Rotor Balancing Using High Breakdown-Point and Bounded-Influence Estimators

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ABSTRACT

In industrial field, one of the most important practical problems of rotating machinery concerns rotor balancing. Many different methods are used for rotor balancing. Traditional influence coefficient method is often employed along with weighted least squares in order to reduce vibration amplitude, typically at selected rotating speeds like critical or operating ones. Usually the selection of the weights of the least squares algorithm is manually made by a skilled operator that can decide in which speed range the vibration reduction is more effective. Several methods have been proposed in order to avoid operator's arbitrariness and an automatic procedure based on robust regression is introduced in this paper. In particular, the analysis is focused on high breakdown-point and bounded-influence estimators. Theoretical aspects and properties of these methods are investigated. The effectiveness and robustness of the proposed balancing procedure are shown by means of an experimental case using a test-rig.

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1 INTRODUCTION

As well known in rotating machinery theory, the rotor deforms itself as a consequence of unbalances. This could be very dangerous for the efficiency of the machine in which the rotor is placed [1].

Rotor balancing is realized by means of the application of balancing masses in suitable planes of the rotor. The balancing masses are often selected as solution of an optimization problem, in which weighted least squares (WLS) are mainly used. Obviously, the goal is the reduction of vibrations in some planes of the rotor, usually but not necessarily in the measuring planes. The weighted method allows reducing the amplitude of vibrations at selected rotating speeds as in flexural critical speeds or in operating conditions. It also permits reducing the effect of possible outliers due to noise, biases, systematic errors and so on in the experimental data or due to inaccuracies of the model. Main drawback of WLS is that the criterion of weight attribution is not univocal and normally expert's operator inspection of the data is required. In this paper, an automatic procedure for the balancing masses estimation based on robust regression methods is introduced. In particular, the analysis is focused on *high breakdown-point* (HBP) and *bounded-influence* (BI) estimators, which are very robust with respect to outliers [9]. Theoretical aspects and properties of these methods are investigated.

The effectiveness and robustness of the estimators here introduced have been tested in the identification of the balancing masses of the test-rig of Politecnico di Milano (PdM).

For each method, the results are compared to those of the least squares method and of the M-estimator method, used in a previous study, for the same test-rig [19]. The paper is organized starting from basic concepts of rotor balancing. In the second part, several robust regression methods are analyzed. Finally the experimental application of the proposed methods to the test-rig is presented.

1.1 Rotor Balancing Approach

The *influence coefficient method* [1][5][17] is the most commonly used approach for rotor balancing due to its simplicity and experimental nature. In general, the monitoring data are collected for many rotating speeds and on several measuring planes (which often correspond to the bearings). On each measuring plane, vibrations are measured along one direction or two orthogonal ones depending on the type of the machine. The rotating speeds, at which the measures are available, are organized as a vector of n_s elements

$$\boldsymbol{\Omega} = \ \Omega_1 \quad \dots \quad \Omega_j \quad \dots \quad \Omega_{n_s} \ ^{\mathrm{T}} \qquad (1)$$

For a flexible rotor, the balancing masses depend on the rotating speed and are obtained as a solution of the overdetermined system

$$\mathbf{y} + \ \mathbf{X} \ \boldsymbol{\theta} = \mathbf{e} \qquad (2)$$

- $\mathbf{y}$ is an $2n_m n_s \times 1$ complex (magnitude and phase) vector of experimentally measured vibrations for n_s rotating speeds, organized as follows

$$\mathbf{y} = \ \ \xi(\Omega_1)^{\mathrm{T}} \quad \dots \quad \xi(\Omega_j)^{\mathrm{T}} \quad \dots \quad \xi(\Omega_{n_s})^{\mathrm{T}} \ ^{\mathrm{T}} \qquad (3)$$

where each term $\xi(\Omega_j)$ is a $2n_m \times 1$ complex vector of experimental vibrations measured at *j*th rotating speed for all n_m measuring planes

$$\xi(\Omega_j) = \left\{ \ \xi^{(1)}(\Omega_j)^{\mathrm{T}} \quad \dots \quad \xi^{(k)}(\Omega_j)^{\mathrm{T}} \quad \dots \quad \xi^{(n_m)}(\Omega_j)^{\mathrm{T}} \right\}^{\mathrm{T}} \qquad (4)$$

and $\xi^{(k)}(\Omega_j) = \ \xi_V^{(k)}(\Omega_j) \quad \xi_H^{(k)}(\Omega_j) \ ^{\mathrm{T}}$ is the 2×1 complex vector of vibrations measured in *k*th measuring plane, at *j*th rotating speed and along the vertical and the horizontal directions.

