{f}\right| U\left|\psi_{i}\right\rangle$. In order to extract an information metric, we must expand about $D_{B}^{2}=0$, but for $U \neq 1$, this does not necessarily occur when $\left|\psi_{i}\right\rangle=\left|\psi_{f}\right\rangle$.

With density matrices $\rho_{i}=\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$ and $\rho_{f}=\left|\psi_{f}\right\rangle\left\langle\psi_{f}\right|$, the following Bures distance contains the transition amplitude.

$$
\begin{equation*}
D_{B}\left(U^{\dagger} \rho_{i} U, \rho_{f}\right)^{2}=2\left(1-\sqrt{U^{\dagger} \rho_{i} U \rho_{f}}\right) \tag{3.5.1}
\end{equation*}
$$

Suppose $U=e^{-i \lambda H}$ for some dimensionless hermitian operator $H$.

$$
\begin{equation*}
\left.D_{B}\left(U^{\dagger} \rho_{i} U, \rho_{f}\right)^{2}=2\left(1-\left|\left\langle\psi_{f}\right| U\right| \psi_{i}\right\rangle \mid\right) \tag{3.5.2}
\end{equation*}
$$

Expanding in $\lambda$,

$$
\begin{align*}
D_{B}\left(U^{\dagger} \rho_{i} U, \rho_{f}\right)^{2} \approx & \approx\left(1-\left(\left|\left\langle\psi_{f} \mid \psi_{i}\right\rangle\right|^{2}-2 \lambda \operatorname{Im}\left(\left\langle\psi_{f}\right| H\left|\psi_{i}\right\rangle\left\langle\psi_{i} \mid \psi_{f}\right\rangle\right)\right.\right. \\
& \left.\left.\left.\left.+\left.\lambda^{2}\left(\left|\left\langle\psi_{i}\right| H\right| \psi_{f}\right\rangle\right|^{2}-\frac{1}{2}\left|\left\langle\psi_{i}\right| H^{2}\right| \psi_{f}\right\rangle \mid\left(\left\langle\psi_{i} \mid \psi_{f}\right\rangle+\left\langle\psi_{f} \mid \psi_{i}\right\rangle\right)\right)\right)^{1 / 2}\right) \tag{3.5.3}
\end{align*}
$$

where $\operatorname{Im}(a+i b) \equiv b$. If we choose $\left|\psi_{i}\right\rangle=\left|\psi_{f}\right\rangle$, the order $\lambda$ term above becomes zero. Expanding in $\lambda$ then gives

$$
\begin{equation*}
D_{B}\left(U^{\dagger} \rho_{i} U, \rho_{f}\right)^{2} \approx \lambda^{2}\left(\left\langle H^{2}\right\rangle-\langle H\rangle^{2}\right) . \tag{3.5.4}
\end{equation*}
$$

At $\left|\psi_{f}\right\rangle=\left|\psi_{i}\right\rangle$, the transition amplitude therefore admits the information metric at order $\mathcal{O}\left(\lambda^{2}\right)$

$$
\begin{equation*}
d s^{2}=d \lambda^{2}\left(\left\langle H^{2}\right\rangle-\langle H\rangle^{2}\right) \tag{3.5.5}
\end{equation*}
$$

For $\left|\psi_{i}\right\rangle \neq\left|\psi_{f}\right\rangle$, the states that have $D_{B}^{2}=0$ are $\left|\psi_{f}\right\rangle=U\left|\psi_{i}\right\rangle$. If we allow the states to vary independently, the full metric is the sum of (3.5.5) and the $\lambda=0$ metric.

To understand this discussion more explicitly, consider the following two qubit system.

$$
\begin{equation*}
\left|\psi_{i}\right\rangle=\binom{\cos \theta}{\sin \theta} \otimes\binom{\cos \phi}{\sin \phi}, \quad \quad\left|\psi_{f}\right\rangle=\binom{\cos \theta^{\prime}}{\sin \theta^{\prime}} \otimes\binom{\cos \phi^{\prime}}{\sin \phi^{\prime}} \tag{3.5.6}
\end{equation*}
$$

For $\lambda=0$,

$$
\begin{equation*}
F\left(U^{\dagger} \rho_{i} U, \rho_{f}\right)=\left|\cos \left(\theta^{\prime}-\theta\right) \cos \left(\phi^{\prime}-\phi\right)\right|^{2} \tag{3.5.7}
\end{equation*}
$$

Expanding with $\theta^{\prime}=\theta+d \theta, \phi^{\prime}=\phi+d \phi$, the information metric is

$$
\begin{equation*}
d s^{2}=d \theta^{2}+d \phi^{2} \tag{3.5.8}
\end{equation*}
$$

Turning on the interaction $H=\sigma_{z}^{1} \otimes \sigma_{z}^{2}$ gives

$$
\begin{equation*}
D_{B}\left(U^{\dagger} \rho_{i} U, \rho_{f}\right)^{2} \approx 2\left(1-\sqrt{\left(\cos \left(\theta^{\prime}-\theta\right) \cos \left(\phi^{\prime}-\phi\right)\right)^{2}+\lambda^{2}\left(\cos \left(\theta^{\prime}+\theta\right) \cos \left(\phi^{\prime}+\phi\right)\right)^{2}}\right) \tag{3.5.9}
\end{equation*}
$$

Because the states and $H$ chosen were real, there is no $\mathcal{O}(\lambda)$ term above. Nevertheless, we still must check whether expanding about the point $\theta^{\prime}=\theta+d \theta, \phi^{\prime}=\phi+d \phi$ gives a consistent information metric.

$$
\begin{align*}
D_{B}\left(U^{\dagger} \rho_{i} U, \rho_{f}\right)^{2} \approx & \lambda^{2}\left(1-\cos ^{2} 2 \theta \cos ^{2} 2 \phi\right) \\
& +d \phi\left(\lambda^{2} \cos ^{2} 2 \theta \sin 4 \phi\right)+d \theta\left(\lambda^{2} \sin 4 \theta \cos ^{2} 2 \phi\right) \\
& -d \theta d \phi\left(\lambda^{2} \sin 4 \theta \sin 4 \phi\right)+d \theta^{2}\left(1+\frac{\lambda^{2}}{4}(3 \cos 4 \theta-1) \cos ^{2} 2 \phi\right) \\
& +d \phi^{2}\left(1+\frac{\lambda^{2}}{4}(3 \cos 4 \theta-1) \cos ^{2} 2 \theta\right) . \tag{3.5.10}
\end{align*}
$$

The $\mathcal{O}(d \theta, d \phi)$ terms are first order, proportional to $\lambda$, and non-zero. This was expected from the fact that $D_{B}^{2} \neq 0$ at $\theta=\theta^{\prime}, \phi=\phi^{\prime}$ once the interaction $H$ is
included. We therefore have no meaningful information metric in $d \phi, d \theta$ at higher order in $\lambda$. The expansion of the information metric in $\lambda$ terminates,

$$
\begin{equation*}
d s^{2}=d \theta^{2}+d \phi^{2}+d \lambda^{2}\left(1-\cos ^{2} 2 \theta \cos ^{2} 2 \phi\right) \tag{3.5.11}
\end{equation*}
$$

which agrees with (3.5.5) with $d \theta=d \phi=0$.
Applying this approach to the S-matrix may require some modification. Consider a unitary matrix $S$ written as $S=1+i T$. To obtain a form similar to (3.5.5), we write

$$
\begin{equation*}
S=e^{-i H_{S}} \tag{3.5.12}
\end{equation*}
$$

where $H_{S}^{\dagger}=H_{S}$. Suppose $H_{S}$ can be expanded in a small parameter, $H_{S}=$ $\sum_{n} \lambda^{n} H_{S}^{(n)}$. This leads to

$$
\begin{equation*}
d s^{2}=d \lambda^{2}\left(\left\langle\left(H_{S}^{(1)}\right)^{2}\right\rangle-\left\langle H_{S}^{(1)}\right\rangle^{2}\right), \tag{3.5.13}
\end{equation*}
$$

of which (3.5.5) is a special case. 3.5.13 corresponds to a transition amplitude with identical initial and final states. $\sqrt{5}$ According to the quantum Cramer-Rao theorem, the error in estimating $\lambda$ from measuring the states is bounded from below by $\left(\left\langle\left(H_{S}^{(1)}\right)^{2}\right\rangle-\left\langle H_{S}^{(1)}\right\rangle^{2}\right)^{-1}$.
(3.5.13) is a completely general formula for transition amplitudes. It applies to transition amplitudes in position space as well as momentum space. Transition amplitudes have been studied in AdS/CFT [74, 75, 76, 77, 78, 79, 65, 80. The modular Hamiltonian $K_{A}=\log \rho_{A}$ generates unitary evolution within subsystem $A$, where states are defined on slices of constant modular time. (3.5.13) therefore applies to transition amplitudes within the domain of dependence of $A$. It may be interesting to note that the quantity $\left\langle K_{A}^{2}\right\rangle-\left\langle K_{A}\right\rangle^{2}$ has been studied recently 81 , 82, 83].

### 3.6 Future directions

We found that factorization, the OPE, and the $1 / N$ expansion are encoded by information metric of correlators. It would be natural to flesh out this information geometry description: one can explore higher points, odd points, Lorentzian signature, operators with spin, twist operators, and so on. The interplay between $1 / N$ corrections and quantum information ideas can be explored in this context. Special cases of our results yield the information metric of the two-point function in

[^8]excited states, and of multitrace operators. Applications of more sophisticated ideas in quantum information geometry may produce new constraints on CFT data. We conclude by discussing a few directions in more detail.

The information metric in principle encodes some or all of the same information as the original correlator. In this way, the information metric geometrizes the correlator in a seemingly novel fashion. It would be interesting if this description served as a useful organizing tool for CFT data. However, note that the information metric is derived from the normalized four-point function, which is a ratio of correlators and not a correlator itself. It would be interesting to understand this object better, though its appearance may suggest that quantities that are natural in information geometry are obscured in standard correlator language. In this spirit, it may be useful to understand what CFT features are encoded by the curvature scalar and tensors of the information metric. As multiple OPE channels are encoded by the information metric, can we impose crossing as a condition on the information geometry? If so, can this be used to derive new constraints on OPE data? It would be interesting to identify the information geometry of a single conformal block. On a more basic level, how do conformal transformations of the correlator act on the information metric?

We have shown that cross terms in the metric signal a failure of factorization. They also represent non-trivial interplay between several different parts of the information geometry boundary, each of which is asymptotically AdS. It may be interesting to develop a better understanding of the full geometry. One could also ask whether higher-genus manifolds are allowed, and if so, what this would imply for the correlator. More modestly, can information manifolds of correlators have conjugate points, which appear in studies of complexity [84, 85]?

The relationship between OPE data and complexity in holographic CFTs is not yet well-understood, though is natural to explore in light of recent work 86, 87]. Ideas used in our work may be useful for studies of CFT complexity. Computing $1 / N$ corrections to complexity may clarify its possible bulk dual 88 .

In short, we have shown that information geometry provides a new representation of a large class of CFT correlators. While the usefulness of this representation remains to be seen, many new avenues are now open for exploration.

## Chapter 4 Classical codes, Chiral CFTs and Lattices

This chapter is essentially identical to:
Classical codes and chiral CFTs at higher genus [22]

### 4.1 Introduction: Chiral CFTs and classical error correcting codes

Two-dimensional conformal field theories are among the best understood quantum field theories, and yet a lot remains unknown. This is especially true for theories with central charge $c>1$, where unitarity allows for infinite-dimensional representations of the Virasoro algebra. The resulting theories are, in general, much more complicated than those theories with $c<1$. One powerful method for studying 2 d conformal field theories (CFTs) is the modular bootstrap [89], which constrains the torus partition function based on the requirement that it be invariant under the $\operatorname{PSL}(2, \mathbb{Z})$ group of modular transformations. The constraints from genus 1 modular invariance have been used to derive universal bounds on the spectrum, including bounds on the dimension gap, twist gap, and operator degeneracies $17,90,91,92$, 93, 94, 95, 96, 97, 98, 99, 100, 101, 20.

The torus partition function fully specifies the spectrum of the theory, but contains no information about the dynamics, i.e. the OPE coefficients. For this reason, it is unable to completely define a theory. This is clear from specific cases where different theories have the same genus 1 partition function, the most famous of which arise from Milnor's example of two isospectral self-dual even lattices in 16 dimensions which define isospectral CFTs with $c=16$ 102. Perhaps more surprisingly, there exist examples of modular invariant functions, decomposable in Virasoro characters with non-negative integer multiplicities, which do not correspond to any known CFT; for an example in the non-chiral case, see section 6 of [19]. The dynamical information captured by the OPE coefficients can, in principle, be addressed using the full conformal bootstrap method [103] to bound sphere four-point functions using crossing symmetry, however these constraints are not easily combined with those of modular invariance.

Considering higher genus partition functions might offer a way to bound the spectrum and OPE coefficients at the same time. It was argued in 104 that consistency of 2d CFTs with crossing symmetry and modular invariance implies that the theory can be consistently defined on Riemann surfaces of any genus. The same Riemann
surface can be obtained by sewing together simple three-holed spheres in different ways - higher genus consistency essentially means that the resulting partition function must not depend on the sewing procedure. This leads to the requirement that the partition function is invariant under the full genus- $g$ modular group $\operatorname{Sp}(2 g, \mathbb{Z})$. Unlike the torus partition function, the higher genus partition functions do contain information about the OPE coefficients. This leads to the natural question: to what degree does the set of higher genus partition functions characterize a 2d CFT? The view that a 2d CFT may be defined by its vacuum amplitudes for all genera, originally advanced in [105], has been addressed more recently in 106, 107. Other recent interest in deriving universal bounds from higher genus modular invariance include 108, 109, 110.

In this chapter, we will use 2d CFTs defined by error-correcting codes (ECCs) to study higher genus modular invariance. Simply speaking, an $n$-dimensional code $\mathcal{C}$ is a collection of binary vectors, "codewords," of length $n$. These vectors can be visualized as vertices on a unit hypercube. Importantly, this means that an error correcting code can be used to define a lattice, simply by identifying the vectors in the code with the unit cell of the lattice (this is the so-called "Construction A," of Leech and Sloane [111]). This, in turn, allows us to define a chiral CFT with central charge $c=n$ as the compactification of $n$ free bosons on this lattice [18]. The relationship between code and CFTs has been known for a long time 112, 113]. Recently it has been shown [21, 19] that the correspondence between classical codes and chiral CFTs has a generalization to quantum error correcting codes and full non-chiral CFTs.

Code CFTs provide an interesting setting for studying higher genus modular invariance for two main reasons. The first is that a number of codes are known explicitly, allowing us to construct higher genus partition functions of true CFTs directly by computing their higher-weight enumerator polynomial. The second is that the higher genus modular transformations act linearly on partition functions which take the form of code enumerator polynomials ${ }^{1}$ This means that we can actually solve the transformations explicitly. By requiring that the enumerator polynomial has positive integer coefficients, we can list every partition function which could possibly derive from an error-correcting code. Finally, we impose a further constraint, called "factorization limits:" when the higher genus Riemann surface degenerates into two lower-genus Riemann surfaces connected by an infinitely thin, long tube, then

[^9]the higher genus partition function must factorize into the products of lower-genus partition functions.

For cases where the full set of codes is known, such as for $c=24$, we find that this procedure correctly reproduces every code partition function, plus a number of "fake" partition functions, which are known not to correspond to codes. Whether or not they correspond to CFTs at all is an open question. Interestingly, imposing constraints from higher genus decreases the number of fake partition functions. For example, at $c=24$, there are only 9 codes. At genus 1 , there are 190 partition functions consistent with modular invariance. However, we find that only 29 of these partition functions can arise from modular invariant genus 2 partition functions via factorization. Only 21 can arise from genus 3 partition functions. We speculate that performing the algorithm laid out in this chapter for arbitrarily high genus would eliminate every partition function except those arising from ECCs.

For genus $g \leqslant 3$ and central charge $c=24$, we can look at the location of the true theories inside the space of all allowed theories, revealing an interesting geometric structure. The requirement that the partition functions have positive integer coefficients means that the allowed partition functions form a discrete set indexed by a few coefficients which obey simple linear inequalities. More specifically, these coefficients live in a polytope whose dimension is equal to one less than the number of independent Siegel modular forms with weight $c / 2$. This dimension grows as the genus grows, but the higher genus polytope can always be projected to the lower-genus space, and its projection must be inside the polytope defined by the lower-genus constraints. This means that each genus gives stricter constraints than the previous one, at least for $g \leqslant 3$.

The outline of the chapter is as follows. In section 4.2, we review the correspondence described above in the case of chiral CFTs at genus 1. In section 4.3, we show how to extend the correspondence between codes and CFTs to higher genus, focusing on the higher-weight enumerator polynomial and how it is constrained by the symmetries and factorization properties required by the CFT. We also comment on some issues related to genus $g>3$. In section 4.4, we show explicitly how to use these constraints to determine every allowed enumerator polynomial for a given genus $g$ and dimension $c$. We do this for $c$ up to 48 for genus 2 and $c$ up to 24 for genus 3. We show where the real ECCs lie in the space of allowed enumerator polynomials. We also use our results to fix the genus 2 Siegel modular forms in terms of the polynomial variables of ECCs.

Table 4.1: Dictionary relating error correcting codes and CFTs.

| Code $\mathcal{C}$ | Lattice $\Lambda$ | Chiral CFT |
| :--- | :--- | :--- |
| code dimension $n$ | dimension $n$ of $\mathbb{R}^{n}$ | central charge $c=n$ |
| codewords $c \in \mathbb{F}_{2}^{n}$ | lattice vectors $\vec{v}$ | states |
| weight $g$ enumerator <br> polynomial $W_{\mathcal{C}}^{(g)}$ | genus g theta series $\Theta_{\Lambda}^{g}$ | genus $g$ partition func- <br> tion $Z^{(g)}$ |
| code variables $x_{i}=$ <br> $x_{0}, x_{1}, \ldots$ | theta constants $\vartheta_{i}(\Omega)$ <br> 4.3 .20 | theta constants $\vartheta_{i}(\Omega)$ |
| Hamming weight $w(c)$ | vector length $\\|v\\|$ | dimension $h$ |
| doubly-even self dual | even self-dual | modular invariant |

### 4.2 Review: Classical Codes and Chiral CFTs

In this section, we will review the relation between classical error-correcting codes and two-dimensional chiral CFTs. See table 4.1 for a summary.

## Error-correcting codes and enumerator polynomials

In this chapter, we are considering the problem of listing all possible enumerator polynomials by generating all polynomials consistent with a set of particular symmetries. So let us first review how classical error-correcting codes define enumerator polynomials, and their relation to CFTs. For a more extensive recent review of these topics, see [19. For an earlier review, see 114, 115].

A binary code $\mathcal{C}$ is a set of codewords $c \in \mathbb{F}_{2}^{n}$, which are length $n$ vectors over $\mathbb{F}_{2}$, the finite field of two elements. The idea of an error-correcting code is to facilitate communication over a noisy channel: if a message is encoded, the message may still be readable even when a certain number of flipped bits (errors) occur between the sender and receiver. The simplest example is the repetition code. The length 3 repetition code is defined by the encoding

$$
\begin{align*}
(0) & \rightarrow(000)  \tag{4.2.1}\\
(1) & \rightarrow(111)
\end{align*}
$$

Suppose that each bit has a $10 \%$ chance of being flipped when transmitting the message. If the original message is encoded as (1), then it has a $90 \%$ chance of being
read correctly by the receiver. But if the sender instead transmits (111), then the receiver may interpret the original message (1) from any of (111) (no errors) or (110), (101), or (011) (one error). If there are two or three errors, however, the receiver will incorrectly reconstruct the original message. So the receiver has a $97.2 \%$ chance of correctly interpreting the message.

In the example above, the receiver reconstructed the message by replacing a three-bit signal, (e.g. (110)), with the "nearest" codeword (e.g. (111)). Nearest, in this case, means with the lowest Hamming distance $d$. For two vectors $c_{1}, c_{2} \in \mathbb{F}_{2}^{n}$, the Hamming distance is defined as the number of bits in $c_{1}$ and $c_{2}$ which are different, equivalently $\left|c_{1}-c_{2}\right|$. The Hamming distance for a code $\mathcal{C}$ is defined as the minimum of the Hamming distance between each distinct elements $c \in \mathcal{C}$. The Hamming distance gives a measure of error-correcting abilities; a code with Hamming distance $d$ can correct errors in up to $t=\lfloor(d-1) / 2\rfloor$ errors. A code which has $2^{k}$ codewords living in $\mathbb{F}_{2}^{n}$, and which has Hamming distance $d$, is called a $[n, k, d]$ code. For example, the repetition code defined in (4.2.1) is a $[3,1,3]$ code, and is thus able to correct up to 1 error.

The key to error-correcting codes is redundancy - the more redundancy there is, the more errors are allowed before the receiver will misinterpret the message. For example, if we used (00000) and (11111) as the codewords, we would find $99.144 \%$ chance of guessing the correct message. This would correspond to a $[5,1,5]$ code, and could therefore detect up to 2 errors. A central question in coding theory is how to design codes which give the highest error-correcting ability, represented by $d$, for fixed values of $n$ and $k$.

## Doubly-even self-dual codes

To proceed, we will need to invoke a few more definitions from coding theory. A code is linear if the sum of two codewords is a codeword. From here on, we will only consider linear codes. For a linear code $\mathcal{C}$, we can define the dual code $\mathcal{C}^{\perp}$

$$
\begin{equation*}
\mathcal{C}^{\perp}=\{\tilde{c} \mid \tilde{c} \cdot c \equiv 0(\bmod 2) \forall c \in \mathcal{C}\} . \tag{4.2.2}
\end{equation*}
$$

If $\mathcal{C}$ is a type $[n, k, d]$ code, then its dual $\mathcal{C}^{\perp}$ will be of type $\left[n, n-k, d^{\prime}\right]$. A code for which $\mathcal{C}=\mathcal{C}^{\perp}$ is called self-dual. Clearly, this is only possible when $n=2 k$.

We also need to define the Hamming weight: for a vector $c \in \mathbb{F}_{2}^{n}$, the Hamming weight $w(c)$ is simply the number of 1 s in $c: w(c)=|c|$. A linear code is called even if the Hamming weight of all codewords is divisible by two. It is doubly-even if all Hamming weights are divisible by four. Looking ahead, doubly-even self-dual
codes are particularly relevant since they define lattices which are even and self-dual. Such lattices define chiral CFTs according to a construction by Dolan, Goddard and Montague 112, 113.

## Enumerator polynomials

A convenient quantity for describing codes is the (weight one) enumerator polynomial, defined by

$$
\begin{equation*}
W_{\mathcal{C}}\left(x_{0}, x_{1}\right)=\sum_{c \in \mathcal{C}} x_{0}^{n-w(c)} x_{1}^{w(c)} \tag{4.2.3}
\end{equation*}
$$

The enumerator polynomial essentially counts all of the codewords with given Hamming weight. Linear codes always contain the codeword $\overrightarrow{0}$, so the enumerator polynomial always contains the monomial $x_{0}^{n}$ with coefficient 1 . The coefficients of each monomial are all positive integers because they are degeneracies, and they must add up to $2^{k}$. For example, the simplest self-dual double even code, the Hamming [8, 4, 4] code has the enumerator polynomial

$$
\begin{equation*}
W_{\text {Hamming }}\left(x_{0}, x_{1}\right)=x_{0}^{8}+14 x_{0}^{4} x_{1}^{4}+x_{1}^{8} \tag{4.2.4}
\end{equation*}
$$

The enumerator polynomial describes a code in terms the Hamming weights of its elements much like the partition function describes a CFT in terms of the dimensions of its states; later, we will describe how this analogy is precise for lattice CFTs. This analogy also serves to emphasize that it is not, in general, possible to extract the entire code its (genus 1) enumerator polynomial.

Code duality $\mathcal{C} \mapsto \mathcal{C}^{\perp}$ can be expressed as a transformation on the enumerator polynomials according to the MacWilliams identity [116]:

$$
\begin{equation*}
x_{0} \mapsto \frac{x_{0}+x_{1}}{\sqrt{2}}, \quad x_{1} \mapsto \frac{x_{0}-x_{1}}{\sqrt{2}} . \tag{4.2.5}
\end{equation*}
$$

Enumerator polynomials for self-dual codes must therefore be invariant under this transformation. Furthermore, doubly-even codes must be invariant under

$$
\begin{equation*}
x_{1} \mapsto i x_{1} \tag{4.2.6}
\end{equation*}
$$

The transformations (4.2.5) and 4.2.6 are directly related to the $S$ and $T$ modular transformations on CFT partition functions. They represent a powerful set of constraints on the possible enumerator polynomials. This idea is central to the method of this chapter. At a given polynomial degree, we can solve these constraints to find
the most general set of polynomials which are invariant under these transformations. If we require that the coefficients are positive and integer, there are finite number of solutions which may be enumerated explicitly. This will give a list which must include the enumerator polynomials of every real code. However, in general it will also include a number of "fake" enumerator polynomials. We cannot tell if a polynomial corresponds to a real code without additional information. This motivates our study of higher-weight enumerator polynomials, which are related to the higher genus partition functions.

## Lattices and CFTs from codes: Construction A

An error-correcting code can be viewed as a collection of points in $\mathbb{F}_{2}^{n}$. Let us imagine embedding this cube into $\mathbb{Z}^{n}$, and then requiring symmetry under translating any direction by two. The result is a lattice in $\mathbb{Z}^{n}$. This is known as Construction A, originally due to Leech and Sloane [117]. More precisely, the lattice $\Lambda$ associated to a code $\mathcal{C}$ by Construction A is defined as

$$
\begin{equation*}
\Lambda(\mathcal{C})=\left\{v / \sqrt{2} \mid v \in \mathbb{Z}^{n}, v \equiv c(\bmod 2), c \in \mathcal{C}\right\} \tag{4.2.7}
\end{equation*}
$$

Now we may explain why we are concerned with doubly-even self-dual codes. First, consider a doubly-even code. The Hamming weight of all codewords must be divisible by four. If $v \equiv c(\bmod 2)$, and $c$ is a codeword whose weight is divisible by 4 , then $v^{2}$ is also divisible by 4 . The element of the lattice, $v / \sqrt{2}$, therefore has a square-length that is divisible by 2 . A lattice where every vector has an even norm is, by definition, an even lattice.

Next consider the lattice of the dual of code, $\Lambda\left(\mathcal{C}^{\perp}\right)$. Consider an element $\tilde{v} / \sqrt{2}$ of this lattice, and take its inner product with an element $v / \sqrt{2}$ of the lattice $\Lambda(\mathcal{C})$.

$$
\begin{align*}
\frac{\tilde{v}}{\sqrt{2}} \cdot \frac{v}{\sqrt{2}} & =\frac{1}{2}(\tilde{c}+2 \vec{a}) \cdot(c+2 \vec{b}) \\
& =\frac{1}{2} \tilde{c} \cdot c+\vec{a} \cdot c+\vec{b} \cdot \tilde{c}+2 \vec{a} \cdot \vec{b} \tag{4.2.8}
\end{align*}
$$

where in the first line, we have used the definition of an element of a code lattice, with $\vec{a}$ some element of $\mathbb{Z}^{n}$. The dual of a lattice is defined as the lattice of points with integer inner products with points in the original lattice. The inner product in (4.2.8) is an integer if and only if $\tilde{c} \cdot c$ is even. But this is the definition of the dual code $\mathcal{C}^{\perp}$. Therefore the lattice of the dual of a code is the same as the dual of the lattice of a code:

$$
\begin{equation*}
\Lambda\left(\mathcal{C}^{\perp}\right)=\Lambda(\mathcal{C})^{*} \tag{4.2.9}
\end{equation*}
$$

In summary, each doubly-even self-dual error-correcting code defines an even selfdual lattice via Construction A. These lattices are particulary interesting because they are the precisely the lattices which define conformal field theories. This idea was originally discussed by Narain [118, 119] for the case of Lorentzian lattices. The idea is to consider a free CFT compactified on a lattice. Then requiring modular invariance of the partition function amounts to the requirement that the lattice is even and self-dual. In the present case, we are discussing Euclidean lattices. Compactifying $c$ free bosons on a $c$-dimensional lattice results in a chiral CFT with central charge $c$.

## Lattice theta-functions

We will need one more element which will make the connection between codes and CFTs clearer - the lattice theta function. On one hand, the lattice theta-function can be related to the partition function of the CFT corresponding to free bosons compactified on that lattice. On the other hand, for a lattice described by a code, the lattice theta function will be directly related to the enumerator polynomial. An integral self-dual lattice $\Lambda$ of rank $n=0(\bmod 8)$, equipped with a Euclidean metric can be associated with a lattice theta series, or theta function, $\Theta_{\Lambda}$. The (genus 1) theta-function of a lattice is defined by

$$
\begin{equation*}
\Theta_{\Lambda}(\tau)=\sum_{v \in \Lambda} q^{v^{2} / 2}, \quad q=e^{2 \pi i \tau} \tag{4.2.10}
\end{equation*}
$$

The theta-function of a $c$-dimensional even self-dual lattice is related to the partition function of a CFT by

$$
\begin{equation*}
Z(\tau)=\frac{\Theta_{\Lambda}(\tau)}{\eta(\tau)^{c}} \tag{4.2.11}
\end{equation*}
$$

Here $\eta(\tau)$ denotes the Dedekind eta function

$$
\begin{equation*}
\eta(\tau)=e^{\frac{2 i \pi \tau}{24}} \prod_{k=1}^{\infty}\left(1-e^{2 \pi i k \tau}\right) \tag{4.2.12}
\end{equation*}
$$

Partition functions of 2d CFTs are required to be invariant under the modular transformations

$$
\begin{equation*}
T: \quad \tau \rightarrow \tau+1, \quad \quad S: \quad \tau \rightarrow-\frac{1}{\tau} . \tag{4.2.13}
\end{equation*}
$$

The theta-function for the dual lattice may be related to the the original thetafunction by

$$
\begin{equation*}
\Theta_{\Lambda^{*}}(\tau)=\mu(-i \tau)^{c / 2} \Theta(-1 / \tau), \tag{4.2.14}
\end{equation*}
$$

where $\mu$ is a volume factor which is one for self-dual lattices. Using the fact that $\eta(-1 / \tau)=\sqrt{-i \tau} \eta(\tau)$, it is clear that self-dual lattices correspond to partition functions which are invariant under $S$. If the lattice is even, then $\Theta(\tau)=\Theta(\tau+1)$. However the Dedekind eta function picks up a phase under $T$ transformations: $\eta(\tau+1)=\exp (2 \pi i / 24) \eta(\tau)$. This phase cancels when the number of eta functions is a multiple of 24 . The result is that modular invariance of chiral CFTs requires that the central charge is divisible by 24.

## Theta-functions and enumerator polynomials

We are now ready to complete the correspondence between error-correcting codes and conformal field theories. The key element is the relation between the enumerator polynomial and the lattice theta-function (and thus the CFT partition function). The relation is

$$
\begin{equation*}
\Theta_{\Lambda(\mathcal{C})}(\tau)=W_{\mathcal{C}}\left(\theta_{3}\left(q^{2}\right), \theta_{2}\left(q^{2}\right)\right) \tag{4.2.15}
\end{equation*}
$$

where $\theta_{3}$ and $\theta_{2}$ are two of Jacobi's theta functions. These are defined by ${ }^{2}$

$$
\begin{array}{ll}
\theta_{1}(q)=\sum_{n=-\infty}^{\infty}(-1)^{n} q^{\frac{(n+1 / 2)^{2}}{2}}=0, & \theta_{2}(q)=\sum_{n=-\infty}^{\infty} q^{\frac{(n+1 / 2)^{2}}{2}} \\
\theta_{3}(q)=\sum_{n=-\infty}^{\infty} q^{\frac{n^{2}}{2}}, & \theta_{4}(q)=\sum_{n=-\infty}^{\infty}(-1)^{n} q^{\frac{n^{2}}{2}} . \tag{4.2.17}
\end{array}
$$

Essentially, 4.2.15) states that the theta function of a lattice corresponding to a code is computed by the enumerator polynomial of the code with the Jacobi $\theta$ functions as arguments. As we will be generalizing equation 4.2.15) to higher genus in section 4.3, it will be useful to present its derivation. First, consider that each vector in the lattice lies in the equivalence class defined by a specific codeword. So we may rewrite the sum as the sum over all vectors as

$$
\begin{equation*}
\vec{v}=\frac{\vec{c}+2 \vec{a}}{\sqrt{2}}, \quad \vec{c} \in \mathcal{C}, \quad \vec{a} \in \mathbb{Z}^{n} \tag{4.2.18}
\end{equation*}
$$

So we can rewrite the theta function:

$$
\begin{align*}
\Theta(q) & =\sum_{\vec{c}, \vec{a}} q^{\frac{\bar{c}^{2}}{4}} q^{\vec{c} \cdot \vec{a}+\vec{a}^{2}} \\
& =\sum_{\vec{c}} q^{\frac{\vec{c}^{2}}{4}}\left(\sum_{a_{1}} q^{c_{1} a_{1}+a_{1}^{2}}\right)\left(\sum_{a_{2}} q^{c_{2} a_{2}+a_{2}^{2}}\right) \cdots\left(\sum_{a_{n}} q^{c_{n} a_{n}+a_{n}^{2}}\right) \tag{4.2.19}
\end{align*}
$$

[^10]where we have expanded using the components $c_{i}$ and $a_{i}$ of the vectors $\vec{c}$ and $\vec{a}$. Now, $\vec{c}$ has $w(c)$ entries equal to 1 , and $n-w(c)$ entries equal to 0 , so
\[

$$
\begin{align*}
\Theta(q) & =\sum_{\vec{c}} q^{\frac{w(c)}{4}}\left(\sum_{j} q^{j+j^{2}}\right)^{w(c)}\left(\sum_{j} q^{j^{2}}\right)^{n-w(c)} \\
& =\sum_{\vec{c}}\left(\sum_{j} q^{(j+1 / 2)^{2}}\right)^{w(c)}\left(\sum_{j} q^{j^{2}}\right)^{n-w(c)}  \tag{4.2.20}\\
& =\sum_{\vec{c}}\left(\theta_{2}\left(q^{2}\right)\right)^{w(c)}\left(\theta_{3}\left(q^{2}\right)\right)^{n-w(c)} \\
& =W_{\mathcal{C}}\left(\theta_{3}\left(q^{2}\right), \theta_{2}\left(q^{2}\right)\right),
\end{align*}
$$
\]

which finishes the proof.
For convenience when we consider the higher genus case, let us introduce the following notation

$$
\begin{equation*}
\vartheta_{0}(\tau):=\theta_{3}\left(q^{2}\right), \quad \vartheta_{1}(\tau):=\theta_{2}\left(q^{2}\right), \quad q=e^{2 \pi i \tau} . \tag{4.2.21}
\end{equation*}
$$

With this definition, we can write

$$
\begin{equation*}
\Theta_{\Lambda(\mathcal{C})}(\tau)=W_{\mathcal{C}}\left(\vartheta_{0}(\tau), \vartheta_{1}(\tau)\right), \tag{4.2.22}
\end{equation*}
$$

where the right-hand side is $W_{\mathcal{C}}$ evaluated at $x_{A}=\vartheta_{A}(\tau)$. Note that the standard modular transformations of the Jacobi theta functions imply that the $\vartheta_{A}(\tau)$ transform as

$$
\begin{align*}
T: & \vartheta_{0}(\tau) \mapsto \vartheta_{0}(\tau), \quad \vartheta_{1}(\tau) \mapsto i \vartheta_{1}(\tau),  \tag{4.2.23}\\
S: & \vartheta_{0}(\tau) \mapsto \sqrt{-i \tau} \frac{\vartheta_{0}(\tau)+\vartheta_{1}(\tau)}{\sqrt{2}}, \quad \vartheta_{1}(\tau) \mapsto \sqrt{-i \tau} \frac{\vartheta_{0}(\tau)-\vartheta_{1}(\tau)}{\sqrt{2}} \tag{4.2.24}
\end{align*}
$$

The $S$ transformation mimics the MacWilliams identity 4.2.5) and thus guarantees the transformation 4.2.14 of the lattice theta function.

### 4.3 Chiral CFTs at Higher Genus

Having reviewed the relation between error-correcting codes and CFTs, we are now ready to develop the central points of this chapter, which is how this correspondence works at higher genus. The results of this are summarized in table 4.2. At genus 2 and 3 , this is relatively straightforward. After developing this correspondence, we will be able to show, in section 4.4, how the modular invariance at genus 2 and 3 effectively constrain the space of possible enumerator polynomials, and therefore of code CFTs. At genus $g>3$, the situation is more complicated, as we will review in section 4.3 ,

Table 4.2: Summary of the differences between genus 1 and higher genus.

| Genus $g=1$ | General genus |
| :--- | :--- |
| Enumerator polynomial $W\left(x_{0}, x_{1}\right)$, | Higher-weight enumerator polynomial |
|  | $W\left(x_{0}, \ldots, x_{2^{g}-1}\right)$, |
| Modular parameter $\tau$, | Period matrix $\Omega$, |
| Modular group PSL $(2, \mathbb{Z})$ acting on | Modular group PSp $(2 g, \mathbb{Z})$ act- |
| $\{\operatorname{Im} \tau>0\}$, | ing on the Siegel upper half-plane |
|  | $\left\{\Omega=\Omega^{T}, \operatorname{Im} \Omega \succ 0\right\}$, |
| $\vartheta_{A}(\tau)$ expressible in terms of Jacobi | $\vartheta_{A}(\Omega)$ expressible in terms of theta |
| theta functions $\theta_{2,3}\left(q^{2}\right)$, | constants of second order characteris- |
|  | tic $\theta\left[\begin{array}{c}c_{i} / 2 \\ \mathbf{0}\end{array}\right](0,2 \Omega)$, |
|  |  |
| Evaluation at $\vartheta_{A}: W\left(\vartheta_{0}(\tau), \vartheta_{1}(\tau)\right)$, | Map Th: $x_{A} \mapsto \vartheta_{A}(\Omega)$. |

## Higher genus partition function from codes

The central question answered by this chapter is how the correspondence reviewed in the previous section can be generalized to higher genus Riemann surfaces. The higher genus partition function for a 2d CFT on a lattice can be expressed schematically as

$$
\begin{equation*}
Z^{(g)}(\Omega)=\frac{\hat{Z}^{(g)}(\Omega)}{\Phi_{g}(\Omega)} \tag{4.3.1}
\end{equation*}
$$

The partition function $Z^{(g)}$ depends on the period matrix $\Omega$, which describes the Riemann surface in an analogous way to $\tau$ in the genus 1 case. The numerator $\hat{Z}^{(g)}$ of the partition function is equal to the higher genus lattice theta function for theories defined by a self-dual even lattice $\Lambda$,

$$
\begin{equation*}
\hat{Z}^{(g)}(\Omega)=\Theta_{\Lambda}(\Omega) \tag{4.3.2}
\end{equation*}
$$

As we will discuss, this function has simple transformations under the higher genus modular group (it is a Siegel modular form of weight $c / 2$ ).

We will primarily be interested in the numerator, but let us make some comments about the denominator, $\Phi_{g}(\Omega)=F_{g}(\Omega)^{c}$, which corresponds to a sum over oscillator modes. At each genus, $F_{g}(\Omega)$ is universal and can in principle be evaluated by considering just one representative lattice CFT. At genus $1, F_{1}(\tau)=\eta(\tau)$. At higher genus, no simple compact expression is known for the denominator. There exist some formal expressions 120 and at genus 2 a useful series expansion 107. While
these constructions give a denominator with the correct weight to cancel the weight of the modular form in the numerator, the resulting partition function picks up phases under the modular transformations that generalize the $T$ transformation in (4.2.13). At genus 1 , one needs $c=24 k$ for integer $k$ to cancel phases and render $Z^{(1)}$ completely modular invariant, and indeed, $\Delta_{12}=\eta(\tau)^{24}$ is a Siegel modular form of degree 12 , see section 4.3 . For higher genus, it is not generally possible to find a nowhere vanishing denominator that is also a Siegel modular form $\sqrt{3}$ meaning that, in general, the partition function will always be defined up to phases.$^{4}$

## Lattice theta functions at higher genus

We consider a compact Riemann surface $\Sigma$ with genus $g$ with $2 g$ cycles $a_{i}, b_{i}$ where $i \in\{1, \ldots, g\}$. The canonical choice for the cycles is for their intersection numbers $\iota($,$) to satisfy$

$$
\begin{equation*}
\iota\left(a_{i}, a_{j}\right)=\iota\left(b_{i}, b_{j}\right)=0, \quad \iota\left(a_{i}, b_{j}\right)=\delta_{i j} . \tag{4.3.3}
\end{equation*}
$$

A choice of $a$ cycles fixes the normalization of the $g$ holomorphic 1 forms $\omega_{j}$ associated with the surface according to $\oint_{a_{i}} \omega_{j}=\delta_{i j}$. The $b$ cycles fix the matrix defined by

$$
\begin{equation*}
\oint_{b_{i}} \omega_{j}=\Omega_{i j} \tag{4.3.4}
\end{equation*}
$$

$\Omega_{i j}$ is called the Riemann period matrix associated with $\Sigma$ and is a symmetric matrix with positive definite imaginary part: $\operatorname{Im} \Omega \succ 0$.

The genus $g$ lattice theta series is defined as a sum over $g$-tuples of lattice vectors by

$$
\begin{equation*}
\Theta_{\Lambda}(\Omega)=\sum_{\vec{v}_{1}, \ldots, \vec{v}_{g} \in \Lambda} e^{\pi i \vec{v}_{i} \cdot \vec{v}_{j} \Omega_{i j}} \tag{4.3.5}
\end{equation*}
$$

For instance, at genus 2 this gives

$$
\begin{equation*}
\Theta_{\Lambda}^{g=2}(\Omega)=\sum_{\vec{v}_{1}, \overrightarrow{v_{2}} \in \Lambda} e^{\pi i\left(\vec{v}_{1} \cdot \vec{v}_{1} \Omega_{11}+2 \vec{v}_{1} \cdot \vec{v}_{2} \Omega_{12}+\vec{v}_{2} \cdot \vec{v}_{2} \Omega_{22}\right)} \tag{4.3.6}
\end{equation*}
$$

which can also be rewritten as

$$
\begin{equation*}
\Theta_{\Lambda}^{g=2}(\Omega)=\sum_{\vec{v}_{1}, \vec{v}_{2} \in \Lambda} q^{\frac{\vec{v}_{1} \cdot \vec{v}_{1}}{2}} r^{\vec{v}_{1} \cdot \vec{v}_{2}} s^{\frac{\vec{v}_{2} \cdot \vec{v}_{2}}{2}} \tag{4.3.7}
\end{equation*}
$$

with the the modular parameters $q, r, s$ are defined as

$$
\begin{equation*}
q=e^{2 \pi i \Omega_{11}}, \quad r=e^{2 \pi i \Omega_{12}}, \quad s=e^{2 \pi i \Omega_{22}} . \tag{4.3.8}
\end{equation*}
$$

[^11]
## Higher-weight enumerator polynomials

Above we have seen that Construction A can be used to define a lattice from a code, and that the lattice theta function can be computed from the code enumerator polynomial by replacing the code variables with theta constants according to

$$
\begin{equation*}
\Theta_{\Lambda(\mathcal{C})}^{g=1}(q)=W_{\mathcal{C}}\left(\vartheta_{0}(\tau), \vartheta_{1}(\tau)\right) . \tag{4.3.9}
\end{equation*}
$$

Now we would like to construct the analogous function at higher genus, which reproduces the higher genus lattice theta series from the code. The appropriate function is the higher-weight enumerator polynomial and has been known in the math literature for some time [122], but to our knowledge its connection to higher genus CFTs has never been pointed out.