- $\mathbf{X}$ is the $2n_m n_s \times n_b$ global *matrix of influence coefficients* at different rotating speed

$$\mathbf{X} = \left[\ \mathbf{C}(\Omega_1)^{\mathrm{T}} \quad \dots \quad \left[\mathbf{C}(\Omega_j)\right]^{\mathrm{T}} \quad \dots \quad \left[\mathbf{C}(\Omega_{n_s})\right]^{\mathrm{T}} \right]^{\mathrm{T}} \qquad (5)$$

where $\mathbf{C}$ is the $2n_m \times n_b$ matrix of influence coefficients at generic *j*th rotating speed

$$\left[\mathbf{C}(\Omega_j)\right] = \begin{bmatrix} \boldsymbol{\alpha}_{1,1} & \cdots & \boldsymbol{\alpha}_{1,w} & \cdots & \boldsymbol{\alpha}_{1,n_b} \\ \vdots & \ddots & & & \vdots \\ \boldsymbol{\alpha}_{k,1} & & \boldsymbol{\alpha}_{k,w} & & \boldsymbol{\alpha}_{k,n_b} \\ \vdots & & & \ddots & \vdots \\ \boldsymbol{\alpha}_{n_m,1} & \cdots & \boldsymbol{\alpha}_{n_m,w} & \cdots & \boldsymbol{\alpha}_{n_m,n_b} \end{bmatrix} \qquad (6)$$

and $\boldsymbol{\alpha}_{k,w}$ is the 2×1 generic complex influence coefficient vector, that gives the vibrations at generic *k*th measuring plane (along the two directions) for an unitary balancing mass placed at generic *w*th balancing plane.

- $\boldsymbol{\theta}$ is the vector of generalized balancing masses (amplitudes and phases) placed at n_b balancing planes

$$\boldsymbol{\theta} = \left[\boldsymbol{\theta}_1^* \quad \ldots \quad \boldsymbol{\theta}_w^* \quad \ldots \quad \boldsymbol{\theta}_{n_b}^* \right]^{\mathrm{T}} \tag{7}$$

where $\boldsymbol{\theta}_w^* = r_w m_w \mathrm{e}^{\mathrm{i}\psi_w}$ is the *w*th generalized mass placed at a pre-established distance from the rotating axis (r_w is the eccentricity of the true balancing mass m_w)

- $\mathbf{e}$ is the vector of errors at measuring planes for all n_s analyzed rotating speeds.

1.2 Regression Model

Equation (2) represents the model of a multiple linear regression problem [3][16], that could be rewritten as

$$y_i = x_{i,1}\theta_1 + \ldots + x_{i,p}\theta_p + e_i = \mathbf{x}_i^{\mathrm{T}}\boldsymbol{\theta} + e_i \tag{8}$$

for $i = 1, \ldots, n$, where $n = 2 \cdot n_m \cdot n_s$ is the number of observations (experimentally measured response) and $p = n_b$ the number of unknown coefficients or *predictors* (generalized balancing masses).

In statistical terms $\mathbf{y}$ is called vector of *response variables*, $\mathbf{X}$ matrix of predictors, $\boldsymbol{\theta}$ vector of predictors. For a given parameters estimate $\hat{\boldsymbol{\theta}}$, the residuals are given by

$$r_i = y_i - \mathbf{x}_i^{\mathrm{T}}\hat{\boldsymbol{\theta}} \tag{9}$$

or in matrix notation by $\mathbf{r} = \mathbf{y} - \mathbf{X}\,\hat{\boldsymbol{\theta}}$, where $\hat{\mathbf{y}} = \mathbf{X}\,\hat{\boldsymbol{\theta}}$ is the *estimated response*.

Object of the estimation process is to identify the balancing masses $\hat{\boldsymbol{\theta}}$ that, in general, minimize a function of residuals.