The higher-weight enumerator polynomial generalizes the weight 1 enumerator polynomial by comparing more than one codeword at a time. We will see later that this is related to the genus $g$ partition function. Specifically, the $g^{\text {th }}$ weight polynomial is defined by summing over $g$-tuples of codewords, and can be represented compactly as 123

$$
\begin{equation*}
W_{\mathcal{C}}^{(g)}\left(x_{i}\right)=\sum_{\mathbf{M} \in \mathcal{C}^{g}} \prod_{i=1}^{n} x_{\mathrm{row}_{i}(\mathbf{M})} \tag{4.3.10}
\end{equation*}
$$

In this expression, the sum over all $g$-tuples of codewords is represented by the sum over all choices of the $n \times g$ matrix $\mathbf{M}$. Each column of $\mathbf{M}$ is a codeword, which is vector of length $n$ for an $n$-dimensional code. $\operatorname{row}_{i}(\mathbf{M})$ denotes the $i$ th row of M. When writing $x_{\text {row }_{i}}$, we interpret the index of $x$ as the binary representation of the index. Specifically $x_{0}=x_{[0 \ldots 00]}, x_{1}=x_{[0 \ldots 01]}, x_{2}=x_{[0 \ldots 10]}$, etc. We can see from this that the weight $g$ enumerator polynomial will be a homogeneous degree $n$ polynomial in the $2^{g}$ variables $x_{0}, x_{1}, \ldots x_{2^{g}-1}$.

For a specific case of this abstract definition, consider the $g=2$ enumerator polynomial, sometimes called the biweight polynomial. It can be written as

$$
\begin{equation*}
W_{\mathcal{C}}^{(2)}\left(x_{0}, x_{1}, x_{2}, x_{3}\right)=\sum_{c_{1}, c_{2} \in \mathcal{C}} x_{0}^{n+c_{1} \cdot c_{2}-w\left(c_{1}\right)-w\left(c_{2}\right)} x_{1}^{w\left(c_{2}\right)-c_{1} \cdot c_{2}} x_{2}^{w\left(c_{1}\right)-c_{1} \cdot c_{2}} x_{3}^{c_{1} \cdot c_{2}} \tag{4.3.11}
\end{equation*}
$$

We sum over all pairs of codewords. The exponents of the variables now involve the dot product of $c_{1}$ and $c_{2}$, as well as their individual weights. The result is a homogeneous degree $n$ polynomial of four variables. For example, the biweight polynomial of the $[8,4,4]$ Hamming code is given by

$$
\begin{align*}
W_{\text {Hamming }}^{(2)} & =x_{0}^{8}+x_{1}^{8}+x_{2}^{8}+x_{3}^{8}+168 x_{0}^{2} x_{1}^{2} x_{2}^{2} x_{3}^{2} \\
& +14 x_{0}^{4} x_{1}^{4}+14 x_{0}^{4} x_{2}^{4}+14 x_{0}^{4} x_{3}^{4}+14 x_{1}^{4} x_{2}^{4}+14 x_{1}^{4} x_{3}^{4}+14 x_{2}^{4} x_{3}^{4} \tag{4.3.12}
\end{align*}
$$

In fact, we will see below that this is the unique degree eight genus two polynomial which satisfies some appropriate symmetries which we will derive in the next subsection.

Proving that this polynomial captures the theta function amounts to repeating the proof presented in 4.2.19) and 4.2.20 for our higher genus theta function. The key is to use the fact that each lattice vector is in the equivalence class of a codeword,

$$
\begin{equation*}
\vec{v}_{i}=\frac{\vec{c}_{i}+2 \vec{m}_{i}}{\sqrt{2}} \tag{4.3.13}
\end{equation*}
$$

to rewrite the lattice theta function (4.3.5) as

$$
\begin{equation*}
\Theta_{\Lambda(\mathcal{C})}(\Omega)=\sum_{\left\{\vec{c}_{1}, \ldots, \vec{c}_{g}\right\} \in \mathcal{C}} \sum_{\left\{\vec{m}_{1}, \ldots, \vec{m}_{g}\right\} \in \mathbb{Z}^{n}} e^{\frac{1}{2} \pi i\left(\vec{c}_{i}+2 \vec{m}_{i}\right) \cdot\left(\vec{c}_{j}+2 \vec{m}_{j}\right) \Omega_{i j}} . \tag{4.3.14}
\end{equation*}
$$

Now we want to consider the elements $\vec{c}_{i}$ entry-by-entry. So we define $c_{(i) k}$ to denote the $k^{t h}$ entry of $\vec{c}_{i}$. Then the lattice theta function equals

$$
\begin{align*}
\Theta_{\Lambda}(\Omega)= & \sum_{\left\{\vec{c}_{1}, \ldots, \vec{c}_{g}\right\} \in \mathcal{C}^{g}}\left(\sum_{\left\{m_{(1) 1}, \ldots, m_{(g) 1}\right\} \in \mathbb{Z}^{g}} e^{\frac{1}{2} \pi i\left(c_{(i) 1} c_{(j) 1}+2 c_{(i) 1} m_{(j) 1}+2 m_{(i) 1} c_{(j) 1}+4 m_{(i) 1} m_{(j) 1}\right) \Omega_{i j}}\right) \\
& \times\left(\sum_{\left\{m_{(1) 2}, \ldots, m_{(g) 2}\right\} \in \mathbb{Z}^{g}} e^{\frac{1}{2} \pi i\left(c_{(i) 2} c_{(j) 2}+2 c_{(i) 2} m_{(j) 1}+2 m_{(i) 2} c_{(j) 2}+4 m_{(i) 2} m_{(j) 2}\right) \Omega_{i j}}\right) \times \\
& \cdots \times\left(\sum_{\left\{m_{(1) n}, \ldots, m_{(g) n}\right\} \in \mathbb{Z}^{g}} e^{\frac{1}{2} \pi i\left(c_{(i) n} c_{(j) n}+2 c_{(i) n} m_{(j) n}+2 m_{(i) n} c_{(j) n}+4 m_{(i) n} m_{(j) n}\right) \Omega_{i j}}\right) \tag{4.3.15}
\end{align*}
$$

Next we will define $\mathbf{M}=\left\{\vec{c}_{1}, \ldots, \vec{c}_{g}\right\} \in \mathcal{C}^{g}$. Then the $c_{(i) k}$ appearing in the $k^{\text {th }}$ term in parentheses form the $k^{\text {th }}$ row of $\mathbf{M}$. Let us define a vector $2 \boldsymbol{a}_{k}=c_{(i) k}$. Furthermore, for any $k$, the range of $m_{(i) k}$ in the sum is the same, so let us define $\boldsymbol{m}=\boldsymbol{m}_{(i) k}$. With these replacements, the lattice theta function becomes

$$
\begin{equation*}
\Theta_{\Lambda}(\Omega)=\sum_{\mathbf{M} \in \mathcal{C}^{g}}\left(\sum_{\boldsymbol{m} \in \mathbb{Z}^{g}} e^{\pi i\left(\boldsymbol{a}_{1}+\boldsymbol{m}\right) \cdot(2 \Omega) \cdot\left(\boldsymbol{a}_{1}+\boldsymbol{m}\right)}\right) \times \cdots \times\left(\sum_{\boldsymbol{m} \in \mathbb{Z}^{g}} e^{\pi i\left(\boldsymbol{a}_{n}+\boldsymbol{m}\right) \cdot(2 \Omega) \cdot\left(\boldsymbol{a}_{n}+\vec{m}\right)}\right) . \tag{4.3.16}
\end{equation*}
$$

At this point, we need to introduce the higher genus versions of the Jacobi theta functions. These are conventionally defined by (see e.g. 124 )

$$
\theta\left[\begin{array}{l}
\boldsymbol{a}  \tag{4.3.17}\\
\boldsymbol{b}
\end{array}\right](\boldsymbol{z}, \Omega)=\sum_{\boldsymbol{m} \in \mathbb{Z}^{g}} \exp (\pi i(\boldsymbol{m}+\boldsymbol{a}) \cdot \Omega \cdot(\boldsymbol{m}+\boldsymbol{a})+2 \pi i(\boldsymbol{m}+\boldsymbol{a}) \cdot(\boldsymbol{z}+\boldsymbol{b}))
$$

We can see that each of the terms in parentheses in 4.3.16 is equal to a theta constant. So we find

$$
\Theta_{\Lambda}(\Omega)=\sum_{\mathbf{M} \in \mathcal{C}^{g}}\left(\theta\left[\begin{array}{c}
\boldsymbol{a}_{1}  \tag{4.3.18}\\
\mathbf{0}
\end{array}\right](0,2 \Omega)\right) \times \cdots \times\left(\theta\left[\begin{array}{c}
\boldsymbol{a}_{n} \\
\mathbf{0}
\end{array}\right](0,2 \Omega)\right) .
$$

Now recall that $2 \boldsymbol{a}_{i}=\boldsymbol{c}_{i}$ are the rows of $\mathbf{M}$. So this formula is equal to the genus $g$ enumerator polynomial with the $x_{i}$ replaced by a corresponding theta expression. Following [123], we call this replacement the theta map:

$$
\text { Th }: x_{\left[c_{1}, c_{2}, \ldots\right]} \mapsto \theta\left[\begin{array}{c}
\boldsymbol{c} / 2  \tag{4.3.19}\\
\mathbf{0}
\end{array}\right](0,2 \Omega)
$$

More precisely, the specific functions that appear on the right-hand side of (4.3.19) are known as theta constants of second order characteristic 123. We will make more comments on the theta map in section 4.3, where we will see that it is injective only for $g<3$ and surjective only for $g<4$ (125].

Now recall that $x_{A}=x_{\left[a_{1}, a_{2}, \ldots\right]}$ if $\left[a_{1}, a_{2}, \ldots\right]$ is the binary representation of $A$. Then it is convenient to introduce the following short form of the theta constants of second order characteristic,

$$
\vartheta_{A}(\Omega):=\theta\left[\begin{array}{c}
\boldsymbol{a} / 2  \tag{4.3.20}\\
\mathbf{0}
\end{array}\right](0,2 \Omega),
$$

allowing us to write the theta map as

$$
\begin{equation*}
\mathrm{Th}: x_{A} \mapsto \vartheta_{A}(\Omega) . \tag{4.3.21}
\end{equation*}
$$

This leads to one of the central equations we will use in this chapter:

$$
\begin{equation*}
\Theta_{\Lambda}(\Omega)=W_{\mathcal{C}}^{(g)}\left(\vartheta_{0}(\Omega), \vartheta_{1}(\Omega), \ldots, \vartheta_{2^{g}-1}(\Omega)\right) \tag{4.3.22}
\end{equation*}
$$

In order to use this relationship to constrain code CFTs, we will need to determine the higher-weight MacWilliams identities obeyed by the enumerator polynomials. These transformations were already known in the math literature on error-correcting codes [122], and the connection to modular forms was stressed in [126, 127, 128, 129]. Here, we will review how to derive these relations, taking the viewpoint that they arise as a consequence of modular invariance of the higher genus partition function.

## Modular transformations at higher genus

Two-dimensional chiral CFTs on a Riemann surface of genus $g$ enjoy more symmetries than just the modular transformations of a genus-one surface. The higher genus
modular group is $\operatorname{Sp}(2 g, \mathbb{Z})$, which arises from the fact that there are different ways of choosing the cycles satisfying (4.3.3) on a given Riemann surface. Specifically, for an element

$$
\left(\begin{array}{ll}
A & B  \tag{4.3.23}\\
C & D
\end{array}\right) \in \operatorname{Sp}(2 g, \mathbb{Z})
$$

the period matrix $\Omega_{i j}$ transforms as

$$
\begin{equation*}
\Omega \mapsto \tilde{\Omega}=(A \Omega+B)(C \Omega+D)^{-1} \tag{4.3.24}
\end{equation*}
$$

These transformations act on the lattice theta function according to

$$
\begin{equation*}
\Theta_{\Lambda}(\tilde{\Omega})=\operatorname{det}(C \Omega+D)^{\frac{n}{2}} \Theta_{\Lambda}(\Omega) \tag{4.3.25}
\end{equation*}
$$

The determinant factor in front of this is canceled by the transformation of the denominator factor to ensure that the partition factor is invariant (up to phase).

In [130], an explicit basis for the generators of $\operatorname{Sp}(2 g, \mathbb{Z})$ is given. It is generated by three matrices for $g=2,3$ and only two matrices for $g>3$. We give the generators for $g=2$ and $g=3$, which we call $T_{g}, R_{g}$, and $D_{g}$. These will allow us to determine the transformations of the code variables $x_{i}$. For $g=2$, we have

$$
T_{g=2}=\left(\begin{array}{cccc}
1 & 0 & 1 & 0  \tag{4.3.26}\\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), \quad R_{g=2}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{array}\right), \quad D_{g=2}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array}\right) .
$$

For $g=3$, the generators are $6 \times 6$ matrices:

$$
\begin{gather*}
T_{g=3}=\left(\begin{array}{cccccc}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right), \quad R_{g=3}=\left(\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right),  \tag{4.3.27}\\
D_{g=3}=\left(\begin{array}{cccccc}
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0
\end{array}\right) .
\end{gather*}
$$

These generators determine how the transformations act on the period matrix $\Omega$ through (4.3.24). This, in turn, determines the transformation of the theta constants. The derivation requires use of the Poisson resummation formula. As it is rather tedious, we will merely record the answers.

$$
\begin{align*}
& T_{g=2}: \quad \vartheta_{0}(\Omega) \mapsto \vartheta_{0}(\Omega), \quad \vartheta_{1}(\Omega) \mapsto \vartheta_{1}(\Omega), \quad \vartheta_{2}(\Omega) \mapsto i \vartheta_{2}(\Omega), \quad \vartheta_{3}(\Omega) \mapsto i \vartheta_{3}(\Omega), \\
& R_{g=2}: \quad \vartheta_{0}(\Omega) \mapsto \vartheta_{0}(\Omega), \quad \vartheta_{1}(\Omega) \mapsto \vartheta_{3}(\Omega), \quad \vartheta_{2}(\Omega) \mapsto \vartheta_{2}(\Omega), \quad \vartheta_{3}(\Omega) \mapsto \vartheta_{1}(\Omega), \\
& D_{g=2}: \quad \vartheta_{0}(\Omega) \mapsto \sqrt{-i \Omega_{11}} \frac{\vartheta_{0}(\Omega)+\vartheta_{2}(\Omega)}{\sqrt{2}}, \quad \vartheta_{1}(\Omega) \mapsto \sqrt{-i \Omega_{11}} \frac{\vartheta_{0}(\Omega)-\vartheta_{2}(\Omega)}{\sqrt{2}}, \\
& \vartheta_{2}(\Omega) \mapsto \sqrt{-i \Omega_{11}} \frac{\vartheta_{1}(\Omega)+\vartheta_{3}(\Omega)}{\sqrt{2}}, \quad \vartheta_{3}(\Omega) \mapsto \sqrt{-i \Omega_{11}} \frac{\vartheta_{1}(\Omega)-\vartheta_{3}(\Omega)}{\sqrt{2}} . \tag{4.3.28}
\end{align*}
$$

It is important that $\operatorname{det}(C \Omega+D)$ is equal to 1 for $T_{g=2}$ and $R_{g=2}$ and $\Omega_{11}$ for $D_{g=2}$. This ensures

$$
\begin{equation*}
D_{g=2}: \quad W_{\mathcal{C}}^{(2)}\left(\vartheta_{0}(\Omega), \ldots\right) \mapsto \operatorname{det}(C \Omega+D)^{\frac{n}{2}} W_{\mathcal{C}}^{(2)}\left(\frac{\vartheta_{0}(\Omega)+\vartheta_{2}(\Omega)}{\sqrt{2}}, \ldots\right) \tag{4.3.29}
\end{equation*}
$$

and likewise for any other $\operatorname{Sp}(4, \mathbb{Z})$ transformation. Therefore we see that the transformation of the theta function 4.3.25) requires that the enumerator polynomial satisfy the higher-weight MacWilliams identities:

$$
\begin{array}{llll}
T_{g=2}: & x_{0} \mapsto x_{0}, & x_{1} \mapsto x_{1}, & x_{2} \mapsto i x_{2}, \\
R_{g=2}: & x_{0} \mapsto x_{0}, & x_{1} \mapsto x_{3} \mapsto i x_{3}, & x_{2} \mapsto x_{2}, \quad x_{3} \mapsto x_{1}, \\
D_{g=2}: & x_{0} \mapsto \frac{x_{0}+x_{2}}{\sqrt{2}}, & x_{1} \mapsto \frac{x_{0}-x_{2}}{\sqrt{2}}, & x_{2} \mapsto \frac{x_{1}+x_{3}}{\sqrt{2}}, \tag{4.3.30}
\end{array} x_{3} \mapsto \frac{x_{1}-x_{3}}{\sqrt{2}}, ~ \$
$$

For genus 3 the analysis is the same. We skip the transformations of the theta constants and simply present

$$
\begin{array}{ll}
T_{g=3}: & x_{4} \rightarrow i x_{4}, \quad x_{5} \rightarrow i x_{5}, \quad x_{6} \rightarrow i x_{6}, \quad x_{7} \rightarrow i x_{7} \\
R_{g=3}: & x_{2} \rightarrow x_{6}, \quad x_{3} \rightarrow x_{7}, \quad x_{6} \rightarrow x_{2}, \quad x_{7} \rightarrow x_{3} \\
D_{g=3}: & x_{0} \rightarrow \frac{x_{0}+x_{4}}{\sqrt{2}}, \quad x_{1} \rightarrow \frac{x_{0}-x_{4}}{\sqrt{2}}, \quad x_{2} \rightarrow \frac{x_{1}+x_{5}}{\sqrt{2}}, \quad x_{3} \rightarrow \frac{x_{1}-x_{5}}{\sqrt{2}}, \\
& x_{4} \rightarrow \frac{x_{2}+x_{6}}{\sqrt{2}}, \quad x_{5} \rightarrow \frac{x_{2}-x_{6}}{\sqrt{2}}, \quad x_{6} \rightarrow \frac{x_{3}+x_{7}}{\sqrt{2}}, \quad x_{7} \rightarrow \frac{x_{3}-x_{7}}{\sqrt{2}} . \tag{4.3.31}
\end{array}
$$

These transformations will be crucial to the algorithm we describe in section 4.4. For genus 2 they were already known $\sqrt{122}$. We do not know if they have been
written down in the code literature for genus 3. They effectively express a very complicated set of transformations, i.e. the higher genus modular transformations of the partition function, in terms of a small set of linear transformations. We use the term "invariant polynomials" to refer to those polynomials which are unchanged by these transformations.

## Factorization limits

Further constraints may be imposed on the higher genus partition function based on the limit where the Riemann surface becomes singular 105. For a genus $g$ Riemann surface $\Sigma_{g}$ we may smoothly deform $\Sigma_{g}$ so that it consists of two Riemann surfaces of genus $h$ and $g-h$, connected by a long, infinitely thin tube. In this limit, the genus $g$ partition function must behave as the product of a genus $h$ partition function, and a genus $g-h$ partition function. We refer to these deformations as "factorization limits". The case of $h=1$ is depicted in figure 4.1.


Figure 4.1: Factorization of a genus $g$ Riemann surface.
In the factorization limit, the period matrix becomes block-diagonal. For instance, for $h=1$,

$$
\begin{equation*}
\Omega \rightarrow \tau \oplus \Omega^{\prime} \tag{4.3.32}
\end{equation*}
$$

where $\tau=\Omega_{11}$ and $\Omega_{i j}^{\prime}=\Omega_{i+1, j+1}$ (i.e. we have $\Omega_{1 J}=\Omega_{J 1}=0$ ). From the CFT perspective, in the factorizatin limit only the identity operator flows across the thin tube, and the partition function is expected to factorize into a product of the lower genus partition functions. Equivalently, for a lattice theta function,

$$
\begin{equation*}
\Theta_{\Lambda}^{(g)}(\Omega) \rightarrow \Theta_{\Lambda}^{(g)}\left(\tau \oplus \Omega^{\prime}\right)=\Theta_{\Lambda}^{(1)}(\tau) \Theta_{\Lambda}^{(g-1)}\left(\Omega^{\prime}\right) \tag{4.3.33}
\end{equation*}
$$

The case of arbitrary $h$ is completely analogous.

We shall see that factorization imposes strong constraints on which enumerator polynomials can correspond to CFT partition functions. There are a number of higher-weight polynomials which are modular invariant but which do not factorize properly. This allows us to eliminate them from the list of possible partition functions. Let us see how the factorization property (4.3.33) acts on the level of the polynomial variables $x_{i}$. First recall the binary form of the genus $g$ polynomial variables,

$$
\begin{equation*}
x_{\left[i_{g-1} i_{g-2} \cdots i_{0}\right]}^{(g)} \longleftrightarrow x_{i}^{(g)}, \quad i=\sum_{n=0}^{g-1} i_{n} 2^{n} . \tag{4.3.34}
\end{equation*}
$$

Then (4.3.33) is consistent with the factorization map

$$
\begin{equation*}
x_{\left[i_{g-1} i_{g-2} \cdots i_{0}\right]}^{(g)} \mapsto x_{i_{g-1}}^{(1)} y_{\left[i_{g-2} \cdots i_{0}\right]}^{(g-1)}, \tag{4.3.35}
\end{equation*}
$$

or, equivalently

$$
x_{i} \mapsto \begin{cases}x_{0} y_{i}, & 0 \leqslant i<2^{g-1}  \tag{4.3.36}\\ x_{1} y_{i-2^{g-1}}, & 2^{g-1} \leqslant i<2^{g}\end{cases}
$$

Here $x_{i}$ and $y_{i}$ refer to the polynomial variables of the two resulting Riemann surfaces. It can be seen that these definitions are consistent with the factorization property of code enumerator polynomials,

$$
\begin{equation*}
W_{\mathcal{C}}^{(g)}\left(x_{i}\right) \mapsto W_{\mathcal{C}}^{(1)}\left(x_{i}\right) W_{\mathcal{C}}^{(g-1)}\left(y_{i}\right) \tag{4.3.37}
\end{equation*}
$$

under (4.3.35).

Genus 2 example: It is easy to see that this is correct requirement with the simple example of genus 2 . On the genus 2 Riemann surface one can choose $a$ cycles and $b$ cycles as indicated in figure 4.2, and introduce the variables $q=e^{2 \pi i \Omega_{11}}, r=e^{2 \pi i \Omega_{12}}$ and $s=e^{2 \pi i \Omega_{22}}$ according to 4.3.8). In the degeneration limit, $q$ and $s$ reduce to the modular parameters $e^{2 \pi i \tau}$ and $e^{2 \pi i \tau^{\prime}}$ of the respective tori. Moreover, the one-form $\omega_{2}$ dual to the $a_{2}$ cycle on the right torus becomes approximately zero on the left torus (and vice versa), showing that $\Omega_{12} \rightarrow 0$, or equivalently, the $r \rightarrow 1$ limit ${ }^{5}$ For the lattice theta function in this example, we have (see 4.3.7)

$$
\begin{equation*}
\Theta_{\Lambda}^{g=2}(\Omega)=\sum_{\vec{v}_{1}, \vec{v}_{2} \in \Lambda} q^{\frac{\vec{v}_{1} \cdot \vec{v}_{1}}{2}} r^{\vec{v}_{1} \cdot \vec{v}_{2}} s^{\frac{\vec{v}_{2} \cdot \vec{v}_{2}}{2}} \tag{4.3.38}
\end{equation*}
$$

[^12]

Figure 4.2: Factorization of a genus 2 Riemann surface. The $a$ cycles are depicted in red, $b$ cycles in green. Under the factorization, the one-form $\omega_{2}$ dual to $a_{2}$ vanishes on the left torus. Hence $\Omega_{12}=\oint_{b_{1}} \omega_{2}=0$, so that $r=e^{2 \pi i \Omega_{12}} \rightarrow 1$.

In the limit $r \rightarrow 1$, the resulting lattice theta function then breaks into the product of two sums,

$$
\begin{equation*}
\Theta_{\Lambda}^{g=2}\left(\Omega_{12}=0\right)=\sum_{\vec{v}_{1} \in \Lambda} q^{\frac{\vec{v}_{1} \cdot \vec{v}_{1}}{2}} \sum_{\vec{v}_{2} \in \Lambda} s^{\frac{\vec{v}_{2} \cdot \vec{v}_{2}}{2}}, \tag{4.3.39}
\end{equation*}
$$

which are each equal to a genus 1 lattice theta function. For the polynomial variables, it is easy to see from the definition of $\theta$

$$
x_{[0,0]}=\theta\left[\begin{array}{l}
0  \tag{4.3.40}\\
0 \\
0 \\
0
\end{array}\right](0,2 \Omega) \xrightarrow{\Omega_{12} \rightarrow 0}\left(\theta\left[\begin{array}{l}
0 \\
0
\end{array}\right]\left(0,2 \Omega_{11}\right)\right)\left(\theta\left[\begin{array}{l}
0 \\
0
\end{array}\right]\left(0,2 \Omega_{22}\right)\right)=x_{0} y_{0}
$$

etc.
It is also sometimes convenient to consider the Siegel map $\Phi$. This maps the polynomial variables according to

$$
\Phi\left(x_{i}^{(g)}\right)= \begin{cases}x_{g}^{(g-1)}, & 0 \leqslant i<2^{g-1}  \tag{4.3.41}\\ 0, & 2^{g-1} \leqslant i<2^{g}\end{cases}
$$

The siegel $\Phi$ map is equivalent to taking first the factorization $Z^{(g)} \rightarrow Z^{(1)} Z^{(g-1)}$ and then taking the limit $\tau \rightarrow i \infty$, which sets $x_{0}=1, x_{1}=0$.

## Genus $g>3$ and Siegel modular forms

In most of this chapter we are interested in the case where $g=2$ or $g=3$, mainly because these are the cases where the enumerator polynomials are small enough to
make explicit calculations tractable. Here we would like to broaden our discussion to any genus, which will require some additional mathematical formalism. The reader primarily interested in our results at genus 2 and 3 may proceed directly to section 4.4

In particular, we are interested in properties of the evaluation map Th, which sends polynomial variables (functions of $x_{i}$ ) to theta expressions (functions of $\Omega$ ). The Th map is a ring homomorphism between invariant polynomials and modular covariant functions in the Siegel upper half plane

$$
\begin{equation*}
\mathcal{H}_{g}=\left\{\Omega=\Omega^{T}, \operatorname{Im} \Omega \succ 0\right\} \tag{4.3.42}
\end{equation*}
$$

Such covariant forms are known in the mathematics literature as Siegel modular forms. In section 4.3 we review the most important facts about Siegel modular forms in order to explain the properties of the Th map at higher genus. Starting at genus 3, the Th map ceases to be one-to-one; there exist a degree 16 polynomial $j_{8}$ which maps to 0 . This implies a non-trivial relation among the genus-3 theta functions. More complications arise at even higher genus, which we will discuss in section 4.3

## Siegel modular forms

A Siegel modular form (for the group $\operatorname{Sp}(2 g, \mathbb{Z})$ ) of degree $k$ is a function $f_{k}: \mathcal{H}_{g} \rightarrow \mathbb{C}$ which transforms covariantly with weight $k$. This is to say that under the transformation

$$
\Omega \mapsto \Omega^{\prime}=(A \Omega+B)(C \Omega+D)^{-1}, \quad\left(\begin{array}{ll}
A & B  \tag{4.3.43}\\
C & D
\end{array}\right) \in \operatorname{Sp}(2 g, \mathbb{Z})
$$

$f_{k}$ transforms as

$$
\begin{equation*}
f_{k}\left(\Omega^{\prime}\right)=\operatorname{det}(C \Omega+D)^{k} f_{k}(\Omega) \tag{4.3.44}
\end{equation*}
$$

The space of Siegel modular forms at genus $g$ defines a ring graded by degree, $M_{g}=$ $\bigoplus M_{g}^{k}$. If we let $R_{g}$ be the ring of invariant polynomials, i.e. those satisfying the generalized MacWilliams identities, the map Th : $R_{g} \rightarrow M_{g}$ defined by 4.3.19) becomes a ring homomorphism compatible with the grading. In fact, we have the commuting diagram 123

where $\Phi$ is the Siegel map (4.3.41). Here $R_{g}$ has the obvious grading by degree. The ring $M_{g}$ of modular forms (of even degree) is known for $g \leqslant 3$, and has the following generators

$$
\begin{array}{ll}
g=1 & G_{4}, G_{6} \\
g=2 & E_{4}, E_{6}, \chi_{10}, \chi_{12}, \\
g=3 & \alpha_{4}, \alpha_{6}, \alpha_{10}, \alpha_{12}, \alpha_{12}^{\prime}, \beta_{14}, \alpha_{16}, \beta_{16}, \chi_{18}, \alpha_{18},  \tag{4.3.48}\\
\alpha_{20}, \gamma_{20}, \beta_{22}, \beta_{22}^{\prime}, \alpha_{24}, \gamma_{24}, \gamma_{26}, \chi_{28}, \alpha_{30}
\end{array}
$$

where the indices denote the degree of the respective generator. The result at genus 2 is due to Igusa 132 . At genus 3, an overcomplete list of 34 generators was given in [133], and was recently reduced to the minimal set of 19 generators listed in (4.3.48) [134] [6 At genus 1 and 2, the ring is generated by the holomorphic Eisenstein series: $G_{k}, k=4,6$ (genus 1) and $E_{k}, k=4,6,10,12$ (genus 2). See appendix A.5 for relations between the generators at genus 2 .

Since we have $c$ divisible by 8, only Siegel modular forms of degree divisible by four will enter our discussion. At genus 1, this subring can be generated by $G_{4}$ and any degree 12 modular form. A convenient choice of the latter is $\Delta_{12}=\frac{1}{1728}\left(G_{4}^{3}-G_{6}^{2}\right)$. In the next section, we will find the following expressions for these forms: $7^{7}$

$$
\begin{align*}
G_{4}(\tau) & =\vartheta_{0}(\tau)^{8}+\vartheta_{1}(\tau)^{8}+14 \vartheta_{0}(\tau)^{4} \vartheta_{1}(\tau)^{4} \\
\Delta_{12} & =\frac{1}{16} \vartheta_{0}(\tau)^{4} \vartheta_{1}(\tau)^{4}\left(\vartheta_{0}(\tau)^{4}-\vartheta_{1}(\tau)^{4}\right)^{4} \tag{4.3.50}
\end{align*}
$$

Likewise, at genus 2, it is convenient to introduce the notation $\psi_{12}=\frac{1}{1728}\left(E_{4}^{3}-E_{6}^{2}\right)$.
The map Th : $R_{g} \rightarrow M_{g}$ is one-to-one for genus $g=1$ and $g=2$. However, at genus 3 , there is a degree 16 polynomial $j_{8}^{(3)}\left(x_{0}, \ldots x_{7}\right)$ which maps to a non-trivial relation among the genus 3 theta constants. It turns out that this is the only such relation at genus 3 126]. In summary, for $g \leqslant 3$ we have

$$
\begin{equation*}
M_{1} \cong R_{1}, \quad M_{2} \cong R_{2}, \quad M_{3} \cong R_{3} /\left\langle j_{8}^{(3)}\right\rangle, \tag{4.3.51}
\end{equation*}
$$

where $\left\langle j_{8}^{(3)}\right\rangle$ denotes the ideal generated by $j_{8}^{(3)}$. So we see that at $g=3$, Th fails to be injective, but the quotient in (4.3.51) makes the situation easy to deal with. The

[^13]situation at higher genus is further complicated by additional relations, and by the fact that Th is no longer surjective either, as we discuss in section 4.3.

Siegel modular forms under factorization In the factorization limit, defined in section 4.3, genus $g$ Siegel modular forms factor into products of lower-genus forms. For the purposes of this chapter, we will only need to consider the factorization of genus 2 Siegel modular forms, which is given by (with $q=e^{2 \pi i \tau_{1}}, s=e^{2 \pi i \tau_{2}}$ ):

$$
\begin{align*}
& E_{4}(q, r, s) \longrightarrow G_{4}(q) G_{4}(s) \\
& E_{6}(q, r, s) \longrightarrow G_{6}(q) G_{6}(s) \\
& \chi_{10}(q, r, s) \longrightarrow 0  \tag{4.3.52}\\
& \chi_{12}(q, r, s) \longrightarrow \Delta_{12}(q) \Delta_{12}(s) \\
& \psi_{12}(q, r, s) \longrightarrow G_{4}(q)^{3} \Delta_{12}(s)+\Delta_{12}(q) G_{4}(s)^{3}-1728 \Delta_{12}(q) \Delta_{12}(s)
\end{align*}
$$

Siegel modular forms under degeneration limits The Siegel $\Phi$ operator relates modular forms at different genus according to equation 4.3.41). For the forms of degree up to 12 , it acts as follows:

$$
\begin{align*}
\alpha_{4} & \mapsto E_{4} \mapsto G_{4} \mapsto 1, \\
\alpha_{6} & \mapsto E_{6} \mapsto G_{6} \mapsto 1, \\
\alpha_{10} & \mapsto \chi_{10} \mapsto 0,  \tag{4.3.53}\\
\alpha_{12} & \mapsto \chi_{12} \mapsto 0, \\
\alpha_{12}^{\prime} & \mapsto 0 .
\end{align*}
$$

Forms that map to zero under $\Phi$ are called cusp forms. $\chi_{10}, \chi_{12}$ and $\alpha_{12}^{\prime}$ are cusp forms. At genus 1 , the modular discriminant $\Delta_{12}$ is a cusp form.

## Beyond Genus 3

The method employed in this chapter is suitable for genus $g \leqslant 3$. Apart from the polynomial $j_{8}\left(x_{i}\right)$ at genus 3 , which maps onto a nontrivial relation among the theta functions, there is a one-to-one map between enumerator polynomals, Siegel modular forms, and CFT partition functions:

$$
\begin{equation*}
W^{(g)}\left(x_{i}\right)\left(\bmod j_{8}\right) \longleftrightarrow W^{(g)}\left(\vartheta_{i}(\Omega)\right) \longleftrightarrow Z^{(g)}(\Omega) \tag{4.3.54}
\end{equation*}
$$

Moreover, for genus $g \leqslant 3$ the ring $M_{g}$ of genus $g$ Siegel modular forms (for $\operatorname{Sp}(2 g, \mathbb{Z})$ ) has been fully characterized, and one can search for partition functions in this space.

Table 4.3: Summary of the differences between genus $g \leqslant 3$ and higher genus.

Genus $g \leqslant 3$
Th : $R_{g} \rightarrow M_{g}$ surjective (one-to-one $\quad \mathrm{Th}: R_{g} \rightarrow M_{g}$ neither surjective nor for $g \leqslant 2$ )

Ring $M_{g}$ of Siegel modular forms known

Moduili space $\mathcal{M}_{g}$ of Riemann sur- Locus of $\mathcal{M}_{g}$ inside $\mathcal{A}_{g}$ is non-trivial, faces dense inside moduli space $\mathcal{A}_{g}$ of ppavs

Genus $g \geqslant 4$ injective

Ring $M_{g}$ unknown and not known for $g>4$

However, as summarized in table 4.3, starting from genus 4 there are complications to each of these statements.

Consider first the ring homomorphism $\mathrm{Th}: R_{g} \rightarrow M_{g}$ mapping invariant polynomials to Siegel modular forms. At genus 1 and 2 it is completely invertible. At genus 3 it is surjective but injectivity fails; the kernel is generated by $j_{8}$, a degree 16 polynomial, see 4.3.51. At higher genus there exist more relations like $\operatorname{Th}\left(j_{8}^{(3)}\right)=0$. For instance, at genus 4, the first element of the kernel of Th is a degree 24 polynomial [135], and there are additional relations, one at degree 28 and five at degree 32 [123]. (It is not known if additional relations exist at higher degree.) Starting from genus 4, moreover, it has been shown that the map Th is not even surjective 125,8 meaning that there are Siegel modular forms that do not descend from enumerator polynomials. For instance, as shown already in [136], at genus 5 , the unique degree 6 Siegel modular form (which maps to $\alpha_{6} \mapsto E_{6} \mapsto G_{6}$ under successive applications of the Siegel $\Phi$ operator) has no pre-image in $R_{5}$ - no polynomial maps to it under Th. Note, however, that there is a polynomial which maps to the genus 5 counterpart of $G_{4}$, as expected on the grounds that each code has an enumerator polynomial at every genus ${ }^{9}$ While non-surjectivity of Th does not prevent us from finding code CFTs using our method, it means that our method will not be able to find other CFTs whose high-genus partition function derives from Siegel modular forms but not from any enumerator polynomial.

[^14]Next, consider the fact that the ring of Siegel modular forms is unknown at higher genus. As reviewed above, the complete set of relations for at genus 3 was only recently determined [134], although the generators have been known for a while [133]. At genus 4 the complete set of generators is not even known. Combined with non-surjectivity of the map Th, the lack of a complete set of generators poses another obstacle of extending our program beyond code CFTs at high genus.

Finally, consider the last obstacle, namely the construction of a partition function given a Siegel modular form $f \in M_{g}$. For concreteness, focus on the numerator of the partition function, $\hat{Z} \in \lambda^{c / 2}\left(\mathcal{M}_{g}\right){ }^{10}$ The relation to the Siegel upper half plane $\mathcal{H}_{g}$, on which the Siegel modular forms are functions, is as follows. Consider first the quotient space

$$
\begin{equation*}
\mathcal{A}_{g}=\mathcal{H}_{g} / \operatorname{Sp}(2 g, \mathbb{Z}) \tag{4.3.55}
\end{equation*}
$$

denoted the moduli space of principally polarized abelian varieties (ppavs). This space generalizes the fundamental region $-\frac{1}{2}<\operatorname{Re} \tau<\frac{1}{2},|\tau|>1$ to higher genus. By modular covariance, Siegel modular forms $f \in M_{g}$ of weight $k$ are mapped to sections of the $k$ th power of the determinant line bundle on $\mathcal{A}_{g}$. By the Torelli theorem, there is an embedding $\mathcal{M}_{g} \rightarrow \mathcal{A}_{g}$ mapping a Riemann surface to its Jacobian. ${ }^{11}$ For $g \leqslant 3$, the moduli space $\mathcal{M}_{g}$ of genus $g$ Riemann surfaces is dense in $\mathcal{A}_{g}$, meaning that almost every point in $\mathcal{A}_{g}$ represents a genus $g$ Riemann surface. However, as the dimension formulas

$$
\operatorname{dim} \mathcal{M}_{g}=\left\{\begin{array}{ll}
1 & g=1,  \tag{4.3.56}\\
3 g-3 & g>1,
\end{array} \quad \operatorname{dim} \mathcal{A}_{g}=\frac{1}{2} g(g+1)\right.
$$

show, for $g \geqslant 4$ the embedding of $\mathcal{M}_{g}$ inside $\mathcal{A}_{g}$ has a non-zero co-dimension. At genus 4 , with $\operatorname{dim} \mathcal{M}_{4}=9, \operatorname{dim} \mathcal{A}_{4}=10$, the locus of (the closure of) $\mathcal{M}_{4}$ inside $\mathcal{A}_{4}$ is known, and given by the vanishing of a specific degree eight modular form $J_{8}^{(4)}$ :

$$
\begin{equation*}
\overline{\mathrm{Jac} \mathcal{M}_{4}}=\left\{J_{8}^{(4)}=0\right\} \subset \mathcal{A}_{4} . \tag{4.3.57}
\end{equation*}
$$

At higher genus, the embedding is not known. The modular form $J_{8}^{(4)}$ is called the Schottky form, and is related to the polynomial $j_{8}^{(3)}$, as we will discuss in detail below.

[^15]There are two consequences of $\overline{\mathrm{Jac} \mathcal{M}_{g}} \neq \mathcal{A}_{g}$ for $g \geqslant 4$. Firstly, the existence of non-zero Siegel modular forms $f$ which vanish on the moduli space, means that the $\operatorname{map} f \mapsto \hat{Z}$ is not injective. More importantly, there may be sections $\hat{Z}$ of $\lambda^{c / 2}\left(\mathcal{M}_{g}\right)$ (i.e. numerators of partition function) that do not lift to sections of $\lambda^{c / 2}\left(\mathcal{A}_{g}\right)$ at sufficiently high genus. This means that when considering candidate CFTs with enumerator polynomial at $g \leqslant g_{\max }$ and which lack enumerator polynomial at $g>$ $g_{\max }$, if $g_{\max } \geqslant 3$ we cannot rule out the existence of a CFT corresponding to this set of enumerator polynomials.

## The Schottky form and the codes associated to $e_{8} \oplus e_{8}$ and $\boldsymbol{d}_{16}^{+}$Let us

 now consider in more detail an example of the consequences of the considerations discussed here. This is Milnor's example [102] of isospectral lattices mentioned in the introduction, which is based on a result by Witt [138. Consider the codes associated via Construction A to the lattices $e_{8} \oplus e_{8}$ and $d_{16}^{+}$, which are the root lattices of $E_{8} \times E_{8}$ and $\operatorname{spin}(32) / \mathbb{Z}_{2} \sqrt{138}{ }^{12}$ In fact, $e_{8}=d_{8}^{+}$and $d_{16}^{+}$are the first two elements of a general class of lattices $d_{8 k}^{+}$with the following description (see e.g. [139]). Let $\Lambda_{0}=\left\{\left(v_{1}, \ldots v_{8 k}\right) \in \mathbb{Z}^{8 k} \mid v_{1}+\ldots+v_{8 k}=0(\bmod 2)\right\}$. Then the lattice for $d_{8 k}^{+}$is given by$$
\begin{equation*}
\Lambda_{d_{8 k}^{+}}=\Lambda_{0} \cup\left(\frac{1}{2}(1,1, \ldots, 1)+\Lambda_{0}\right) \tag{4.3.58}
\end{equation*}
$$

In the sum defining the lattice theta function, the restriction to $v_{1}+\ldots+v_{8 k}=$ $0(\bmod 2)$ can be implemented by $1 \rightarrow \frac{1+(-1)^{v_{1}+\ldots+v_{8 k}}}{2}$, and using the definition 4.3.17) of the higher genus theta functions we find that ${ }^{13}$

$$
\begin{equation*}
\Theta_{d_{8 k}^{+}}=\frac{1}{2^{g}} \sum_{\mathbf{A}} \theta[\mathbf{A}](\Omega)^{8 k} \tag{4.3.59}
\end{equation*}
$$

Here the sum is over all $2^{g-1}\left(2^{g}+1\right)$ vectors $\mathbf{A} \in\left\{\binom{0}{0},\binom{1 / 2}{0},\binom{0}{1 / 2},\binom{1 / 2}{1 / 2}\right\}^{g}$ that give a non-zero theta constant ${ }^{14}$

Define $j_{8}^{(g)}$ to be the difference between the enumerator polynomials of the codes corresponding to the lattices $e_{8} \oplus e_{8}$ and $d_{16}^{+}$,

$$
\begin{equation*}
j_{8}^{(g)}=\left(W_{e 8}^{(g)}\right)^{2}-W_{d_{16}^{+}}^{(g)} \tag{4.3.60}
\end{equation*}
$$

[^16]For the case of genus 3, we will recover an expression for $j_{8}^{(3)}$ in the next section, see (4.4.30). The polynomial $j_{8}^{(g)}$ evaluates under Th to the Schottky form $J_{8}^{(g)}$ :

$$
\begin{equation*}
\operatorname{Th}\left(j_{8}^{(g)}\right)=J_{8}^{(g)}:=\frac{1}{2^{g}}\left(\left(\sum_{\mathbf{A}} \theta[\mathbf{A}](\Omega)^{8}\right)^{2}-2^{g} \sum_{\mathbf{A}} \theta[\mathbf{A}](\Omega)^{16}\right) \tag{4.3.61}
\end{equation*}
$$

It was first written down at genus 4 in a different format by Schottky in 1888 [140] and was later found by Igusa 141 to be proportional to the expression above. As advertised in the previous section, at genus 4 the Schottky form has the important property that its vanishing solves the Schottky problem, that is to determine the locus of $\mathcal{M}_{g}$ inside $\mathcal{A}_{g}$.

Now return to the partition functions corresponding to the codes $e_{8} \oplus e_{8}$ and $d_{16}^{+}$. For genus 1 and 2, their enumerator polynomials agree: $W_{e_{8} \oplus e_{8}}^{(g)}=W_{d_{16}^{+}}^{(2)}, g=1,2$. At genus 3, their enumerator polynomials are different, $W_{d_{16}^{+}}^{(3)}=W_{e_{8} \oplus e_{8}}^{(3)}-j_{8}^{(3)}$, but the associated Siegel modular forms are equal: $\operatorname{Th}\left(W_{d_{16}^{+}}^{(3)}\right)=\operatorname{Th}\left(W_{e_{8} \oplus e_{8}}^{(3)}\right)$. At genus 4 the associated Siegel modular forms are different, but coincide on $\mathcal{M}_{4}$, so that $\hat{Z}_{d_{16}^{+}}^{(4)}=\hat{Z}_{e_{8} \oplus e_{8}}^{(4)} \in \lambda^{8}\left(\mathcal{M}_{4}\right) .^{15}$ Finally, at genus 5, it was shown in 142 that at genus 5 the partition functions in fact do differ,

$$
\begin{equation*}
\hat{Z}_{d_{16}^{+}}^{(g)}=\hat{Z}_{e_{8} \oplus e_{8}}^{(g)}, \quad g \leqslant 4, \quad \hat{Z}_{d_{16}^{+}}^{(g)} \neq \hat{Z}_{e_{8} \oplus e_{8}}^{(g)}, \quad g \geqslant 5 . \tag{4.3.62}
\end{equation*}
$$

An alternative proof of this fact was given in [106]. In conclusion, the example we have considered shows that at $c=16$ there are two chiral conformal field theories, corresponding to the codes/lattices $e_{8} \oplus e_{8}$ and $d_{16}^{+}$, that share partition functions for $g \leqslant 4$ and are only distinguished at genus 5 .

In fact, the non-vanishing of $J_{8}$ on $\mathcal{M}_{5}$ was found in [142] with the motivation of computing the chiral superstring measure, and implies the non-vanishing of the cosmological constant at genus 5 in type II and heterotic superstring theory. This follows from an all-genus ansatz of the superstring measure proposed in [143] based on results at lower genus $144,145,146$. This ansatz, when summed over even spin structures (equivalent to the summation variables in 4.3.58), becomes proportional to $J_{8}^{(g)}$ (see 147 for an overview) ${ }^{16}$

[^17]
### 4.4 Counting Higher Genus Enumerator Polynomials

In section 4.3, we examined how the correspondence between classical errorcorrecting codes and 2d chiral CFTs extends to higher genus. Now we would like demonstrate the utility of this correspondence by explicitly showing how the higher genus modular symmetries, combined with the required factorization properties, drastically shrinks the space of allowed enumerator polynomials.

We aim to present the algorithm used in this section in a fairly self-contained way. However we will not include many details, such as the action of the modular group and the factorization limits, which are discussed in section 4.3.

## Method

In practice, our method is very simple. There are four steps:

1. write down the most general genus $g$ polynomial,
2. reduce undetermined coefficients by imposing the symmetries,
3. list solutions where all coefficients are positive integers,
4. eliminate all solutions which don't factorize.

The logic behind these steps has been considered at length in the previous section, so let us just remind the reader of a few important points. Then we will turn to some specific examples, which should make the procedure clear.

In the first step, the polynomial we start with depends on both the genus $g$ and the dimension $c$ of the code. Specifically, it will be the most general homogeneous degree $c$ polynomial of $2^{g}$ variables. Recall the definition of the enumerator polynomial of a code: the coefficients of each term represent degeneracies of codewords. True enumerator polynomials must have positive integer coefficients, thereby motivating step 3.

The second step amounts to imposing modular invariance. This is a strict constraint on the polynomials which can possibly correspond to codes. The size of the modular group grows, leading to stricter constraints. However, the size of the polynomials in step 1 also increases quickly with the genus, so the number of solutions in step 2 will increase quickly with the genus. It is only by requiring the correct factorization limits, which we do in step 4 , that we find a reduction in the number of potential codes.

## Example: enumerating $c=24$ polynomials

Let us now show how this works for the specific example of 24 -dimensional codes. It was found by Conway (unpublished, see note in [153]) that there are 9 doubly-even self-dual codes in 24 dimensions.

We will see that our algorithm restricts the number of potential enumerator polynomials. At genus 1, there are 190 possible polynomials which satisfy steps 1, 2, and 3. At genus 2, we will have an explosion in the number of possible polynomials which satisfy 1,2 and 3 . However only a tiny subset consisting of 29 polynomials, will properly factorize into genus 1 polynomials. This effectively rules out $190-29=161$ of the genus 1 polynomials. They cannot correspond to codes because if they did, they would lead to a factorizing genus 2 polynomial. Repeating the procedure at genus 3 leads to a further reduction from 29 to 21 possible polynomials, as we will see.

## Genus 1

The genus $1, c=24$ partition function is an order-24 polynomial of $x_{0}$ and $x_{1}$. We start by writing the most general polynomial with one for the leading coefficient (which is required because the identity is always a codeword).

$$
\begin{equation*}
P_{\mathrm{gen}}^{(1)}=x_{0}^{24}+a_{1} x_{0}^{23} x_{1}+a_{2} x_{0}^{22} x_{1}^{2}+a_{3} x_{0}^{21} x_{1}^{3}+\ldots . \tag{4.4.1}
\end{equation*}
$$

Next, we recall the invariances of this polynomial

$$
\begin{equation*}
x_{0} \mapsto \frac{x_{0}+x_{1}}{\sqrt{2}}, \quad x_{1} \mapsto \frac{x_{0}-x_{1}}{\sqrt{2}} \tag{4.4.2}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{1} \mapsto i x_{1} . \tag{4.4.3}
\end{equation*}
$$

Imposing these conditions places a number of constraints on the allowed values for $a_{i}$. The resulting polynomial therefore depends on far fewer coefficients:

$$
\begin{align*}
P_{\mathrm{inv}}^{(1)}=x_{0}^{24}+a_{4} x_{0}^{20} x_{1}^{4}+ & \left(759-4 a_{4}\right) x_{0}^{16} x_{1}^{8}+\left(2576+6 a_{4}\right) x_{0}^{12} x_{1}^{12} \\
& +\left(759-4 a_{4}\right) x_{0}^{8} x_{1}^{16}+a_{4} x_{0}^{4} x_{1}^{20}+x_{1}^{24} . \tag{4.4.4}
\end{align*}
$$

Now that we have determined the most general invariant polynomial, we must impose that all of the coefficients are positive integers. It is easy to see that this is the case when $a_{4}$ is an integer satisfying

$$
\begin{equation*}
0 \leq a_{4} \leq 189 \tag{4.4.5}
\end{equation*}
$$

So we find that there are 190 possible genus 1 polynomials. Any possible ECC must have one of these enumerator polynomials, because no other polynomial satisfies all of the necessary conditions. Note that in the case of genus 1 , there is no step 4.

Also note that we have only a single undetermined coefficient. This reflects the fact that the ring of genus 1 Siegel modular forms has only two independent elements at degree 12: $G_{4}^{3}$ and $G_{6}^{2}$. This gives one degree of freedom once the leading coefficient is set to one. We shall find a similar story for genus 2 and genus 3 .

To relate this result to the physical spectrum of the theory, we can write the partition function,

$$
\begin{equation*}
Z_{c=24}^{(1)}=\frac{P_{\mathrm{inv}}^{(1)}}{\eta(\tau)^{24}}, \tag{4.4.6}
\end{equation*}
$$

and expand in small $q$ to find ${ }^{17}$

$$
\begin{equation*}
Z_{c=24}^{(1)}(\tau)=\frac{1}{q}\left(1+\left(72+16 a_{4}\right) q+196884 q^{2}+21493760 q^{3}+\ldots\right) . \tag{4.4.7}
\end{equation*}
$$

From this expansion we can see that the number of conserved currents $N_{\text {currents }}=$ $72+16 a_{4}$. The genus 1 allowed polynomials, therefore, could have between 72 and 3,096 currents.

## Genus 2

The algorithm for genus 2 is largely the same for the first three steps. First we have

$$
\begin{equation*}
P_{\mathrm{gen}}^{(2)}=x_{0}^{24}+a_{1,0,0} x_{0}^{23} x_{1}+a_{0,1,0} x_{0}^{23} x_{2}+a_{0,0,1} x_{0}^{23} x_{3}+a_{2,0,0} x_{0}^{22} x_{1}^{2}+\ldots . \tag{4.4.8}
\end{equation*}
$$

The result is a 2,925 -term polynomial in $x_{0}, x_{1}, x_{2}, x_{3}$. Next we impose the symmetries 4.3.30. The expression for $P_{\text {inv }}^{2}$ is very long, but it has a shorter set of unique

[^18]coefficients, which must be positive integers. The result is
\[

$$
\begin{align*}
a_{4,0,0} & \geqslant 0, \\
759-4 a_{4,0,0} & \geqslant 0, \\
2576+6 a_{4,0,0} & \geqslant 0, \\
2 a_{2,2,2}+5 a_{4,0,0} & \geqslant 0, \\
-2 a_{2,2,2}+250 a_{4,0,0} & \geqslant 0, \\
22770+36 a_{2,2,2}-540 a_{4,0,0} & \geqslant 0, \\
a_{2,2,2} & \geqslant 0,  \tag{4.4.9}\\
-12 a_{2,2,2}+480 a_{4,0,0} & \geqslant 0, \\
22 a_{2,2,2}+2112 a_{4,0,0} & \geqslant 0, \\
340032-52 a_{2,2,2}-2432 a_{4,0,0} & \geqslant 0, \\
212520+76 a_{2,2,2}-3480 a_{4,0,0} & \geqslant 0, \\
1275120+36 a_{2,2,2}-1530 a_{4,0,0} & \geqslant 0, \\
4080384-8 a_{2,2,2}+29952 a_{4,0,0} & \geqslant 0 .
\end{align*}
$$
\]

This list of unique coefficients implies some redundant inequalities, but we have include the entire set for completeness. Using Mathematica, we can easily extract the set of integer solutions to these inequalities. The result is a set of 135,065 solutions. Note also that there are two undetermined parameters after imposing $a_{0,0,0}=1$. This is again a reflection that there are three independent modular forms at genus 2: $E_{4}^{3}, E_{6}^{2}$ and $\chi_{12}$ (see section 4.3).

Here we can demonstrate how step 4, the factorization limit, can be imposed to further restrict the number of polynomials. It is important to note the this will provide a restriction on the genus 1 polynomials as well: we will rule out all genus 2 polynomials which don't factorize properly, and we will rule out all genus 1 polynomials which don't arise from factorizing a genus 2 polynomial. The logic here is simple: a real code must have a genus 2 polynomial, and it must factorize into the square of that code's genus 1 polynomial.

Recall how the factorization limit acts on our genus 2 polynomials. ${ }^{18}$

$$
\begin{equation*}
x_{0} \rightarrow x_{0}^{2}, \quad x_{1} \rightarrow x_{0} x_{1}, \quad x_{2} \rightarrow x_{0} x_{1}, \quad x_{3} \rightarrow x_{1}^{2} \tag{4.4.10}
\end{equation*}
$$

So we merely need to check each of our 135,065 genus 2 polynomials. If the factorization limit turns it into the square of a genus 1 polynomial, then it is allowed.

[^19]Remarkably, there are only 29 such polynomials on our entire list! Each of these polynomials is the square of one of our 190 genus 1 polynomials. Therefore we have shown that only 29 of our genus 2 polynomials can possibly come from codes, and only 29 of our genus 1 polynomials can possibly come from codes. See table 4.4 for a summary.

For genus 2 at $c=24$, the set of constraints defines a two-dimensional polytope, and it is interesting to consider the way that the allowed polynomials sit inside it. This is displayed in figure 4.3. We see that a single genus 1 polynomial is excluded near the origin, and then a large number of polynomials are excluded in the upper range for the $a_{2,2,2}$. It is also interesting to note that $a_{4,0,0}$ satisfies $759 \geq 4 a_{4,0,0} \geq 0$, the same bound that we found on $a_{4}$ at genus 1 . However the rest of the constraints eventually give stronger constraints on $a_{4,0,0}$, meaning that if we project figure 4.3 (a) to the $y$ axis, and identify $a_{4,0,0} \sim a_{4}$, we find that the space of allowed genus 2 polynomials lies inside the space of allowed genus 1 polynomials. We believe this pattern will persist to higher $g$ and $c$. A final interesting feature is that the real theories display a quadratic relationship between $a_{2,2,2}$ and $a_{4,0,0}$. This appears to be related to the fact that, in equation (4.19) of [108], there are two undetermined coefficients $b_{2}$ and $b_{3}$ appearing in the genus 2 partition function which are linear and quadratic in the number of currents $N_{c}$, respectively. It would be interesting to see if any version of this pattern persists - already at $c=32$, it is known that the undetermined coefficients can depend on the number of currents and the OPE coefficients of low-lying states 108.

## Genus 3

The procedure for genus 3 follows the same idea as genus 1 and 2 , however some tricks are required to deal with the large number of terms in the polynomial $P_{g \text { gen }}^{(3)}$ and the large number of solutions, i.e. polynomials of the form $P_{\mathrm{inv}}^{(3)}$ with positive coefficients.

First, we do not start with the most general polynomial; instead we eliminate from the beginning a large number of terms which could never appear (analogous to how terms with odd powers could never appear at genus 1 , due to the $x_{1} \rightarrow i x_{1}$ transformation). The precise procedure for doing this is laid out in 129 in the discussion on "admissible polynomials."

Then we can impose the symmetries on this smaller polynomial to construct $P_{\text {inv }}^{(3)}$. We find that it depends on four independent parameters. This should reflect five independent modular forms at this order. However there are actually only four!


Figure 4.3: Allowed polytope for the genus 2 coefficients. Green dots represent polynomials which lie inside the bounds, and red outside. Plot (a) shows all 29 allowed genus 2 polynomials with integer coefficients. In (b), we zoomed into the bottom left, and allowed for half-integer coefficients in $P_{\text {inv }}^{1}$ allowing a larger number of solutions. Black stars represent lattices. We can see the $k=4, k=5$, and $k=6$ lattices outside the allowed region.

This is due to the existence of a "theta relation," discussed in section 4.3. This is a non-trivial combination of theta functions which equals zero, i.e. an element of the kernel of the theta-map. The conclusion is that, at genus 3 , the space of polynomials is larger than the space of modular forms.

Another trick we use is that we do not enumerate every solution. In general each coefficient of $P_{\text {inv }}^{(3)}$ must be positive, giving an inequality. For genus 3 (or genus 2, at large $c$ ), it becomes impossible to enumerate every solution. However it is possible to determine which genus 1 polynomials can arise as the factorization limit of genus 3 polynomials without determining the full set of genus 3 polynomials.

Our upgraded approach is thus to search through each of the 29 genus 1 polynomials which are allowed by genus 2 factorization, and determine if it can also arise from genus 3 factorization. Specifically, we look at the factorization limit of $P_{\text {inv }}^{(3)}$ and then we solve for the undetermined $a$-coefficients for each of the genus 1 polynomials. In every case, we find solutions, but for some of the genus 1 polynomials, all of the solutions result in disallowed (negative coefficient) genus 3 polynomials. In this way, we are able to rule out more genus 1 polynomials from our list.

The final result from this procedure is given in table 4.4. From our original list of 190 polynomials, we first reduce to 29 genus 2 polynomials (this number was already known in [129]). Then we find that demanding the existence of a consistent genus 3 polynomial further reduces this to a set of 21 polynomials. As far as we know, we are the first to count this number. It is easy to see from table 4.4 that each of the polynomials which corresponds to a real code is allowed by our procedure.

It is possible to continue for higher genus. At genus 3 (for $c=24$ ), we are at the edge of what is possible with our desktop computers. Probably to address this problem at higher genus, a new computational algorithm is necessary. The positivity conditions are linear, and the resulting spaces of solutions are convex polytopes in the space of $a_{i, j, k, . .}$ 's. This suggests that some sort of linear programming approach could drastically improve the efficiency of our rather basic method by searching for the edges of the space rather than enumerating every solution inside.

## Finding lattice theta functions

Above we have seen that our method may be used to find enumerator polynomials for every code that exists at $c=24$. These define lattices, so our method is able to find some lattice theta functions. However there are other theta functions, deriving from non-code lattices, which are not captured by our approach. In general it occurs for lattices whose theta functions correspond to enumerator polynomials with halfinteger coefficients and, in the case of $u(1)^{24}$, negative coefficients. We would like to understand how these lattice theta functions fit into our discussion so far.

By examining explicitly some of the theta functions for non-code lattices, we find that they can be written in enumerator polynomial form, but have fractional and

Table 4.4: Polynomials organized by number of spin 1 currents $N_{\text {currents }}=24 k$. The number of currents may be computed from the genus 1 polynomial using $N_{\text {currents }}=$ $72+16 a_{4}$. See [106] for the list of lattices organized by the number of currents.

| $k$ | $g=2$ | $g=3$ | lattice | code |
| :---: | :---: | :---: | :---: | :---: |
| 1 |  |  | $u(1)^{24}$ |  |
| 3 | $\checkmark$ | $\checkmark$ | $(a 1)^{24}$ | $\checkmark$ |
| 4 |  |  | $(a 2)^{12}$ |  |
| 5 |  |  | $(a 3)^{8}$ |  |
| 6 |  |  | $(a 4)^{6}$ |  |
| 7 | $\checkmark$ | $\checkmark$ | $(d 4)^{6},(a 5)^{4} d 4$ | $\checkmark$ |
| 8 |  |  | $(a 6)^{4}$ |  |
| 9 | $\checkmark$ |  | $(a 7)^{2}(d 5)^{2}$ |  |
| 10 |  |  | $(a 8)^{3}$ |  |
| 11 | $\checkmark$ | $\checkmark$ | $(d 6)^{4},(a 9)^{2} d 6$ | $\checkmark$ |
| 13 | $\checkmark$ | $\checkmark$ | $(e 6)^{4}, a 11 d 7 e 6$ |  |
| 14 |  |  | $(a 12)^{2}$ |  |
| 15 | $\checkmark$ | $\checkmark$ | $(d 8)^{3}$ | $\checkmark$ |
| 17 | $\checkmark$ | $\checkmark$ | $a 15 d 9$ |  |
| 19 | $\checkmark$ | $\checkmark$ | $a 17 e 7, d 10(e 7)^{2}$ | $\checkmark$ |
| 21 | $\checkmark$ | $\checkmark$ |  |  |
| 23 | $\checkmark$ | $\checkmark$ | $(d 12)^{2}$ | $\checkmark$ |
| 25 | $\checkmark$ | $\checkmark$ |  |  |
| 26 |  |  | $a 24$ |  |
| 27 | $\checkmark$ | $\checkmark$ |  |  |
| 29 | $\checkmark$ | $\checkmark$ |  |  |
| 31 | $\checkmark$ | $\checkmark$ | $d 16 e 8,(e 8)^{3}$ | $\checkmark \cdot 2$ |
| 33 | $\checkmark$ | $\checkmark$ |  |  |
| 35 | $\checkmark$ | $\checkmark$ |  |  |
| 37 | $\checkmark$ | $\checkmark$ |  |  |
| 39 | $\checkmark$ | $\checkmark$ |  |  |
| 41 | $\checkmark$ | $\checkmark$ |  |  |
| 43 | $\checkmark$ | $\checkmark$ |  |  |
| 45 | $\checkmark$ | $\checkmark$ |  |  |
| 47 | $\checkmark$ | $\checkmark$ | $d 24$ | $\checkmark$ |
| 49 | $\checkmark$ |  |  |  |
| 51 | $\checkmark$ |  |  |  |
| 53 | $\checkmark$ |  |  |  |
| 55 | $\checkmark$ |  |  |  |
| 57 | $\checkmark$ |  |  |  |
| 59 | $\checkmark$ |  |  |  |
| 61 | $\checkmark$ |  |  |  |

negative coefficients. The specifics are quite interesting. For genus 1 , there is a single lattice with negative coefficients. This is the $u(1)^{24}$ lattice, corresponding to $k=1$ in our table. At genus 2, we find that the $k=4,5$, and 6 lattices also yield enumerator polynomials with negative coefficients. The lattices are visible as the black stars 4.3, (b), where it is easy to see that these three polynomials are ruled out. So it appears that even if we repeat our procedure and allow for fractional coefficients, this will still not be enough to enumerate all lattices due to this negativity.

It seems likely that the coefficients are not allowed to be arbitrarily negative, because they still need to lead to a positive state expansion. For example, for the $u(1)^{24}$ theory, has the following theta-function in polynomial form

$$
\begin{equation*}
\Theta_{u(1)^{24}}\left(x_{0}, x_{1}\right)=x_{0}^{24}-3 x_{0}^{20} x_{1}^{4}+771 x_{0}^{16} x_{1}^{8}+2558 x_{0}^{12} x_{1}^{12}+771 x_{0}^{8} x_{1}^{16}-3 x_{0}^{4} x_{1}^{20}+x_{1}^{24} . \tag{4.4.11}
\end{equation*}
$$

We see that $a_{4}=-3$, corresponding to 24 currents. This is the smallest value of all lattices; every other lattice has $a_{4}>0$. It seems that what happens is that the enumerator polynomial, which represents the numerator of the partition function, is able to be slightly negative because as long as it can be compensated by the denominator (e.g. $\eta(q)^{c}$ in the genus 1 case) to ensure that the expansion in $q$, which counts the actual state degeneracies, is purely positive. Thus it seems that arbitrary amounts of negativity are probably not allowed in the enumerator polynomial.

It is known that there are 24 even self-dual Euclidean lattices in dimension 24 [154]. These are a subset of the 71 meromorphic chiral CFTs 155 . ${ }^{19}$ It would be very interesting to find the correct generalization of our procedure to enumerate lattices or general meromorphic theories, rather than just codes.

## Summary of results

## Counting polynomials with positive integer coefficients

Having performed this procedure, we may now list our results for a variety of different values of $c$.

- $c=8$ :
- there is a single polynomial at genus 1,2 , and 3 ,
- these all correspond to the Hamming [8,4,4] code.

[^20]- $c=16$ :
- at genus 1 and 2, there is a single polynomial,
- at genus 3 , we find 1,681 polynomials, which all factorize into the unique genus 1 polynomial. These 1,681 polynomials are a linear combination of the enumerator polynomials of two different codes the codes associated to the $e_{8} \otimes e_{8}$ and $d_{16}^{+}$lattices - which may be averaged in 1,681 ways to form positive integer polynomials ${ }^{20}$
- $c=24$ :
- there are 190 genus 1 polynomials. 29 come from consistent genus 2 polynomials, and 21 at genus 3 ,
- these come from 9 codes. Note however that the genus 1 partition functions of two of the code theories agree, so that there are eight unique genus 1 code theory partition functions,
- we find 135,065 invariant polynomials at genus 2.
- $c=32$ :
- we find 284 genus 1 polynomials,
- at genus 2, there are 210,116 polynomials with the correct factorization limits, but they only result in 161 unique genus 1 polynomials,
- There are 85 codes 156,157 , but only 37 unique genus 1 enumerator polynomials. 80 have Hamming distance $d=4$, and 5 have $d=8$.
- $c=40$ :
- there are 400 genus 1 polynomials,
- 246 of these come from genus 2 polynomials,
- we are not able to compute the total number of genus 2 polynomials
- there are 94,343 codes, 77,873 with Hamming distance $d=4$, the rest with $d=8158$.
- $c=48$ :
- there are $14,381,890$ genus 1 polynomials,

[^21]- 2,580,972 come from genus 2 polynomials,
- we are not able to enumerate all genus 2 polynomials.


## Siegel modular forms in the code variables

We have seen that the lattice theta series $\Theta_{\Lambda}(\Omega)$ transforms as modular form of weight $\frac{c}{2}$

$$
\begin{equation*}
\Theta_{\Lambda}(\tilde{\Omega})=\operatorname{det}(C \Omega+D)^{\frac{c}{2}} \Theta_{\Lambda}(\Omega) \tag{4.4.12}
\end{equation*}
$$

Modular forms of even weight are generated by a ring (4.3.46) which allows us fix $\Theta_{\Lambda}$ in terms of the generators of the ring. This is possible for genus 2 and 3 as well since the generators of the ring of Siegel modular forms is known 4.3.47). In this section, we constrain the space of code genus 2 CFT partitions functions and reproduce the results established in the previous subsection in this manner. The motivation for doing so is that the constraints of factorization on $\Theta_{\Lambda}$ can be imposed by exploiting the known factorization properties of the Siegel modular forms (4.3.52). This was precisely the strategy used in [108] to constrain the space of genus 2 chiral CFTs. In this section we show that by expressing genus 2 Siegel modular forms in terms of code variables, factorization constraints can be easily put on code CFT's.

## Siegel forms in code basis

In order write the Siegel forms in terms of the code variables, one only needs to know the number of generators in the ring and the action of the Siegel $\Phi$ operator 4.3.53). We first consider how this works at genus 1 . For $c=8$, the only allowed modular form of weight 4 is $G_{4}$, which fixes the associated $\Theta_{\Lambda}$. The associated Construction A lattice arises from the unique code of dimension 8 - the Hamming [8, 4, 4] code. Applying the theta map to this code gives us an expression for $G_{4}$ in terms of code variables.

$$
\begin{equation*}
G_{4} \cong x_{0}^{8}+14 x_{0}^{4} x_{1}^{4}+x_{1}^{8} . \tag{4.4.13}
\end{equation*}
$$

Similarly, at $c=24$, the only the most general $\Theta_{\Lambda}$ of weight 12 is characterized by the number of spin 1 currents $N_{1}$

$$
\begin{equation*}
\Theta_{\Lambda}^{g=1}=G_{4}^{3}+\left(N_{1}-744\right) \Delta_{12} \tag{4.4.14}
\end{equation*}
$$

This statement in terms of enumerator polynomials is a well known result in coding theory and is called Gleasons theorem for binaray self dual codes. It states that the
enumerator polynomial for any self dual code can be written as a linear combination of the expressions for $G_{4}$ and $\Delta$ written in the code basis. We can work backwards and express $\Delta_{12}$ in code variables since it must be given by a linear combination of $G_{4}^{3}$ in code variables and the enumerator polynomial of any even self dual $c=24$ code of choice. We then fix the overall normalization of $\Delta_{12}$ by demanding that it maps to zero under the Siegel theta map $\Phi$. The identity $\Delta_{12}=\frac{1}{1728}\left(G_{4}^{3}-G_{6}^{2}\right)$ fixes $G_{6}$ in terms of code variables as well

$$
\begin{align*}
G_{6} & \cong x_{0}^{12}-33 x_{0}^{8} x_{1}^{4}-33 x_{0}^{4} x_{1}^{8}+x_{1}^{12}  \tag{4.4.15}\\
\Delta_{12} & \cong \frac{1}{16} x_{0}^{4} x_{1}^{4}\left(x_{0}^{4}-x_{1}^{4}\right)^{4} . \tag{4.4.16}
\end{align*}
$$

We proceed in a completely analogous fashion at genus 2 . In this case again, at $c=8$ the unique weight 4 Siegel form, $E_{4}$, must be the biweight enumerator polynomial of the Hamming $[8,4,4]$ code. At $c=24$, there is more freedom and the most general lattice theta series that can arise is:

$$
\begin{equation*}
\Theta_{\Lambda}^{g=2}=E_{4}^{3}+a_{1} \psi_{12}+a_{2} \chi_{12} . \tag{4.4.17}
\end{equation*}
$$

We can take any 2 distinct $\Theta_{\Lambda}^{g=2}$ to be ones arising from the biweight enumerator polynomials of any two distinct $c=24$ codes. This gives us 2 equations of the form 4.4.17) which we can use to solve for $\psi_{12}$ and $\chi_{12}$. Demanding the normalization (4.3.53) allows us to fix the coefficients $a_{1}$ and $a_{2}$ and the end result is that can express them in code variables A.5.2) (A.5.5). The remaining Siegel form $\chi_{10}$ first appears at $c=32$ and it is fixed by repeating this procedure by taking the biweight enumerator polynomial of any self dual $c=32$ code and expressing it terms of known Siegel forms in the code basis according to

$$
\begin{equation*}
\Theta_{\Lambda}^{g=2}=E_{4}^{4}+a_{1} E_{4} \psi_{12}+a_{2} E_{4} \chi_{12}+a_{3} E_{6} \chi_{10} \tag{4.4.18}
\end{equation*}
$$

The unknown constants $a_{1}, a_{2}, a_{3}$ are fixed by demanding that $\chi_{10} \mapsto 0$ under the action of the Siegel $\Phi$ operator.

## Factorization constraints

We have seen that under the factorization limit the period matrix becomes block diagonal and the lattice theta function factorizes

$$
\begin{equation*}
\Theta_{\Lambda}^{g=2}(\Omega) \rightarrow \Theta_{\Lambda}^{g=2}\left(\tau_{1} \oplus \tau_{2}\right)=\Theta_{\Lambda}^{g=1}\left(\tau_{1}\right) \Theta_{\Lambda}^{g=1}\left(\tau_{2}\right) \tag{4.4.19}
\end{equation*}
$$

Using the expressions for code variables (A.5.2)-A.5.5), we can write down putative enumerator polynomials or equivalently code CFT partition functions. For these to
come from codes, we now demand that these polynomials have positive and integer degeneracy of codewords and that they factorize into squares of well defined lower genus enumerator polynomials in the factorization limit.
$\boldsymbol{c}=\mathbf{2 4}$ 4.4.17 must devolve to the square of the genus 1 lattice theta function given by (4.4.14). This is straightforward to implement since factorization properties of the Siegel forms are known (4.3.52). The result is

$$
\begin{equation*}
\Theta_{\Lambda}^{g=2}=E_{4}^{3}+\left(N_{1}-744\right) \psi_{12}+\left(\left(N_{1}-744\right)\left(N_{1}+984\right)\right) \chi_{12} \tag{4.4.20}
\end{equation*}
$$

It is important to emphasise that the above constraint is true in any meromorphic $c=24$ CFT and these constraints for various $c$ are discussed in 108. Here, we use it to constrain the space of allowed codes factorize because we know how to express the Siegal modular forms in terms of code variables. In fact, demanding that 4.4.20) has positive integer coefficients when written in code variables lets us recover the result that there are only 29 enumerator polynomials which factorize at genus 2 out of the 190 consistent genus 1 code polynomials.
$\boldsymbol{c}=32$ The allowed enumerator polynomial is parametrized by the number of spin 1 currents $N_{1}$ :

$$
\begin{equation*}
\Theta_{\Lambda}^{g=1}=G_{4}^{4}+\left(N_{1}-992\right) \Delta G_{4} . \tag{4.4.21}
\end{equation*}
$$

Demanding factorization gives us

$$
\begin{equation*}
\Theta_{\Lambda}^{g=2}=E_{4}^{4}+\left(N_{1}-992\right) \Delta E_{4} \psi_{12}+\left(N_{1}-992\right)\left(N_{1}+736\right) E_{4} \chi_{12}+c_{1} E_{6} \chi_{10} . \tag{4.4.22}
\end{equation*}
$$

Here $c_{1}$ is an unknown constant which cannot be determined by factorization constraints since $\chi_{10} \rightarrow 0$ under factorization. For $\Theta_{\Lambda}^{g=1}$ to possibly arise from a code, we must have $N_{1}=16 k$ for $6 \leqslant k \leqslant 289$ and $k \in \mathbb{Z}$. In order for $\Theta_{\Lambda}^{g=2}$ to possibly arise from a code, the range of allowed reduces to $6 \leqslant k$
leqslant166. We should note however that in general each of these $k$ have multiple consistent $c_{1}$ associated with them obeying inequalities. The different $c_{1}$ correspond to allowed values of the 3 point structure coefficient $c_{i, j, k}$ of light primary operators. Imposing unitarity does not rule out any code CFT's.
$\boldsymbol{c}=40$ The allowed enumerator polynomial is parametrized by $N_{1}$,

$$
\begin{equation*}
\Theta_{\Lambda}^{g=1}=G_{4}^{5}+\left(N_{1}-1240\right) \Delta_{12} G_{4}^{2} . \tag{4.4.23}
\end{equation*}
$$

For $\Theta_{\Lambda}^{g=1}$ to possibly arise from a code, we must have $N_{1}=120+16 k$ for $0 \leqslant k \leqslant 399$ and $k \in \mathbb{Z}$. Demanding a factorizable genus 2 enumerator polynomial gives:

$$
\begin{align*}
\Theta_{\Lambda}^{g=2}= & E_{4}^{5}+\left(N_{1}-1240\right) E_{4}^{2} \psi_{12}+\left(N_{1}-1240\right)\left(N_{1}+488\right) E_{4}^{2} \chi_{12}  \tag{4.4.24}\\
& +c_{1} E_{4} E_{6} \chi_{10}+c_{2} \chi_{10}^{2} .
\end{align*}
$$

where $c_{1}$ and $c_{2}$ are unknown coefficients. In order for $\Theta_{\Lambda}^{g=2}$ to possibly arise from a code, the range of allowed reduces to $0 \leqslant k \leqslant 246$ and these are not further constrained by demanding unitarity.
$c=48$ The allowed genus 1 partition function is

$$
\begin{equation*}
\Theta_{\Lambda}^{g=1}=G_{4}^{6}+\left(N_{1}-1488\right) G_{4}^{3} \Delta_{12}+\left(N_{2}-743 N_{1}+159769\right) \Delta_{12}^{2} \tag{4.4.25}
\end{equation*}
$$

For this arise from a code, we must have $N_{1}=144+16 k_{1}$ and $N_{2}=10199+2160 k_{1}+$ $256 k_{2}$ with $k_{1}$ and $k_{2}$ satisfying

$$
\begin{equation*}
0 \leqslant k_{1} \leqslant 374 \quad 0 \leqslant k_{2} \leqslant \frac{1}{8}\left(17296+733 k_{1}\right) \tag{4.4.26}
\end{equation*}
$$

or

$$
\begin{equation*}
375 \leqslant k_{1} \leqslant 766 \quad 0 \leqslant k_{2} \leqslant \frac{1}{28}\left(1997688-2607 k_{1}\right) \tag{4.4.27}
\end{equation*}
$$

This gives $14,381,890$ genus 1 polynomials.

$$
\begin{align*}
\Theta_{\Lambda}^{g=2}= & E_{4}^{6}+E_{4}^{3}\left(N_{1}\left(N_{1}+238\right)-2\left(N_{2}+338329\right)\right) \chi_{12}  \tag{4.4.28}\\
& +E_{4}^{3}\left(N_{1}-1488\right) \psi_{12}-\left(N_{1}+1968\right)\left(743 N_{1}-N_{2}-159769\right) \chi_{12} \psi_{12} \\
& +\left(-743 N_{1}+N_{2}+159769\right)\left(985 N_{1}+N_{2}+574489\right) \chi_{12}^{2} \\
& +\left(-743 N_{1}+N_{2}+159769\right) \psi_{12}^{2}+c_{1} E_{4}^{2} E_{6} \chi_{10}+c_{2} E_{4} \chi_{10}^{2} .
\end{align*}
$$

Demanding that the genus 1 polynomials arise from genus 2 code polynomials gives us $2,580,972$ polynomials.

Genus 3 As reviewed in sections 4.3 and 4.3 , starting from genus 3 , there are invariant polynomials that map to zero under Th. This is due to non-trivial relations among the higher genus theta functions. At genus 3 , the kernel of Th is generated by the degree 16 polynomial $j_{8}$. This means that we can write down a unique pre-image of the rank 4 form $\alpha_{4}$

$$
\begin{align*}
\alpha_{4} \cong W_{e_{8}}^{(3)}= & \sum_{i=0}^{7} x_{i}^{8}+14 \sum_{i<j} x_{i}^{4} x_{j}^{4}+1344 \prod_{i=0}^{7}+168 \sum_{\substack{i<j<k<l \\
i+j+k+l=6,14,22}} x_{i}^{2} x_{j}^{2} x_{k}^{2} x_{l}^{2} \\
& +168\left(x_{0}^{2} x_{1}^{2} x_{4}^{2} x_{5}^{2}+x_{0}^{2} x_{2}^{2} x_{4}^{2} x_{6}^{2}+x_{1}^{2} x_{3}^{2} x_{5}^{2} x_{7}^{2}+x_{2}^{2} x_{3}^{2} x_{6}^{2} x_{7}^{2}\right) \tag{4.4.29}
\end{align*}
$$

but at $c \geqslant 16$ the modular forms in the code variables will only be defined up to adding terms proportional to $j_{8}$. For instance, the 1,681 polynomials found at $c=16$ are all of the form

$$
\begin{equation*}
P=\left(W_{e_{8}}^{(3)}\right)^{2}+\frac{a}{1344} j_{8}^{(3)} \tag{4.4.30}
\end{equation*}
$$

for integer $a$ in the range $-1344 \leqslant a \leqslant 366$. The code $d_{16}^{+}$corresponds to $a=-1344$. An explicit expression for $j_{8}$ is given in A.5.8.

### 4.5 Discussion

In this chapter, we have shown how code CFTs provide a simple setup to explore higher genus modular invariance: the weight- $g$ enumerator polynomial evaluated at theta constants is the numerator of the genus- $g$ partition function of a code CFT. The higher genus modular transformations take the form of linear transformations on the enumerator polynomial. It is possible to solve the constraints completely, yielding the entire set of possible code enumerator polynomials. These are further reduced by requiring that the partition function factorizes as the genus-g Riemann surface factorizes into two lower-genus Riemann surfaces. The result is that the higher genus constraints are much more restrictive than the lower genus constraints. In the case of $c=24$, we find 190 candidate polynomials at genus 1,29 candidates at genus 2 , and 21 candidates at genus 3 (see table 4.4). There are exactly 9 doublyeven self-dual codes at $c=24$ so we speculate that this process, if pushed to even higher genus, will eventually converge to 9 .

So what to make of the ruled out theories, for instance the "fake" genus 1 partition functions which have no corresponding partition function at genus 2 or 3? Indeed, our work was largely motivated by this question, posed in 19 for non-chiral CFTs but with a direct counterpart in the chiral case. If the fake partition functions indeed do not correspond to any CFT, their existence poses a clear limitation of the genus 1 modular bootstrap. Here we have shown that extending the modular bootstrap to higher genus gives a significant improvement.

It is possible, however, is that the "fake" genus 1 partition functions may correspond to CFTs which are not derived from error-correcting codes. Therefore it would be desirable to enlarge the class of theories that can be captured by our approach. In section 4.4, we investigated what relaxations are needed to find lattice theories, which may or may not come from codes. These theories have partition functions which can be written in "enumerator polynomial form", meaning that they are homogeneous polynomials of theta-constants. However, unlike code theories, the
polynomials have coefficients which may be fractional and slightly negative. The negativity, in particular, makes it difficult to devise a finite algorithm to enumerate them. However, they do not seem to be allowed to be arbitrarily negative, which leaves open the possibility that it may be possible to enumerate them by only slightly relaxing the positivity constraint. It is natural to try this at $c=24$, where there are 24 even self-dual Euclidean lattices [154] and 71 meromorphic theories in total (155.

Another interesting direction for future work would be to try to make contact with the traditional bootstrap program [103]. Higher genus partition functions contain some information about OPE coefficients, so some constraints on dynamical information are implied by higher genus modular invariance. It would be interesting to try to understand the extent of this - in particular, are the constraints implied by modular invariance at all genera equivalent to those implied by consistency of sphere $n$-point functions with crossing, or is one constraint stronger than the other? It would also be interesting to see in detail how the factorization limits, which go beyond the symmetries of the theory, might be related to OPE data. Specifically, do the higher genus partition functions which do not factorize correctly lead to OPE data which is somehow pathological? In this chapter, we considered the complete factorization limit, where $\Omega$ is made strictly block diagonal. However one can consider subleading corrections to this factorization, taking off-diagonal elements of $\Omega$ to be small expansion parameters. This yields information about averages of OPE coefficients for light operators 108. We leave exploring this systematically to future work.

Our work opens avenues to explore the relationship between quantum errorcorrecting codes, Lorentzian lattices, and (non-chiral) CFTs [21, 19, 159] at higher genus. As we will discuss in [160], this relationship admits a higher genus generalization similar to the one presented in this chapter. The enumerator polynomial of a quantum error-correcting code at genus $g$ is a polynomial in $2^{g-1}\left(2^{g}+1\right)=$ $3,10,36, \ldots$ variables. This polynomial evaluated at theta constants gives the higher genus partition function of a putative non-chiral CFT. The constraints on higher genus partition functions from modular invariance and factorization are interesting and largely unexplored areas, and the techniques developed in this chapter give clear directions to understand them. This will be explored in a future chapter 160 .

Another interesting byproduct of our work is a set of explicit expressions for genus 2 and 3 Siegel modular forms in terms of the polynomials of theta constants. These expressions facilitate easy manipulation of the forms - for example, it is obvious
in our basis if a form is a product of two other forms. In principle, one could use our method to determine the full ring of modular forms for genus $g \leqslant 3$. We believe that this approach would be particularly interesting in the case of quantum error-correcting codes, which correspond to non-chiral CFTs and where the theory of modular forms is much less developed.

Interesting recent work has given a possible holographic bulk interpretation to the average over Narain lattice CFT's [121, 161, which are computed using the Siegel-Weil formula. The chiral version of this formula averages even unimodular Euclidean lattices of dimension $c$, to give a (holomorphic) Siegel modular form at every genus, e.g. see 162,

$$
\begin{equation*}
\sum_{\Lambda} \frac{1}{\operatorname{Aut}(\Lambda)} \Theta_{\Lambda}=m_{\frac{c}{2}} E_{\frac{c}{2}}^{(g)} \tag{4.5.1}
\end{equation*}
$$

where $E_{k}^{(g)}$ is the genus $g$ holomorphic Eisenstein series. Here the sum is weighted by the number of automorphisms of each lattice and $m_{k}=\frac{B_{k}}{2 k} \frac{B_{2}}{4} \frac{B_{4}}{8} \cdots \frac{B_{2 k-2}}{4 k-4}$ where $B_{k}$ denotes the $k^{t h}$ Bernoulli number. The appearance of the Siegel modular form suggests that a bulk interpretation of the chiral average may also be possible [163]. In this context, it would be interesting to see if our discussion on factorization of individual lattice partition functions provides a simple setting to study statistical properties of the averaged partition function.

## Chapter 5 Quantum codes, lattices and CFTs

This chapter is essentially identical to:
Narain CFTs and Quantum Codes at Higher Genus [23]

### 5.1 Introduction: Partition functions of Narain CFTs and quantum error correcting codes

There is a fascinating relationship between error-correcting codes (ECCs) and twodimensional conformal field theories. This relationship was first explored in the context of classical binary codes - codes over $\mathbb{F}_{2}$ - which are related to chiral meromorphic CFTs. Any doubly-even self-dual code defines an even self-dual lattice by "Construction A" of Leech and Sloane [117], and such a lattice can be used to define a free meromorphic CFT by the work of Dolan, Goddard and Montague 112, 113, 18]. In this relation, also reviewed in [19], a code of length $n$ provides an $n$-dimensional lattice, which gives rise to a meromorphic CFT of central charge $c=n$. The construction makes it extremely easy to compute the partition function of the resulting CFT. It takes the form

$$
\begin{equation*}
Z^{(g=1)}(\tau)=\frac{W\left(\theta_{3}\left(q^{2}\right), \theta_{2}\left(q^{2}\right)\right)}{\eta(\tau)^{n}} \tag{5.1.1}
\end{equation*}
$$

Here $q=e^{2 \pi i \tau}$ for the modular parameter $\tau, \eta(\tau)$ is the Dedekind eta function and $\theta_{i}\left(q^{2}\right)$ are Jacobi theta functions. The function $W\left(x_{0}, x_{1}\right)$ is the so-called enumerator polynomial, a central object of this chapter. It is trivially computable given the codewords defining the code. The conditions required for the partition function to be modular invariant (up to phases) follow immediately from simple properties of the code. These are the so-called MacWilliams identities $116{ }^{1}$

$$
\begin{equation*}
W\left(x_{0}, x_{1}\right)=W\left(x_{0}, i x_{1}\right), \quad W\left(x_{0}, x_{1}\right)=W\left(\frac{x_{0}+x_{1}}{\sqrt{2}}, \frac{x_{0}-x_{1}}{\sqrt{2}}\right) . \tag{5.1.2}
\end{equation*}
$$

The construction above is naturally extended to higher genus, where the generalized MacWilliams identities acting on the higher-weight enumerator polynomial in $2^{g}$ variables guarantee genus $g$ modular invariance of the corresponding partition function [160]. The upshot is two-fold: firstly, since the enumerator polynomial

[^22]is directly computable from the code, one gets immediate access to higher-genus partition functions of code CFTs; secondly, since the classification of the involved entities - codes, lattices and CFTs - is incomplete, ${ }^{2}$ the enumerator polynomial form provides a simple way to analyze modular invariance as linear relations.