1.3 Least Squares (LS)

Classical *least squares* regression consists of minimizing the sum of squared residuals

$$\hat{\boldsymbol{\theta}}_{LS} = \min_{\boldsymbol{\theta}} \left\{ \sum_{i=1}^{n} r_i^2 \right\} = \min_{\boldsymbol{\theta}} \ \mathbf{r}^{*\mathrm{T}}\mathbf{r} \tag{10}$$

where $\mathbf{r}^{*\mathrm{T}}$ denotes the conjugate vector transpose. The minimization is obtained equating to zero the first derivative of the objective function with respect to $\boldsymbol{\theta}$ for all cases, by means of the Moore-Penrose's pseudo-inverse matrix $\mathbf{X}^{\dagger}$

$$\hat{\boldsymbol{\theta}}_{LS} = \mathbf{X}^{\dagger}\mathbf{y} = \left(\mathbf{X}^{*\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{*\mathrm{T}}\mathbf{y} \tag{11}$$

where $\mathbf{X}^{*\mathrm{T}}$ denotes the Hermitian transpose.

In spite of its computational simplicity, the least squares estimator is characterized by lack of robustness, i.e. one single contaminated data could have an arbitrarily large effect on the estimate.

1.4 MiniMax (L_∞)

The *minimax* estimator [5] minimizes the worst residual:

$$\hat{\boldsymbol{\theta}}_{L_\infty} = \min_{\boldsymbol{\theta}} \max_{i} \left[r_i^2\left(\boldsymbol{\theta}\right) \right] \tag{12}$$

The robustness is weak as in LS method, because outlier points are not identified.

If complex data values are taken into account, the minimization is performed both for the real part and for the imaginary part.

2 ROBUST REGRESSION

The points that breakdown the estimate are called *influential points* [26] and could be classified as:

- *vertical outliers* (also *regression outliers* or simply *outliers*) when they are far from the linear pattern of the majority of the data but whose $\mathbf{x}_i$ is not outlying;
- *leverage points* (or *influential predictors*) when $\mathbf{x}_i$ is outlying. A point $\left(\mathbf{x}_i, y_i\right)$ is a good leverage point if it follows the pattern of the majority and a bad leverage point otherwise.

In the first case, the *breakdown-point* (BP), introduced by Hampel in 1971 [12] as an asymptotic concept and developed by Donoho and Huber [11] for the corresponding finite sample notion, is a typical measure of robustness. For finite sample, it is defined as the smallest percentage of contaminated data (outliers) that could cause the estimator to take on arbitrarily large values. Given the regression method T, the breakdown-point for a sample data set Z is defined as

$$\varepsilon_n^*\left(T, Z\right) = \min\left\{ \frac{m}{n} : \sup_{Z^*} \left\| T(Z) - T(Z^*) \right\| = \infty \right\} \tag{13}$$

where the supremum is for all samples Z^* obtained arbitrarily changing m original observations in the set Z. On the contrary, the *asymptotic breakdown-point* ε^* is defined as the limit of the finite sample breakdown-point when n goes to infinity. *High breakdown-point* (HBP) estimators are able to increase the robustness in regression cases. Obviously a breakdown-point cannot exceed 50%: if more than half of the observations are contaminated, it is not possible to distinguish between the underlying distribution and the contaminating distribution.

For leverage points, the so called *influence function* (IF) [13][18], describes the local effect on the estimator T of an additional observation. In this case a standard measure of *leverage* is the size of the diagonal elements of the so called *prediction* or *hat matrix* $\mathbf{H}$, and many estimators use this quantity to detect and downweight leverage values. In general this measure is given by h_{ii}, the *i*th diagonal term of the hat matrix

$$\mathbf{H} = \mathbf{X}\left(\mathbf{X}^{*\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{*\mathrm{T}} \tag{14}$$

Estimators that bound the influence of any single element of row of $\mathbf{X}$ are able to guard against leverage points as well as regression outliers. These are usually called *bounded-influence* (BI) estimators.