Solving the constraints from higher-genus modular invariance amounts to characterizing the ring of invariant polynomials, which for genus $g \leqslant 3$ determines the space of partition functions for meromorphic chiral CFTs via the theory of Siegel modular forms ${ }^{3}$ In |160], we showed that by imposing higher-genus modular invariance, the number of possible genus 1 partition functions of code CFT is greatly reduced. We also noted that there exist conformal field theories that do not derive from codes, but which still admit the "code enumerator form" of (5.1.1), however with non-negative and non-integer coefficients. This holds for instance for all of the 71 meromorphic CFTs at $c=24$ classified by Schellekens 155 .

In this chapter, we turn to an interesting generalization of the above relation, namely between quantum error-correcting codes and non-chiral CFTs. This relation was spelled out in detail in [19]. The starting point is a class of quantum errorcorrecting codes known as stabilizer codes. These have an equivalent description in terms of classical codes over the finite field $\mathbb{F}_{4}$. In general, one can consider codes defined over any finite field $F$, and it has been shown recently that ternary codes, defined over $F=\mathbb{F}_{3}$, define $\mathcal{N}=1$ supersymmetric 2d CFTs 164. Other work on the relation between error-correcting codes and conformal field theories include [21, $165,163,166,159,167$. Interestingly, by the Gray map $\mathbb{F}_{4}$ can be related to $\mathbb{F}_{2} \oplus \mathbb{F}_{2}$, which is the first case of a series of constructions of Narain CFTs from codes over $\mathbb{F}_{p} \oplus \mathbb{F}_{p}$ for $p$ prime 168.

Via the "New Construction A," a code over $\mathbb{F}_{4}$ defines a Lorentzian lattice, on which a 2d non-chiral ("full") CFT can be defined by the construction of Narain [118, 119. Compared to the chiral case discussed above, there are now several differences. In the New Construction A, the central charge $n=c=\bar{c}$ can take any positive integer value, and the phases showing up in the modular constraints always cancel, rendering completely modular invariant partition functions of the form

$$
\begin{equation*}
Z_{\mathcal{C}}(q, \bar{q})=\frac{W_{\mathcal{C}}\left(\theta_{3}(q) \theta_{3}(\bar{q})+\theta_{4}(q) \theta_{4}(\bar{q}), \theta_{3}(q) \theta_{3}(\bar{q})-\theta_{4}(q) \theta_{4}(\bar{q}), \theta_{2}(q) \theta_{2}(\bar{q})\right)}{2^{n}|\eta(\tau)|^{2 n}} \tag{5.1.3}
\end{equation*}
$$

[^23]where $q=e^{2 i \pi \tau}$ and $\bar{q}=e^{-2 i \pi \bar{\tau}}$. $W_{\mathcal{C}}\left(x_{0}, x_{1}, x_{2}\right)$ is called the refined enumerator polynomial, to be defined below. On the other hand, this construction gives rise to a discrete set of CFTs, which lies inside but is not dense in the continuous Narain moduli space. For instance, in the simplest case of central charge $n=1$, the unique equivalence class of quantum codes gives rise to the Narain CFT of the compact boson at the radius 1 , whereas at $c=1$, Narain CFTs can be defined for any compactification radius.

As in the chiral case, the partition function will be constructed by a sum over vectors in a lattice defined by the code, and it will be described in terms of highergenus theta functions with known modular properties. The theory of such functions is less developed compared the chiral case, where indeed the theory of Siegel modular forms has led to strong constraints on meromorphic CFTs at low $c$ 106, 107, 108, 109, 110. Nevertheless, we shall see that the set of partition functions in the form dictated by (5.1.3) does capture some interesting theories also outside the class of code CFTs, $4^{4}$ to be discussed more in section 5.5 , which may hint at the possibility to develop a theory of non-chiral modular forms.

The main novelty of this chapter is to generalize the relation between quantum ECCs and full CFTs to higher-genus partition functions. This amounts to writing down higher-weight enumerator polynomials suitable for codes over general fields. In general, for a field $F$, there is a natural construction of a weight- $g$ enumerator polynomial in $|F|^{g}$ variables as a sum over $g$-tuples of codewords, see 5.3.28 below. For the case at hand, where $F=\mathbb{F}_{4}$, the genus 1 version was called the complete enumerator polynomial in [19]. However, we will see that the identical vanishing of some modular functions - at genus 1 manifested by $\theta_{1}(q)=0$ - suggests the use of a refined enumerator polynomial in $2^{g-1}\left(2^{g}+1\right)=3,10,36, \ldots$ variables. (In fact, the counting is the same as the number of even spin structures on the Riemann surface, familiar from higher-loop superstring computations.) We will give a complete description of this construction, and give many details at genus 2 .

## A puzzle with fake partition functions

One of our motivations to study the relation between quantum ECCs and full CFTs, was to resolve the following puzzle that emerged from the considerations at genus 1 in [19]. At genus 1, the refined enumerator polynomial for a code $\mathcal{C}$ is homogeneous

[^24]degree $n$, in three variables, and can be written as a sum over all codewords
\[

$$
\begin{equation*}
W_{\mathcal{C}}\left(x_{0}, x_{1}, x_{2}\right)=\sum_{c \in \mathcal{C}} x_{0}^{n-w_{X}(c)-w_{Y}(c)-w_{Z}(c)} x_{1}^{w_{Y}(c)} x_{2}^{w_{X}(c)+w_{Z}(c)}, \tag{5.1.4}
\end{equation*}
$$

\]

where we think of the codewords $c \in \mathcal{C}$ as vectors with entries in the set $\{\mathbb{1}, X, Y, Z\}$, and $w_{t}(c)$ counts the number of entries $c_{i}=t .5$ For codes to define CFTs, the enumerator polynomial defined in 5.1.4 must satisfy the generalized MacWilliams identities $116,169,170,171,172$, which is to say that $W_{\mathcal{C}}\left(x_{0}, x_{1}, x_{2}\right)$ is invariant under

$$
\begin{equation*}
x_{0} \mapsto \frac{1}{2}\left(x_{0}+x_{1}+2 x_{2}\right), \quad x_{1} \mapsto \frac{1}{2}\left(x_{0}+x_{1}-2 x_{2}\right), \quad x_{2} \mapsto \frac{1}{2}\left(x_{0}-x_{1}\right), \tag{5.1.5}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{1} \mapsto-x_{1} . \tag{5.1.6}
\end{equation*}
$$

The work of [19] proposed to study solutions $W\left(x_{0}, x_{1}, x_{2}\right)$ of (5.1.5-5.1.6) without explicitly constructing any quantum code. Combining these equations with the assumption that $W\left(x_{0}, x_{1}, x_{2}\right)$ has non-negative integer coefficients, which must be the case for code CFTs, one finds a discrete set of solutions for each degree $n$. Only some of these solutions correspond to known codes. The simplest non-trivial example is at $n=3$. In this case, 19 found 10 solutions, however only four of them correspond to known codes:

$$
\begin{align*}
W_{1}^{3} & =\left(x_{0}+x_{2}\right)^{3},  \tag{5.1.7}\\
W_{1} W_{2} & =\left(x_{0}+x_{2}\right)\left(x_{0}^{2}+x_{1}^{2}+2 x_{2}^{2}\right)  \tag{5.1.8}\\
W_{3} & =x_{0}^{3}+3 x_{0} x_{2}^{2}+3 x_{1}^{2} x_{2}+x_{2}^{3},  \tag{5.1.9}\\
\tilde{W}_{3} & =x_{0}^{3}+3 x_{0} x_{1}^{2}+4 x_{2}^{3} . \tag{5.1.10}
\end{align*}
$$

The six remaining polynomials also give rise to seemingly consistent partition functions with $n=3$, which we denote as "fake partition functions. ${ }^{76}$ These fake partition functions have enumerator polynomials in $x_{i}$ with non-negative integer coefficients and could therefore correspond to code enumerator polynomials. Furthermore, they

[^25]have character decompositions with non-negative integer coefficients - both in Virasoro characters and $U(1)^{c} \times U(1)^{c}$ characters. The latter decomposition takes the form
\[

$$
\begin{equation*}
Z(\tau, \bar{\tau})=\sum_{h, \bar{h}} d_{h, \bar{h}} \frac{q^{h} \bar{q}^{\bar{h}}}{\eta(\tau)^{c} \eta(\bar{\tau})^{c}} \tag{5.1.11}
\end{equation*}
$$

\]

For instance, consider one of the fake theories,

$$
\begin{equation*}
W_{\text {fake }}=\frac{2}{3} W_{1}^{3}+\frac{1}{3} \tilde{W}_{3} \tag{5.1.12}
\end{equation*}
$$

The first few degeneracies of $U(1)^{c} \times U(1)^{c}$ primaries for $W_{\text {fake }}$ read

$$
\begin{align*}
& d_{0,0}=1, \quad d_{\frac{1}{8}, \frac{1}{8}}=4, \quad d_{\frac{1}{4}, \frac{1}{4}}=12, \quad d_{\frac{3}{4}, \frac{3}{4}}=8  \tag{5.1.13}\\
& d_{1,1}=12, \quad d_{1,0}=d_{0,1}=0, \quad \ldots
\end{align*}
$$

Since this fake theory, like all fake theories found by examining solutions to the Macwilliams identities, has all non-negative integer degeneracies, we cannot prove that it does not correspond to a CFT by genus 1 considerations alone. However, in this chapter, we find that by considering the constraints from genus 2 modular invariance on code CFT partition functions, we can prove that a number of fake partition functions cannot be defined by error-correcting codes through New Construction A. This includes all six fake theories at $n=3$. Continuing to higher values of central charge $n$, we find that the higher-genus constraints will not rule out all fake partition functions, but drastically reduce their number.

## Structure of this chapter

The rest of this chapter is structured as follows. In section 5.2, we review the construction of 2d CFTs from quantum error-correcting codes. This includes a number of elements. First we review classical codes over general fields, and how they can be used to define lattices and enumerator polynomials. Then we present the generalization to quantum codes. After giving a description of quantum error-correcting codes and an explanation of how error correction is achieved, we show how quantum ECCs define Lorentzian lattices and therefore Narain CFTs. A brief overview and an explicit example of the construction are given in section 5.2 .

Section 5.3 is where we explain how this construction may be extended to higher genus. This is essentially the main result of the chapter. We show how higher-genus partition functions are related to the so-called higher-weight enumerator polynomials of the codes. These need to be evaluated with higher-weight Jacobi theta functions
as arguments. We determine how modular invariance acts on these higher-weight enumerator polynomials.

We use these observations to aid the classification of code CFTs. In section 5.4, we describe how to characterize the ring of polynomials defined by invariance under higher-genus modular transformations. This allows us to count the number of potentially valid genus 1 and genus 2 partition functions explicitly for $n \leqslant 6$. We show that only a fraction of the valid genus 1 partition functions arise as the factorization limit of valid genus 2 partition functions, which underlies our claim that genus 2 modular invariance is a strong constraint on the space of theories. We also discuss how we can resolve several sets of isospectral theories - such theories have the same genus 1 partition function, but we find different genus 2 partition functions. Finally, in section 5.5 we point out the "enumerator polynomial form" is a useful ansatz for the partition function that applies to many non-code theories as well. We use it to construct modular invariant functions with a large gap in primaries.

### 5.2 Quantum codes and Narain CFTs

The purpose of this section is to review the construction of CFTs from errorcorrecting codes. The original construction, due to 112,113 , associates chiral CFTs to binary codes. In this chapter, we will be primarily interested in the more recent construction, due to [21, 19], which associates non-chiral CFTs to classical codes over $\mathbb{F}_{4}$, or equivalently, quantum stabilizer codes. None of the material presented in this section is new: it is merely meant as a review of that work. A large part of the discussion also follows $\sqrt{173}$ - see that textbook for a more thorough introduction.

We have included a much briefer version of this discussion in section 5.2. The reader who wants to proceed more quickly towards the results of this chapter may prefer to start there.

## Classical codes over general fields

Error-correcting codes are designed to encode information redundantly to protect against corruption. The classical example is the repetition code, where 0 is encoded as 000 and 1 is encoded as 111. In the information-theoretic context then, errorcorrecting codes should be thought of as a map from length- $k$ vectors to length- $n$ vectors, where $n>k$. However for our purposes, we shall simply think of them as a collection of codewords, i.e. as the image of this map. For classical codes, these
elements may be vectors over $\mathbb{F}_{2}$ (for the common case of binary codes) or any other finite field $F$. For quantum codes, the elements are spins, or qubits. Before turning to quantum codes, let us briefly review some facts about codes. These ideas are also reviewed in 114, 115, and more recently for a physics audience by 19, 160.

Additive codes are those where the sum of any two codewords is a codeword. These may be easily specified by a generator matrix $G$ :

$$
\begin{equation*}
c=G x, \quad c \in \mathcal{C} \subset F^{n}, \quad x \in F^{k} . \tag{5.2.1}
\end{equation*}
$$

Here we use $\mathcal{C}$ to denote the code and $c$ to denote its element codewords. It is clear then that $G$ must be an $n \times k$ matrix, and that there are $|F|^{k}$ codewords. Alternatively, a code may be specified by its parity check matrix $H$, defined to satisfy

$$
\begin{equation*}
H c=0 \quad \text { if and only if } c \in \mathcal{C} . \tag{5.2.2}
\end{equation*}
$$

The parity check matrix is an $(n-k) \times n$ matrix, and also satisfies $H G=0$. The parity check matrix is directly useful in error correction. If a codeword is corrupted, $c \rightarrow c^{\prime}=c+e$, this can be easily detected by applying the parity check

$$
\begin{equation*}
H c^{\prime}=H(c+e)=H e \tag{5.2.3}
\end{equation*}
$$

We shall see that this step has a direct analogue when we discuss the case of quantum error-correcting codes.

The error-correction ability of a code is directly related to how far apart its codewords are, which is measured by the Hamming distance. The Hamming distance $d\left(c_{1}, c_{2}\right)$ for two codewords is defined as the $\ell^{0}$ norm, or the number of entries which are different. The Hamming distance for a code is defined as the minimum distance between any two codewords. An error-correcting code with length $n,|F|^{k}$ elements, and Hamming distance $d$ is denoted as an $[n, k, d]$ code.

Enumerator polynomials A coarse description of a code is provided by its enumerator polynomial, which counts the degeneracy of codewords. The most general such object is the complete enumerator polynomial, defined by

$$
\begin{equation*}
W_{\mathcal{C}}\left(x_{0}, \ldots\right)=\sum_{c \in \mathcal{C}}\left(\prod_{i=1}^{n} x_{c_{i}}\right) . \tag{5.2.4}
\end{equation*}
$$

This is a function of $|F|$ variables, $x_{0}, \ldots, x_{|F|-1}$. The coefficient of each monomial $x_{0}^{n_{0}} x_{1}^{n_{1}}, \ldots$ is the number of codewords with $n_{0} 0 \mathrm{~s}, n_{1}$ copies of the first non-zero
element of $F$, and so on. Here we imagine $x_{0}$ to correspond to $0 \in F$, and the other $x_{i}$ to the other elements of $F$ in a fixed but otherwise arbitrary order.

For codes over $\mathbb{F}_{2}$, the enumerator polynomial simply counts the non-zero elements in each codeword. The number of non-zero elements is called the Hamming weight, defined by $w(c)=d(c, 0)$. In this case, we can write the enumerator polynomial as

$$
\begin{equation*}
W_{\mathcal{C}}\left(x_{0}, x_{1}\right)=\sum_{c \in \mathcal{C}} x_{0}^{n-w(c)} x_{1}^{w(c)} \tag{5.2.5}
\end{equation*}
$$

Error-correcting codes can be used to define lattices by embedding them into a bigger vector space, and identifying lattice vectors as living in cosets defined by each codeword. The classic example of this is Construction A of Leech and Sloane 117, which associates lattices in $\mathbb{R}^{n}$ to binary codes via

$$
\begin{equation*}
\Lambda(\mathcal{C})=\left\{\left.\frac{v}{\sqrt{2}} \right\rvert\, v \in \mathbb{Z}^{n}, v \equiv c(\bmod 2) \text { for some } c \in \mathcal{C}\right\} \tag{5.2.6}
\end{equation*}
$$

This identification leads to a relationship between the code enumerator polynomial and lattice theta function: 7

$$
\begin{equation*}
\Theta_{\Lambda}(\tau)=W_{\mathcal{C}}\left(\theta_{3}\left(q^{2}\right), \theta_{2}\left(q^{2}\right)\right) \tag{5.2.7}
\end{equation*}
$$

i.e. the lattice theta function can be found by substituting the polynomial variables $x_{0}, \ldots$ with Jacobi theta functions. This formula has generalizations for other fields, including the one relevant for our purposes: $\mathbb{F}_{4}$. It has elements $0,1, \omega, \omega^{2}$ which satisfy

$$
\begin{equation*}
\omega * \omega^{2}=1, \quad \omega+1=\omega^{2}, \quad 1+1=\omega+\omega=\omega^{2}+\omega^{2}=0 \tag{5.2.8}
\end{equation*}
$$

We may think of $\omega$ as being a third-root of unity (in which case the last requirement, $x+x=0$, must be input by hand).

We shall primarily be interested in self-dual codes, since these will lead to modular-invariant partition functions. The dual of a code over a field $F$ is given by

$$
\begin{equation*}
C^{\perp} \equiv\left\{a \in F^{n} \mid(a, c)=0, \text { for all } c \in \mathcal{C}\right\} \tag{5.2.9}
\end{equation*}
$$

Here the algebra is over $F$. For fields with characteristic 2 , such as $\mathbb{F}_{2}$ or $\mathbb{F}_{4}$, this amounts to the requirement that $(a, c)=0(\bmod 2)$. The definition of duality depends on the definition of the inner product (.) used. We will be interested in the

[^26]inner-product defined by
\[

$$
\begin{equation*}
a \cdot c=\sum_{i=1}^{n} \bar{a}_{i} c_{i}+a_{i} \bar{c}_{i} \tag{5.2.10}
\end{equation*}
$$

\]

where $\bar{a}$ is the complex conjugate of $a$ when viewing $\omega$ as a third root of unity.
For a general field, code-duality acts on the Hamming enumerator polynomial by (116)

$$
\begin{equation*}
W_{\mathcal{C}^{\perp}}\left(x_{0}, x_{1}\right)=\frac{1}{|\mathcal{C}|} W_{\mathcal{C}}\left(x_{0}+(|F|-1) x_{1}, x_{0}-x_{1}\right) . \tag{5.2.11}
\end{equation*}
$$

Self-dual codes are for which $\mathcal{C}=\mathcal{C}^{\perp}$. As a result, their enumerator polynomials are unchanged under duality and $W_{\mathcal{C}}\left(x_{0}, x_{1}\right)=W_{\mathcal{C}^{\perp}}\left(x_{0}, x_{1}\right)$.

Example: extended Hamming [8,4,4] code The extended Hamming [8,4,4] code is a classical binary code defined by the generator matrix

$$
G^{T}=\left(\begin{array}{llllllll}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1  \tag{5.2.12}\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}\right)
$$

This code has $2^{4}=16$ codewords, which are the length 8 vectors defined by multiplying all 16 of the length 4 binary vectors by $G$. The Hamming code is the unique doubly-even self-dual binary code of length 8 . Its enumerator polynomial is

$$
\begin{equation*}
W_{\text {Hamming }}\left(x_{0}, x_{1}\right)=x_{0}^{8}+14 x_{0}^{4} x_{1}^{4}+x_{1}^{8}, \tag{5.2.13}
\end{equation*}
$$

were we used the formula (5.2.5).

## Quantum codes

In direct analog to classical codes, quantum error-correcting codes are designed to protect quantum information, denoted by a state $|\psi\rangle$, from corruption. Errors in quantum computation can take the form of an operator $E$ acting on the $|\psi\rangle$. These errors are taken to be in the Pauli group $\mathcal{P}_{n}$. The $n$-qubit Pauli group consists of tensor products of $I, X, Y$ and $Z$ (these are the usual Pauli operators, also known by $I, \sigma_{x}, \sigma_{y}, \sigma_{z}$ ) and an overall phase of $\pm i$ or $\pm 1$. Let $\mathcal{E}$ denote the linear space of errors acting on the Hilbert space. A subspace $C$ of the $n$-qubit Hilbert space is said to form a code iff

$$
\begin{equation*}
\langle\psi| E^{\dagger} E|\psi\rangle=c(E), \tag{5.2.14}
\end{equation*}
$$

for all $E \in \mathcal{E}$ where $c(E)$ does not depend on the state $|\psi\rangle$. This is referred to as the Knill-Laflamme condition 174.

Example: three qubit flip code Let us consider a very simplified example where the only possible errors are acting by $X$ which flips the states $|0\rangle$ (spin up) and $|1\rangle$ (spin down). We can protect against such errors by by encoding

$$
\begin{equation*}
a|0\rangle+b|1\rangle \quad \rightarrow \quad a|000\rangle+b|111\rangle . \tag{5.2.15}
\end{equation*}
$$

The space of states spanned by $|000\rangle$ and $|111\rangle$ is called the code subspace, a subspace of the bigger three qubit Hilbert space. Now we transmit the message, and want to be able to correct any bit flips that may have happened. This process has two steps: 1) syndrome diagnosis, where we determine the errors, and 2) recovery, where we return the system to its initial state.

Syndrome diagnosis is done via measurements of the four projection operators:

$$
\begin{align*}
& P_{0}=|000\rangle\langle 000|+|111\rangle\langle 111|, \\
& P_{1}=|100\rangle\langle 100|+|011\rangle\langle 011|, \\
& P_{2}=|010\rangle\langle 010|+|101\rangle\langle 101|,  \tag{5.2.16}\\
& P_{3}=|001\rangle\langle 001|+|110\rangle\langle 110| .
\end{align*}
$$

If no bit is flipped, then measuring $P_{0}$ will give 1 . If only the $i^{\text {th }}$ bit is flipped, then only $P_{i}$ will give 1 . This code is constructed to detect and correct a maximum of 1 error on any qubit and will fail if there are two or three errors. Importantly, measuring these operators does not change the state (which is required for us to perform all four measurements).

The second step is recovery. In this case it is very simple - if a bit has been flipped, we apply $X$ to that bit to flip it back.

These steps are directly analogous to the classical case: syndrome diagnosis for classical codes is performed by applying the parity check matrix $H$, and recovery is simply the step of interpreting the message as the closest codeword. This example was rather artificial, because we only allowed for particular kinds of errors, but it illustrates the error-detection and recovery steps. In particular, it shows that there are methods of detecting and correcting errors without destroying the state. Next we shall discuss a broader class of codes, which will be relevant to our interest in CFTs.

## Stabilizer codes

A simple way to specify the code subspace is to specify a set of operators which stabilize that subspace. For example, the state

$$
\begin{equation*}
|\psi\rangle=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) \tag{5.2.17}
\end{equation*}
$$

is stabilized by the operators $Z_{1} Z_{2}$ and $X_{1} X_{2}$, because these do not change $|\psi\rangle$. In fact, up to a phase, $|\psi\rangle$ is the unique such state. Therefore specifying the stabilizer, in this case, completely specifies the state $|\psi\rangle$.

There are two basic requirements for a group to stabilize a non-trivial set of states: 1) that it does not include the element $-I$ (which obviously cannot stabilize any state) and 2) that all the elements commute. The latter follows because operators in $G_{n}$ either commute or anti-commute. If $g_{1}$ and $g_{2}$ anti-commute, then any element $|\psi\rangle$ which is stabilized by $g_{1} g_{2}$ cannot be stabilized by $g_{2} g_{1}$.

Now consider an operator $S$ which is in the stabilizer of the code subspace - it has an eigenvalue +1 for any state $|\psi\rangle$. Then measuring $S$ will detect any errors $E$ which anticommute with $S$, because $S E|\psi\rangle=-E|\psi\rangle$. Each set of errors $E$ will anticommute with a set of operators $S$, and this set of operators $S$ defines a stabilizer code. If $S$ is an Abelian subgroup of order $k$ of the Pauli group $\mathcal{P}_{n}$ and $-I \notin S$, then the space of states stabilized by all elements of $S$ is an $[[n, n-k, d]]$ quantum stabilizer code. Here, $d$ is the quantum Hamming distance of the code and is defined as the minimum weight of an operator which commutes with $S$ but is not in $S$. The weight of an operator is the number of $X \mathrm{~s}, Y \mathrm{~s}$, and $Z \mathrm{~s}$ comprising it.

Stabilizer codes are related to classical codes over $\mathbb{F}_{4}$, discovered by Calderbank, Rains, Shor, and Sloane [175]. The key to this relation is the Gray map between $\mathbb{F}_{4}$ and $\mathbb{F}_{2}^{2}$, which associates

$$
\begin{align*}
0 \leftrightarrow(0,0), & & 1 \leftrightarrow(1,1),  \tag{5.2.18}\\
\omega \leftrightarrow(1,0), & & \bar{\omega} \leftrightarrow(0,1) .
\end{align*}
$$

It is a $\mathbb{F}_{2}$-linear map that also relates the inner product $(5.2 .10$ to the symplectic inner product

$$
\begin{equation*}
(a, b) \cdot\left(a^{\prime}, b^{\prime}\right)=\sum_{i} a_{i} b_{i}^{\prime}+a_{i}^{\prime} b_{i} . \tag{5.2.19}
\end{equation*}
$$

This map can be used to relate classical codes over $\mathbb{F}_{4}$ with quantum stabilizer codes in the following way. Consider a classical code $\mathcal{C}$ over $\mathbb{F}_{4}$. Then $c \in \mathcal{C}$ is a length- $n$ vector whose entries are $0,1, \omega, \omega^{2}$. Through the Gray map, this can be
related to a pair of vectors, $\alpha$ and $\beta$, with entries in $\mathbb{F}_{2}$,

$$
\begin{equation*}
c \leftrightarrow(\alpha, \beta) . \tag{5.2.20}
\end{equation*}
$$

The result is that each codeword $c \in \mathcal{C}$ can be used to specify a stabilizer in the related quantum code, $\mathcal{C}^{*}$ through the relation

$$
\begin{equation*}
g=i^{\alpha \cdot \beta}\left(X_{1}^{\alpha_{1}} X_{2}^{\alpha_{2}} \ldots X_{n}^{\alpha_{n}}\right)\left(Z_{1}^{\beta_{1}} Z_{2}^{\beta_{2}} \ldots Z_{n}^{\beta_{n}}\right) . \tag{5.2.21}
\end{equation*}
$$

We have specified each generator by the position of its $X \mathrm{~s}$ and $Z \mathrm{~s}$, which are packaged into the binary vectors $\alpha$ and $\beta$. The relationship goes both ways and a set of stabilizers can be used to specify a set of codewords in $\left(\mathbb{F}_{4}\right)^{n}$, or equivalently elements of $\left(\mathbb{F}_{2}\right)^{2 n}$.

Representing each generator as a pair of (row) vectors lets us represent the full set as the matrix $H=[\alpha \mid \beta]$. This is broken into two $n \times(n-k)$ submatrices: A " 1 " in $\alpha_{i j}$ means that generator $g_{i}$ includes $X_{j}$. A " 1 " in $\beta_{i j}$ means that the generator $g_{i}$ includes $Z_{j}$. The presence of a $Y_{j}$ is indicated by a " 1 " in both $\alpha$ and $\beta$.

Example: Steane $[7,1,3]$ code This is entirely specified by the stabilizer group generated by

$$
\begin{align*}
& g_{1}=X_{4} X_{5} X_{6} X_{7}, \\
& g_{2}=X_{2} X_{3} X_{6} X_{7}, \\
& g_{3}=X_{1} X_{3} X_{5} X_{7},  \tag{5.2.22}\\
& g_{4}=Z_{4} Z_{5} Z_{6} Z_{7}, \\
& g_{5}=Z_{2} Z_{3} Z_{6} Z_{7}, \\
& g_{6}=Z_{1} Z_{3} Z_{5} Z_{7}
\end{align*}
$$

As an example, the parity check matrix of the Steane code is

$$
H=\left(\begin{array}{lllllll|lllllll}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{5.2.23}\\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}\right)
$$

Clearly $H$ is the quantum version of the classical parity check matrix. The condition that the stabilizers form an Abelian subgroup equivalent to

$$
\begin{equation*}
H g H^{\mathrm{T}}=0, \tag{5.2.24}
\end{equation*}
$$

where

$$
g=\left(\begin{array}{cc}
0 & I_{\mathrm{n} \times \mathrm{n}}  \tag{5.2.25}\\
I_{\mathrm{n} \times \mathrm{n}} & 0
\end{array}\right) .
$$

This equation is equivalent to

$$
\begin{align*}
g_{i} g_{j}-g_{j} g_{i}=0 & \Leftrightarrow \alpha_{i} \cdot \beta_{j}-\alpha_{j} \cdot \beta_{i} \equiv 0(\bmod 2)  \tag{5.2.26}\\
& \Leftrightarrow \bar{c}_{i} \cdot c_{j}-c_{i} \cdot \bar{c}_{j} \equiv 0\left(\text { over } \mathbb{F}_{4}\right) . \tag{5.2.27}
\end{align*}
$$

The second line here amounts to the requirement that the code over $\mathbb{F}_{4}$ is selforthogonal, meaning $\mathcal{C} \subset \mathcal{C}^{\perp}$. Self-duality $\left(\mathcal{C}=\mathcal{C}^{\perp}\right)$ is a stronger requirement. A vector in $\mathbb{F}_{4}^{n}$ can be thought of as having real dimension $2 n$ through the Gray map. Therefore if $\mathcal{C}$ is an $[n, m, d]$ code, then its dual will be an $[n, 2 n-m, d]$ code. As a result, self-dual codes over $\mathbb{F}_{4}$ must be $[n, n, d]$ codes. In fact, classical $[n, m, d]$ codes define quantum $[[n, n-m, \tilde{d}]]$ codes, so we see that self-dual codes over $\mathbb{F}_{4}$ define quantum codes which cannot actual transmit any information. $\sqrt[8]{7}$ From here on out, we will consider $k$ to define the size of the quantum $[[n, k, \tilde{d}]]$ code. For self-dual codes, $k=0$.

The codewords are the vectors stabilized by the generators defined by $H$, and the space of all linear combinations of codewords is called the "code subspace." Just as in the classical case, the codewords are given by the kernel of $H$, with multiplication defined in (5.2.19).

To make this more precise, we can define a binary "generator matrix" $G$ of dimension $2 n \times(n+k)$ whose columns form a basis of codewords. It must be defined to satisfy

$$
\begin{equation*}
H g G=0, \tag{5.2.28}
\end{equation*}
$$

for all $g$. $G$ can be chosen so that its first $n-k$ rows coincide with those of $H$. The remaining rows, spanning logical operations on the code subspace, will not matter for our purposes because we have $k=0$.

For the Steane code, the logical operators (operators which commute with the stabilizers but are not in the code subspace) are

$$
\begin{align*}
X_{L} & =X_{1} X_{2} X_{3} X_{4} X_{5} X_{6} X_{7} \\
Z_{L} & =Z_{1} Z_{2} Z_{3} Z_{4} Z_{5} Z_{6} Z_{7} \tag{5.2.29}
\end{align*}
$$

[^27]The logical states or eigenstates of these logical operators are

$$
\begin{align*}
|0\rangle_{\mathrm{L}}= & \frac{1}{\sqrt{8}}(|0000000\rangle+|1010101\rangle+|0110011\rangle+|1100110\rangle \\
& +|0001111\rangle+|1011010\rangle+|0111100\rangle+|1101001\rangle) \\
|1\rangle_{\mathrm{L}}= & \frac{1}{\sqrt{8}}(|1111111\rangle+|0101010\rangle+|1001100\rangle+|0011001\rangle  \tag{5.2.30}\\
& +|1110000\rangle+|0100101\rangle+|1000011\rangle+|0010110\rangle)
\end{align*}
$$

The Steane code is an example of Calderbank-Shor-Steane (CSS) codes. These codes will be interesting to us because they may be constructed from classical codes. In particular, consider two classical binary codes, an $\left[n, k_{1}\right] \operatorname{code} \mathcal{C}_{1}$, and an $\left[n, k_{2}\right]$ code $\mathcal{C}_{2}$, and which satisfy $\mathcal{C}_{2} \subseteq \mathcal{C}_{1}$. Then we can form an $\left[n, k_{1}-k_{2}\right]$ code, denoted $\operatorname{CSS}\left(\mathcal{C}_{1}, \mathcal{C}_{2}\right)$ in the following way: For a given codeword of $x \in \mathcal{C}_{1}$, we define

$$
\begin{equation*}
\left|x+\mathcal{C}_{2}\right\rangle=\frac{1}{\sqrt{\left|\mathcal{C}_{2}\right|}} \sum_{y \in \mathcal{C}_{2}}|x+y\rangle \tag{5.2.31}
\end{equation*}
$$

If we do this for each codeword, we will end up with $k_{1}$ codewords in the quantum code. But many of these may be the same - this will happen for two codewords $x$ and $x^{\prime} \in \mathcal{C}_{1}$ whenever $x-x^{\prime} \in \mathcal{C}_{2}$. So in fact, the code $\mathcal{C}_{1}$ breaks into cosets determined by the structure of $\mathcal{C}_{2}$, and the resulting code has $2^{k_{1}-k_{2}}$ unique codewords.

The Steane code is able to correct "arbitrary single-qubit errors." This includes a phase flip (applying $Z$ to a single bit) and a single bit flip (applying $X$ to a single bit). We will define $e_{1}$ to be a vector with a single 1 , which denotes the position of the phase flip, and a similar vector $e_{2}$ to denote the position of the bit flip.

If the codewords becomes corrupted, then $|\psi\rangle \rightarrow\left|\psi^{\prime}\right\rangle$ becomes

$$
\begin{equation*}
\frac{1}{\sqrt{\left|\mathcal{C}_{2}\right|}} \sum_{y \in \mathcal{C}_{2}}|x+y\rangle \rightarrow \frac{1}{\sqrt{\left|\mathcal{C}_{2}\right|}} \sum_{y \in \mathcal{C}_{2}}(-1)^{(x+y) \cdot e_{1}}\left|x+y+e_{2}\right\rangle \tag{5.2.32}
\end{equation*}
$$

Syndrome diagnosis for the bit flip is accomplished by first adding auxiliary qubits to the system, i.e. to write $\left|x+y+e_{2}\right\rangle$ as $\left|x+y+e_{2}\right\rangle|0\rangle_{\text {aux }}$ and then mapping ${ }^{99}$

[^28]
## Enumerator polynomials for quantum codes

Using the Gray map, we can define various types of enumerator polynomials for quantum codes. For classical codes, the enumerator polynomial will count the number of $0 \mathrm{~s}, 1 \mathrm{~s}, \omega \mathrm{~s}$, and $\omega^{2} \mathrm{~s}$, of each codeword. For the quantum code, this corresponds to counting the number of $I \mathrm{~s}, Y \mathrm{~s}, X \mathrm{~s}$, and $Z \mathrm{~s}$ in each stabilizer. Therefore we define the weights

$$
\begin{equation*}
w_{x}(c)=\overrightarrow{1} \cdot \alpha, \quad w_{y}(c)=\alpha \cdot \beta, \quad w_{z}(c)=\overrightarrow{1} \cdot \beta \tag{5.2.33}
\end{equation*}
$$

As a result, the complete enumerator polynomial can be written as

$$
\begin{equation*}
W_{\mathcal{C}}\left(x_{0}, x_{1}, x_{2}, x_{3}\right)=\sum_{c \in \mathcal{C}} x_{0}^{n-w_{x}(c)-w_{y}(c)-w_{z}(c)} x_{1}^{w_{x}(c)} x_{2}^{w_{y}(c)} x_{3}^{w_{z}(c)} . \tag{5.2.34}
\end{equation*}
$$

However we will not need this. By an argument from [19], which we will revisit in section 5.3, it is convenient to instead study the refined enumerator polynomial,

$$
\begin{equation*}
W_{\mathcal{C}}\left(x_{0}, x_{1}, x_{2}\right)=\sum_{c \in \mathcal{C}} x_{0}^{n-w_{x}(c)-w_{y}(c)-w_{z}(c)} x_{1}^{w_{y}(c)} x_{2}^{w_{x}(c)+w_{z}(c)} . \tag{5.2.35}
\end{equation*}
$$

Self-duality, on the level of the refined enumerator polynomial, takes the form of the requirement that $W_{\mathcal{C}}\left(x_{0}, x_{1}, x_{2}\right)$ is invariant under ${ }^{10}$

$$
\begin{equation*}
x_{0} \rightarrow \frac{1}{2}\left(x_{0}+x_{1}+2 x_{2}\right), \quad x_{1} \rightarrow \frac{1}{2}\left(x_{0}+x_{1}-2 x_{2}\right), \quad x_{2} \rightarrow \frac{1}{2}\left(x_{0}-x_{1}\right) . \tag{5.2.36}
\end{equation*}
$$

We need another criterion beyond self-duality: a stabilizer code is called real if the stabilizers are all real, i.e. the number of $Y \mathrm{~s}$ in each stabilizer is even. What happens if we just multiply each $Y$ by $i$ ? This requires $w_{y}(c)$ to be an even number, which means that real codes are invariant under

$$
\begin{equation*}
x_{1} \rightarrow-x_{1} . \tag{5.2.37}
\end{equation*}
$$

## CFTs from quantum codes

Having reviewed most of the required elements, we are now ready to explain the socalled New Construction A, due to Dymarsky and Shapere, which is a construction of non-chiral, "full," CFTs from stabilizer codes, or equivalently from codes over $\mathbb{F}_{4}$. The central point is that each code defines a lattice through

$$
\begin{equation*}
\Lambda(\mathcal{C})=\left\{\left.\frac{v}{\sqrt{2}} \right\rvert\, v \in \mathbb{Z}^{2 n}, v \equiv(\alpha, \beta)(\bmod 2) \text { for some }(\alpha, \beta)=c \in \mathcal{C}\right\} \tag{5.2.38}
\end{equation*}
$$

[^29]To define a CFT, we would like to think of this as a Lorentzian lattice. This can be done by embedding $\Lambda(\mathcal{C})$ in $\mathbb{R}^{2 n}$. If we use $v \leftrightarrow(a, b)$, which is the same as the $c \leftrightarrow(\alpha, \beta)$ basis for the codewords, then we use the symplectic metric

$$
g=\left(\begin{array}{ll}
0 & I  \tag{5.2.39}\\
I & 0
\end{array}\right)
$$

leading to $|v|^{2}=2 a \cdot b$. This can be transformed to coordinates where we have the usual Lorentzian metric, $|v|^{2}=p_{L}^{2}-p_{R}^{2}$, by defining

$$
\begin{equation*}
p_{L}=\frac{a+b}{\sqrt{2}}, \quad p_{R}=\frac{a-b}{\sqrt{2}} \tag{5.2.40}
\end{equation*}
$$

When considered as a Lorentzian lattice, the following result 19 follows:

- The lattice defined by a code will be self-dual if and only if the code is self-dual.
- A lattice defined by a code will be even if and only if the code is real.

Lattices may be characterized by their theta functions, which for a Lorentzian lattice takes the form 11

$$
\begin{equation*}
\Theta_{\Lambda}(\tau, \bar{\tau})=\sum_{v \in \Lambda} q^{p_{L}^{2} / 2} \bar{q}^{p_{R}^{2} / 2}, \quad q=e^{2 \pi i \tau}, \quad \bar{q}=e^{-2 \pi i \bar{\tau}} \tag{5.2.41}
\end{equation*}
$$

Just as in the case of Euclidean lattices, the theta function for a Lorentzian lattice is related to the enumerator polynomial of its defining code. In this case, the relationship takes the form [19]:

$$
\begin{equation*}
\Theta_{\Lambda(\mathcal{C})}(\tau, \bar{\tau})=W_{\mathcal{C}}\left(\Theta_{3}+\Theta_{4}, \Theta_{3}-\Theta_{4}, \Theta_{2}\right) \tag{5.2.42}
\end{equation*}
$$

where we have defined $\Theta_{m}(\tau, \bar{\tau})=\theta_{m}\left(e^{2 \pi i \tau}\right) \theta_{m}\left(e^{-2 \pi i \bar{\tau}}\right)$. We will give the derivation of this formula in section 5.3 as a special case of the general-genus result.

## Narain CFTs

Consider now the theory of $n$ free bosons compactified on a lattice $\Gamma$ [118, 119, i.e. moving freely in $\mathbb{R}^{n} / \Gamma$. This theory is described by the action

$$
\begin{equation*}
S=-\frac{1}{4 \pi \alpha^{\prime}} \int d t d \sigma \sqrt{-g}\left(\partial_{\mu} \Phi^{I} \partial^{\mu} \Phi^{I}+\epsilon^{\mu \nu} B_{I J} \partial_{\mu} \Phi^{I} \partial_{\nu} \Phi^{J}\right) . \tag{5.2.43}
\end{equation*}
$$

[^30]The antisymmetric field $B$ is required to construct the most general theory of this type.

Consider now the case where spacetime is also a 2 d torus. Then periodicity requires that $\vec{\Phi}(t, \sigma) \sim \vec{\Phi}(t, \sigma+2 \pi)$. But the lattice compactification implies we have also identified $\vec{\Phi}(t, \sigma) \sim \vec{\Phi}(t, \sigma)+2 \pi \vec{\lambda}$, where $\vec{\lambda} \in \Gamma$ (note that $\Gamma$ is different from $\Lambda$, which will be formed from $\Gamma$ and $\Gamma^{*}$ ). So the most general possibility is

$$
\begin{equation*}
\vec{\Phi}(t, \sigma+2 \pi)=\vec{\Phi}(t, \sigma)+2 \pi \vec{\lambda} \tag{5.2.44}
\end{equation*}
$$

where $\vec{\lambda}$ is zero or any other element of $\Gamma$. Now consider the following solution to the equations of motion:

$$
\begin{align*}
& \vec{\Phi}_{L}(t+\sigma)=\frac{1}{2} \vec{\Phi}(0,0)+\frac{1}{2} \alpha^{\prime}(t+\sigma) \vec{p}_{L}+\frac{1}{2} i \sum_{n \neq 0} \frac{\vec{a}_{n}}{n} e^{-i n(t+\sigma)},  \tag{5.2.45}\\
& \vec{\Phi}_{R}(t+\sigma)=\frac{1}{2} \vec{\Phi}(0,0)+\frac{1}{2} \alpha^{\prime}(t-\sigma) \vec{p}_{R}+\frac{1}{2} i \sum_{n \neq 0} \frac{\vec{b}_{n}}{n} e^{-i n(t+\sigma)} \tag{5.2.46}
\end{align*}
$$

where $\vec{\Phi}(t, \sigma)=\vec{\Phi}_{L}(t+\sigma)+\vec{\Phi}_{R}(t-\sigma)$. From the solution, we see that if $\sigma \rightarrow \sigma+2 \pi$, then

$$
\begin{equation*}
\vec{\Phi} \rightarrow \vec{\Phi}+\pi \alpha^{\prime}\left(\vec{p}_{L}-\vec{p}_{R}\right) \tag{5.2.47}
\end{equation*}
$$

Thus if the periodicity condition of (5.2.44) is to be satisfied, we must have

$$
\begin{equation*}
\frac{1}{2} \alpha^{\prime}\left(\vec{p}_{L}-\vec{p}_{R}\right)=\vec{\lambda} \in \Gamma \tag{5.2.48}
\end{equation*}
$$

Furthermore, compactification on the lattice $\Gamma$ implies that the momenta $\vec{P}$ is in the dual lattice $\Gamma^{*}$. By computing the canonical momentum, we find that

$$
\begin{equation*}
\vec{V}=\alpha^{\prime} \vec{P}+B \vec{\lambda} \tag{5.2.49}
\end{equation*}
$$

where $\vec{V}$ is defined as the coefficient multiplying $t$ in the solution $\vec{\Phi}(t, \sigma)$. From (5.2.46), it must be

$$
\begin{equation*}
\vec{V}=\frac{1}{2} \alpha^{\prime}\left(\vec{p}_{L}+\vec{p}_{R}\right) \tag{5.2.50}
\end{equation*}
$$

Solving for $\vec{p}_{L}$ and $\vec{p}_{R}$, we find

$$
\begin{equation*}
\vec{p}_{L}=\vec{P}+\frac{1}{\alpha^{\prime}}(B+I) \vec{\lambda}, \quad \vec{p}_{R}=\vec{P}+\frac{1}{\alpha^{\prime}}(B-I) \vec{\lambda} . \tag{5.2.51}
\end{equation*}
$$

The set of all $\left(\vec{p}_{L}, \vec{p}_{L}\right)$ in this parametrization forms the lattice $\Lambda$. From here on, we will set $\alpha^{\prime}=2$ to keep $\vec{p}_{L}$ and $\vec{p}_{R}$ dimensionless, as they are in the previous subsection.

The Narain theories have a $U(1)^{n} \times U(1)^{n}$ symmetry, corresponding to moving $\vec{\Phi}_{L}$ or $\vec{\Phi}_{R}$ around the compact directions. The primary operators with respect to this symmetry are

$$
\begin{equation*}
V_{p_{L}, p_{R}}=e^{i \vec{p}_{L} \vec{\Phi}_{L}} e^{i \vec{p}_{R} \vec{\Phi}_{R}} \tag{5.2.52}
\end{equation*}
$$

Since the elements $v$ of the lattice $\Lambda$ are labeled by $\vec{p}_{L}$ and $\vec{p}_{R}$, we see that we have a single primary for each lattice vector. The weights of these primaries are simply $h=p_{L}^{2} / 2, \bar{h}=p_{R}^{2} / 2$.

The characters of the $U(1)^{n} \times U(1)^{n}$ symmetry group are

$$
\begin{equation*}
\chi_{h, \bar{h}}(\tau, \bar{\tau})=\frac{q^{h} \bar{q}^{\bar{h}}}{\eta(\tau)^{n} \eta(\bar{\tau})^{n}} . \tag{5.2.53}
\end{equation*}
$$

Therefore, the final result for the partition function takes the form

$$
\begin{equation*}
Z(\tau, \bar{\tau})=\sum_{v \in \Lambda} \frac{q^{p_{L}^{2} / 2} \bar{q}^{p_{R}^{2} / 2}}{\eta(\tau)^{n} \eta(\bar{\tau})^{n}} . \tag{5.2.54}
\end{equation*}
$$

The numerator of this sum is precisely the lattice theta function introduced in (5.2.41). Combining this with the result (5.2.42) yields a formula for the genus 1 partition function:

$$
\begin{equation*}
Z_{\mathcal{C}}=\frac{W_{\mathcal{C}}\left(\Theta_{3}+\Theta_{4}, \Theta_{3}-\Theta_{4}, \Theta_{2}\right)}{2^{n}|\eta(\tau)|^{2 n}} \tag{5.2.55}
\end{equation*}
$$

for the CFT defined by the code. The numerator in this expression is simply the refined enumerator polynomial, evaluated at combinations of the Jacobi theta functions in the notation $\Theta_{m}(\tau, \bar{\tau})=\theta_{m}\left(e^{2 \pi i \tau}\right) \theta_{m}\left(e^{-2 \pi i \bar{\tau}}\right)$.

## Code theories

We have now reviewed how a code $\mathcal{C}$ defines a Lorentzian lattice $\Lambda(\mathcal{C})$, via 5.2.38), and how a lattice $\Lambda$ defines a CFT, essentially through the definition of the vertex operators (5.2.52), where $\left(\vec{p}_{L}, \vec{p}_{R}\right) \in \Lambda$. Thus it is clear how to associate a CFT to a code.

In practice, however, different codes may define the same theory, and it may be useful to have a way of classifying all distinct theories. There is a large group of code equivalences, which are transformations between codes which define the same theory. They include permuting the components of codewords and swapping $X_{i}$ and $Z_{i}$ for any $i$. These lead to different lattices which are related by T-dualities, so the corresponding CFT will ultimately be the same. In fact, all T-dualities which relate
two code theories are of this form, i.e. permutations of components and swaps of $X$ and $Z$ (19].

Recall now that self-dual real codes can be specified by $n$ pairs of (row) vectors $\left(\alpha_{i}, \beta_{i}\right)$, which define the code's generator matrix:

$$
G^{T}=\left(\begin{array}{c|c}
\alpha_{1} & \beta_{1}  \tag{5.2.56}\\
\alpha_{2} & \beta_{2} \\
\vdots & \vdots \\
\alpha_{n} & \beta_{n}
\end{array}\right)
$$

One of the main results of [19] is that, due to code equivalences, every code theory defined from New Construction A can be described by a code whose generator matrix has the form

$$
\begin{equation*}
G^{T}=(B \mid I) \tag{5.2.57}
\end{equation*}
$$

where $B$ is an antisymmetric binary matrix. Codes in this form are called $B$-form codes. Therefore the result can be stated in the following way: any real self-dual code is equivalent to a $B$-form code. The generators for a $B$-form code take the form

$$
\begin{equation*}
g_{i}=Z_{i} \prod_{j=1}^{n}\left(X_{j}\right)^{B_{i j}} \tag{5.2.58}
\end{equation*}
$$

One way to organize the set of possible matrices $B$ is through graphs. A graph can be defined by an adjacency matrix $M$ where the entry $M_{i j}$ contains information about the link between node $i$ to node $j$. $B$ is a binary antisymmetric matrix, which is equivalent $(\bmod 2)$ to a symmetric matrix with zeroes on the diagonal. Hence the resulting graph is undirected and has no self-links. The end result is that, due to code equivalences, each code theory can be represented by a binary $n \times n$ matrix $B$ or by an undirected graph with $n$ nodes. We shall use both throughout this chapter ${ }^{12}$

## Recap and overview

Since this section includes a number of diverse elements, we will conclude with an executive summary.

[^31]Codes A stabilizer code is defined by its codewords, which take the form

$$
\begin{equation*}
g=i^{\alpha \cdot \beta}\left(X_{1}^{\alpha_{1}} X_{2}^{\alpha_{2}} \ldots X_{n}^{\alpha_{n}}\right)\left(Z_{1}^{\beta_{1}} Z_{2}^{\beta_{2}} \ldots Z_{n}^{\beta_{n}}\right) \tag{5.2.59}
\end{equation*}
$$

Here $n$ is the dimension of the code space. The number of generators is $k$, so there are $2^{k}$ codewords. The set of codewords can then be specified by the length- $n$ binary vectors $\alpha$ and $\beta$. There is one pair $(\alpha, \beta)$ for each codeword.

The Gray map 5.2 .18 associates tuples in $\mathbb{F}_{2}$ with elements of $\mathbb{F}_{4}$. This allows a pair $(\alpha, \beta)$ of binary vectors to be combined into a vector over $\mathbb{F}_{4}$, which we denote $c$. We then think of the collection of $c s$ as elements of a code over $\mathbb{F}_{4}$.

Enumerator polynomials There is a natural definition of enumerator polynomials for codes over general fields, so the Gray map allows us to define an enumerator polynomial for our quantum codes. First we define the weights

$$
\begin{equation*}
w_{x}(c)=\overrightarrow{1} \cdot \alpha, \quad w_{y}(c)=\alpha \cdot \beta, \quad w_{z}(c)=\overrightarrow{1} \cdot \beta \tag{5.2.60}
\end{equation*}
$$

The enumerator polynomial of interest for us is the refined enumerator polynomial

$$
\begin{equation*}
W_{\mathcal{C}}\left(x_{0}, x_{1}, x_{2}\right)=\sum_{c \in \mathcal{C}} x_{0}^{n-w_{x}(c)-w_{y}(c)-w_{z}(c)} x_{1}^{w_{y}(c)} x_{2}^{w_{x}(c)+w_{z}(c)} \tag{5.2.61}
\end{equation*}
$$

The abelian group structure of the stabilizer codes already ensures that all $\mathbb{F}_{4}$ codes defined this way are additive and self-orthogonal $\left(\mathcal{C} \subset \mathcal{C}^{\perp}\right)$. The $\mathbb{F}_{4}$ code will be even provided that the quantum codes are real, meaning all generators $g$ in the form (5.2.59) are real. Self-duality is equivalent to the requirement that the code is self-orthogonal, which is automatically satisfied, plus the requirement that $|\mathcal{C}|=$ $\left|\mathcal{C}^{\perp}\right|$. This means that the $\mathbb{F}_{4}$ code is an $[n, n, d]$ code, so the quantum code has $n$ generators. The enumerator polynomials of such codes are invariant under the transformations

$$
\begin{equation*}
x_{0} \rightarrow \frac{1}{2}\left(x_{0}+x_{1}+2 x_{2}\right), \quad x_{1} \rightarrow \frac{1}{2}\left(x_{0}+x_{1}-2 x_{2}\right), \quad x_{2} \rightarrow \frac{1}{2}\left(x_{0}-x_{1}\right) \tag{5.2.62}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{1} \rightarrow-x_{1} \tag{5.2.63}
\end{equation*}
$$

Lattices Codes over $\mathbb{F}_{4}$ define lattices via

$$
\begin{equation*}
\Lambda(\mathcal{C})=\left\{\left.\frac{v}{\sqrt{2}} \right\rvert\, v \in \mathbb{Z}^{2 n}, v \equiv(\alpha, \beta)(\bmod 2) \text { for some }(\alpha, \beta)=c \in \mathcal{C}\right\} \tag{5.2.64}
\end{equation*}
$$

It follows from the definition that the lattice will be self-dual if the code is self-dual (with respect to the symplectic metric), and it will be even if the code is real.

Lattices are characterized by their theta functions, defined in (5.2.41). The lattice theta function can be directly computed from the error-correcting code via the formula

$$
\begin{equation*}
\Theta_{\Lambda(\mathcal{C})}(\tau, \bar{\tau})=W_{\mathcal{C}}\left(\Theta_{3}+\Theta_{4}, \Theta_{3}-\Theta_{4}, \Theta_{2}\right) \tag{5.2.65}
\end{equation*}
$$

with $\Theta_{m}(\tau, \bar{\tau})=\theta_{m}\left(e^{2 \pi i \tau}\right) \theta_{m}\left(e^{-2 \pi i \bar{\tau}}\right)$.

CFT partition function The code CFT is defined as the Narain theory associated to the lattice,

$$
\begin{equation*}
Z_{\mathrm{CFT}}=\sum_{h, \bar{h}} \chi_{h, \bar{h}} \tag{5.2.66}
\end{equation*}
$$

for the $U(1)^{n} \times U(1)^{n}$ characters 5.2 .53 . Via the lattice construction, we related this to the enumerator polynomial, and arrived at the expression

$$
\begin{equation*}
Z_{\mathcal{C}}=\frac{W_{\mathcal{C}}\left(\Theta_{3}+\Theta_{4}, \Theta_{3}-\Theta_{4}, \Theta_{2}\right)}{2^{n}|\eta(\tau)|^{2 n}} \tag{5.2.67}
\end{equation*}
$$

for the CFT defined by the code. The numerator in this expression is simply given by the refined enumerator polynomial; the denominator is a universal factor, corresponding to a one-loop determinant.

The conformal weights in 5.2.66 are given by

$$
\begin{equation*}
(h, \bar{h})=\left(\frac{1}{2} \vec{p}_{L}^{2}, \frac{1}{2} \vec{p}_{R}^{2}\right), \quad\left(\vec{p}_{L}, \vec{p}_{R}\right)=\left(\frac{a+b}{2}, \frac{a-b}{2}\right), \quad a=2 \vec{P}+B \vec{\lambda}, \quad b=\vec{\lambda} \tag{5.2.68}
\end{equation*}
$$

where $\vec{\lambda}$ and $\vec{P}$ take all values in the integer lattice $\mathbb{Z}^{n}$. A parametrization in terms of the codewords $c \leftrightarrow(\alpha, \beta)$ (using the Gray map), is

$$
\begin{equation*}
\left(\vec{p}_{L}, \vec{p}_{R}\right)=\left(a+\frac{\alpha}{2}, b+\frac{\beta}{2}\right), \quad a, b \in \mathbb{Z}^{n}, \quad(\alpha, \beta) \leftrightarrow c . \tag{5.2.69}
\end{equation*}
$$

The sum over all $\left(\vec{p}_{L}, \vec{p}_{R}\right)$ then becomes a sum over all codewords $c$ and all integer values for $a$ and $b$.

Modular invariance of the partition function 5.2.67) follows from MacWilliams identities 5.2.62 and 5.2.63. One can check that the involved Jacobi theta functions in $\Theta_{m}(\tau, \bar{\tau})$ satisfies transformation identities under the modular $T$ and $S$ transformations. These may introduce powers of $|\tau|$. These factors are compensated for by the corresponding transformations of $|\eta(\tau)|^{2}$.
$B$-form codes There are a number of code equivalences which relate different codes to the same theory. These equivalences act as T-dualities at the level of the CFT. The result is that all real self-dual codes are equivalent to a code with the form

$$
G^{T}=\left(\begin{array}{c|c}
\alpha_{1} & \beta_{1}  \tag{5.2.70}\\
\alpha_{2} & \beta_{2} \\
\vdots & \vdots \\
\alpha_{n} & \beta_{n}
\end{array}\right)=\left(\begin{array}{ll}
B & I
\end{array}\right)
$$

where $B$ is an antisymmetric binary matrix. The generators 5.2.59) of codes in this form are simply given by

$$
\begin{equation*}
g_{i}=Z_{i} \prod_{j=1}^{n}\left(X_{j}\right)^{B_{i j}} \tag{5.2.71}
\end{equation*}
$$

Therefore every code theory can be specified by (at least) one binary antisymmetric $n \times n$ matrix $B$. Such matrices can be used to define undirected graphs with $n$ nodes and no self-links

Example: $n=3$ code Let us illustrate these elements with a simple example. Take

$$
B=\left(\begin{array}{lll}
0 & 1 & 1  \tag{5.2.72}\\
1 & 0 & 1 \\
1 & 1 & 0
\end{array}\right)
$$

This corresponds to the graph complete graph on 3 vertices (see figure 5.1).


Figure 5.1: Graph defining the matrix in equation (5.2.72).

We can use this to compute the generators:

$$
\begin{align*}
g_{1} & =Z_{1} X_{2} X_{3}  \tag{5.2.73}\\
g_{2} & =X_{1} Z_{2} X_{3}  \tag{5.2.74}\\
g_{3} & =X_{1} X_{2} Z_{3} \tag{5.2.75}
\end{align*}
$$

The full set of elements is then $G=\left\{I, g_{1}, g_{2}, g_{3}, g_{1} g_{2}, g_{1} g_{3}, g_{2} g_{3}, g_{1} g_{2} g_{3}\right\}$. This allows us to compute the refined enumerator polynomial using (5.2.61). The result is

$$
\begin{equation*}
\tilde{W}_{3}\left(x_{0}, x_{1}, x_{2}\right)=x_{0}^{3}+3 x_{0} x_{1}^{2}+4 x_{2}^{3} \tag{5.2.76}
\end{equation*}
$$

This agrees with equation (6.21) of [19], and is equal to the $\tilde{W}_{3}$ used in the introduction. Here the tilde denotes that this is the extremal code at $n=3$. We will return to this topic in section 5.4

### 5.3 Narain lattices and code theories at higher genus

The goal of this section is to spell out the relation between quantum error-correcting codes and CFT partition functions at higher genus. Recall that a quantum errorcorrecting code can be specified by a collection of generators $g$ which satisfies certain properties. Each codeword is essentially a string of Pauli matrices, described by the symbols $\mathbb{1}, X, Y, Z$, which can also be specified by a pair of binary vectors $(\alpha, \beta)$. By relating the Pauli matrices to the elements of $\mathbb{F}_{4}=\left\{0, \omega, 1, \omega^{2}\right\}$ via the Gray map, there is a relation to classical codes over $\mathbb{F}_{4}$.

In this section we will give a construction at higher-genus that is the exact counterpart of (5.2.67). This will give a genus $g$ partition function of the form

$$
\begin{equation*}
Z^{(g)}(\Omega, \bar{\Omega})=\frac{f_{n}\left(\Theta_{m}(\Omega, \bar{\Omega})\right)}{\left|\Phi_{g}\right|} \tag{5.3.1}
\end{equation*}
$$

Here $\Omega$ is the period matrix of the genus- $g$ Riemann surface, which is the direct higher-genus analog of the complex structure parameter $\tau$. $\left|\Phi_{g}\right|$ generalizes the contribution of $2^{n}|\eta(\tau)|^{2 n}$ in the genus 1 case and takes into account the contribution to the partition function due to oscillator modes. Formally, $\left|\Phi_{g}\right|=\left|\operatorname{det}^{\prime} \bar{\partial}\right|^{n}$, which corresponds to the determinant of the Laplacian operator $\bar{\partial}$ on the genus- $g$ Riemann surface with zero modes removed 176]. This does not depend on the choice of Narain Lattice so it will not factor into our discussion.

Finally, $f_{n}$ is a degree $n$ homogeneous polynomial in $2^{g-1}\left(2^{g}+1\right)$ theta functions $\Theta_{m}$. In this chapter, we will relate $f_{n}\left(\Theta_{m}\right)$ to the refined higher-weight enumerator polynomial, which we will define below.

## Higher-weight theta functions

Given a Euclidean lattice $\Lambda$, it is natural to define the higher-weight lattice theta series

$$
\begin{equation*}
\Theta_{\Lambda}(\Omega)=\sum_{v_{1} \in \Lambda} \cdots \sum_{v_{g} \in \Lambda} \exp \left(2 \pi i v_{i} \Omega_{i j} v_{j}\right) \tag{5.3.2}
\end{equation*}
$$

which is an analytic function of the Siegel upper half plane $\mathcal{H}_{g}$, defined by

$$
\begin{equation*}
\mathcal{H}_{g}=\left\{\Omega \in \operatorname{Mat}_{g \times g}(\mathbb{C}) \mid \Omega=\Omega^{T}, \operatorname{Im} \Omega \succ 0\right\} \tag{5.3.3}
\end{equation*}
$$

with known modular transformation properties. In general, $\Theta_{\Lambda}$ will evaluate to a combination of higher-weight theta functions, see [177, 178, 179, 124, or 123. Their general definition is through the sum

$$
\theta\left[\begin{array}{c}
\boldsymbol{m}  \tag{5.3.4}\\
\boldsymbol{n}
\end{array}\right](\boldsymbol{z}, \Omega)=\sum_{\boldsymbol{k} \in \mathbb{Z}^{g}}(-1)^{2(\boldsymbol{k}+\boldsymbol{m}) \cdot(\boldsymbol{z}+\boldsymbol{n})} \exp (i \pi(\boldsymbol{k}+\boldsymbol{m}) \cdot \Omega(\boldsymbol{k}+\boldsymbol{m}))
$$

where all quantities in bold font are length- $g$ (column) vectors. The theta functions (5.3.4) have well-known transformation properties under the genus $g$ modular maps, which will be discussed in detail in section 5.3 ,

A number of the theta functions have a zero at $\boldsymbol{z}=0$, generalizing the statement that $\theta_{1}(q)=0$ in the genus 1 case. To see this at higher genus, we first introduce the notation

$$
\begin{equation*}
\Theta_{\boldsymbol{m}}(\Omega, \bar{\Omega})=\theta_{\boldsymbol{m}}(\Omega) \theta_{\boldsymbol{m}}(\Omega), \quad \boldsymbol{m} \in \operatorname{even}_{g} . \tag{5.3.5}
\end{equation*}
$$

For $\Theta_{\boldsymbol{m}}$ in (5.3.5) to be non-zero, $\boldsymbol{m}$ ranges over the set of genus- $g$ even characteristics - the subset of $\{1,2,3,4\}^{g}$ that contains an even number of " 1 "s. There are $2^{g-1}\left(2^{g}+\right.$ 1) such even characteristics. For example, at genus 1 and 2 , the theta functions are indexed by

$$
\begin{equation*}
\operatorname{even}_{1}=\{2,3,4\}, \quad \text { even }_{2}=\{11,22,23,24,32,33,34,42,43,44\} \tag{5.3.6}
\end{equation*}
$$

The holomorphic function $\theta_{\boldsymbol{m}}(\Omega)$ in 5.3 .5 is defined by

$$
\theta_{\boldsymbol{m}}(\Omega)=\theta\left[\begin{array}{l}
\boldsymbol{a}(\boldsymbol{m})  \tag{5.3.7}\\
\boldsymbol{b}(\boldsymbol{m})
\end{array}\right](0, \Omega),
$$

where $\boldsymbol{a}(\boldsymbol{m})$ and $\boldsymbol{b}(\boldsymbol{m})$ have entries according to

$$
\left.\left.\begin{array}{lll}
m_{i}=1: & \left(a_{i}, b_{i}\right)=\left(\frac{1}{2}, \frac{1}{2}\right), & m_{i}=2: \\
m_{i}=3: & \left(a_{i}, b_{i}\right)=(0,0), & m_{i}=4: \tag{5.3.9}
\end{array}\right)\left(a_{i}, b_{i}\right)=\left(\frac{1}{2}, 0\right), \frac{1}{2}\right) .
$$

For instance, at genus 1, we have

$$
\theta_{2}(\tau)=\theta\left[\begin{array}{c}
1 / 2  \tag{5.3.10}\\
0
\end{array}\right](0, \tau), \quad \theta_{3}(\tau)=\theta\left[\begin{array}{l}
0 \\
0
\end{array}\right](0, \tau), \quad \theta_{4}(\tau)=\theta\left[\begin{array}{c}
0 \\
1 / 2
\end{array}\right](0, \tau),
$$

which are the usual Jacobi theta functions.

Factorization limit In the analysis of higher-genus partition functions we will make use of the "factorization limit," where a genus $g$ Riemann surface degenerates into two parts of genus $g-h$ and $h$ respectively, connected by an infinitely long thin tube. In this limit, $\Omega$ becomes block-diagonal,

$$
\begin{equation*}
\Omega_{g} \rightarrow \Omega_{g-h} \oplus \Omega_{h} \tag{5.3.11}
\end{equation*}
$$

and it is easy to verify that

$$
\begin{equation*}
\Theta_{a_{1} a_{2} \cdots a_{g}}\left(\Omega_{g}\right) \rightarrow \Theta_{a_{1} a_{2} \cdots a_{g}}\left(\Omega_{g-h} \oplus \Omega_{h}\right)=\Theta_{a_{1} \cdots a_{g-h}}\left(\Omega_{g-h}\right) \Theta_{a_{g-h+1} \cdots a_{g}}\left(\Omega_{h}\right) . \tag{5.3.12}
\end{equation*}
$$

## From code to lattice theta series

The lattice theta series is a function of the period matrices. At genus $g=1$, there is a definition of a theta function on a Lorentzian lattice $\Lambda$, defined by

$$
\begin{equation*}
\Theta_{\Lambda}^{(1)}(q, \bar{q})=\sum_{\left(\vec{p}_{L}, \vec{p}_{R}\right) \in \Lambda} q^{\frac{\vec{p}_{L} \cdot \vec{p}_{L}}{2}} \bar{q}^{\frac{\vec{p}_{R^{\prime}} \cdot \vec{p}_{R}}{2}} . \tag{5.3.13}
\end{equation*}
$$

With this definition, the generalization of to genus 2 is

$$
\begin{equation*}
\Theta_{\Lambda}^{(2)}(q, \bar{q}, r, \bar{r}, s, \bar{s})=\sum_{\left(\vec{p}_{L}, \vec{p}_{R}\right),\left(\vec{k}_{L}, \vec{k}_{R}\right) \in \Lambda} q^{\frac{\vec{p}_{L} \cdot \vec{p}_{L}}{2}} q^{\frac{\vec{p}_{R} \cdot \vec{p}_{R}}{2}} r^{\vec{p}_{L} \cdot \vec{k}_{L}} \bar{r}^{\vec{p}_{R}} \cdot \vec{k}_{R} S^{\frac{\vec{k}_{L} \cdot \vec{k}_{L}}{2}} \bar{s}^{\frac{\vec{k}_{R} \cdot \vec{k}_{R}}{2}} . \tag{5.3.14}
\end{equation*}
$$

Here $q, r$, and $s$ are defined by

$$
\begin{equation*}
q=e^{2 i \pi \Omega_{11}}, r=e^{2 i \pi \Omega_{12}}, s=e^{2 i \pi \Omega_{22}}, \quad \bar{q}=e^{-2 i \pi \bar{\Omega}_{11}}, \bar{r}=e^{-2 i \pi \bar{\Omega}_{12}}, \bar{s}=e^{-2 i \pi \bar{\Omega}_{22}} . \tag{5.3.15}
\end{equation*}
$$

## Derivation at genus 1

Consider the (complete) genus 1 enumerator polynomial, introduced in (5.2.34),

$$
\begin{equation*}
W_{\mathcal{C}}^{(1)}\left(x_{0}, x_{1}, x_{2}, x_{3}\right)=\sum_{c \in \mathcal{C}} x_{0}^{n-w_{x}(c)-w_{y}(c)-w_{z}(c)} x_{1}^{w_{x}(c)} x_{2}^{w_{y}(c)} x_{3}^{w_{z}(c)} . \tag{5.3.16}
\end{equation*}
$$

We will now write it in a different form, which is suitable for generalizations both to arbitrary fields and to higher genus,

$$
\begin{equation*}
W_{\mathcal{C}}^{(1)}\left(x_{[1]}, x_{[X]}, x_{[Y]}, x_{[Z]}\right)=\sum_{\vec{c} \in \mathcal{C}} \prod_{i=1}^{n} x_{\left[c_{i}\right]} \tag{5.3.17}
\end{equation*}
$$

where $x_{[1]}=x_{0}, x_{[X]}=x_{1}, x_{[Y]}=x_{2}$ and $x_{[Z]}=x_{3}$. Each code-word $c$ has entries $c_{i}$, taking the value in a size four set. We have given three equivalent formulations, where this set is

$$
\begin{equation*}
\{\mathbb{1}, X, Y, Z\} \simeq\left\{0, \omega, 1, \omega^{2}\right\} \simeq\{(0,0),(1,0),(1,1),(0,1)\} \tag{5.3.18}
\end{equation*}
$$

The first of these equivalences correspond to relating the quantum code to a classical code over $\mathbb{F}_{4}$. The second corresponds to using the Gray map, and is what will be used when relating the quantum error-correcting code to the theta function of a Lorentzian lattice. We write

$$
\begin{equation*}
c_{i} \leftrightarrow\left(\alpha_{i}, \beta_{i}\right) \tag{5.3.19}
\end{equation*}
$$

The genus 1 theta series of the Lorentzian lattice $\Lambda(\mathcal{C})$ associated to the code $\mathcal{C}$ by (5.2.38), is given by

$$
\begin{equation*}
\Theta_{\Lambda(\mathcal{C})}=\sum_{\vec{c} \in \mathcal{C}} \prod_{i=1}^{n} \sum_{a_{i} \in \mathbb{Z}} \sum_{b_{i} \in \mathbb{Z}} q^{\frac{1}{2}\left(a_{i}+\frac{\alpha_{i}}{2}+b_{i}+\frac{\beta_{i}}{2}\right)^{2}} \bar{q}^{\frac{1}{2}\left(a_{i}+\frac{\alpha_{i}}{2}-b_{i}-\frac{\beta_{i}}{2}\right)^{2}} \tag{5.3.20}
\end{equation*}
$$

where we substituted (5.2.40) into 5.3.13).
Let us study the contribution in 5.3 .20 from a given $\vec{c}$ in the outermost sum and a given $i$ in the product. This corresponds to determining

$$
\begin{equation*}
\operatorname{Th}\left(x_{\left[c_{i}\right]}\right):=\sum_{a_{i} \in \mathbb{Z}} \sum_{b_{i} \in \mathbb{Z}} q^{\frac{1}{2}\left(a_{i}+\frac{\alpha_{i}}{2}+b_{i}+\frac{\beta_{i}}{2}\right)^{2}} \bar{q}^{\frac{1}{2}\left(a_{i}+\frac{\alpha_{i}}{2}-b_{i}-\frac{\beta_{i}}{2}\right)^{2}} \tag{5.3.21}
\end{equation*}
$$

We would like to simplify (5.3.21 so that it splits into two terms, in such a way that in each term the two infinite sums separate. This is achieved by introducing the summation variables

$$
\begin{equation*}
\mu=a_{i}+b_{i}, \quad \nu=a_{i}-b_{i} \tag{5.3.22}
\end{equation*}
$$

and summing over even values of $\mu+\nu$. The restriction to even $\mu+\nu$ can be implemented by inserting the "complicated unit" $\frac{1+(-1)^{\mu+\nu}}{2}$. With this substitution,

$$
\begin{align*}
\operatorname{Th}\left(x_{\left[c_{i}\right]}\right)= & \frac{1}{2}\left(\sum_{\mu \in \mathbb{Z}} q^{\frac{1}{2}\left(\mu+\frac{\alpha_{i}+\beta_{i}}{2}\right)^{2}}\right)\left(\sum_{\nu \in \mathbb{Z}} \bar{q}^{\frac{1}{2}\left(\nu+\frac{\alpha_{i}-\beta_{i}}{2}\right)^{2}}\right) \\
& +\frac{1}{2}\left(\sum_{\mu \in \mathbb{Z}}(-1)^{\mu} q^{\frac{1}{2}\left(\mu+\frac{\alpha_{i}+\beta_{i}}{2}\right)^{2}}\right)\left(\sum_{\nu \in \mathbb{Z}}(-1)^{\nu} \bar{q}^{\frac{1}{2}\left(\nu+\frac{\alpha_{i}-\beta_{i}}{2}\right)^{2}}\right) . \tag{5.3.23}
\end{align*}
$$

Depending on the values of $\alpha_{i}$ and $\beta_{i}$, each of the factor in 5.3.23) evaluates to one of the Jacobi theta functions

$$
\begin{array}{ll}
\left(\alpha_{i}, \beta_{i}\right)=(0,0), & \operatorname{Th}\left(x_{[1]}\right)=\frac{1}{2}\left(\theta_{3}(q) \theta_{3}(\bar{q})+\theta_{4}(q) \theta_{4}(\bar{q})\right), \\
\left(\alpha_{i}, \beta_{i}\right)=(1,0), & \operatorname{Th}\left(x_{[X]}\right)=\frac{1}{2}\left(\theta_{2}(q) \theta_{2}(\bar{q})+\theta_{1}(q) \theta_{1}(\bar{q})\right), \\
\left(\alpha_{i}, \beta_{i}\right)=(1,1), & \operatorname{Th}\left(x_{[Y]}\right)=\frac{1}{2}\left(\theta_{3}(q) \theta_{3}(\bar{q})-\theta_{4}(q) \theta_{4}(\bar{q})\right), \\
\left(\alpha_{i}, \beta_{i}\right)=(0,1), & \operatorname{Th}\left(x_{[Z]}\right)=\frac{1}{2}\left(\theta_{2}(q) \theta_{2}(\bar{q})-\theta_{1}(q) \theta_{1}(\bar{q})\right) . \tag{5.3.27}
\end{array}
$$

Now we can see why we introduced the refined enumerator polynomial with $x_{[X]}=$ $x_{[Z]}$, since $\operatorname{Th}\left(x_{[X]}\right)=\operatorname{Th}\left(x_{[Z]}\right)$ by the vanishing of $\theta_{1}(q)$.

## Derivation at higher genus

It is natural to consider the higher-genus enumerator polynomial of a code over $F$,

$$
\begin{equation*}
W_{\mathcal{C}}^{(g)}\left(x_{[A]}\right):=\sum_{\mathbf{M} \in \mathcal{C}^{g}} \prod_{i=1}^{n} x_{\left[\mathrm{row}_{i}(\mathbf{M})\right]} \tag{5.3.28}
\end{equation*}
$$

This is a polynomial in $|F|^{g}$ variables $x_{[A]}$, where $A \in F^{g}$. Here the sum is over all possible $g$-tuples of codewords, packaged into the $n \times g$ matrix $\mathbf{M}$.

Having worked out the derivation of the genus 1 lattice theta series from a code, the generalization to higher-genus is straightforward.

The higher-weight enumerator polynomial (5.3.28) is a sum over $g$-tuples of codewords, where each summand is a product over the entries indexed by $i$. We will now study each factor in such a product, i.e. for a fixed set of codewords $\vec{c}_{(1)}, \ldots, \vec{c}_{(g)}$ and index $i$. Define $\boldsymbol{c}_{i}$ with components $\left(c_{(1), i}, \ldots, c_{(g) i}\right)$, and further $\boldsymbol{c}_{i} \sim\left(\boldsymbol{\alpha}_{i}, \boldsymbol{\beta}_{i}\right)$ by (5.3.19). Then

$$
\begin{align*}
\operatorname{Th}\left(x_{\left[\boldsymbol{c}_{i}\right]}\right)=\sum_{\boldsymbol{a} \in \mathbb{Z}^{g}} \sum_{\boldsymbol{b} \in \mathbb{Z}^{g}} & \exp \left(\frac{2 \pi i}{2}\left(\boldsymbol{a}+\frac{\boldsymbol{\alpha}_{i}}{2}+\boldsymbol{b}+\frac{\boldsymbol{\beta}_{i}}{2}\right) \Omega\left(\boldsymbol{a}+\frac{\boldsymbol{\alpha}_{i}}{2}+\boldsymbol{b}+\frac{\boldsymbol{\beta}_{i}}{2}\right)\right) \\
& \times \exp \left(\frac{2 \pi i}{2}\left(\boldsymbol{a}+\frac{\boldsymbol{\alpha}_{i}}{2}-\boldsymbol{b}-\frac{\boldsymbol{\beta}_{i}}{2}\right) \bar{\Omega}\left(\boldsymbol{a}+\frac{\boldsymbol{\alpha}_{i}}{2}-\boldsymbol{b}-\frac{\boldsymbol{\beta}_{i}}{2}\right)\right) . \tag{5.3.29}
\end{align*}
$$

Now let $\boldsymbol{\mu}=\boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{\nu}=\boldsymbol{a}-\boldsymbol{b}$, and write the sum as

$$
\begin{equation*}
\sum_{a \in \mathbb{Z}^{g}} \sum_{\boldsymbol{b} \in \mathbb{Z}^{g}}(\cdots)=\frac{1}{2^{g}} \sum_{\boldsymbol{\mu} \in \mathbb{Z}^{g}} \sum_{\boldsymbol{\nu} \in \mathbb{Z}^{g}} \prod_{h=1}^{g}\left(1+(-1)^{\mu_{(h)}+\nu_{(h)}}\right)(\cdots) \tag{5.3.30}
\end{equation*}
$$

It is now clear that the resulting expression,

$$
\begin{align*}
\operatorname{Th}\left(x_{\left[\boldsymbol{c}_{i}\right]}\right)=\frac{1}{2^{g}} \sum_{\boldsymbol{\mu} \in \mathbb{Z}^{g}} & \sum_{\boldsymbol{\nu} \in \mathbb{Z}^{g}} \exp \left(\pi i\left(\boldsymbol{\mu}+\frac{\boldsymbol{\alpha}_{i}+\boldsymbol{\beta}_{i}}{2}\right) \Omega\left(\boldsymbol{\mu}+\frac{\boldsymbol{\alpha}_{i}+\boldsymbol{\beta}_{i}}{2}\right)\right) \\
& \times \exp \left(\pi i\left(\boldsymbol{\nu}+\frac{\boldsymbol{\alpha}_{i}-\boldsymbol{\beta}_{i}}{2}\right) \bar{\Omega}\left(\boldsymbol{\nu}+\frac{\boldsymbol{\alpha}_{i}-\boldsymbol{\beta}_{i}}{2}\right)\right) \prod_{h=1}^{g}\left(1+(-1)^{\mu_{(h)}+\nu_{(h)}}\right), \tag{5.3.31}
\end{align*}
$$

will be a linear combination of the $\Theta_{m}(\Omega, \bar{\Omega})$ defined in (5.3.5) above.
In general, the variable $x_{\left[\boldsymbol{c}_{i}\right]}$ for $\boldsymbol{c}_{i} \leftrightarrow\left(\boldsymbol{\alpha}_{i}, \boldsymbol{\beta}_{i}\right)$ maps to

$$
\begin{equation*}
\operatorname{Th}\left(x_{\left[c_{i}\right]}\right)=\sum_{r_{1}} \sum_{r_{2}} \cdots \sum_{r_{g}}(-1)^{\sigma_{1}\left(r_{1}\right)+\sigma_{2}\left(r_{2}\right)+\ldots+\sigma_{g}\left(r_{g}\right)} \Theta_{r_{1} r_{2} \cdots r_{g}}(\Omega, \bar{\Omega}), \tag{5.3.32}
\end{equation*}
$$

where the sum over $r_{h}$ is determined by the following

$$
\begin{array}{lll}
\left(\alpha_{(h), i}, \beta_{(h), i}\right)=(0,0) & r_{h}=3,4, & \sigma_{h}(3)=\sigma_{h}(4)=0, \\
\left(\alpha_{(h), i}, \beta_{(h), i}\right)=(1,0) & r_{h}=2,1, & \sigma_{h}(2)=\sigma_{h}(1)=0, \\
\left(\alpha_{(h), i}, \beta_{(h), i}\right)=(1,1) & r_{h}=3,4, & \sigma_{h}(3)=0, \sigma_{h}(4)=1, \\
\left(\alpha_{(h), i}, \beta_{(h), i}\right)=(0,1) & r_{h}=2,1, & \sigma_{h}(2)=0, \sigma_{h}(1)=1 . \tag{5.3.36}
\end{array}
$$

## Explicit formulas at genus 2

Like the case at genus 1, some of the variables in the complete enumerator polynomial map to identical theta functions. At genus two, there are ten non-zero theta functions $\Theta_{\boldsymbol{m}}$, while six theta functions identically vanish: $\Theta_{12}=\Theta_{13}=\Theta_{14}=\Theta_{21}=\Theta_{31}=$ $\Theta_{41}=0$. Taking this into account, we find that

$$
\begin{align*}
y_{0}:=x_{[11]} & \mapsto \Theta_{33}+\Theta_{34}+\Theta_{43}+\Theta_{44},  \tag{5.3.37}\\
y_{1}:=x_{[1 Y]} & \mapsto \Theta_{33}-\Theta_{34}+\Theta_{43}-\Theta_{44},  \tag{5.3.38}\\
y_{2}:=x_{[Y 1]} & \mapsto \Theta_{33}+\Theta_{34}-\Theta_{43}-\Theta_{44},  \tag{5.3.39}\\
y_{3}:=x_{[Y Y]} & \mapsto \Theta_{33}-\Theta_{34}-\Theta_{43}+\Theta_{44},  \tag{5.3.40}\\
y_{4}:=x_{[1 X]}=x_{[1 Z]} & \mapsto \Theta_{32}+\Theta_{42},  \tag{5.3.41}\\
y_{5}:=x_{[X 1]}=x_{[Z 1]} & \mapsto \Theta_{23}+\Theta_{24},  \tag{5.3.42}\\
y_{6}:=x_{[X X]}=x_{[Z Z]} & \mapsto \Theta_{11}+\Theta_{22},  \tag{5.3.43}\\
y_{7}:=x_{[Y X]}=x_{[Y Z]} & \mapsto \Theta_{32}-\Theta_{42},  \tag{5.3.44}\\
y_{8}:=x_{[X Y]}=x_{[Z Y]} & \mapsto \Theta_{23}-\Theta_{24},  \tag{5.3.45}\\
y_{9}:=x_{[X Z]}=x_{[Z Z]} & \mapsto-\Theta_{11}+\Theta_{22} . \tag{5.3.46}
\end{align*}
$$

We see that compared to the original 16 code variables $x_{\left[c_{i} c_{i}^{\prime}\right]}$, we are now considering a subspace spanned by ten variables $y_{i}$. This is the genus 2 version of going from the complete enumerator polynomial to the refined enumerator polynomial.

One may also write an expression that gives the genus two enumerator polynomial directly in terms of the $y_{i}$, but the resulting formula is not particularly illuminating:

$$
\begin{align*}
& W_{g=2}\left(y_{0}, \ldots, y_{9}\right)=\sum_{c \in \mathcal{C}} \sum_{\tilde{c} \in \mathcal{C}} y_{0}^{|(1-\alpha) \wedge(1-\beta) \wedge(1-\tilde{\alpha}) \wedge(1-\tilde{\beta})|} y_{1}^{|(1-\alpha) \wedge(1-\beta) \wedge \tilde{\alpha} \wedge \tilde{\beta}|} y_{2}^{|\alpha \wedge \beta \wedge(1-\tilde{\alpha}) \wedge(1-\tilde{\beta})|} \\
& \quad \times y_{3}^{|\alpha \wedge \beta \wedge \tilde{\alpha} \wedge \tilde{\beta}|} y_{4}^{|(1-\alpha) \wedge(1-\beta) \wedge \tilde{\alpha} \wedge(1-\tilde{\beta})|+|(1-\alpha) \wedge(1-\beta) \wedge(1-\tilde{\alpha}) \wedge \tilde{\beta}|} \\
& \times y_{5}^{|\alpha \wedge(1-\beta) \wedge(1-\tilde{\alpha}) \wedge(1-\tilde{\beta})|+|(1-\alpha) \wedge \beta \wedge(1-\tilde{\alpha}) \wedge(1-\tilde{\beta})|} \\
& \quad \times y_{6}^{|\alpha \wedge(1-\beta) \wedge \tilde{\alpha} \wedge(1-\tilde{\beta})|+|(1-\alpha) \wedge \beta \wedge(1-\tilde{\alpha}) \wedge \tilde{\beta}|} y_{7}^{|\alpha \wedge \beta \wedge \tilde{\alpha} \wedge(1-\tilde{\beta})|+|\alpha \wedge \beta \wedge(1-\tilde{\alpha}) \wedge \tilde{\beta}|} \\
& \times y_{8}^{|\alpha \wedge(1-\beta) \wedge \tilde{\alpha} \wedge \tilde{\beta}|+|(1-\alpha) \wedge \beta \wedge \tilde{\alpha} \wedge \tilde{\beta}|} y_{9}^{|(1-\alpha) \wedge \beta \wedge \tilde{\alpha} \wedge(1-\tilde{\beta})|+|\alpha \wedge(1-\beta) \wedge(1-\tilde{\alpha}) \wedge \tilde{\beta}|} \tag{5.3.47}
\end{align*}
$$

where $\wedge$ denotes the component-wise "AND" operator.
We are also interested in the factorization limit (5.3.12) in terms of code variables. For the case of genus 2, it takes the form

$$
\begin{array}{lllll}
y_{0} \mapsto x_{0} x_{0}^{\prime}, & y_{1} \mapsto x_{0} x_{1}^{\prime}, & y_{2} \mapsto x_{1} x_{0}^{\prime}, & y_{3} \mapsto x_{1} x_{1}^{\prime}, & y_{4} \mapsto x_{0} x_{2}^{\prime}, \\
y_{5} \mapsto x_{2} x_{1}^{\prime}, & y_{6} \mapsto x_{2} x_{2}^{\prime}, & y_{7} \mapsto x_{1} x_{2}^{\prime}, & y_{8} \mapsto x_{2} x_{1}^{\prime}, & y_{9} \mapsto x_{2} x_{2}^{\prime} \tag{5.3.48}
\end{array}
$$

where the $x_{i}$ refer to variables on the left genus 1 Riemann surface and $x_{i}^{\prime}$ to the variables on the right genus 1 Riemann surface.

## From theta relations to polynomial relations

The goal now will be to study the transformation properties of the higher-weight theta functions in order to determine how they lift to transformations of the enumerator polynomials. The modular transformations are

$$
\Omega \mapsto \Omega^{\prime}=(A \Omega+B)(C \Omega+D)^{-1}, \quad \Gamma=\left(\begin{array}{ll}
A & B  \tag{5.3.49}\\
C & D
\end{array}\right) \in \operatorname{Sp}(2 g, \mathbb{Z})
$$

The corresponding transformations of the higher-genus theta functions is given by [177] (see also e.g. 124])

$$
\theta\left[\begin{array}{l}
\boldsymbol{a}^{\prime}  \tag{5.3.50}\\
\boldsymbol{b}^{\prime}
\end{array}\right]\left(0, \Omega^{\prime}\right)=\epsilon(\Gamma) \exp (-i \pi \phi(\boldsymbol{a}, \boldsymbol{b}, \Gamma)) \sqrt{\operatorname{det}(C \Omega+D)} \theta\left[\begin{array}{l}
\boldsymbol{a} \\
\boldsymbol{b}
\end{array}\right](0, \Omega)
$$

where

$$
\binom{\boldsymbol{a}^{\prime}}{\boldsymbol{b}^{\prime}}=\left(\begin{array}{cc}
D & -C  \tag{5.3.51}\\
-B & A
\end{array}\right)\binom{\boldsymbol{a}}{\boldsymbol{b}}+\frac{1}{2}\binom{\left(C D^{T}\right)_{\mathrm{diag}}}{\left(A B^{T}\right)_{\mathrm{diag}}}
$$

where $\epsilon(\Gamma)$ is a phase which is an eighth root of unity, $\phi(\boldsymbol{a}, \boldsymbol{b}, \Gamma)=\boldsymbol{a} \cdot D^{T} B \boldsymbol{a}+\boldsymbol{b}$. $C^{T} A \boldsymbol{b}-2 \boldsymbol{a} \cdot B^{T} C \boldsymbol{b}+\left(\boldsymbol{a} \cdot D^{T}-\boldsymbol{b} \cdot C^{T}\right)\left(A B^{T}\right)_{\text {diag }}$, and $M_{\text {diag }}$ denotes the diagonal entries of a matrix $M$, seen as a column vector.