2.1 Weighted Least Squares (WLS)

The first step in robustness improving could be obtained manually by removing outlier data, or by assigning weights to several observations. In this way some measured vibrations are downweighted in order to give more importance to the corresponding rotating speed such as at rated speed or at critical speeds. In other words, it is necessary to minimize the quantity

$$\hat{\boldsymbol{\theta}}_{WLS} = \min_{\boldsymbol{\theta}}\left\{\sum_{i=1}^{n} w_i r_i^2\right\} = \min_{\boldsymbol{\theta}}\left(\mathbf{r}^{*\mathrm{T}}\,\mathbf{W}\,\mathbf{r}\right) \tag{15}$$

where $\mathbf{W}$ is the diagonal matrix of weights w_i. The solution is obtained by means of the weighted pseudo-inverse matrix $\mathbf{X}_W^{\dagger}$

$$\hat{\boldsymbol{\theta}}_{WLS} = \mathbf{X}_W^{\dagger}\,\mathbf{y} = \left(\mathbf{X}^{*\mathrm{T}}\,\mathbf{W}\,\mathbf{X}\right)^{-1}\mathbf{X}^{*\mathrm{T}}\,\mathbf{W}\,\mathbf{y} \tag{16}$$

2.2 M-estimators

One of the most important robust regression method is the *M-estimator*, a generalization of *maximum likelihood estimators* (MLEs) introduced by Huber in 1981 [15] based on the minimization of a symmetric function ρ of the standardized residuals, with a unique minimum at zero

$$\hat{\boldsymbol{\theta}}_M = \min_{\boldsymbol{\theta}} \left\{ \sum_{i=1}^{n} \rho\left(\frac{r_i}{\hat{\sigma}} \right) \right\} = \min_{\boldsymbol{\theta}} \left\{ \sum_{i=1}^{n} \rho\left(\frac{y_i - \mathbf{x}_i{}^{\mathrm{T}} \boldsymbol{\theta}}{\hat{\sigma}} \right) \right\} \tag{17}$$

where $\hat{\sigma}$ is a robust estimation of the scale parameter σ of the error distribution. Parameter estimation in mechanical system often uses vibration data that are complex number. Since complex numbers are bivariate data, they can be conceived as the simplest case of multidimensional distribution for which the estimation can be realized considering the concept of *data depth*: Tukey's [23] median absolute deviation (TMAD) could be then used [18]

$$\hat{\sigma}_{\mathbb{C}} = \mathrm{TMAD}(\mathbf{r}) = \mathrm{med} \left| \mathbf{r}\ \boldsymbol{\theta}\ - \mathrm{T}^{*}\left[\mathbf{r}\ \boldsymbol{\theta}\ \right] \right| \tag{18}$$

where $\mathrm{T}^{*}(\cdot)$ is the Tukey's median operator. The M-estimate of the regression parameter $\boldsymbol{\theta}$, solution of problem (17), is obtained equating to zero the first derivative of ρ with respect to θ_j

$$\sum_{i=1}^{n} x_{ij} \psi \left(\frac{y_i - \mathbf{x}_i{}^{\mathrm{T}} \hat{\boldsymbol{\theta}}_M}{\hat{\sigma}} \right) = 0 \tag{19}$$

where the function $\psi = \mathrm{d}\rho / \mathrm{d}\theta_j$ is proportional to the influence function that describes the behavior of M-estimators [13].

The solution $\hat{\boldsymbol{\theta}}_M$ is computed by means of an iterative algorithm called *iteratively reweighted least-squares* (IRLS) [7][18][29].

The convergence of the values of $\hat{\boldsymbol{\theta}}$ is achieved upon a stated criterion, for instance

$$\frac{\left| \delta^{k} - \delta^{k-1} \right|}{\left| \delta^{k-1} \right|} \le \varepsilon \tag{20}$$

where ε is a suitable convergence value and δ^{k} is the *relative residual* at *k*th iteration between the experimental data $\mathbf{y}$ and the estimated response $\hat{\mathbf{y}} = \mathbf{X}\ \hat{\boldsymbol{\theta}}$, that has the following general expression

$$\delta = \sqrt{\frac{(\mathbf{y}-\hat{\mathbf{y}})^{*\mathrm{T}}(\mathbf{y}-\hat{\mathbf{y}})}{\mathbf{y}^{*\mathrm{T}}\mathbf{y}}} \tag{21}$$

Making use of weights, the behavior of an M-estimator is represented by the function ψ. Several M-estimators with different ρ functions have been proposed in the last years. Huber's M-estimator method has a good behavior, so it will be used to make comparison with the other methods introduced in this paper. Its definition is

$$\rho(r_i) = \begin{cases} \frac{1}{2}\left(\frac{r_i}{\hat{\sigma}}\right)^2 & \text{if} \quad \left|\frac{r_i}{\hat{\sigma}}\right| \le c \\ c\left(\left|\frac{r_i}{\hat{\sigma}}\right| - \frac{c}{2}\right) & \text{if} \quad \left|\frac{r_i}{\hat{\sigma}}\right| > c \end{cases} \tag{22}$$

The tuning parameter c for Huber's estimator is equal to 1.345 and gives 95% of asymptotic efficiency with respect to a normal distribution [22].