In our construction, the theta functions always come in pairs $\theta(\Omega) \theta(\bar{\Omega})$, so any phases appearing from the modular transformations will cancel between the holomorphic and antiholomorphic parts. Furthermore, under the modular transformations 5.3.49 the denominator $\left|\Phi_{g}\right|$ transforms covariantly with modular weights $\left(\frac{n}{2}, \frac{n}{2}\right)$. This is required to cancel the square-root factor in 5.3.50.

Consider now the set of $\operatorname{Sp}(2 g, \mathbb{Z})$ transformation acting on the theta functions $\Theta_{m}$. Up to the weights imposed by the square-root factor in 5.3.50, such transformations amount to mapping the $\Theta_{m}$ among each other. We now wish to lift these relations to the polynomial variables. It is clear that we only need to exhibit this lift for the generators of $\operatorname{Sp}(2 g, \mathbb{Z})$.

As a warm-up, consider the case $g=1$. We can take as generators

$$
\begin{equation*}
T: \quad \tau \mapsto \tau+1, \quad S: \tau \mapsto-\frac{1}{\tau} \tag{5.3.52}
\end{equation*}
$$

The corresponding transformations of the Jacobi theta functions induce

$$
T:\left\{\begin{array}{l}
\Theta_{2}(\tau+1)=\Theta_{2}(\tau),  \tag{5.3.53}\\
\Theta_{3}(\tau+1)=\Theta_{4}(\tau), \\
\Theta_{4}(\tau+1)=\Theta_{3}(\tau),
\end{array} \quad S: \quad\left\{\begin{array}{l}
\Theta_{2}(-1 / \tau)=|\tau| \Theta_{4}(\tau) \\
\Theta_{3}(-1 / \tau)=|\tau| \Theta_{3}(\tau) \\
\Theta_{4}(-1 / \tau)=|\tau| \Theta_{2}(\tau)
\end{array}\right.\right.
$$

These relations lift to

$$
T:\left\{\begin{array}{l}
x_{0} \mapsto x_{0},  \tag{5.3.54}\\
x_{1} \mapsto-x_{1}, \\
x_{2} \mapsto x_{2},
\end{array} \quad S: \quad\left\{\begin{array}{l}
x_{0} \mapsto \frac{1}{2}\left(x_{0}+x_{1}+2 x_{2}\right), \\
x_{1} \mapsto \frac{1}{2}\left(x_{0}+x_{1}-2 x_{2}\right), \\
x_{2} \mapsto \frac{1}{2}\left(x_{0}-x_{1}\right),
\end{array}\right.\right.
$$

which are exactly the MacWilliams identities (5.2.36) and (5.2.37) given in section 5.2.

To generalize to higher genus, it is convenient to use the generators of $\operatorname{Sp}(2 g, \mathbb{Z})$ as given by [130] and reviewed in [160]. At genus $g=2$ and $g=3$, there are three generators; in all other cases there are only two. At genus 2, the generators can be taken to be $T, R$, and $D$, with the corresponding matrices $\Gamma$ of the form
$T: \Gamma=\left(\begin{array}{cccc}1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{array}\right), \quad R: \quad \Gamma=\left(\begin{array}{cccc}1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1\end{array}\right), \quad D: \Gamma=\left(\begin{array}{cccc}0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0\end{array}\right)$.

These are related to the more familiar set of genus 2 generators, $T_{1}, T_{2}, U, S_{1}, S_{2}$, by the following:

$$
\begin{equation*}
T_{1}=T, \quad T_{2}=D^{-1} T D, \quad U=D R D^{-1}, \quad S_{1}=T D^{2} T D^{2} T, \quad S_{2}=D R D^{2} R D^{2} R \tag{5.3.56}
\end{equation*}
$$

where $D^{-1}=D^{7}$. Using 5.3.50, the transformations of the theta functions $\Theta_{m}(\Omega, \bar{\Omega})$ can be determined, and the corresponding equations for the polynomial variables are

$$
\begin{gather*}
T_{1}:\left\{\begin{array}{l}
y_{2} \mapsto-y_{2}, \\
y_{3} \mapsto-y_{3}, \\
y_{7} \mapsto-y_{7},
\end{array}, T_{2}:\left\{\begin{array}{l}
y_{1} \mapsto-y_{1}, \\
y_{3} \mapsto-y_{3}, \\
y_{8} \mapsto-y_{8},
\end{array}, U:\left\{\begin{array}{l}
y_{7} \mapsto-y_{7}, \\
y_{8} \mapsto-y_{8}, \\
y_{9} \mapsto-y_{9},
\end{array}\right.\right.\right.
\end{gather*}, \begin{aligned}
& S_{1}:\left\{\begin{array}{l}
y_{0} \mapsto \frac{1}{2}\left(y_{0}+y_{2}+2 y_{5}\right), \\
y_{1} \mapsto \frac{1}{2}\left(y_{1}+y_{3}+2 y_{8}\right), \\
y_{2} \mapsto \frac{1}{2}\left(y_{0}+y_{2}-2 y_{5}\right), \\
y_{3} \mapsto \frac{1}{2}\left(y_{1}+y_{3}-2 y_{8}\right), \\
y_{4} \mapsto \frac{1}{2}\left(y_{4}+y_{6}+y_{7}+y_{9}\right), \\
y_{5} \mapsto \frac{1}{2}\left(y_{0}-y_{2}\right), \\
y_{6} \mapsto \frac{1}{2}\left(y_{4}+y_{6}-y_{7}-y_{9}\right), \\
y_{7} \mapsto \frac{1}{2}\left(y_{4}-y_{6}+y_{7}-y_{9}\right), \\
y_{8} \mapsto \frac{1}{2}\left(y_{1}-y_{3}\right), \\
y_{9} \mapsto \frac{1}{2}\left(y_{4}-y_{6}-y_{7}+y_{9}\right)
\end{array} \quad S_{2}:\left\{\begin{array}{l}
y_{0} \mapsto \frac{1}{2}\left(y_{0}+y_{1}+2 y_{4}\right), \\
y_{1} \mapsto \frac{1}{2}\left(y_{0}+y_{1}-2 y_{4}\right), \\
y_{2} \mapsto \frac{1}{2}\left(y_{2}+y_{3}+2 y_{7}\right), \\
y_{3} \mapsto \frac{1}{2}\left(y_{2}+y_{3}-2 y_{7}\right), \\
y_{4} \mapsto \frac{1}{2}\left(y_{0}-y_{1}\right), \\
y_{5} \mapsto \frac{1}{2}\left(y_{5}+y_{6}+y_{8}+y_{9}\right), \\
y_{6} \mapsto \frac{1}{2}\left(y_{5}+y_{6}-y_{8}-y_{9}\right), \\
y_{7} \mapsto \frac{1}{2}\left(y_{2}-y_{3}\right), \\
y_{8} \mapsto \frac{1}{2}\left(y_{5}-y_{6}+y_{8}-y_{9}\right), \\
y_{9} \mapsto \frac{1}{2}\left(y_{5}-y_{6}-y_{8}+y_{9}\right) .
\end{array}\right.\right.
\end{aligned}
$$

### 5.4 Partition functions for code theories

In the previous section, we have described how code CFTs are constrained by modular invariance for general genus, and by factorization limits, which relate different genera. Now we shall explicitly demonstrate how to use these requirements to constrain
the space of possible code theories. This is essentially a primitive example of the modular bootstrap - because modular invariance is so simple for the code theories, we can enumerate all of its possible solutions. Genus 1 modular invariance completely determines all of the $n=1$ and $n=2$ code theories. We shall see that genus 2 considerations are enough to fix the space of $n=3$ code theories, so this will be our primary example, given in section 5.4. At $n>3$, genus 2 constraints greatly reduce the space of theories but do not entirely fix them. We will summarize the classification of invariant polynomials for general $n$ in section 5.4, and for $n \leqslant 6$ we will compare the number of valid (modular invariant polynomial with positive integer coefficients) genus 1 partition functions, genus 2 partition functions, and actual code theories in section 5.4.

## Finding invariant polynomials: example at $n=3$

Let us start by presenting the case $n=3$ in full detail. For genus 1 , the most general homogeneous degree 3 polynomial is

$$
\begin{equation*}
P_{\mathrm{gen}}^{(1)}[3]=x_{0}^{3}+a_{2,1,0} x_{0}^{2} x_{1}+a_{2,0,1} x_{0}^{2} x_{2}+\ldots, \tag{5.4.1}
\end{equation*}
$$

which has 10 terms in total. Now recall that for genus 1, modular invariance implies that the polynomial is invariant under

$$
\begin{array}{ll}
S: & x_{0} \rightarrow \frac{1}{2}\left(x_{0}+x_{1}+2 x_{2}\right), \quad x_{1} \rightarrow \frac{1}{2}\left(x_{0}+x_{1}-2 x_{2}\right), \quad x_{2} \rightarrow \frac{1}{2}\left(x_{0}-x_{1}\right), \\
T: & x_{1} \rightarrow-x_{1} . \tag{5.4.2}
\end{array}
$$

This fixes all but two of the undetermined coefficients, so we are left with

$$
\begin{align*}
P_{\mathrm{inv}}^{(1)}[3]=x_{0}^{3}+ & \left(4-a_{0,0,3}\right) x_{0} x_{2}^{2}+\left(-1+a_{0,0,3}\right) x_{0} x_{1}^{2}  \tag{5.4.3}\\
& +\left(4-a_{0,0,3}-a_{0,2,1}\right) x_{0}^{2} x_{2}+a_{0,2,1} x_{1}^{2} x_{2}+a_{0,0,3} x_{2}^{3}
\end{align*}
$$

Requiring that all of these coefficients are non-negative leads to the inequalities

$$
\begin{equation*}
0 \leq a_{0,2,1} \leq 3, \quad 1 \leq a_{0,0,3} \leq 4-a_{0,2,1} \tag{5.4.4}
\end{equation*}
$$

which can be displayed as the two-dimensional region shown in figure 5.2 (a). Finally, we require that each of these coefficients is an integer, which leads to 10 solutions, the dots in figure 5.2(a).

Now consider genus two. We again write the most general polynomial and then impose modular invariance using (5.3.57)-(5.3.58). The result is a large polynomial

$$
\begin{equation*}
P_{\mathrm{inv}}^{(2)}[3]=y_{0}^{3}+a_{0,0,0,0,0,0,0,1,1,1} y_{3} y_{7} y_{8}+\ldots, \tag{5.4.5}
\end{equation*}
$$



Figure 5.2: Allowed regions for $n=3$. Plot (a): 2d region for genus 1. Dots represent polynomials with integer coefficients. Red dots, appearing at $(0,1),(3,1),(1,2)$, and $(0,4)$, represent factorization limits of genus 2 polynomials, as well as physical code theories. Plot (b): rough 3d plot of genus 2 allowed region. Remarkably, the factorizing, positive integer solutions lie on the vertices of this polytope!
which has a total of 48 terms and 3 undetermined parameters. Requiring positivity of all coefficients again gives a finite region, which we have displayed in figure 5.2 (b). If we require that the coefficients are all integers, we find 11 solutions. So there are 11 potential genus 2 partition functions.

However code theories must have genus 2 partition functions which factorize into genus 1 partition functions in the limit where the genus 2 Riemann surface becomes degenerate. Recall that this limit gives

$$
\begin{array}{lllll}
y_{0} \mapsto x_{0} x_{0}^{\prime}, & y_{1} \mapsto x_{0} x_{1}^{\prime}, & y_{2} \mapsto x_{1} x_{0}^{\prime}, & y_{3} \mapsto x_{1} x_{1}^{\prime}, & y_{4} \mapsto x_{0} x_{2}^{\prime} \\
y_{5} \mapsto x_{2} x_{1}^{\prime}, & y_{6} \mapsto x_{2} x_{2}^{\prime}, & y_{7} \mapsto x_{1} x_{2}^{\prime}, & y_{8} \mapsto x_{2} x_{1}^{\prime}, & y_{9} \mapsto x_{2} x_{2}^{\prime} \tag{5.4.6}
\end{array}
$$

Doing this, one finds that only four factorize of the genus 2 polynomials factorize into genus 1 polynomials, and in fact these are precisely the four the genus 1 polynomials which come from actual codes via New Construction A, given in (5.1.7)-(5.1.10)! The factorizing polynomials, which correspond to the set of real theories in this case, are displayed in red in figure 5.2 (a). They sit at the vertices of the polytope in figure 5.2 (b).

This is the general procedure we will follow for $n=3,4,5,6$ in the later sections. Actually, the case of $n=3$ is a little special. We could have determined all real theories by only requiring factorization, without positivity at genus 1 or genus 2 . We believe this is related to the fact that at $n=3$, genus 2 is enough to eliminate all fake theories - essentially, it fully fixes the partition function (given the assumption that it has enumerator polynomial form). We conclude that at $n=3$, there are no non-code theories with EP form because only the four code theories are consistent with factorization at genus 2 .

At $n=4$, where genus 2 does not remove all of the fake theories, factorization without positivity is not enough to find all of the modular invariant polynomials. Furthermore, just like $n=3$ at genus 1 , the real $n=4$ theories do not all lie at the vertices of the genus 1 and genus 2 polytopes; some lie on the edges, faces, and in the bulk. Thus we find two special properties obeyed by $n=3$ at genus $2: 1$ ) the set of real theories is fixed only by factorization, and 2) the real theories lie on the corners of the polytope. As such, it would be very interesting to push our calculation to genus 3, to see if these two patterns persist, but we leave this for future work.

## Invariant polynomial ring

Now let us be more general, and describe the method of classifying invariant polynomials, valid for any $n$. We will be interested in the ring $R_{g}$ of polynomials invariant under the genus $g$ modular relations. $R_{g}$ admits a grading by degree

$$
\begin{equation*}
R_{g}=\bigoplus_{n=0}^{\infty} R_{g}^{[n]} \tag{5.4.7}
\end{equation*}
$$

where $\operatorname{dim} R_{g}^{[n]}$ can be computed by Molien's formula, which we will now review (see e.g. 180 ). For a matrix group $G$ acting on $n$ variables $x_{0}, \ldots x_{n-1}$, Molien's formula gives the dimension of the space $R_{g}^{[n]}$ of degree $n$ invariant polynomials

$$
\begin{equation*}
M(r)=\sum_{n=0}^{\infty} r^{n} \operatorname{dim}\left(R_{g}^{[n]}\right)=\frac{1}{|G|} \sum_{g \in G} \operatorname{det}(1-r g)^{-1}, \tag{5.4.8}
\end{equation*}
$$

where $|G|$ is the number of (distinct) elements in $G$, and $g$ are the individual matrices in the representation in question.

Genus 1: Recall that invariance under the modular group $S L(2, \mathbb{Z})=S P(2, \mathbb{Z})$ acting on the three-dimensional vector space of $\left\{x_{0}, x_{1}, x_{2}\right\}$, is given by equation (5.4.2). Using (5.4.8), we find that Molien series for this group acting on 3 variables is given by

$$
\begin{equation*}
M(r)=\frac{1}{(1-r)\left(1-r^{2}\right)\left(1-r^{3}\right)}=1+r+2 r^{2}+3 r^{3}+4 r^{4}+5 r^{5}+7 r^{6}+8 r^{7}+\ldots \tag{5.4.9}
\end{equation*}
$$

The number of solutions to the transformations (5.4.2) for general degree- $n$ homogeneous polynomials is given by the $r^{n}$ coefficient in the Molien series in 5.4.9). This lets us infer the number of new generators at each order. The whole ring $R_{1}$ is generated by three polynomials $p_{n}^{(g=1)}\left(x_{0}, x_{1}, x_{2}\right)$, of degree $n$ for $n=1,2,3$ 21.

Genus 2: At genus 2, we have a 10 -dimensional representation of $S P(4, \mathbb{Z})$ furnished by the $10 y$-variables. The transformations are given in (5.3.57) (5.3.58) and we find that $|G|=720$. This allows us to calculate the Molien series

$$
\begin{equation*}
M(r)=\frac{f(r)}{(1-r)^{2}\left(1-r^{2}\right)\left(1-r^{3}\right)^{2}\left(1-r^{4}\right)^{2}\left(1-r^{5}\right)^{2}\left(1-r^{6}\right)}, \tag{5.4.10}
\end{equation*}
$$

where

$$
\begin{align*}
f(r)= & 1+r+r^{4}+r^{5}+3 r^{6}+2 r^{7}+3 r^{8}+3 r^{9}+5 r^{10}+3 r^{11}+10 r^{12}+6 r^{13} \\
& +9 r^{14}+7 r^{15}+4 r^{16}+2 r^{17}+r^{19} . \tag{5.4.11}
\end{align*}
$$

From (5.4.10), the number of generators is calculated. In principle, there could appear non-trivial relations between the generators. We find that there are no non-trivial relations among the terms constructed from generators at lower degree. Therefore, the generators organize as given in table 5.1.

Table 5.1: Number of linearly independent invariant polynomials, and number of generators of degree $n$ at genus $g=2$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| invariant polynomials | 1 | 2 | 4 | 8 | 14 | 27 | 46 | 82 | 140 | 237 | 386 | 630 |
| new generators | 1 | 1 | 2 | 3 | 4 | 6 | 7 | 9 | 11 | 10 | 4 | 0 |

There are new generators only up to $n=11$. A general method to completely specify this full set is to use enumerator polynomials of actual code theories as a set of generators. At each $n$ from $n=1$ to $n=11$ one can choose the new generators arising at each $n$ in Table 5.1 to be those from actual (inequivalent) code theories.

## Finding invariant polynomials: results for various $n$

Now let us describe the results of counting invariant polynomial for general values of $n$. There are too many genus 2 solutions to count above $n=6$, but in principle our method will work for any order $n$.

There are 58 generators, as described in table 5.1 above. Rather than list all of them, we will list the full set of adjacency graphs associated with matrices $B$ which can be used to calculate the complete ring of invariant polynomials using (5.3.47). The set of generators is given in appendix A. 6 .
$\boldsymbol{n}=\mathbf{1}$ At genus 1, we have a single polynomial:

$$
\begin{equation*}
W_{1}=x_{0}+x_{2} . \tag{5.4.12}
\end{equation*}
$$

At genus 2, we also find one polynomial,

$$
\begin{equation*}
W_{1}^{(2)}=y_{0}+y_{4}+y_{5}+y_{6}, \tag{5.4.13}
\end{equation*}
$$

which reproduces $W_{1}\left(x_{0}, x_{1}, x_{1}\right) W_{1}\left(x_{0}^{\prime}, x_{1}^{\prime}, x_{2}^{\prime}\right)$ in the factorization limit in 5.3.48). These polynomials correspond to the single unique code with $B=0$, or equivalently the graph with one node and zero vertices .
$\boldsymbol{n}=\mathbf{2}$ At genus 1, we have two polynomials:

$$
\begin{align*}
W_{2} & =x_{0}^{2}+x_{1}^{2}+2 x_{2}^{2}  \tag{5.4.14}\\
\left(W_{1}\right)^{2} & =\left(x_{0}+x_{2}\right)^{2}
\end{align*}
$$

There are two genus 2 polynomials which reproduce each of these polynomials in the factorization limit. They are

$$
\begin{align*}
W_{2}^{(2)} & =y_{0}^{2}+y_{1}^{2}+y_{2}^{2}+y_{3}^{3}+2\left(y_{4}^{2}+y_{5}^{2}+y_{6}^{2}+y_{7}^{2}+y_{8}^{2}+y_{9}^{2}\right), \\
\left(W_{1}^{(2)}\right)^{2} & =\left(y_{0}+y_{4}+y_{5}+y_{6}\right)^{2} . \tag{5.4.15}
\end{align*}
$$

These polynomials correspond to two unique codes corresponding to

$$
B=\left(\begin{array}{ll}
0 & 1  \tag{5.4.16}\\
1 & 0
\end{array}\right)
$$

and

$$
B=\left(\begin{array}{ll}
0 & 0  \tag{5.4.17}\\
0 & 0
\end{array}\right)
$$

These polynomials can be taken to be the new generators at this order.
$\boldsymbol{n}=\mathbf{3}$ At genus 1, we have 10 polynomials, out of which only 4 arise from inequivalent codes.

At genus 2, there are 11 polynomials but only 4 factorize. There are only $4 B$ form codes, so the set of codes is entirely determined by consistency with genus 2 modular invariance.

Since there are 2 new generators at this order, we can choose the following code generator matrices to define them:

$$
B=\left(\begin{array}{lll}
0 & 1 & 1  \tag{5.4.18}\\
1 & 0 & 1 \\
1 & 1 & 0
\end{array}\right), \quad B=\left(\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{array}\right)
$$

to get

$$
\begin{aligned}
W_{3}^{(2)}= & y_{0}^{3}+3 y_{1}^{2} y_{4}+3 y_{0} y_{4}^{2}+y_{4}^{3}+3 y_{2}^{2} y_{5}+3 y_{0} y_{5}^{2}+y_{5}^{3}+3 y_{3}^{2} y_{6}+6 y_{4} y_{5} y_{6}+3 y_{0} y_{6}^{2}+y_{6}^{3}+3 y_{5} y_{7}^{2} \\
& +3 y_{6} y_{7}^{2}+6 y_{3} y_{7} y_{8}+3 y_{4} y_{8}^{2}+3 y_{6} y_{8}^{2}+6 y_{2} y_{7} y_{9}+6 y_{1} y_{8} y_{9}+3 y_{4} y_{9}^{2}+3 y_{5} y_{9}^{2} \\
\tilde{W}_{3}^{(2)}= & y_{0}^{3}+3 y_{0} y_{1}^{2}+3 y_{0} y_{2}^{2}+6 y_{1} y_{2} y_{3}+3 y_{0} y_{3}^{2}+4 y_{4}^{3}+4 y_{5}^{3}+4 y_{6}^{3}+12 y_{4} y_{7}^{2}+12 y_{5} y_{8}^{2}+12 y_{6} y_{9}^{2}
\end{aligned}
$$

It is straightforward to check that the the most general invariant polynomial can be written as a linear combination of $W_{3}^{(2)}, \tilde{W}_{3}^{(2)}, W_{2}^{(2)} W_{1}^{(2)}$, and $\left(W_{1}^{(2)}\right)^{3}$. For convenience, these are collected in appendix A.7. This procedure of using explicit codes to construct generators is carried on to $n=11$ to obtain the full set. The results are given in graph form in appendix A.6.
$\boldsymbol{n}=\mathbf{4} \quad$ At genus 1, we have 20 polynomials.
At genus 2, there are 45 polynomials but only 10 factorize.
9 of these polynomials derive from real codes, leaving only one fake polynomial.
$\boldsymbol{n}=\mathbf{5} \quad$ At genus 1, we have 395 polynomials.
At genus 2 there are 1078 polynomials, but only 23 factorize.
21 of these polynomials derive from real codes, leaving only two fake polynomials.
$\boldsymbol{n}=\mathbf{6} \quad$ At genus 1, we have 27,280 polynomials.
At genus 2, 79 polynomials factorize to genus 1 polynomials. We are unable to count the total number of polynomials.

64 of these polynomials derive from real codes.
$\boldsymbol{n}=\mathbf{7} \quad$ At genus 1, we have 2,224,626 polynomials.
We are unable to count the number of genus 2 polynomials.
There are 218 polynomials from actual codes.

## Isospectral CFTs differ at genus 2

The partition function is a coarse observable. At genus 1, it only contains information about the spectrum of the theory, and it is possible that different theories may have the same genus 1 partition function $\sqrt{13}$ Higher-genus partition functions contain information about averages of OPE coefficients. In principle increasing the genus increases the amount of information extractable, though it is hoped 105 that with enough partition functions, the theory will be completely specified. A more precise understanding of this idea is one of the primary motivations of the present work.

For chiral CFTs, a demonstration is provided by Milnor's example of isospectral lattices. These correspond to chiral CFTs defined by compactification on the isospectral $d_{16}$ and $E_{8}^{2}$ lattices. It has been known since 142 that these two theories can actually be distinguished by going to genus 5, i.e. the partition functions are the same for $g \leqslant 4$.

In [19] a non-chiral version of this phenomenon was discovered. This example consists of two different CFTs at $n=7$ which have the same genus 1 partition function. These CFTs have the same spectrum, but it is clear from their definition

[^32]through inequivalent $B$-form codes that they must be different. The $B$-forms are given as
\[

B_{1}=\left($$
\begin{array}{lllllll}
0 & 1 & 0 & 0 & 0 & 1 & 0  \tag{5.4.19}\\
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0
\end{array}
$$\right), \quad B_{2}=\left($$
\begin{array}{lllllll}
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0
\end{array}
$$\right) .
\]

An explicit computation shows that these matrices yield the same genus 1 enumerator polynomial,

$$
\begin{align*}
W= & y_{0}^{7}+y_{0}^{5} y_{1}^{2}+5 y_{0}^{4} y_{1}^{2} y_{2}+5 y_{0}^{2} y_{1}^{4} y_{2}+y_{0}^{5} y_{2}^{2}+12 y_{0}^{3} y_{1}^{2} y_{2}^{2}+9 y_{0} y_{1}^{4} y_{2}^{2}+4 y_{0}^{4} y_{2}^{3}  \tag{5.4.20}\\
& +22 y_{0}^{2} y_{1}^{2} y_{2}^{3}+4 y_{1}^{4} y_{2}^{3}+5 y_{0}^{3} y_{2}^{4}+25 y_{0} y_{1}^{2} y_{2}^{4}+11 y_{0}^{2} y_{2}^{5}+11 y_{1}^{2} y_{2}^{5}+10 y_{0} y_{2}^{6}+2 y_{2}^{7} .
\end{align*}
$$

Likewise, the genus 2 enumerator polynomials $W_{1}^{(2)}$ and $W_{2}^{(2)}$ can be constructed from $B_{1}$ and $B_{2}$ using (5.3.47). We will not record them here since they are quite lengthy expressions, but they are distinct. One can further check that under factorization, both $W_{1}^{(2)}$ and $W_{2}^{(2)}$ factorize to $(W)^{2}$. Holomorphic modular forms at higher genus that degenerate in the factorization limit are referred to as cusp forms. The nonchiral analogs of such cusp forms would be $W_{1}^{(2)}-W_{2}^{(2)}$.

At $c=8$, the situation is more interesting as there are 61 pairs of isospectral theories and 5 isospectral triples. We find that the genus 2 partition functions are different for all pairs of isospectral theories. We also find that the genus 2 partition functions are different for all 5 isospectral triples.

### 5.5 Beyond code theories

Throughout this chapter, we have used the enumerator polynomial to simplify the form of the partition function. The primary benefit of this is that the constraints of modular invariance become very simple, allowing us to solve them exactly, i.e. to write the most general partition function (in enumerator polynomial form) which satisfies the constraints.

In our previous work [160], we pointed out that in fact, the partition function of every meromorphic CFT should have enumerator polynomial form, albeit with potentially negative or fractional coefficients. This follows directly from the fact
that the (numerator of the) partition functions must be linear combinations of Siegel modular forms, combined with standard results relating these modular forms to code enumerator polynomials (e.g. [129]). Non-chiral CFTs are much richer however, and to our knowledge, it is not known what subset of these CFTs might have partition functions with enumerator polynomial form. Let us explore a few examples.

## Minimal models

The first example is the 2D Ising CFT, which has the following genus 1 partition function:

$$
\begin{equation*}
Z_{4,3}^{(g=1)}=\frac{1}{2|\eta(\tau)|}\left(\sqrt{\theta_{2}(q) \theta_{2}(\bar{q})}+\sqrt{\theta_{3}(q) \theta_{3}(\bar{q})}+\sqrt{\theta_{4}(q) \theta_{4}(\bar{q})}\right) . \tag{5.5.1}
\end{equation*}
$$

The Ising CFT has $c=1 / 2$, so we see that

$$
\begin{equation*}
Z_{4,3}^{(g=1)}=\frac{W\left(x_{0}, x_{1}, x_{2}\right)}{|\sqrt{2} \eta(\tau)|^{2 c}} \tag{5.5.2}
\end{equation*}
$$

where

$$
\begin{equation*}
W\left(x_{0}, x_{1}, x_{2}\right)=\sqrt{\frac{x_{0}+x_{1}}{2}}+\sqrt{\frac{x_{0}-x_{1}}{2}}+\sqrt{x_{2}} . \tag{5.5.3}
\end{equation*}
$$

The Ising model can be written as a sum of square roots of the code variables. It is easy to verify that the $S$ transformation merely cycles these terms. The potential implications of the form (5.5.1) for the Ising CFT partition function has previously been discussed in [181], and the generalization to higher genus in [182].

One might hope that this extends to other minimal models as well, but it appears that it does not. Consider for instance the Lee-Yang CFT, a non-unitary minimal model with $c=-22 / 5$. The genus 1 partition function is known ${ }^{14}$

$$
\begin{equation*}
Z_{5,2}^{(g=1)}=\left|q^{-1 / 60} G(q)\right|^{2}+\left|q^{11 / 60} H(q)\right|^{2} . \tag{5.5.4}
\end{equation*}
$$

The functions $G(q)$ and $H(q)$ are the Rogers-Ramanujan functions, defined by

$$
\begin{equation*}
G(q)=\sum_{n=0}^{\infty} \frac{q^{n^{2}+\frac{1}{24}}}{\eta(q)}, \quad H(q)=\sum_{n=0}^{\infty} \frac{q^{n^{2}+n+\frac{1}{24}}}{\eta(q)} \tag{5.5.5}
\end{equation*}
$$

This is not equal to $\sum_{i=2}^{4}\left|\theta_{i}(q) / \eta(\tau)\right|^{c / 2}$, as the naive pattern would suggest. It remains possible that some non-trivial identity relates this partition function to Jacobi theta functions, but we were not able to find it. As far as we can tell,

[^33]the "enumerator polynomial form" displayed by the Ising CFT is an accident. Or perhaps it is related to the fact that the theory can be realized as a single free fermion. Perhaps the other minimal models are obtainable from a suitable generalization of New Construction A, such as the one of 168. It would be interesting to understand this better in the future.

## Chiral CFTs revisited

As a motivating example, let us consider a simple case of meromorphic, or chiral CFTs. We use "meromorphic" in the sense of [155], where it is taken to mean theories where $Z=\chi(\tau) \chi(\bar{\tau})$, with $\chi(\tau)=\chi(-1 / \tau)$. This is a strong constraint on the form of the partition function, and it leads to the requirement that $c$ be a multiple of 8 . Furthermore, if $c$ is a multiple of 24 , then $\chi(\tau)$ will be individually modular invariant and therefore can be the full partition function of a chiral CFT.

The connection between ECCs and chiral CFTs was considered in [160], where it was observed that every meromorphic CFT should have a partition function with enumerator polynomial form. This simply follows from the requirement that every chiral character $\chi(\tau)$ must be a modular form. For $g<4$, the ring of modular forms is completely captured by the ring of invariant polynomials after the standard substitutions $x_{i} \rightarrow \theta_{i}(q)$ [129, 125].

Let us now see some concrete examples of these observations. One interesting set of examples are the three "extremal $c=24$ theories." The first is the theory coming from the "Golay code:"

$$
\begin{equation*}
W_{\text {Golay }}\left(x_{0}, x_{1}\right)=x_{0}^{24}+759 x_{0}^{16} x_{1}^{8}+2576 x_{0}^{12} x_{1}^{12}+759 x_{0}^{8} x_{1}^{16}+x_{1}^{24} . \tag{5.5.6}
\end{equation*}
$$

The Golay code is the classical binary length 24 code which maximizes the Hamming distance. Relatedly, its enumerator polynomial has the largest gap between consecutive powers of $x_{0}$ (i.e. there is no $x_{0}^{20}$ term). The next example is the Leech lattice, with

$$
\begin{equation*}
W_{\text {Leech }}\left(x_{0}, x_{1}\right)=x_{0}^{24}-3 x_{0}^{20} x_{1}^{4}+771 x_{0}^{16} x_{1}^{8}+2558 x_{0}^{12} x_{1}^{12}+771 x_{0}^{8} x_{1}^{16}-3 x_{0}^{4} x_{1}^{20}+x_{1}^{24} . \tag{5.5.7}
\end{equation*}
$$

This theory does not derive from a code through Construction A. However, it is related to the Golay code by the twisting procedure [112, 113, reviewed in [19], which relates a self-dual even lattice to a new self-dual even lattice. The Leech lattice is "extremal" in the sense that it is known to provide the densest possible sphere packing in 24 dimensions [184].

The final interesting example is the Monster CFT, the theory whose automorphism group is the Monster group. Its partition function is given in enumerator polynomial form by

$$
\begin{equation*}
W_{\text {Monster }}\left(x_{0}, x_{1}\right)=x_{0}^{24}-\frac{9}{2} x_{0}^{20} x_{1}^{4}+777 x_{0}^{16} x_{1}^{8}+2549 x_{0}^{12} x_{1}^{12}+777 x_{0}^{8} x_{1}^{16}+\frac{9}{2} x_{0}^{4} x_{1}^{20}+x_{1}^{24} \tag{5.5.8}
\end{equation*}
$$

The Monster is the CFT which maximizes the spectral gap - the gap in Virasoro primaries - for $c=24$. It is neither a code nor a lattice theory, but it is related to the Leech lattice by $\mathbb{Z}_{2}$ orbifolding. So we see that these three theories are related.

The extremality of the Leech lattice can also be phrased in CFT language - it is the CFT which maximizes the gap in $U(1)^{c}$ primaries. This remarkable connection was explored in [20], where it was used to provide exact analytic functionals for the modular bootstrap. In fact, it turns out that the extremality of the Golay code theory also has a CFT interpretation as the maximization of the gap in $S U(2)^{c}$ characters. This will be explored in a future paper [185].

We can push this further. For the Monster theory, the extremal theory (the theory with the maximium gap in Virasoro primaries) at $c=24$, we have constructed the genus 2 and genus 3 partition function. We can also construct the genus 1 and genus 2 partition functions of the conjectured extremal $c=48$ theory, analogous to the Monster. We have collected a few of these lengthy expressions in appendix A.7.

## Non-chiral CFTs and the maximal gap

Next let us consider the more general, non-chiral case. The general strategy will be to still assume enumerator polynomial form for the partition function, but to go beyond code theories by allowing negative polynomial coefficients. The goal will be to identify interesting theories, such as the "extremal theories" which maximize the gap in Virasoro primaries $\Delta_{\text {gap }}$. We will do this by considering first the most general (modular invariant) enumerator polynomial, and then simply choosing the coefficients $a_{i, j, k}$ to maximize the gap. We will focus on the genus 1 partition function but in a few cases with low central charge $n$, we will be able to provide genus 2 expressions as well.
$\boldsymbol{n}=1$ In this case, $W_{1}=x_{0}+x_{2}$ is the only invariant polynomial. Its gap is $\Delta_{\text {gap }}=1 / 4$
$\boldsymbol{n}=\mathbf{2}$ Now we have a one parameter family of solutions. The gap is maximized by $W_{2}=x_{0}^{2}+x_{1}^{2}+2 x_{2}^{2}$, which has $\Delta_{\text {gap }}=1 / 2$
$\boldsymbol{n}=\mathbf{3}$ In this case, the maximal gap is $\Delta_{\text {gap }}=3 / 4$, coming from the polynomial

$$
\begin{equation*}
\tilde{W}_{3}=x_{0}^{3}+3 x_{0} x_{1}^{2}+4 x_{2}^{3} \tag{5.5.9}
\end{equation*}
$$

We can recognize that this is the enumerator polynomial of (5.2.76), which came from the $B$-form code corresponding to the complete graph $K_{3}$, the fully-connected graph with three nodes.

There is a unique genus 2 enumerator polynomial which factorizes into this genus 1, fixing the genus 2 partition function of this theory to be

$$
\begin{align*}
\tilde{W}_{3}^{(2)}= & y_{0}^{3}+3 y_{0} y_{1}^{2}+3 y_{0} y_{2}^{2}+3 y_{0} y_{3}^{2}+6 y_{1} y_{2} y_{3} \\
& +4 y_{4}^{3}+4 y_{5}^{3}+4 y_{6}^{3}+12 y_{4} y_{7}^{2}+12 y_{5} y_{8}^{2}+12 y_{6} y_{9}^{2} \tag{5.5.10}
\end{align*}
$$

$\boldsymbol{n}=4$ Here we find the same thing: the extremal theory, which now has $\Delta_{\text {gap }}=1$, also corresponds to a code theory. It has

$$
\begin{equation*}
\tilde{W}_{4}=x_{0}^{4}+6 x_{0}^{2} x_{1}^{2}+x_{1}^{4}+8 x_{2}^{4} \tag{5.5.11}
\end{equation*}
$$

This theory, also realizable as the $S O(8)$ WZW with level 1 , or the theory of 8 free fermions with diagonal GSO projection, was shown to saturate the modular bootstrap constraints at $n=495$. So in this case the extremal theory with enumerator polynomial form is the same as the most general extremal theory (in contrast to the cases of $n=1,2$, and 3, where the bounds of 95 are not saturated by our EP-form partition functions). $\tilde{W}_{4}$ is the $B$-form code deriving from the complete graph $K_{4}$. So in both of these cases, the extremal theories are given by the complete graphs.

Again, we can fix the genus 2 partition function from this:

$$
\begin{align*}
\tilde{W}_{4}^{(2)}=y_{0}^{4}+ & 6 y_{0}^{2} y_{1}^{2}+y_{1}^{4}+6 y_{0}^{2} y_{2}^{2}+6 y_{0}^{2} y_{2}^{2}+y_{2}^{4}+6 y_{0}^{2} y_{3}^{2}+6 y_{1}^{2} y_{3}^{2} \\
& +y_{3}^{4}+24 y_{0} y_{1} y_{2} y_{3}+8 y_{4}^{4}+8 y_{5}^{4}+8 y_{6}^{4}+8 y_{7}^{4}  \tag{5.5.12}\\
& +8 y_{8}^{4}+8 y_{9}^{4}+48 y_{4}^{2} y_{7}^{2}+48 y_{5}^{2} y_{8}^{2}+48 y_{6}^{2} y_{9}^{2} .
\end{align*}
$$

$\boldsymbol{n}=\mathbf{5} \quad$ Here something new happens. From the Molien series, the general $n=5$ polynomial has 4 undetermined coefficients, but if we use this to cancel out the maximum number of states, the resulting "extremal theory" has negative degeneracies.

Instead, we may only cancel 3 , rather than 4 undetermined degeneracies. This results a function with $\Delta_{\text {gap }}=1$, and 1 remaining undetermined coefficient,

$$
\begin{equation*}
\tilde{W}_{5}=x_{0}^{5}+\frac{16-a_{0,0,5}}{7} x_{1}^{4} x_{2}-4 \frac{16-a_{0,0,5}}{7} x_{0}^{2} x_{1}^{2} x_{2}+\ldots \tag{5.5.13}
\end{equation*}
$$

If we demand that the remaining degeneracies are positive integers, we find that $a_{0,0,5}$ must be an integer satisfying $0 \leq a_{0,0,5} \leq 16$. If we set $a_{0,0,5}=16$, we recover the code theory corresponding to the complete graph $K_{5}$. None of the other values of $a_{0,0,5}$ can derive from a code via New Construction A due to the signs in 5.5.13)
$\boldsymbol{n}=8 \quad$ Noting that $n=6$ and $n=7$ are similar to $n=5$, we proceed to $n=8$. In this case, we find that we can cancel enough states to make $\Delta_{\text {gap }}=7 / 4$, leading to a partition function with negative integer degeneracies. If we require non-negative degeneracies ${ }^{[15}$ the maximum gap is $\Delta_{\text {gap }}=5 / 4$. This does not saturate the bound $\Delta_{\text {gap }}=c / 8+1 / 2$ of 95 .

Alternatively, we may try to only cancel scalar degeneracies. If we try to cancel the maximum number, we find $\Delta_{\text {gap }}^{s=0}=5 / 2$, but then a number of other states have negative degeneracies. More interestingly, if we cancel the maximum number while requiring unitarity, we find a unique theory with $\Delta_{\text {gap }}^{s=0}=2$. This theory is described by the enumerator polynomial

$$
\begin{equation*}
\tilde{W}_{8}^{s=0}=x_{0}^{8}+60 x_{0}^{6} x_{1}^{2}+60 x_{1}^{6} x_{2}^{2}+134 x_{0}^{4} x_{1}^{4}-32 x_{0}^{4} x_{2}^{4}-32 x_{1}^{4} x_{2}^{4}-192 x_{0}^{2} x_{1}^{2} x_{2}^{4}+256 x_{2}^{8} \tag{5.5.14}
\end{equation*}
$$

Current modular bootstrap bounds require that $\Delta_{\text {gap }}^{s=0}<2$ for $n<8$, with a kink at $\Delta_{\text {gap }}^{s=0}=2$ for $n=8$ which is conjectured to be saturated by the $E_{8}$ WZW model at level 1 [95]. This theory holomorphically factorizes, $Z(q, \bar{q})=Z_{E_{8}}(q) Z_{E_{8}}(\bar{q})$ into two copies of the chiral theory which results from from compactifying 8 chiral bosons on the $E_{8}$ lattice. This theory is also related to the Hamming code via Construction A.

This gives another example for which a theory which is extremal in some sense can be written in EP form but does not arise from a code. That it does not derive from a code is obvious because of the negative coefficients appearing in (5.5.14). This may be thought of as analogous to the enumerator-polynomial form representation of the Leech lattice and Monster partition functions.

A third interesting example exists for $n=8$ - the theory described in [186] with $O_{10}^{+}(2) .2$ automorphism group. This theory is completely fixed from the general

[^34]invariant EP form by requiring that our partition functions have only integer scaling dimensions, and that the degeneracy of scalars at $\Delta=1$ is 496. The resulting enumerator polynomial is
\[

$$
\begin{equation*}
W_{n=8}^{O_{10}^{+}(2) \cdot 2}=x_{0}^{8}-2 x_{0}^{6} x_{1}^{2}+10 x_{0}^{4} x_{1}^{4}-2 x_{0}^{2} x_{1}^{6}+x_{1}^{8}+30 x_{0}^{4} x_{2}^{4}+180 x_{0}^{2} x_{1}^{2} x_{2}^{4}+30 x_{1}^{4} x_{2}^{4}+8 x_{2}^{8} \tag{5.5.15}
\end{equation*}
$$

\]

In principle, it should be possible to check if these genus 1 partition functions correspond to a single genus 2 partition function. Since these theories do not derive from codes, it may be that their genus 1 partition functions have EP form but their genus 2 partition functions do not - we do not know if this is possible. For the chiral CFTs, we found that extremal genus 2 partition functions at $c=24$ is entirely fixed by factorization, so it would be interesting to try that here. However, the most general genus 2 polynomial is beyond what we are able to do with our desktop computers right now, so we will have to leave exploring them to the future.
$\boldsymbol{n}=\mathbf{2 4}$ In this case, by allowing for negative degeneracies we can construct a modular invariant function with $\Delta_{\text {gap }}=5$. As an interesting side note, it is possible to identify the theory $Z=Z_{\text {Monster }}(q) Z_{\text {Monster }}(\bar{q})$ by setting quarter integer degeneracies to 0 and requiring a gap. However, this modular-invariant partition function has negative degeneracies from the point of view of a non-chiral CFT because the $1-q$ in the vacuum character of $Z_{\text {Monster }}(q)$ has a minus sign, which multiplies all of the characters in $Z_{\text {Monster }}(\bar{q})$, and vice-versa.

## Asymptotics at large $c$

Finally, let us make a few comments about theories in the limit of large central charge.

## Complete graphs

The complete graphs provide a special class of code theories, which includes every positive integer $n$. We can study them by explicitly constructing their enumerator polynomials from their $B$-matrices, which have a 1 in every entry except on the diagonal. For $c \geqslant 4$, the gap of very complete-graph theory has $\Delta_{\text {gap }}=1$. In fact, this is the upper bound on the gap of any code theory [19].

As we noted before, the $n=4$ deriving from the complete graph $K_{4}$ is equivalent to the $S O(8)$ WZW model. It appears that this pattern continues beyond $n=4$. We have checked for $n$ up to 15 that the number of currents is always given by
$N_{n}=n(2 n-1)$. Extra currents require extra symmetries, and $n(2 n-1)$ is the number of currents one would find in the presence of symmetry enhancement to $S O(2 n)$. In fact, for large enough $n$, no other rank $n$ Lie group contains this many currents, so $S O(2 n)$ is the only possibility. As a result, we conjecture that the complete graph codes give a family of CFTs with symmetry enhancement to $S O(2 n)$. It would be interesting to study these theories further, to try to prove this conjecture or understand their holographic duals, in the future.

## Estimate for non-unitarity theories

It is also possible to provide an estimate of the maximal gap possible by comparing the number of degeneracies we need to cancel with the number of coefficients we can choose. That is, if we have $k$ coefficients, then we should be able to set $k$ degeneracies to zero ${ }^{16}$

First we consider the number of possible operators up to a given scaling dimension. Because these theories have quarter-integer scaling dimensions, the maximal number of possible operators up to $\Delta$ given by

$$
\begin{equation*}
\rho(\Delta)=4 \sum_{k=0}^{\Delta}(k+1)=2(\Delta+2)(\Delta+1) . \tag{5.5.16}
\end{equation*}
$$

So to create a gap $\Delta$, one must cancel $2(\Delta+2)(\Delta+1)$ theories.
Next we determine the number of coefficients which we can pick to cancel degeneracies. This is given by the coefficients appearing in the Molien series. The $n^{\text {th }}$ such coefficient is given by

$$
\begin{equation*}
\omega(n)=\frac{1}{72}\left(47+9(-1)^{n}+6 n(6+n)+16 \cos \frac{2 n \pi}{3}\right) . \tag{5.5.17}
\end{equation*}
$$

At large $n, \omega(n) \sim n^{2} / 12$ and $\rho(\Delta) \sim 2 \Delta^{2}$. Thus we can create a gap up to $\Delta$ at central charge $c=n$ as long as $\omega(n)=\rho(\Delta)$, leading to the estimate

$$
\begin{equation*}
\Delta_{\text {gap }}=\frac{n}{\sqrt{24}} . \tag{5.5.18}
\end{equation*}
$$

This is a huge gap, and is inconsistent with the strongest current asymptotic bounds given in [101. However, those bounds require unitarity. So we expect that if such large gaps are possible then they must come from modular invariant functions with

[^35]negative degeneracies of Virasoro characters. This is exactly what we find for $n=24$. In that case, $\Delta_{\text {gap }}=5$ but the function with that gap has negative degeneracies. Thus our estimate on gap avoids the bootstrap bounds because the modular invariant functions satisfying it will be non-unitarity.

### 5.6 Discussion

This chapter is an invitation to study the consequences of higher-genus modular invariance, using the example of theories defined from quantum error-correcting codes through the New Construction A of [19]. The guiding idea is that higher-genus modular invariance should be more constraining than genus 1. Error-correcting code theories provide a simple playground to explore this because their partition functions conform to the polynomial ansatz central to this chapter. In this simple "enumeratorpolynomial form," the action of the modular group $S P(2 g, \mathbb{Z})$ becomes a set of simple linear transformations on the polynomial variables. This allows for a simple algorithm to solve the constraints and list every partition function which could possibly derive from a code. This also requires that we impose (1) that all polynomial coefficients are positive integers, as they should derive from degeneracies of codewords, and (2) that the higher-genus partition functions factorize into lower-genus partition functions in the limit where Riemann surface they live on becomes singular.

We find that higher-genus modular invariance is very constraining. Specifically, there are a large number of genus 1 expressions which look like consistent partition functions. We have proven that such partition functions cannot arise, via factorization, from genus 2 partition functions with enumerator polynomial form. This means that they cannot be code theories, but, importantly, it does not imply that they cannot be CFTs at all. Since the number of actual $B$-form codes is known, we can compare our results to the true set of code theories. We find that for $n=3$, genus 2 modular invariance rules out all polynomials which do not derive from codes. For $n>3$, the number of fake polynomials is greatly reduced by genus 2 considerations, and presumably pushing our algorithm to higher genus would be required to remove all fake polynomials. It would be interesting to understand exactly what genus is required for a given $n \cdot{ }^{17}$

In [19], a number of isospectral code CFTs were discovered - theories which are known to be different, but which have the same genus 1 partition function and thus the same spectrum. Such theories are non-chiral analogs of Milnor's example

[^36]isospectral Euclidean lattices in 16 dimensions, which define isospectral chiral CFTs. We find that all of the examples given in [19] of isospectral theories up to $n=8$ can be distinguished by their genus 2 partition functions. This is similar to how the theories in Milnor's example can be distinguished by their genus 5 partition functions [142].

One of the main lessons of this chapter is that the enumerator polynomial ansatz for the partition function greatly simplifies a number of questions. One direction we have investigated is the maximum gap of theories with enumerator polynomial form. Theories deriving from actual codes have a maximal gap of 1 . In the case of $n=4$, $\Delta_{\text {gap }}=1$ is the actual upper bound on the gap 95 , and the theory saturating it, the $S O(8)$ WZW model, is known to be a code theory [19]. However it is possible that there are non-code theories with enumerator polynomial form, which occurs for chiral CFTs, for instance, in the case of the Monster CFT. For $n>4$, we found that a larger gap is consistent with enumerator polynomial form. If we allow for negative Virasoro degeneracies, we can construct very large gaps (ex. $\Delta_{\text {gap }}=5$ for $n=24$ ). For the more interesting case of unitary theories, we were not able to say as much. This is because we do not have a simple way to impose positivity over all Virasoro primaries, since they are not related to the code variables in a simple way. If such a method could be found, we believe it could be very interesting to investigate the maximum gap of theories with enumerator polynomial form.

We found a few other examples of known theories with enumerator polynomial form. Two $n=8$ theories saturating various bootstrap bounds were shown to be non-code theories whose partition functions have enumerator polynomial form. It is notable that such theories only appear at $n=4$ and $n=8$. There are known theories saturating the bounds of $[95, \sqrt{186}]$ at other values of $c$ - in particular, these include a number of other WZW models. These cannot have enumerator polynomial form because their scaling dimensions are not all quarter-integers. It would be very interesting to try to see if such theories might derive from some generalization of the New Construction A. An important step in this direction was taken in [168], where a construction was given for error-correcting codes over fields $\mathbb{F}_{p}$ for any prime $p$. See also [187].

Finally, it would be interesting to use our work to address quantum gravity. $\mathrm{AdS}_{3} / \mathrm{CFT}_{2}$ is one of the most useful testing grounds for new ideas about holography. Recently, this has included new insights about the significance of averaging over the moduli space of CFTs to define a particular bulk theory [161, 121]. In a future work [185] we will apply some of these insights to the case of chiral CFTs. In [19], a formula
for the average over $B$-form codes was given, and the holographic interpretation was discussed in 163. It would be interesting to try to discover the genus 2 average for $B$-form codes, and understand its holographic implications.

## Chapter 6 Conclusion

In this thesis, we studied aspects of 2 dimensional CFTs related to integrability, thermalization and quantum information.

In Chapter 2, we obtained the spectrum of quantum KdV charges $Q_{2 n-1}$ up to quadratic order and the semi-classical spectrum up to cubic order in a $1 / c$ expansion. This perturbative expansion in powers of $c$ is inspired by the AdS/CFT duality. The classical kdV charges are first expressed in a perturbative expansion in powers of the classical action variables $I_{k}$. A semi-classical quantization scheme is used to promote these classical action variables to quantum numbers $n_{k}$. The quantization is completely fixed using additional constraints from the knowledge of how $Q_{2 n-1}$ act on primary states from the $\mathrm{ODE} / \mathrm{IM}$ correspondence.

These results are expected to be helpful in the study of the Generalized Eigenstate Thermalization Hypothesis in large central charge CFTs, especially holographic ones. Numerical studies of integrable spin chains, corresponding to $c>1$ minimal model CFTs, that aim to investigate thermalization may benefit from knowledge of the exact spectrum of these charges.

The thermal expectation values of these conserved charges i.e. $\left\langle Q_{2 n-1}\right\rangle_{\Delta} \equiv$ $\operatorname{Tr}_{\Delta}\left(q^{L_{0}-c / 24} Q_{2 n-1}\right)$ are covariant under torus modular transformations. One may speculate that imposing higher genus modular covariance of such an average, by writing down the average in terms of known Siegel modular forms would give enough constraints to completely fix the spectrum of the quantum charges. This is still an open question. Hopefully, the results obtained here can be used to ask further sharp and interesting questions about black hole thermalization in a solvable setting.

In Chapter 3, we studied the information metric as an alternate means to encode some or all of the same information as the original correlator in a holographic CFT. Interestingly, the information metric geometrizes the correlator in a seemingly novel fashion.

The modular bootstrap constrains the torus partition function based on the requirement that it be invariant under the group of modular transformations. The constraints from genus 1 modular invariance have been used to derive interesting universal bounds on the spectrum. In Chapter 4, we used a known construction relating a class of classical error correction codes to chiral meromorphic CFTs to study the consequences of higher genus modular invariance.

We find that a quantity natural to error correcting codes, ie. the weight- $g$ enu-
merator polynomial, evaluated at theta constants is the numerator of the genus- $g$ partition function of the CFT. For the class of theories we study, the constraints of higher genus modular invariance of the partition function can be written as linear transformations on the enumerator polynomial. Further constraints are imposed by requiring that the partition function factorizes as the genus- $g$ Riemann surface factorizes into two lower-genus Riemann surfaces.

In Chapter 5, the same guiding principle is used to show that the weight- $g$ enumerator polynomial of a quantum error correction code is related to the numerator of the genus- $g$ partition function of certain Narain CFTs. Our observations are that a large number of genus 1 expressions which look like consistent partition functions, cannot arise, via factorization, from genus 2 partition functions with enumerator polynomial form. We also find that certain examples of isospectral but distinct Narain theories can be distinguished by their genus 2 partition functions.

By showcasing how some limitations of the genus 1 bootstrap can be overcome by using these larger symmetries, these investigations motivate a further exploration into the higher genus modular bootstrap. There are interesting opportunities to better understand how these symmetries constrain dynamical information in CFTs, i.e. OPE coefficients by formulating such questions in terms of the higher genus partition functions of the "code theories" we discussed.

## Chapter A Appendices

## A. 1 One-zone potentials: details

One-zone potentials $u$ can be found from the condition $\left\{Q_{3}+\alpha Q_{1}, u\right\}=0$ for some constant $\alpha$. From here we immediately find, see section 2.4 of [10],

$$
\begin{align*}
& \lambda_{0}=-\frac{\alpha}{24}-\frac{k^{2}}{12}\left(\theta_{3}(\tau)^{4}+\theta_{4}(\tau)^{4}\right)  \tag{A.1.1}\\
& \lambda_{1}=-\frac{\alpha}{24}-\frac{k^{2}}{12}\left(\theta_{2}(\tau)^{4}-\theta_{4}(\tau)^{4}\right)  \tag{A.1.2}\\
& \lambda_{2}=-\frac{\alpha}{24}+\frac{k^{2}}{12}\left(\theta_{2}(\tau)^{4}+\theta_{3}(\tau)^{4}\right) \tag{A.1.3}
\end{align*}
$$

Pertubatively, i.e. in the limit of small $q=e^{i \pi \tau}$, corresponding potential is

$$
\begin{equation*}
u=h+\frac{32 k^{4}}{k^{2}+h} q^{2}-16 k^{2} q \cos (k \varphi)-32 k^{2} q^{2} \cos (2 k \varphi)+\mathcal{O}\left(q^{3}\right) \tag{A.1.4}
\end{equation*}
$$

There are useful relations involving Jacobi elliptic functions and hypergeometric function,

$$
\begin{aligned}
& m:=\theta_{2}^{4}(\tau) / \theta_{3}^{4}(\tau), \quad F\left(\frac{1}{2}, \frac{1}{2}, 1 ; m\right)=\theta_{3}(\tau)^{2}, \quad \frac{F\left(\frac{1}{2}, \frac{1}{2}, 1 ; 1-m\right)}{F\left(\frac{1}{2}, \frac{1}{2}, 1 ; m\right)}=-\frac{1}{\pi} \log q \\
& \frac{F\left(\frac{3}{2}, \frac{1}{2}, 1 ; m\right)}{F\left(\frac{1}{2}, \frac{1}{2}, 1 ; m\right)}=1+2 \frac{\partial \ln \theta_{3}^{2}(\tau)}{\partial \ln m} \\
& -16 \sum_{n=0}^{\infty} \frac{q^{2 n+1}}{\left(1-q^{2 n+1}\right)^{2}}+2 \theta_{3}(\tau)^{4}-2 \theta_{4}(\tau)^{4} \frac{F\left(\frac{3}{2}, \frac{1}{2}, 1 ; m\right)}{F\left(\frac{1}{2}, \frac{1}{2}, 1 ; m\right)}=0
\end{aligned}
$$

We also give here more terms in the $q$-expansion of $I_{k}$,

$$
\begin{aligned}
I_{k} & =\frac{32 k^{3} q^{2}}{h+k^{2}}+\frac{64 q^{4}\left(3 h^{2} k^{3}+12 h k^{5}+k^{7}\right)}{\left(h+k^{2}\right)^{3}}+\frac{128 k^{3} q^{6}\left(3 h^{4}+42 h^{3} k^{2}+108 h^{2} k^{4}-58 h k^{6}+k^{8}\right)}{\left(h+k^{2}\right)^{5}} \\
& +\frac{128 k^{3} q^{8}\left(7 h^{6}+156 h^{5} k^{2}+1083 h^{4} k^{4}+1232 h^{3} k^{6}-4035 h^{2} k^{8}+788 h k^{10}+k^{12}\right)}{\left(h+k^{2}\right)^{7}}+\mathcal{O}\left(q^{10}\right)
\end{aligned}
$$

which with help of $Q_{1}=h+k I_{k}$ immediately yields

$$
\begin{align*}
Q_{1}= & h+\frac{32 k^{4}}{k^{2}+h} q^{2}+\frac{64 k^{4}\left(3 h^{2}+12 h k^{2}+k^{4}\right)}{\left(k^{2}+h\right)^{3}} q^{4}  \tag{A.1.5}\\
& +\frac{128 k^{4}\left(3 h^{4}+42 h^{3} k^{2}+108 h^{2} k^{4}-58 h k^{6}+k^{8}\right)}{\left(k^{2}+h\right)^{5}} q^{6}+\mathcal{O}\left(q^{7}\right) \tag{A.1.6}
\end{align*}
$$

The relation for $I_{k}$ in terms of $q$ can be solved for $q$ in terms of $I_{k}$ iteratively, which was used in section 2.2 .

$$
\begin{aligned}
q^{2}= & \frac{\left(h+k^{2}\right)}{32 k^{3}} I_{k}-\frac{\left(3 h^{2}+12 h k^{2}+k^{4}\right)}{512 k^{6}} I_{k}^{2}+\frac{\left(15 h^{3}+87 h^{2} k^{2}+105 h k^{4}+k^{6}\right)}{8192 k^{9}} I_{k}^{3} \\
& -\frac{\left(187 h^{4}+1402 h^{3} k^{2}+3012 h^{2} k^{4}+1606 h k^{6}+k^{8}\right)}{262144 k^{12}} I_{k}^{4}+\mathcal{O}\left(I_{k}^{5}\right) .
\end{aligned}
$$

## A. 2 Perturbative calculation for finite-zone potentials

We start with the two-zone case and parametrize corresponding differential $d p$ with help of two infinitesimal parameters $\epsilon_{1}, \epsilon_{2}$ and $\lambda_{0}$,

$$
\begin{align*}
& \lambda_{1}=\lambda_{0}+\frac{k^{2}}{4}+\epsilon_{1}+a_{1} \epsilon_{1}^{2}+b_{1} \epsilon_{1} \epsilon_{2}+c 1 \epsilon_{2}^{2}+\ldots  \tag{A.2.1}\\
& \lambda_{2}=\lambda_{0}+\frac{k^{2}}{4}-a \epsilon_{1}+a_{2} \epsilon_{1}^{2}+b_{2} \epsilon_{1} \epsilon_{2}+c 2 \epsilon_{2}^{2}+\ldots  \tag{A.2.2}\\
& \lambda_{3}=\lambda_{0}+\frac{\ell^{2}}{4}+\epsilon_{2}+a_{3} \epsilon_{1}^{2}+b_{3} \epsilon_{1} \epsilon_{2}+c 3 \epsilon_{2}^{2}+\ldots  \tag{A.2.3}\\
& \lambda_{4}=\lambda_{0}+\frac{\ell^{2}}{4}-b \epsilon_{2}+a_{4} \epsilon_{1}^{2}+b_{4} \epsilon_{1} \epsilon_{2}+c 4 \epsilon_{2}^{2}+\ldots  \tag{A.2.4}\\
& r_{1}=\lambda_{0}+\frac{k^{2}}{4}+d_{1} \epsilon_{1}^{2}+e_{1} \epsilon_{1} \epsilon_{2}+f_{1} \epsilon_{2}^{2}+\ldots  \tag{A.2.5}\\
& r_{2}=\lambda_{0}+\frac{\ell^{2}}{4}+d_{2} \epsilon_{1}^{2}+e_{2} \epsilon_{1} \epsilon_{2}+f_{2} \epsilon_{2}^{2}+\ldots \tag{A.2.6}
\end{align*}
$$

The parametrization is redundant, with different choices related by redefinitions of $\epsilon_{1}, \epsilon_{2}$. We assume $\epsilon_{1} \sim \epsilon_{2}$ are of the same order and in what follows we refer to expansion in $\epsilon_{1}, \epsilon_{2}$ simply as $\epsilon$ expansion. While keeping two-zone case in mind for concreteness, most of the discussion below applies to $m$-zone case with arbitrary $m$.
$a$-cycles
To impose $a_{1}$-cycle constraint 2.2 .9 , we need to integrate from $\lambda_{1}$ to $\lambda_{2}$. By introducing $x$ via

$$
\begin{equation*}
\lambda=\frac{\lambda_{2}+\lambda_{1}}{2}+x \frac{\lambda_{2}-\lambda_{1}}{2}, \tag{A.2.7}
\end{equation*}
$$

and then expanding in powers of $\epsilon$ we reduce the integral to standard integrals of the form

$$
\begin{equation*}
\int_{-1}^{1} \frac{d x x^{2 n}}{\sqrt{1-x^{2}}}=\frac{\sqrt{\pi} \Gamma(n+1 / 2)}{\Gamma(n+1)} \tag{A.2.8}
\end{equation*}
$$

Provided we want to find $Q_{2 n-1}$ in terms of $I_{k}$ by expanding up to $p$-th power, we would need to keep $2 p$ terms in $\epsilon$-expansion, up to and including $\epsilon^{2 p}$. This method works for any $a$-cycle integral and any number of zones.

## b-cycles

We start with the $b_{1}$-cycle, which goes from $\lambda_{0}$ to $\lambda_{1}$, and introduce another variable $x$

$$
\begin{equation*}
\lambda=\lambda_{1}-x\left(\lambda_{1}-\lambda_{0}\right) \tag{A.2.9}
\end{equation*}
$$

We can use the proximity of $\lambda_{4}$ to $\lambda_{3}$ to expand $\sqrt{\left(\lambda-\lambda_{3}\right)\left(\lambda-\lambda_{4}\right)}$ in $\epsilon$. Now the integral of interest reduced to a sum of integrals of the form

$$
\begin{equation*}
\int_{0}^{1} \frac{d x P(x)}{\sqrt{x(1-x)(16 w+x)}(x-c)^{r}} \tag{A.2.10}
\end{equation*}
$$

where $16 w$ is a small parameter of order $\epsilon$,

$$
\begin{equation*}
16 w=\frac{\lambda_{2}-\lambda_{1}}{\lambda_{1}-\lambda_{0}} \tag{A.2.11}
\end{equation*}
$$

$P(x)$ is some polynomial and $c=1-\ell^{2} / k^{2}$ (we assumed $\ell>k$ ). The integral A.2.10) can be related to

$$
\begin{equation*}
J_{n}(c):=\int_{0}^{1} \frac{d x x^{n}}{\sqrt{x(1-x)(16 w+x)}(x-c)} \tag{A.2.12}
\end{equation*}
$$

by differentiating over $c$. To evaluate it, it is helpful to first introduce the integral

$$
\begin{equation*}
I_{n}:=\int_{0}^{1} \frac{d x x^{n}}{\sqrt{x(1-x)(16 w+x)}}=\sum_{m=0}^{\infty} a_{m}(n) w^{m}+\sum_{m=n}^{\infty} b_{m}(n) w^{m} \ln w \tag{A.2.13}
\end{equation*}
$$

which can be expressed as formal series in $w$. Coefficients $a_{m}(n)$ for $n>m$ and $b_{m}(n)$ for any $n, m$ can be found analytically

$$
\begin{align*}
a_{m}(n) & =\frac{(-16)^{m} \Gamma\left(m+\frac{1}{2}\right) \Gamma(n-m)}{\Gamma(m+1) \Gamma\left(-m+n+\frac{1}{2}\right)}, \quad n>m  \tag{A.2.14}\\
b_{m}(n) & =\frac{16^{m}(-1)^{n+1} \Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1) \Gamma(m-n+1) \Gamma\left(-m+n+\frac{1}{2}\right)} \tag{A.2.15}
\end{align*}
$$

To find $a_{m}(n)$ for $m \geq n$ we can use the iterative relation

$$
\begin{equation*}
I_{n}=(1-2 n)\left(I_{n}-I_{n-1}\right)-\frac{1}{8} \partial_{w}\left(I_{n+1}-I_{n}\right), \tag{A.2.16}
\end{equation*}
$$

which follows from the integration by parts, and $a_{m}(0)$ which can be found directly from A.2.13) since the corresponding integral can be evaluated analytically. For example we find the following iterative relation for $a_{n}(n)$,

$$
\begin{equation*}
a_{m+1}(m+1)=\frac{(-1)^{m} 2^{2 m+3} \Gamma(2 m+1)}{\Gamma(m+2)^{2}}-\frac{8(2 m+1) a_{m}(m)}{m+1}, \quad a_{0}(0)=0 .( \tag{A.2.17}
\end{equation*}
$$

So far we are interested only in first $2 p$ powers of $w$, we only need to worry about $a_{m}(n)$ with $m \leq 2 p$. In our case $p=3$ and we simply tabulate values of $a_{m}(n)$ for $0 \leq m \leq 6$ and $m \geq n$ for convenience, $a_{m}(n)=$

$$
\left(\begin{array}{ccccccl}
0 & & & & & & \\
8 & 8 & & & & & \\
-84 & -104 & -112 & & & & \\
\frac{2960}{3} & 1152 & \frac{4288}{3} & \frac{4736}{3} & & \\
-\frac{37310}{3} & -\frac{42040}{3} & -16368 & -\frac{60992}{3} & -\frac{68224}{3} & & \\
\frac{820008}{5} & 180656 & 203584 & \frac{1189248}{5} & \frac{1478144}{5} & \frac{1666048}{5} & \\
-\frac{11153912}{5} & -\frac{12097344}{5} & -\frac{7995904}{3} & -\frac{9011456}{3} & -\frac{17549824}{5} & -\frac{65468416}{15} & -\frac{74166272}{15}
\end{array}\right)
$$

Going back to A.2.12, we can expand $(x-c)$ in the denominator into power series in $x$, thus reducing the integral to a sum of A.2.13). Provided $n>2 p$ and so far we are only interested in terms of order $w^{r}$ and $w^{r} \ln w$ with $r \leq 2 p$, only relevant contributions would come from $a_{m}(n) w^{m}$ term in A.2.13) with $m<n$. Corresponding coefficients are known analytically, (A.2.14), and can be re-summed yielding,
$J_{n}(c)=-\sum_{m=0}^{2 p} \frac{(-16)^{m} \omega^{m} \Gamma\left(m+\frac{1}{2}\right) \Gamma(l-m){ }_{2} F_{1}\left(1, l-m ; l-m+\frac{1}{2} ; \frac{1}{c}\right)}{c \Gamma(m+1)}+\mathcal{O}\left(w^{2 p+1}\right)$.
Here $2 F_{1}$ is regularized hypergeometric function and this expression is only valid for $n>2 p$. To extend it to smaller $n$ we use the iterative relation, which follows from the integration by parts,

$$
\begin{equation*}
J_{n}=\frac{J_{n+1}-I_{n}}{c} . \tag{A.2.18}
\end{equation*}
$$

This completes technical preliminaries as now integral over $b_{1}$ cycle can be reduced to a number of integrals $J_{n}$ and their derivatives, so far we are only interested in terms of order $w^{r}$ with $r \leq 2 p$. Clearly, the approach above can be used to evaluate integrals over $b_{1}$ when there are more than two zones. In this case one would need to evaluate integrals

$$
\begin{equation*}
\int_{0}^{1} \frac{d x x^{n}}{\sqrt{x(1-x)(16 w+x)} \prod_{i=1}^{m-1}\left(x-c_{i}\right)}, \tag{A.2.19}
\end{equation*}
$$

where $m$ is the number of zones. This can be reduced to A.2.12 by noting

$$
\begin{equation*}
\prod_{i=1}^{m-1} \frac{1}{\left(x-c_{i}\right)}=\sum_{i=1}^{m-1} \frac{\alpha_{i}}{x-c_{i}} \tag{A.2.20}
\end{equation*}
$$

with the appropriate coefficients $\alpha_{i}$.
To evaluate the integral over $b_{2}$-cycle from $\lambda_{2}$ to $\lambda_{3}$ is more challenging because in the $\epsilon \rightarrow 0$ limit there are singularities appearing at both boundaries. There is a straightforward but complicated way. By appropriately changing variables and expanding in $\epsilon$ all terms except for $\sqrt{\left(\lambda-\lambda_{1}\right)\left(\lambda-\lambda_{2}\right)\left(\lambda-\lambda_{3}\right)\left(\lambda-\lambda_{4}\right)}$ we reduce the calculation to the integral

$$
\begin{equation*}
\int_{0}^{1} d x \frac{x^{n}}{\sqrt{x(1-x)(16 w+x)(1+16 u-x)}} \tag{A.2.21}
\end{equation*}
$$

for positive small $w, u$. The indefinite integral of this kind can be evaluated analytically. Then the definite integral above can be integrated by expanding it powers of $w, u$ (which both are of order $\epsilon$ ), and keeping terms up to order $2 p$. This is an involved exercise and instead one can use one of the following shortcuts.

In the particular case of two-zone potential, instead of evaluating integral over $b_{2}$, one can combine the integral over $b_{1}$ and $b_{2}$ such that the contour would enclose $\lambda_{0}, \ldots, \lambda_{3}$. Now one can deform the contour to go from $\lambda_{4}$ to infinity, if necessary accompanied by a circle at infinity. At this point integrand can be expanded in $\epsilon$ such that brunch-cut from $\lambda_{1}$ to $\lambda_{2}$ disappears, yielding pole singularities at $\lambda=\lambda_{0}+k^{2} / 4$. At this point corresponding integral can be rewritten as

$$
\begin{equation*}
\oint_{-\infty}^{-16 w} d x \frac{P(x)}{\sqrt{x(1-x)(x+16 w)}(x-c)^{r}} \tag{A.2.22}
\end{equation*}
$$

where $P(x)$ is some polynomial and $0 \leq c \leq 1$. We also emphasize that to render this integral finite, one may need to close the contour at infinity. This integral can be decomposed into a sum of integrals of the form

$$
\begin{equation*}
\oint_{-\infty}^{-16 w} d x \frac{x^{n}}{\sqrt{x(1-x)(x+16 w)}} \tag{A.2.23}
\end{equation*}
$$

and

$$
\begin{equation*}
\oint_{-\infty}^{-16 w} d x \frac{1}{\sqrt{x(1-x)(x+16 w)}(x-c)}, \tag{A.2.24}
\end{equation*}
$$

and its derivatives. First integral can be reduced to A.2.13 by deforming the contour to go from 0 to 1 . Last integral can be reduced to $J_{0}$ and $J_{1}$ with help of modular transformation mapping $\infty$ to $1,-16 w$ to 0 , and 0 to $-16 w$.

$$
\begin{equation*}
x \rightarrow \frac{x+16 w}{x-1} . \tag{A.2.25}
\end{equation*}
$$

This shortcut works for two-zone case, but with more zones present it is not applicable. Neveftheless there is a very simple trick which make evaluation of $b_{2}$ and other $b$-cycles unnesessary. Indeed, to satisfy 2.2 .9 and 2.2 .10 for all cycles, it is sufficient to satisfy 2.2 .9 for all cycles and 2.2 .10 for $b_{1}$ and also impose that the expansion A.2.1 A.2.6), and its generalizations for the case of more than two zones, is invariant under permutation of indexes and $k_{i}$ defined in (2.2.11). Say, for two zones we find

$$
\begin{align*}
& \lambda_{1}=\lambda_{0}+\frac{k^{2}}{4}-\epsilon_{1}+\frac{3 \epsilon_{1}^{2}}{k^{2}}+\frac{4 \epsilon_{2}^{2} k^{2}}{\ell^{2}\left(k^{2}-\ell^{2}\right)}+\mathcal{O}\left(\epsilon^{3}\right)  \tag{A.2.26}\\
& \lambda_{2}=\lambda_{0}+\frac{k^{2}}{4}+\epsilon_{1}  \tag{A.2.27}\\
& r_{1}=\lambda_{0}+\frac{k^{2}}{4}+\frac{\epsilon_{1}^{2}}{2 k^{2}}+\frac{2 \epsilon_{2}^{2} k^{2}}{\ell^{2}\left(k^{2}-\ell^{2}\right)}+\mathcal{O}\left(\epsilon^{3}\right) . \tag{A.2.28}
\end{align*}
$$

and $\lambda_{3,4}, r_{2}$ related to $\lambda_{1,2}, r_{1}$ by the exchange $\epsilon_{1} \leftrightarrow \epsilon_{2}$ and $k \leftrightarrow \ell$. The same logic with the permutation symmetry works for any number of zones.

Above we only explicitly wrote terms up to $\epsilon^{2}$, while evaluating all terms up to $\epsilon^{6}$. The simple form of $\lambda_{2}$ above is a parametrization choice. With this choice taking $\epsilon_{2}=0$ does not close the second zone. One can check that taking

$$
\begin{equation*}
\epsilon_{2}=-\frac{2 \ell^{2} \epsilon_{1}^{2}}{k^{2}\left(k^{2}-\ell^{2}\right)}+\ldots \tag{A.2.29}
\end{equation*}
$$

such that $\lambda_{4}=\lambda_{3}$ would make $I_{\ell}$ dicussed below vanish. Alternatively one could chooe $\epsilon_{i}$ to control the size of $\lambda_{2 i}-\lambda_{2 i-1}$, but with this choice both all $\lambda_{i}$ would depend on all $\epsilon_{i}$.

## Evaluation of $I_{k}, h$ and $Q_{2 n-1}$.

Evaluation of action variables $I_{k}$ as a pertubative series in $\epsilon_{i}$ is straightforward. It is an integral over $a$-cycle and therefore can be evaluated along the lines discussed above. The only difference, in comparision with the discussion in subsection A.2, is the term $\ln \lambda$, which needs to be expanded in powers of $\epsilon$ yielding polynomials in $x$ in the numerator of A.2.8,

$$
\begin{equation*}
I_{k}=\frac{2 \epsilon_{1}^{2}}{k\left(\lambda_{0}+k^{2} / 4\right)}+\mathcal{O}\left(\epsilon^{3}\right) \tag{A.2.30}
\end{equation*}
$$

Again, we only keep terms up to $\epsilon^{2}$ for simplicity.
Evaluation of $h$ is also straightforward. To that end one needs to calculate $p(0)$, given by an integral from 0 to $\lambda_{0}$. After expanding the integrand in powers of $\epsilon$ it
becomes the integral which can be evaluated in a closed form, yielding

$$
\begin{equation*}
h / 4=\lambda_{0}+\lambda_{0}\left(\frac{2 \epsilon_{1}^{2}}{k^{2}\left(\lambda_{0}+k^{2} / 4\right)}+\frac{2 \epsilon_{2}^{2}}{\ell^{2}\left(\lambda_{0}+\ell^{2} / 4\right)}\right)+\mathcal{O}\left(\epsilon^{3}\right) . \tag{A.2.31}
\end{equation*}
$$

Finally, evaluation of $Q_{2 n-1}$ for any given $n$ is also straightforward since $\lambda_{i}$ are known explicitly. As a result we obtain $I_{k}, h, Q_{2 n-1}$ as functions of $\lambda_{0}$ and $\epsilon_{i}$. One can then reverse-engineer coefficients in 2.2.1 such that it is satisfied.

## A. 3 Spectrum of $Q_{2 n-1}$ acting on primaries

In this appendix we outlined calculation of $\mathrm{Q}_{2 n}^{0}$ (2.3.6) following [51]. Starting from the Schrödinger equation 2.3 .2 , one introduces the following change of variables

$$
\begin{equation*}
\Psi(x)=E^{l(l-3 / 2) / 4 \alpha} w^{(l-3 / 2) / 4} y(w), \quad x=E^{\frac{1}{2 \alpha}} w^{\frac{1}{2 l}} \tag{A.3.1}
\end{equation*}
$$

such that (2.3.2) becomes

$$
\begin{equation*}
-\epsilon^{2} \partial_{w}^{2} y+Z(w) y=0, \quad Z(w)=w, \quad \epsilon=E^{-\frac{\alpha+1}{2 \alpha}} \tag{A.3.2}
\end{equation*}
$$

Taking $\epsilon$ as a formal small parameter this equation can be solved via WKB expansion,

$$
\begin{equation*}
y(w)=e^{\frac{1}{\epsilon} S(w)}, \quad-\epsilon S^{\prime \prime}-S^{2}+Z=0, \quad S(w)=\sum_{n=0}^{\infty} \epsilon^{n} S_{n} \tag{A.3.3}
\end{equation*}
$$

The resulting Riccati equation can be rewritten as the iterative relation to find $S_{n}^{\prime}$ with $S_{0}^{\prime}=-\sqrt{Z(w)}$. It is more convenient for what follows to make another change of variables $z=w^{\alpha /(l+1 / 2)}$ and introduce the polynomial ansatz

$$
\begin{equation*}
S_{n}^{\prime}=i \frac{\alpha}{2 l+1} z^{1-\frac{l+1 / 2}{\alpha}} \tilde{S}_{n}, \quad \tilde{S}_{n}=\sum_{k=0}^{n} i^{n} c_{k}^{(n)} z^{-k+(n-1)(1-1 / 2 \alpha)}(1-z)^{k-(3 n-1) / 2} \mathrm{~A} \tag{A.3.4}
\end{equation*}
$$

The Ricatti equation rewritten in terms of $c_{k}^{(n)}$ gives rise to (2.3.4), which can be used together with 2.3.5 , to iteratively find $c_{k}^{(n)}$. The first few $c_{k}^{(n)}$ read

$$
\begin{align*}
& c_{0}^{(2)}=\frac{5}{8} \alpha, \quad c_{1}^{(2)}=\frac{1}{4}(2 \alpha-1), \quad c_{2}^{(2)}=-\frac{1}{8 \alpha}\left(4 u^{2} \alpha^{2}-1\right) .  \tag{A.3.5}\\
& c_{0}^{(3)}=-\frac{15 \alpha^{2}}{8}, \quad c_{1}^{(3)}=-\frac{9 \alpha^{2}}{4}+\frac{9 \alpha}{8}, \quad c_{2}^{(3)}=\frac{1}{2} \alpha^{2}\left(u^{2}-1\right)+\frac{3 \alpha}{4}-\frac{3}{8}, \\
& c_{3}^{(3)}=-\frac{1}{8 \alpha}\left(4 u^{2} \alpha^{2}-1\right) . \tag{A.3.6}
\end{align*}
$$

To obtain $\mathrm{Q}_{2 n-1}^{0}$ one needs to integrate $\tilde{S}_{2 n}(w(z))$ over a Pochhammer contour $\gamma_{P}$,

$$
\begin{align*}
\mathrm{Q}_{2 n-1}^{0} & =(-1)^{n} \frac{\Gamma\left(\frac{3}{2}-n-\frac{2 n-1}{2 \alpha}\right)}{\sqrt{\pi} \Gamma\left(1-\frac{2 n-1}{2 \alpha}\right)} \frac{(2 n-1) \Gamma(n+1)}{4^{n}(\alpha+1)^{n}} \check{I}_{2 n-1}(\alpha, \hat{l})  \tag{A.3.7}\\
\check{I}_{2 n-1} & =\frac{1}{2\left(1-e^{\frac{-i \pi(2 n-1)}{\alpha}}\right)} \int_{\gamma_{P}} d z \tilde{S}_{2 n}(z) \tag{A.3.8}
\end{align*}
$$

This integral can be evaluated using,

$$
\begin{equation*}
\left(1-e^{2 \pi i a}\right)\left(1-e^{2 \pi i b}\right) B(a, b)=\int_{\gamma_{P}} d z z^{a-1}(1-z)^{b-1} \tag{A.3.9}
\end{equation*}
$$

where $B(a . b)$ is the Euler beta function. Combining everything together yields (2.3.6).

Evaluating $\mathbf{Q}_{2 n-1}^{0}$ explicitly, using computer algebra to solve for $c_{k}^{(n)}$ iteratively, for small and moderate $n$ is an easy task. To obtain $1 / c$ expansion of $\lambda_{2 n-1}^{0}$ for arbitrary $n$ requires knowing corresponding $c_{k}^{(n)}$ in $1 / c$ expansion, i.e. in the limit of large $\alpha$. This proved to be a difficult task. We obtained first three non-trivial terms of $\lambda_{2 n-1}^{0}$ in $1 / \tilde{c}$ expansion (2.3.7), with the first two terms 2.3.8 2.3.9) in closed analytical form. Functions $y_{i}$ and $\zeta_{i}$ there are defined as follows

$$
\begin{align*}
y_{1}(j) & =\sum_{\ell=0}^{j} \frac{1}{2 \ell+1}  \tag{A.3.10}\\
y_{2}(j) & =\sum_{\ell=0}^{j} \frac{1}{(2 \ell+1)^{2}},  \tag{A.3.11}\\
\zeta_{2}(j) & =\sum_{j_{1}+j_{2}=j} \zeta\left(-2 j_{1}-1\right) \zeta\left(-2 j_{2}-1\right)  \tag{A.3.12}\\
\zeta_{3}(j) & =\sum_{j_{1}+j_{2}+j_{3}=j} \zeta\left(-2 j_{1}-1\right) \zeta\left(-2 j_{2}-1\right) \zeta\left(-2 j_{3}-1\right) \tag{A.3.13}
\end{align*}
$$

where sum goes only over non-negative $j_{1}, j_{2}, j_{3}$. Third term (2.3.10 was fixed up to one coefficient $p_{j}$, with the first several values for $0 \leq j \leq 17$ given below for $p_{j}=$

$$
\begin{array}{r}
\left(-\frac{31}{224}, \frac{103}{576},-\frac{7883}{21120}, \frac{868487}{748800},-\frac{505639}{100800}, \frac{394694297}{13708800},-\frac{68117454019}{321753600}, \frac{4929720750223}{2540160000},\right. \\
-\frac{199232137825687}{9180864000}, \frac{48745030162337923}{167650560000},-\frac{618684597383137}{134534400}, \frac{7442737871872435019}{87783696000} \\
-\frac{1420749127340184137621}{788237049600}, \frac{46636700018927407368821}{10655512448000},-\frac{198277953077778046100039}{164670105600}, \\
\frac{21869843836862719834306038469}{587058612940800},-\frac{31428771773709445918185916879}{24404109649920}, \\
\left.\frac{4187283526052269558397574465940213}{84663488093184000}, \ldots\right) .
\end{array}
$$

## A. 4 Conventional Genus 2 Transformations

For convenience, we provide the more conventional set of $\operatorname{Sp}(4, \mathbb{Z})$ transformations in terms of their action on our code variables. These are given by six elements which act on the period matrix according to

$$
\begin{gather*}
T_{1}: \Omega_{11} \mapsto \Omega_{11}+1, \quad T_{2}: \Omega_{22} \mapsto \Omega_{22}+1,  \tag{A.4.1}\\
S_{1}:\left\{\begin{array}{lll}
\Omega_{11} & \mapsto & -1 / \Omega_{11}, \\
\Omega_{12} & \mapsto & -\Omega_{12} / \Omega_{11}, \\
\Omega_{22} & \mapsto & \Omega_{22}-\Omega_{12}^{2} / \Omega_{11}
\end{array} \quad S_{2}:\left\{\begin{array}{lll}
\Omega_{22} & \mapsto & -1 / \Omega_{22} \\
\Omega_{12} & \mapsto & -\Omega_{12} / \Omega_{22} \\
\Omega_{11} & \mapsto & \Omega_{11}-\Omega_{12}^{2} / \Omega_{22}, \\
V: \Omega_{12} & \mapsto & -\Omega_{12} \\
U: \Omega_{12} & \mapsto & \Omega_{12}+1 .
\end{array}\right.\right.
\end{gather*}
$$

$\left\langle S_{1}, T_{1}\right\rangle$ and $\left\langle S_{2}, T_{2}\right\rangle$ are the $\operatorname{Sp}(2, Z)$ subgroups which generate modular transformations for the two tori. These transformations lead to simple linear transformations on the code variables:

$$
\begin{align*}
T_{1}: & x_{0} \mapsto i x_{0}, \quad x_{2} \mapsto i x_{2}, \quad x_{1} \mapsto x_{1}, \quad x_{3} \mapsto x_{3} ; \\
T_{2}: & x_{0} \mapsto x_{0}, \quad x_{2} \mapsto x_{2}, \quad x_{1} \mapsto i x_{1}, \quad x_{3} \mapsto i x_{3} ;  \tag{A.4.4}\\
U: & x_{0} \mapsto-x_{0}, \quad x_{2} \mapsto x_{2}, \quad x_{1} \mapsto x_{1}, \quad x_{3} \mapsto x_{3} ;
\end{align*}
$$

and

$$
\begin{array}{llll}
S_{1}: & x_{0} \mapsto \frac{x_{2}-x_{0}}{\sqrt{2}}, & x_{2} \mapsto \frac{x_{2}+x_{0}}{\sqrt{2}}, & x_{1} \mapsto \frac{x_{3}-x_{1}}{\sqrt{2}},
\end{array} x_{3} \mapsto \frac{x_{3}+x_{1}}{\sqrt{2}} ; ~ 子 \begin{array}{lll}
\sqrt{2} \\
S_{2}: & x_{0} \mapsto \frac{x_{1}-x_{0}}{\sqrt{2}}, & x_{1} \mapsto \frac{x_{1}+x_{0}}{\sqrt{2}},  \tag{A.4.6}\\
x_{2} \mapsto \frac{x_{3}-x_{2}}{\sqrt{2}}, & x_{3} \mapsto \frac{x_{3}+x_{2}}{\sqrt{2}} .
\end{array}
$$

## A. 5 Siegel Modular Forms

## Forms in the code basis

We consider Siegel modular forms for the group $\operatorname{Sp}(2 g, \mathbb{Z})$. First recall from section 5.2 that we have

$$
\begin{align*}
G_{4} & \cong x_{0}^{8}+14 x_{0}^{4} x_{1}^{4}+x_{1}^{8} \\
G_{6} & \cong x_{0}^{12}-33 x_{0}^{8} x_{1}^{4}-33 x_{0}^{4} x_{1}^{8}+x_{1}^{12}  \tag{A.5.1}\\
\Delta_{12} & \cong \frac{1}{16} x_{0}^{4} x_{1}^{4}\left(x_{0}^{4}-x_{1}^{4}\right)^{4}
\end{align*}
$$