3 HIGH BREAKDOWN-POINT METHODS (HBP)

In general, when data are corrupted by outliers, a robust method should have a *high breakdown-point* (HBP), a *bounded-influence* function (BI) and good *efficiency*. A desirable property for regression estimates is that the estimate be *equivariant* with respect to *affine*, *regression*, and *scale* transformations [22]. This means that when one of these transformations is applied to the data, the estimates will transform in the "natural" way.

A *regression equivariant estimator* T has a finite sample breakdown-point

$$\varepsilon_n^*(T,Z) \le \frac{\lfloor (n-p)/2 \rfloor + 1}{n} \tag{23}$$

at all samples Z, where samples are in *general position*, i.e. all the subsets p of the data are linearly independent. For least squares and M-estimators the finite sample breakdown-point is equal to $\varepsilon_n^* = 1/n$.

The *efficiency* of an estimator is normally evaluated in relative terms with respect to the *mean squared error* (MSE) of the residuals. The relative efficiency is defined as the ratio of the MSE of the LS method and the MSE of the given robust estimator, under the hypothesis of unbiased data set in the LS method. The *relative efficiency* of two methods depends on the sample size available for the given estimator, but it is also possible to use the *asymptotic efficiency* defined as the limit of the relative efficiency as the sample size n goes to infinity.

Several equivariant estimators for linear regression with asymptotic breakdown-point equal to 0.5 were proposed in the literature: the *least median of squares* (LMS), the *least trimmed mean squares* (LTS) and the *S-estimators*. More efficient estimates with asymptotic breakdown-point equal to 0.5 are given by the *MM-estimators.*

3.1 Least Median of Squares (LMS)

The *least median of squares* (LMS) method, introduced by Rousseeuw in [20] minimizes the median of the squared residuals

$$\hat{\boldsymbol{\theta}}_{LMS} = \min_{\boldsymbol{\theta}} \ \operatorname*{med}_{i} \left[r_i^2 (\boldsymbol{\theta}) \right] \tag{24}$$

From a different point of view, the LMS method is a robust generalization of classical least squares method substituting the sum operator with a more robust median operator.

This method has an asymptotic breakdown-point equal to $\varepsilon^* = 0.5$, it correctly approximates only half of the data. Due to exact-fit property, it is possible to obtain a regression far from the desired one if outliers are aligned with valid data.

The basic resampling algorithm for approximating the LMS, was proposed by Rousseeuw and Leroy in [22] and further developed in [21]. This algorithm considers a trial subset of *p*-observations and calculates the linear fit passing through them. This procedure is repeated many times and the fit with the lowest median of squared residuals is retained. In this paper, the basic form of LMS method given by eq. (24) is applied. This LMS base form uses weights equal to 0 or 1. A disadvantage of the LMS method is its lack of efficiency because of its $n^{-1/3}$ convergence.

3.2 Least Trimmed Sum of Squares (LTS)

An improvement with respect to LMS estimator is given by the *least trimmed squares* (LTS) [20][22] that minimizes the following criterion

$$\hat{\boldsymbol{\theta}}_{LTS} = \min_{\boldsymbol{\theta}} \left\{ \sum_{i=1}^{h} \left[r_i^2 (\boldsymbol{\theta}) \right]_{i:n} \right\} \tag{25}$$

where the value h is called *coverage*. The LTS estimator searches for the optimal subset of size h whose least squares fit has the smallest sum of squared residuals. To reach a

50% breakdown-point when the data are in general position, the coverage h is fixed so that $\lfloor (n+p)/2 \rfloor \le h \le \lfloor (n+p+1)/2 \rfloor$. The objective function of the LTS is more smooth, making LTS less sensitive to local effects than LMS. Furthermore it has a convergence rate equal to $n^{-1/2}$, greater than the LMS one, making it more suitable than the LMS as a starting point for two-step estimators.