Note also the alternative ways (4.3.49) of writing the forms $G_{4}$ and $\Delta_{12}$. In the code basis, the modular forms at genus 2 are given by

$$
\begin{align*}
E_{4} \cong & x_{0}^{8}+14 x_{0}^{4} x_{2}^{4}+14 x_{0}^{4} x_{1}^{4}+14 x_{0}^{4} x_{3}^{4}+168 x_{0}^{2} x_{2}^{2} x_{1}^{2} x_{3}^{2}+x_{2}^{8}+14 x_{2}^{4} x_{1}^{4}+14 x_{2}^{4} x_{3}^{4}+x_{1}^{8} \\
& +14 x_{1}^{4} x_{3}^{4}+x_{3}^{8}  \tag{A.5.2}\\
E_{6} \cong & x_{0}^{12}-33 x_{0}^{8} x_{2}^{4}-33 x_{0}^{8} x_{1}^{4}-33 x_{0}^{8} x_{3}^{4}+792 x_{0}^{6} x_{2}^{2} x_{1}^{2} x_{3}^{2}-33 x_{0}^{4} x_{2}^{8}+330 x_{0}^{4} x_{2}^{4} x_{1}^{4} \\
& +330 x_{0}^{4} x_{2}^{4} x_{3}^{4}-33 x_{0}^{4} x_{1}^{8}+330 x_{0}^{4} x_{1}^{4} x_{3}^{4}-33 x_{0}^{4} x_{3}^{8}+792 x_{0}^{2} x_{2}^{6} x_{1}^{2} x_{3}^{2} \\
& +792 x_{0}^{2} x_{2}^{2} x_{1}^{6} x_{3}^{2}+792 x_{0}^{2} x_{2}^{2} x_{1}^{2} x_{3}^{6}+x_{2}^{12}-33 x_{2}^{8} x_{1}^{4}-33 x_{2}^{8} x_{3}^{4}-33 x_{2}^{4} x_{1}^{8}+330 x_{2}^{4} x_{1}^{4} x_{3}^{4} \\
& -33 x_{2}^{4} x_{3}^{8}+x_{1}^{12}-33 x_{1}^{8} x_{3}^{4}-33 x_{1}^{4} x_{3}^{8}+x_{3}^{12}, \tag{A.5.3}
\end{align*}
$$

$$
\begin{align*}
\chi_{10} \cong \frac{1}{256}( & x_{0}^{14} x_{2}^{2} x_{1}^{2} x_{3}^{2}-x_{0}^{12}\left(x_{2}^{4}\left(x_{1}^{4}+x_{3}^{4}\right)+x_{1}^{4} x_{3}^{4}\right)-x_{0}^{10} x_{2}^{2} x_{1}^{2} x_{3}^{2}\left(x_{2}^{4}+x_{1}^{4}+x_{3}^{4}\right) \\
& +x_{0}^{8}\left(2 x_{2}^{8}\left(x_{1}^{4}+x_{3}^{4}\right)+x_{2}^{4}\left(2 x_{1}^{8}+13 x_{1}^{4} x_{3}^{4}+2 x_{3}^{8}\right)+2 x_{1}^{4} x_{3}^{4}\left(x_{1}^{4}+x_{3}^{4}\right)\right) \\
& -x_{0}^{6} x_{2}^{2} x_{1}^{2} x_{3}^{2}\left(x_{2}^{8}+14 x_{2}^{4}\left(x_{1}^{4}+x_{3}^{4}\right)+x_{1}^{8}+14 x_{1}^{4} x_{3}^{4}+x_{3}^{8}\right) \\
& -x_{0}^{4}\left(x_{2}^{12}\left(x_{1}^{4}+x_{3}^{4}\right)-x_{2}^{8}\left(2 x_{1}^{8}+13 x_{1}^{4} x_{3}^{4}+2 x_{3}^{8}\right)+x_{1}^{4} x_{3}^{4}\left(x_{1}^{4}-x_{3}^{4}\right)^{2}\right. \\
& \left.\quad+x_{2}^{4}\left(x_{1}^{12}-13 x_{1}^{8} x_{3}^{4}-13 x_{1}^{4} x_{3}^{8}+x_{3}^{12}\right)\right)+x_{0}^{2} x_{2}^{2} x_{1}^{2} x_{3}^{2}\left(x_{2}^{12}-x_{2}^{8}\left(x_{1}^{4}+x_{3}^{4}\right)\right. \\
& \left.-x_{2}^{4}\left(x_{1}^{8}+14 x_{1}^{4} x_{3}^{4}+x_{3}^{8}\right)+\left(x_{1}^{4}-x_{3}^{4}\right)^{2}\left(x_{1}^{4}+x_{3}^{4}\right)\right) \\
& \left.-x_{2}^{4} x_{1}^{4} x_{3}^{4}\left(x_{2}^{8}-2 x_{2}^{4}\left(x_{1}^{4}+x_{3}^{4}\right)+\left(x_{1}^{4}-x_{3}^{4}\right)^{2}\right)\right), \tag{A.5.4}
\end{align*}
$$

$$
\begin{align*}
\chi_{12} \cong & \frac{1}{768}\left(x_{0}^{18} x_{2}^{2} x_{1}^{2} x_{3}^{2}+2 x_{0}^{16}\left(x_{2}^{4}\left(x_{1}^{4}+x_{3}^{4}\right)+x_{1}^{4} x_{3}^{4}\right)-12 x_{0}^{14} x_{2}^{2} x_{1}^{2} x_{3}^{2}\left(x_{2}^{4}+x_{1}^{4}+x_{3}^{4}\right)\right. \\
& -2 x_{0}^{12}\left(x_{2}^{8}\left(x_{1}^{4}+x_{3}^{4}\right)+x_{2}^{4}\left(x_{1}^{8}-38 x_{1}^{4} x_{3}^{4}+x_{3}^{8}\right)+x_{1}^{4} x_{3}^{4}\left(x_{1}^{4}+x_{3}^{4}\right)\right) \\
& +2 x_{0}^{10} x_{2}^{2} x_{1}^{2} x_{3}^{2}\left(11 x_{2}^{8}-26 x_{2}^{4}\left(x_{1}^{4}+x_{3}^{4}\right)+11 x_{1}^{8}-26 x_{1}^{4} x_{3}^{4}+11 x_{3}^{8}\right) \\
& -2 x_{0}^{8}\left(x_{2}^{12}\left(x_{1}^{4}+x_{3}^{4}\right)-18 x_{2}^{8}\left(x_{1}^{8}+x_{1}^{4} x_{3}^{4}+x_{3}^{8}\right)+x_{2}^{4}\left(x_{1}^{12}-18 x_{1}^{8} x_{3}^{4}-18 x_{1}^{4} x_{3}^{8}+x_{3}^{12}\right)\right. \\
& \left.+x_{1}^{4} x_{3}^{4}\left(x_{1}^{8}-18 x_{1}^{4} x_{3}^{4}+x_{3}^{8}\right)\right)-4 x_{0}^{6} x_{2}^{2} x_{1}^{2} x_{3}^{2}\left(3 x_{2}^{12}+13 x_{2}^{8}\left(x_{1}^{4}+x_{3}^{4}\right)\right. \\
& \left.+x_{2}^{4}\left(13 x_{1}^{8}+2 x_{1}^{4} x_{3}^{4}+13 x_{3}^{8}\right)+3 x_{1}^{12}+13 x_{1}^{8} x_{3}^{4}+13 x_{1}^{4} x_{3}^{8}+3 x_{3}^{12}\right) \\
& +2 x_{0}^{4}\left(x_{2}^{16}\left(x_{1}^{4}+x_{3}^{4}\right)-x_{2}^{12}\left(x_{1}^{8}-38 x_{1}^{4} x_{3}^{4}+x_{3}^{8}\right)-x_{2}^{8}\left(x_{1}^{12}-18 x_{1}^{8} x_{3}^{4}-18 x_{1}^{4} x_{3}^{8}+x_{3}^{12}\right)\right. \\
& \left.+x_{2}^{4}\left(x_{1}^{16}+38 x_{1}^{12} x_{3}^{4}+18 x_{1}^{8} x_{3}^{8}+38 x_{1}^{4} x_{3}^{12}+x_{3}^{16}\right)+x_{1}^{4} x_{3}^{4}\left(x_{1}^{4}-x_{3}^{4}\right)^{2}\left(x_{1}^{4}+x_{3}^{4}\right)\right) \\
& +x_{0}^{2} x_{2}^{2} x_{1}^{2} x_{3}^{2}\left(x_{2}^{16}-12 x_{2}^{12}\left(x_{1}^{4}+x_{3}^{4}\right)+x_{2}^{8}\left(22 x_{1}^{8}-52 x_{1}^{4} x_{3}^{4}+22 x_{3}^{8}\right)\right. \\
& \left.-4 x_{2}^{4}\left(3 x_{1}^{12}+13 x_{1}^{8} x_{3}^{4}+13 x_{1}^{4} x_{3}^{8}+3 x_{3}^{12}\right)+\left(x_{1}^{4}-x_{3}^{4}\right)^{2}\left(x_{1}^{8}-10 x_{1}^{4} x_{3}^{4}+x_{3}^{8}\right)\right) \\
& \left.+2 x_{2}^{4} x_{1}^{4} x_{3}^{4}\left(x_{2}^{12}-x_{2}^{8}\left(x_{1}^{4}+x_{3}^{4}\right)-x_{2}^{4}\left(x_{1}^{8}-18 x_{1}^{4} x_{3}^{4}+x_{3}^{8}\right)+\left(x_{1}^{4}-x_{3}^{4}\right)^{2}\left(x_{1}^{4}+x_{3}^{4}\right)\right)\right) . \tag{A.5.5}
\end{align*}
$$

The normalization of $\chi_{10}$ has been fixed by matching with the alternative formula

$$
\begin{equation*}
\chi_{10}=2^{-14} \prod_{\mathbf{A}} \theta[\mathbf{A}](\Omega)^{2} \tag{A.5.6}
\end{equation*}
$$

where the product is over the 10 vectors of the form $\left(a_{1}, a_{2}, b_{1}, b_{2}\right)^{T}, a_{i}, b_{i} \in\{0,1 / 2\}$ that give a non-zero theta constant, see comments after 4.3.59).

At genus 3 we find (in the notation of [129]

$$
\begin{align*}
\alpha_{4} \cong(8)+ & 14(4,4)+168(2,2,2,2)+1344(1,1,1,1,1,1,1,1),  \tag{A.5.7}\\
j_{8}=1344[ & 2(9,1,1,1,1,1,1,1)+(8,0,0,0,2,2,2,2)+(6,2,2,2,4,0,0,0) \\
& +4(5,5,1,1,1,1,1,1)-(4,4,4,0,4,0,0,0)-2(4,4,0,0,2,2,2,2)  \tag{A.5.8}\\
& +16(3,3,3,3,1,1,1,1)-72(2,2,2,2,2,2,2,2)]
\end{align*}
$$

## Holomorphic Eisenstein series and modular forms

Here we will provide a few details on the holomorphic Eisenstein series which might be useful for computating the modular forms. At general genus, can be defined as

$$
E_{k}^{(g)}=\sum_{C, D}(\operatorname{det}(C \Omega+D))^{-k}, \quad\left(\begin{array}{ll}
A & B  \tag{A.5.9}\\
C & D
\end{array}\right) \in \operatorname{Sp}(2 g, \mathbb{Z})
$$

however there are different normalization conventions used in the literature.
For genus 1 and 2, the holomorphic Eisenstein series generates the entire ring of modular forms [188]. For genus $g \geqslant 3$ this is no longer the case, since there are modular forms which cannot be written as a polynomial of the Eisenstein series 133]. For the genus $g \leqslant 3$, the ring of modular forms has the following generators,

$$
\begin{array}{ll}
g=1 & G_{4}, G_{6}, \\
g=2 & E_{4}, E_{6}, \chi_{10}, \chi_{12}, \\
g=3 & \alpha_{4}, \alpha_{6}, \alpha_{10}, \alpha_{12}, \alpha_{12}^{\prime}, \beta_{14}, \alpha_{16}, \beta_{16}, \chi_{18}, \alpha_{18},  \tag{A.5.12}\\
& \alpha_{20}, \gamma_{20}, \beta_{22}, \beta_{22}^{\prime}, \alpha_{24}, \gamma_{24}, \gamma_{26}, \chi_{28}, \alpha_{30},
\end{array}
$$

where there are 19 generators at genus $g=3[134$. 133] originally gave 34, however, [134] show that there are 19 generators among these 34 forms forms. The case at genus $g=2$ was demonstrated in [189], and related to codes in [126] (using references in $\sqrt{129} \mid$ ). The general theorem, that the (pseudo)reflections on codes generate the ring of modular form was done in 129. The set of generators at genus 4 is not known 123, 134.

## Degree one

At degree 1 we follow 108 and define

$$
\begin{equation*}
G_{k}\left(e^{2 \pi i \tau}\right)=\frac{1}{2 \zeta(k)} \sum_{(c, d) \in \mathbb{Z}^{2} \backslash\{(0,0)\}} \frac{1}{(c \tau+d)^{k}} \tag{A.5.13}
\end{equation*}
$$

This evaluates to

$$
\begin{equation*}
G_{k}(q)=1+\frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^{n} \tag{A.5.14}
\end{equation*}
$$

where $\sigma_{k}(n)$ is the divisor sum function defined by

$$
\begin{equation*}
\sigma_{k}(n)=\sum_{d \mid n} d^{k} \tag{A.5.15}
\end{equation*}
$$

The basis is generated by two elements, $G_{4}$ and $G_{6}$. In terms of these,

$$
\begin{align*}
G_{8} & =G_{4}^{2}  \tag{A.5.16}\\
G_{10} & =G_{4} G_{6}  \tag{A.5.17}\\
G_{12} & =\frac{441}{691} G_{4}^{3}+\frac{250}{691} G_{6}^{2}  \tag{A.5.18}\\
G_{14} & =G_{4}^{2} G_{6}  \tag{A.5.19}\\
G_{16} & =\frac{1617}{3617} G_{4}^{4}+\frac{2000}{3617} G_{4} G_{6}^{2}  \tag{A.5.20}\\
G_{18} & =\frac{38367}{43867} G_{4}^{3} G_{6}+\frac{5500}{43867} G_{6}^{3} \tag{A.5.21}
\end{align*}
$$

and

$$
\begin{equation*}
\Delta_{12}=\frac{1}{1728}\left(G_{4}^{3}-G_{6}^{2}\right) \tag{A.5.22}
\end{equation*}
$$

Notice that there is a direct relation

$$
\begin{equation*}
\Delta_{12}\left(e^{2 \pi i \tau}\right)=\eta(\tau)^{24} \tag{A.5.23}
\end{equation*}
$$

where $\eta(\tau)$ is the Dedekind eta function defined in 4.2.12).

## Degree two

The ring of degree two modular forms are generated by four elements, $E_{4}, E_{6}, \chi_{10}$, and $\chi_{12}$, which are given in terms of code variables in equations A.5.2-A.5.5). We may write the holomorphic Eisenstein series in terms of these:

$$
\begin{align*}
E_{8}= & E_{4}^{2},  \tag{A.5.24}\\
E_{10}= & E_{4} E_{6}-\frac{9231667200}{43867} \chi_{10},  \tag{A.5.25}\\
E_{12}= & \frac{441}{691} E_{4}^{3}+\frac{250}{691} E_{6}^{2}-\frac{36980665344000}{53678953} \chi_{12},  \tag{A.5.26}\\
E_{14}= & E_{4}^{2} E_{6}-\frac{187420262400}{657931} E_{4} \chi_{10},  \tag{A.5.27}\\
E_{16}= & \frac{1617}{3617} E_{4}^{4}+\frac{2000}{3617} E_{4} E_{6}^{2}-\frac{4600443838734729216000}{6232699579062017} E_{4} \chi_{12} \\
& -\frac{473779992577941504000}{6232699579062017} E_{6} \chi_{10},  \tag{A.5.28}\\
E_{18}= & \frac{38367}{43867} E_{4}^{3} E_{6}+\frac{5500}{43867} E_{6}^{3}-\frac{1688190624014720716800}{6651496075469717} E_{4}^{2} \chi_{10} \\
& -\frac{2177976079791654912000}{6651496075469717} E_{6} \chi_{12} . \tag{A.5.29}
\end{align*}
$$

We shall briefly describe the implementation of the holomorphic Eisenstein series, which was also reviewed in [108] following [190]. These are given in terms of the moduli $\Omega$ by

$$
\begin{equation*}
E_{k}(\Omega)=\sum_{m, n=0}^{\infty} \sum_{p^{2} \leqslant 4 m n} a_{k}(m, n, p) q^{n} r^{p} s^{m}, \tag{A.5.30}
\end{equation*}
$$

where $q, r$, and $s$ are defined in (4.3.8), the sum over $p$ includes all integers (including negative ones) satisfying $p^{2} \leqslant 4 m n$, and the Fourier coefficients $a_{k}$ are given by

$$
\begin{equation*}
a_{k}(m, n, p)=\frac{2}{\zeta(3-2 k) \zeta(1-k)} \sum_{d \mid(n, m, p)} d^{k-1} H\left(k-1, \frac{4 n m-p^{2}}{d^{2}}\right) \tag{A.5.31}
\end{equation*}
$$

where the sum is over the divisors $d$ of the GCD of $n, m$, and $p . H(r, N)$ is the Cohen function, defined in 191 as

$$
H(r, N)= \begin{cases}\sum_{d^{2} \mid N} h\left(r, N / d^{2}\right) & (-1)^{r} N \equiv 0 \text { or } 1(\bmod 4)  \tag{A.5.32}\\ \zeta(1-2 r) & N=0 \\ 0 & \text { otherwise }\end{cases}
$$

The sum in the first term is over all squares $d^{2}$ which divide $N$. The function $h(r, N)$ is then defined by
$h(r, N)= \begin{cases}(-1)^{\lfloor r / 2\rfloor}(r-1)!N^{r-1 / 2} 2^{1-r} \pi^{-r} L\left(r, \chi_{\left.(-1)^{r} N\right)},\right. & (-1)^{r} N \equiv 0 \text { or } 1(\bmod 4), \\ 0, & (-1)^{r} N \equiv 2 \text { or } 3(\bmod 4) .\end{cases}$

Finally, the character $\chi_{D}(d)$ can be written using the Kronecker symbol via

$$
\begin{equation*}
\chi_{D}(d)=\left(\frac{D}{d}\right) \tag{A.5.34}
\end{equation*}
$$

which allows us to compute the $L$-function

$$
\begin{equation*}
L\left(r, \chi_{(-1)^{r} N}\right)=\sum_{n=1}^{\infty}\left(\frac{(-1)^{r} N}{n}\right) n^{-r} \tag{A.5.35}
\end{equation*}
$$

The result of this chain of computations is an algorithm for computing the Eisenstein series $E_{k}$. In practice, we could not compute the infinite sums associated with the $L$-functions exactly, so we computed them to large values of $n$ and determined the formulas A.5.24 A.5.29 numerically.

## A. 6 Generators for invariant ring at genus 2

In section 5.4 we analyzed the generators for ring $R_{g}$ of invariant genus 2 polynomials. It is convenient to organize the generators by degree. Consider the set $Q_{n}^{[2]}$ of generators of degree $n$, then the ring is given by

$$
\begin{equation*}
R_{2}=\left\langle Q_{1}^{[2]} \cup Q_{2}^{[2]} \cup \cdots \cup Q_{11}^{[2]}\right\rangle . \tag{A.6.1}
\end{equation*}
$$

We represent each generator by an undirected graph, and show the generators in figures A.1 A. 9


Figure A.1: $Q_{1}^{[2]}$ (left), $Q_{2}^{[2]}$ (middle), and $Q_{3}^{[2]}$ (right).


Figure A.2: $Q_{4}^{[2]}$


Figure A.3: $Q_{5}^{[2]}$


Figure A.4: $Q_{6}^{[2]}$


Figure A.5: $Q_{7}^{[2]}$


Figure A.6: $Q_{8}^{[2]}$


Figure A.7: $Q_{9}^{[2]}$


Figure A.8: $Q_{10}^{[2]}$


Figure A.9: $Q_{11}^{[2]}$

## A. 7 Genus 2 partition functions in enumerator polynomial form

Here we include a few expressions too big to conveniently include in the main text.
$n=3$ theories
First let us provide the genus 2 partition functions of each of the actual $n=3$ code theories.

$$
\left.\left.\begin{array}{rl}
W_{1}^{3}= & \left(x_{0}+x_{2}\right)^{3} \\
\left(W_{1}^{(2)}\right)^{3}= & y_{0}^{3} \tag{A.7.1}
\end{array}\right)+3 y_{4} y_{0}^{2}+3 y_{5} y_{0}^{2}+3 y_{6} y_{0}^{2}+3 y_{4}^{2} y_{0}+3 y_{5}^{2} y_{0}+3 y_{6}^{2} y_{0}\right)
$$

$$
\left.\begin{array}{rl}
W_{1} W_{2}= & \left(x_{0}+x_{2}\right)\left(x_{0}^{2}+x_{1}^{2}+2 x_{2}^{2}\right) \\
W_{1}^{(2)} W_{2}^{(2)}= & y_{0}^{3}
\end{array}\right)
$$

$$
W_{3}=x_{0}^{3}+3 x_{1}^{2} x_{2}+3 x_{0} x_{2}^{2}+x_{2}^{3}
$$

$$
W_{3}^{(2)}=y_{0}^{3}+3 y_{4}^{2} y_{0}+3 y_{5}^{2} y_{0}+3 y_{6}^{2} y_{0}+y_{4}^{3}+y_{5}^{3}+y_{6}^{3}+3 y_{5} y_{7}^{2}+3 y_{6} y_{7}^{2}+3 y_{4} y_{8}^{2}
$$

$$
\begin{equation*}
+3 y_{6} y_{8}^{2}+3 y_{4} y_{9}^{2}+3 y_{5} y_{9}^{2}+3 y_{1}^{2} y_{4}+3 y_{2}^{2} y_{5}+3 y_{3}^{2} y_{6} \tag{A.7.3}
\end{equation*}
$$

$$
+6 y_{4} y_{5} y_{6}+6 y_{3} y_{7} y_{8}+6 y_{2} y_{7} y_{9}+6 y_{1} y_{8} y_{9}
$$

$$
\tilde{W}_{3}=x_{0}^{3}+3 x_{0} x_{1}^{2}+4 x_{2}^{3}
$$

$$
\begin{equation*}
\tilde{W}_{3}^{(2)}=y_{0}^{3}+3 y_{1}^{2} y_{0}+3 y_{2}^{2} y_{0}+3 y_{3}^{2} y_{0}+4 y_{4}^{3}+4 y_{5}^{3}+4 y_{6}^{3}+12 y_{4} y_{7}^{2} \tag{A.7.4}
\end{equation*}
$$

$$
+12 y_{5} y_{8}^{2}+12 y_{6} y_{9}^{2}+6 y_{1} y_{2} y_{3}
$$

## Chiral theories

The genus 1 partition function for the extremal $n=24$ chiral theory, also known as the Monster CFT, is

$$
\begin{equation*}
\tilde{W}_{24}^{(1)}\left(x_{0}, x_{1}\right)=x_{0}^{24}-\frac{9}{2} x_{1}^{4} x_{0}^{20}+777 x_{1}^{8} x_{0}^{16}+2549 x_{1}^{12} x_{0}^{12}+777 x_{1}^{16} x_{0}^{8}-\frac{9}{2} x_{1}^{20} x_{0}^{4}+x_{1}^{24} \tag{A.7.5}
\end{equation*}
$$

while its genus 2 partition function is

$$
\begin{align*}
& \tilde{W}_{24}^{(2)}\left(x_{0}, x_{1}, x_{2}, x_{3}\right)= \\
& \frac{1}{4}\left(4 x_{0}^{24}-18\left(x_{1}^{4}+x_{2}^{4}+x_{3}^{4}\right) x_{0}^{20}+63 x_{1}^{2} x_{2}^{2} x_{3}^{2} x_{0}^{18}\right. \\
& +12\left(259 x_{1}^{8}+3\left(x_{2}^{4}+x_{3}^{4}\right) x_{1}^{4}+259 x_{2}^{8}+259 x_{3}^{8}+3 x_{2}^{4} x_{3}^{4}\right) x_{0}^{16}-9396 x_{1}^{2} x_{2}^{2} x_{3}^{2}\left(x_{1}^{4}+x_{2}^{4}+x_{3}^{4}\right) x_{0}^{14} \\
& +2\left(5098 x_{1}^{12}-2313\left(x_{2}^{4}+x_{3}^{4}\right) x_{1}^{8}-3\left(771 x_{2}^{8}-152918 x_{3}^{4} x_{2}^{4}+771 x_{3}^{8}\right) x_{1}^{4}+5098 x_{2}^{12}+5098 x_{3}^{12}\right. \\
& \left.-2313 x_{2}^{4} x_{3}^{8}-2313 x_{2}^{8} x_{3}^{4}\right) x_{0}^{12}-6 x_{1}^{2} x_{2}^{2} x_{3}^{2}\left(6105 x_{1}^{8}-233438\left(x_{2}^{4}+x_{3}^{4}\right) x_{1}^{4}+6105 x_{2}^{8}+6105 x_{3}^{8}\right. \\
& \left.-233438 x_{2}^{4} x_{3}^{4}\right) x_{0}^{10}+6\left(518 x_{1}^{16}-771\left(x_{2}^{4}+x_{3}^{4}\right) x_{1}^{12}+6\left(2863 x_{2}^{8}+142508 x_{3}^{4} x_{2}^{4}+2863 x_{3}^{8}\right) x_{1}^{8}\right. \\
& +\left(-771 x_{2}^{12}+855048 x_{3}^{4} x_{2}^{8}+855048 x_{3}^{8} x_{2}^{4}-771 x_{3}^{12}\right) x_{1}^{4}+518 x_{2}^{16}+518 x_{3}^{16}-771 x_{2}^{4} x_{3}^{12} \\
& \left.+17178 x_{2}^{8} x_{3}^{8}-771 x_{2}^{12} x_{3}^{4}\right) x_{0}^{8}-12 x_{1}^{2} x_{2}^{2} x_{3}^{2}\left(783 x_{1}^{12}-116719\left(x_{2}^{4}+x_{3}^{4}\right) x_{1}^{8}-\left(116719 x_{2}^{8}\right.\right. \\
& \left.\left.+1315158 x_{3}^{4} x_{2}^{4}+116719 x_{3}^{8}\right) x_{1}^{4}+783 x_{2}^{12}+783 x_{3}^{12}-116719 x_{2}^{4} x_{3}^{8}-116719 x_{2}^{8} x_{3}^{4}\right) x_{0}^{6}-6\left(3 x_{1}^{20}\right. \\
& -6\left(x_{2}^{4}+x_{3}^{4}\right) x_{1}^{16}+\left(771 x_{2}^{8}-152918 x_{3}^{4} x_{2}^{4}+771 x_{3}^{8}\right) x_{1}^{12}+\left(771 x_{2}^{12}-855048 x_{3}^{4} x_{2}^{8}\right. \\
& \left.+771 x_{3}^{12}\right) x_{1}^{8}-2\left(3 x_{2}^{16}+76459 x_{3}^{4} x_{2}^{12}+427524 x_{3}^{8} x_{2}^{8}+76459 x_{3}^{12} x_{2}^{4}+3 x_{3}^{16}\right) x_{1}^{4}+3\left(x_{2}^{20}-2 x_{3}^{4} x_{2}^{16}\right. \\
& \left.\left.+257 x_{3}^{8} x_{2}^{12}+257 x_{3}^{12} x_{2}^{8}-2 x_{3}^{16} x_{2}^{4}+x_{3}^{20}\right)\right) x_{0}^{4}+3 x_{1}^{2} x_{2}^{2} x_{3}^{2}\left(21 x_{1}^{16}-3132\left(x_{2}^{4}+x_{3}^{4}\right) x_{1}^{12-2\left(6105 x_{2}^{8}\right.}\right. \\
& \left.-233438 x_{3}^{4} x_{2}^{4}+6105 x_{3}^{8}\right) x_{1}^{8}-4\left(783 x_{2}^{12}-116719 x_{3}^{4} x_{2}^{8}-116719 x_{3}^{8} x_{2}^{4}+783 x_{3}^{12}\right) x_{1}^{4}+3\left(7 x_{2}^{16}\right. \\
& -1044 x_{3}^{4} x_{2}^{\left.\left.12-4070 x_{3}^{8} x_{2}^{8}-1044 x_{3}^{12} x_{2}^{4}+7 x_{3}^{16}\right)\right) x_{0}^{2}+2\left(2 x_{1}^{24}-9\left(x_{2}^{4}+x_{3}^{4}\right) x_{1}^{20}+6\left(259 x_{2}^{8}\right.\right.} \\
& \left.+3 x_{3}^{4} x_{2}^{4}+259 x_{3}^{8}\right) x_{1}^{16}+\left(5098 x_{2}^{12}-2313 x_{3}^{4} x_{2}^{8}-2313 x_{3}^{8} x_{2}^{4}+5098 x_{3}^{12}\right) x_{1}^{12}+3\left(518 x_{2}^{16}\right. \\
& \left.-771 x_{3}^{4} x_{2}^{12}+17178 x_{3}^{8} x_{2}^{8}-771 x_{3}^{12} x_{2}^{4}+518 x_{3}^{16}\right) x_{1}^{8}-9\left(x_{2}^{20}-2 x_{3}^{4} x_{2}^{16}+257 x_{3}^{8} x_{2}^{12}+257 x_{3}^{12} x_{2}^{8}\right. \\
& \left.\left.-2 x_{3}^{16} x_{2}^{4}+x_{3}^{20}\right) x_{1}^{4}+2 x_{2}^{24}+2 x_{3}^{24}-9 x_{2}^{4} x_{3}^{20}+1554 x_{2}^{8} x_{3}^{16}+5098 x_{2}^{12} x_{3}^{12}+1554 x_{2}^{16} x_{3}^{8}-9 x_{2}^{20} x_{3}^{4}\right) \\
& \left.-855048 x_{3}^{8} x_{2}^{4} x_{1}^{8}\right) \tag{A.7.6}
\end{align*}
$$

The $c=48$ extremal theory has partition function in EP form given by

$$
\begin{align*}
\tilde{W}_{48}^{(1)}\left(x_{0}, x_{1}\right) & =x_{0}^{48}-9 x_{1}^{4} x_{0}^{44}+\frac{1155}{32} x_{1}^{8} x_{0}^{40}+\frac{41641}{4} x_{1}^{12} x_{0}^{36}+\frac{4314189}{8} x_{1}^{16} x_{0}^{32} \\
& +\frac{16161123}{4} x_{1}^{20} x_{0}^{28}+\frac{121555689}{16} x_{1}^{24} x_{0}^{24}+\frac{16161123}{4} x_{1}^{28} x_{0}^{20} \\
& +\frac{4314189}{8} x_{1}^{32} x_{0}^{16}+\frac{41641}{4} x_{1}^{36} x_{0}^{12}+\frac{1155}{32} x_{1}^{40} x_{0}^{8}-9 x_{1}^{44} x_{0}^{4}+x_{1}^{48} \tag{A.7.7}
\end{align*}
$$

We have also obtained the polynomial expression for the genus 3 partition function of the $c=24$ extremal theory, and for the genus 2 partition function of the conjectured $c=48$ extremal theory. These were too large to included even here but are available upon request.

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## PUBLICATIONS

- Quasi-classical spectrum of KdV charges in Conformal field theory

Anatoly Dymarsky, Ashish Kakkar, Kirill Pavlenko and Sotaro Sugishita JHEP 09 (2022) 169

- Codes, Lattices and CFTs at Higher Genus Johan Henriksson, Ashish Kakkar and Brian McPeak JHEP 05 (2022) 159
- Quantum Codes, Narian Lattices and CFTs at Higher Genus Johan Henriksson, Ashish Kakkar and Brian McPeak arXiv: 2205.00025
- Information geometry and holographic correlators Hardik Bohra, Ashish Kakkar and Allic Sivramakrishnan JHEP 04 (2022) 037
- Flat Holography: Aspects of the dual field theory

Arjun Bagchi, Rudranil Basu, Ashish Kakkar and Aditya Mehra JHEP 12 (2016) 147

- Galilean Yang-Mills Theory Arjun Bagchi, Rudranil Basu, Ashish Kakkar and Aditya Mehra JHEP 04 (2016) 051
- Characterizing Error Mitigation by Symmetry Verification in QAOA Ashish Kakkar, Jeffery Larson, Alexey and Ruslan Shaydulin arXiv: 2204.05852


## ACADEMIC HONOURS

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[^0]:    ${ }^{1}$ Appearance of $n_{k}$ to parametrize the eigenstates can be understood from the Virasoro algebra, which in the large $c$ limit reduces to a product of Heisenberg algebras, with $n_{k}$ being the corresponding quantum numbers 45, 46.

[^1]:    ${ }^{2}$ Since Ref. [29] was working in the regime of both large central charge and thermodynamic limit $Q_{1} \propto c L^{2}$, it only conjectured the term linear in $n_{k}$, as the $n_{k}$-independent term is $1 / L^{2}$ suppressed.

[^2]:    ${ }^{3}$ An implicit assumption here is that $u$ belongs to the regular orbit $\operatorname{diff}\left(\mathbb{S}^{1}\right) / \mathbb{S}^{1}$, which upon quantization, becomes Verma module.

[^3]:    ${ }^{4}$ Our definition of $q$ is aligned with Wolfram Mathematica. In this section $q$ denotes modular parameter of the genus one elliptic curve $y(\lambda)$. In section 2.5 we use $q$ to denote modular parameter of the CFT spacetime torus.

[^4]:    ${ }^{1}$ To highlight the physics involved, we will refer to this metric as the metric of the correlator. Note that there is a one-to-one mapping between the information distance of two $n$-operator states and a certain set of $2 n$-point correlators in the appropriate kinematic configuration. The normalization factor is understood, though note that it is a correlator as well. A ratio of correlators may seem strange, but universal properties will appear nevertheless.

[^5]:    ${ }^{2}$ While it may be true that the absence of cross terms implies factorization into sub-fidelities, we do not claim this. In principle, the fidelity could factorize into two functions that are not themselves fidelities.

[^6]:    ${ }^{3}$ See [73 for discussion.

[^7]:    ${ }^{4}$ This is true only under certain assumptions and has only been studied in generality up to one loop, but these details will not be relevant in this work.

[^8]:    ${ }^{5}$ Note that the information metric is nonzero only when these states are not eigenstates of $H_{S}^{(1)}$.

[^9]:    ${ }^{1}$ In fact, any lattice CFT has partition functions which can be written in the form of a code enumerator polynomial, and thus must transform the same way.

[^10]:    ${ }^{2}$ In Mathematica, EllipticTheta $[m, 0, q]=\theta_{m}\left(q^{2}\right)$.

[^11]:    ${ }^{3}$ At genus 2, there is a nowhere vanishing modular form of degree 10: $\chi_{10}$, see section 4.3 .
    ${ }^{4}$ If the chiral CFT is paired with its complex conjugate to give a full CFT, all phases picked up by modular transformations are canceled and the full CFT is modular invariant for any $c$. This is what happens, for instance, in the case of Narain CFTs 121.

[^12]:    ${ }^{5}$ There are subleading corrections to factorization 131, which are conveniently expressed in plumbing fixture coordinates giving a recipe for stitching together two tori with modular parameter $\tau_{1}, \tau_{2}$ via a tube of radius $\epsilon$. This gives $r=1+O(\epsilon)$. For our purposes we shall always consider the complete factorization $r=1$.

[^13]:    ${ }^{6}$ The normalization of the genus 3 generators differ between 133 and 134 ; we will not need the normalization of the genus 3 modular forms.
    ${ }^{7}$ In the literature, the following compact expressions are also common

    $$
    \begin{equation*}
    G_{4}=\frac{1}{2}\left(\theta_{2}(q)^{8}+\theta_{3}(q)^{8}+\theta_{4}(q)^{8}\right), \quad \Delta_{12}=\eta(\tau)^{24}=\frac{1}{256} \theta_{3}(q)^{8} \theta_{2}(q)^{8} \theta_{4}(q)^{8} \tag{4.3.49}
    \end{equation*}
    $$

    where the Jacobi theta functions $\theta_{2}, \theta_{3}$ and $\theta_{4}$ have arguments $q$ and not $q^{2}$ as in 4.3.50 (see the definition 4.2.21).

[^14]:    ${ }^{8}$ We thank R Salvati Manni for pointing out this reference.
    ${ }^{9}$ To see how this is possible, consider the problem of writing an alternative type of enumerator polynomials at genus 1 in terms of the variables $a=\theta_{3}(q), b=\theta_{2}(q)$ (i.e. in terms of Jacobi theta functions with argument $q$ and not $q^{2}$ ). Then one finds from 4.3.49, and the Jacobi relation $\theta_{2}(q)^{4}-\theta_{3}(q)^{4}+\theta_{4}(q)^{4}=0$, that $E_{4}=a^{8}+b^{8}-a^{4} b^{4}, \Delta_{12}=\frac{1}{256} a^{8} b^{8}\left(a^{4}-b^{4}\right)^{2}$, while $E_{6}=$ $\sqrt{E_{4}^{3}-1728 \Delta_{12}}$ is not a polynomial.

[^15]:    ${ }^{10}$ Here we stressed that the numerator of the partition function is not a function on $\mathcal{M}_{g}$, but instead a section on $\lambda^{c / 2}\left(\mathcal{M}_{g}\right)$, where $\lambda\left(\mathcal{M}_{g}\right)$ denotes the determinant line bundle on $\mathcal{M}_{g}$. This in turn descends from the determinant line bundle on $\mathcal{A}_{g}$, see for instance comments in 137.
    ${ }^{11}$ Concretely, the Jacobian of a Riemann surface with period matrix $\Omega$ is the complex torus given by $\mathbb{C}^{g} /\left(\mathbb{Z}^{g}+\Omega \mathbb{Z}^{g}\right)$

[^16]:    ${ }^{12}$ The code associated with the $e_{8}$ lattice is in fact the Hamming $[8,4,4]$ code, the unique even self-dual binary code at $c=8$.
    ${ }^{13}$ Note that the argument here is $\Omega$ and not $2 \Omega$ as in 4.3.19).
    ${ }^{14}$ At genus $g=1$, this list contains the three elements $A=\binom{0}{0},\binom{1 / 2}{0}$, and $\binom{0}{1 / 2}$, corresponding to the Jacobi theta functions $\theta_{3}(q), \theta_{2}(q)$ and $\theta_{4}(q)$ respectively, see 4.2.16.

[^17]:    ${ }^{15}$ Again, $\hat{Z}$ refers to the numerator of the partition function, see 4.3.1.
    ${ }^{16}$ An alternative ansatz was proposed in 148 , also leading to non-vanishing cosmological constant at genus 5. The two ansätze can in principle be combined to give a vanishing genus 5 cosmological constant, but this approach runs into difficulties at genus 6 , see $147,149,150$. See also $[151, ~ 152]$ for a relation to the bosonic string theory measure.

[^18]:    ${ }^{17}$ Recall that the genus 1 partition function expands as $q^{-\frac{c}{24}}$ times an expansion in state multiplicities.

[^19]:    ${ }^{18}$ Technically we could represent the polynomials on each torus with different variables, e.g. $x_{0} \rightarrow x_{0} y_{0}$, etc. In principle this could lead to stronger constraints on the allowed polynomials, but interestingly we find no difference in these procedures.

[^20]:    ${ }^{19}$ The famous cases of the Leech lattice and Monster module correspond to $N_{\text {currents }}=24$ and $N_{\text {currents }}=0$ respectively. The Golay code gives the lattice $(a 1)^{24}$ through Construction A, however through the twisted Construction A (see e.g. [19]) it gives rise to the Leech lattice.

[^21]:    ${ }^{20}$ As discussed in detail in section 4.3, these two codes are resolvable by the enumerator polynomials at genus 3, while in fact they have the same partition functions for all genus $g<5$. This is due to non-trivial relations between the theta constants.

[^22]:    ${ }^{1}$ Technically the MacWilliams identity of [116] refers to only the second transformation, while the first one follows from doubly-evenness. For convenience, we use the term "MacWilliams identities" to refer to both conditions.

[^23]:    ${ }^{2}$ The codes involved in the construction - binary double-even self-dual codes, also known as type II codes - have length $n$ divisible by 8 and have been classified for $n \leqslant 40$ [158]. The corresponding lattices (even self-dual lattices) and CFTs (meromorphic CFTs) have been classified for $n=c \leqslant 24$ by 154 and 155 respectively.
    ${ }^{3}$ For more details on the case at $g \geqslant 4$ for chiral CFTs, see our previous work 160 .

[^24]:    ${ }^{4}$ It remains possible that they may be code theories defined by some suitable generalization of New Construction A. See [166] for a broader set of Narain CFTs with code counterparts. Throughout this chapter, "code theories" means those defined by New Construction A.

[^25]:    ${ }^{5}$ As explained in detail in section 5.2 , there is a direct relation between the elements $\{\mathbb{1}, X, Y, Z\}$, which represent Pauli matrices, and the elements $\left\{0, \omega, 1, \omega^{2}\right\}$ that represent the finite field $\mathbb{F}_{4}$.
    ${ }^{6}$ In fact, all of these fake partition functions can be constructed from linear combinations of the real theories. Specifically, all partition functions are non-negative linear combinations of the ones deriving from $W_{1}^{3} . W_{3}$ and $\tilde{W}_{3}$, see figure 5.2 below.

[^26]:    ${ }^{7}$ Recall the definition of the lattice theta function of a Euclidean lattice $\Lambda: \Theta_{\Lambda}(\tau)=\sum_{\lambda \in \Lambda} q^{\lambda^{2} / 2}$.

[^27]:    ${ }^{8}$ These are sometimes referred to as error-detection, rather than error-correction, protocols.

[^28]:    ${ }^{9}$ It is a non-trivial fact that the transformation $|x\rangle|0\rangle_{\text {aux }} \rightarrow|x\rangle|H x\rangle_{\text {aux }}$ can always be accomplished using a quantum circuit composed of CNOT gates. $\left|x+y+e_{2}\right\rangle|0\rangle_{\text {aux }} \rightarrow\left|x+y+e_{2}\right\rangle \mid H(x+$ $\left.\left.y+e_{2}\right)\right\rangle_{\text {aux }}=\left|x+y+e_{2}\right\rangle\left|H e_{2}\right\rangle_{\text {aux }}$.

    Detecting the phase flip errors is almost identical, after using a trick: a phase flip, which acts by $|0\rangle \rightarrow|0\rangle,|1\rangle \rightarrow-|1\rangle$, acts the same as a bit flip in the basis $|+\rangle=|0\rangle+|1\rangle,|-\rangle=|0\rangle-|1\rangle$. So we change basis (formally, apply a Hadamard gate) and then we see that we can detect this error using the same procedure as for the bit flip errors.

[^29]:    ${ }^{10}$ This follows from a general identity for the complete enumerator polynomial 169 (Theorem 10 of chapter 5), see also [170], Theorem 8.

[^30]:    ${ }^{11}$ Here we define the theta function the following way. For a Lorentzian lattice in $(n, n)$ signature, move to coordinates where the metric is of the form $g=\mathbb{I}_{n, \times n} \oplus\left(-\mathbb{I}_{n \times, n}\right)$, so that any lattice vector can be written $v=(\ell, r)$. Then the lattice theta function is defined as $\Theta_{\Lambda}=\sum_{v \in \Lambda} q^{\ell^{2} / 2} \bar{q}^{r^{2} / 2}$.

[^31]:    ${ }^{12}$ The $B$-form, and related graph, are not necessarily unique. This is due to the existence of certain T-dualities that relate two different $B$-form codes. These act on the graphs. This issue is important for the classification of code theories in 19 .

[^32]:    ${ }^{13}$ Such theories can also be differentiated by introducing chemical potentials for the $U(1)^{n} \times$ $U(1)^{n}$ currents 21 .

[^33]:    ${ }^{14}$ See also 183 , which includes a calculation of the genus 2 partition function

[^34]:    ${ }^{15}$ We have only checked that all Virasoro degeneracies are positive up to $\Delta=10$, since we do not know a way to enforce it for all degeneracies in practice.

[^35]:    ${ }^{16}$ This is not rigorous because in principle it is possible that the coefficients do not appear in the degeneracies independently, which could prevent us from canceling some degeneracies. In practice we see that this does not happen, at least up to $n=24$. Still, one should take this subsection as a speculation rather than a proof.

[^36]:    ${ }^{17}$ For chiral theories, it is known that the required genus is at most $n / 2-1$. See 129,160 .