The estimation and the breakdown-point, obviously depend on the choice of coverage h (usually and as in the implemented algorithm, $h = 0.75n$ is assumed). In general it is possible to select the coverage as $h = \lfloor n(1-\beta) \rfloor + 1$, where β is the portion parameter. In this case the asymptotic breakdown-point is $\varepsilon^* = \beta$. For $\beta \to 0.5$ the LMS estimator is obtained, whereas the least squares for $\beta \to 0$.

The computation of the LTS estimator is difficult and if the number of observations is higher, the computational load is prohibitive. To overcome this drawback, several approximate algorithms have been proposed by Agulló [2][4] and the fast-LTS algorithm, also used in the following examples, by Rousseeuw and Van Driessen [25] based on the so called *concentration-step* (C-Step).

3.3 S-Estimators

Introduced by Rousseeuw and Yohai [24] and developed by Rousseeuw and Leroy [22], the *S-estimator* of the regression parameter $\boldsymbol{\theta}$ is obtained from the minimization of a scale function $s[\mathbf{r}(\boldsymbol{\theta})]$ of the residuals

$$\hat{\boldsymbol{\theta}}_S = \min_{\boldsymbol{\theta}} s[r_1(\boldsymbol{\theta}), r_2(\boldsymbol{\theta}), \ldots, r_n(\boldsymbol{\theta})] \qquad (26)$$

and the scale estimator $\hat{\sigma}_S$ is

$$\hat{\sigma}_S = s[r_1(\hat{\boldsymbol{\theta}}_S), r_2(\hat{\boldsymbol{\theta}}_S), \ldots, r_n(\hat{\boldsymbol{\theta}}_S)] \qquad (27)$$

The scale function is obtained by means of a M-estimation as solution of Huber's equation

$$\frac{1}{n}\sum_{i=1}^{n} \rho\left(\frac{r_i}{s}\right) = K \qquad (28)$$

where the constant K is given by $\mathrm{E}_N(\rho) = K$, $\mathrm{E}(\cdot)$ is the expected value and N denotes the standard normal distribution.

If $\lambda = K/\rho(c) = 0.5$ the S-estimator has finite breakdown-point $\varepsilon_n^* = ([n/2] - p + 2)/n$ at any sample in general position (obviously the asymptotic breakdown-point in this condition is $\varepsilon^* = 50\%$). Also in this case several implemented algorithms exist. The algorithm used in this paper is based on the fast-S algorithm with $\lambda = 0.5$ (in order to obtain the highest breakdown-point $\varepsilon^* = 50\%$), developed by Salibian-Barrera and Yohai [27].

3.4 MM-Estimators

S-estimators are highly inefficient when the errors are normally distributed [22]. *MM-estimators* introduced by Yohai [31] have instead a high-breakdown-point with high-efficiency. The algorithm consists of three steps:

- first an estimate $\hat{\boldsymbol{\theta}}_0$ of $\boldsymbol{\theta}$ is performed by means of a LMS, LTS or S high breakdown-point estimator (possibly 0.5 breakdown-point). No high efficiency is required at this step.
- Afterwards the M-estimate $\hat{\sigma}$ of scale parameter is evaluated on residuals of the previous estimate $\boldsymbol{\theta}_0$.
- Last step consists of the M-estimate of regression parameters.

In practice, MM-estimates are based on two functions ρ_0 and ρ_1, which determine the breakdown-point and the efficiency of the estimator, respectively.

4 BOUNDED-INFLUENCE METHODS (BI)

M-estimators are not resistant to bad leverage points, due to their unbounded influence function in position x [10]. The *generalized M-estimators* (GM) are estimators resistant to high leverage points, that bound the influence of any single element or row of $\mathbf{X}$, so they guard against leverage points as well as regression outliers. As said before, a standard statistical measure of leverage is the size of the diagonal elements of the hat matrix defined by eq. (14). The *i*th diagonal element of $\mathbf{H}$ is a measure of the potential influence or leverage of the *i*th predictor observation. Matrix $\mathbf{H}$ is symmetric and idempotent, so $0 \le h_{ii} \le 1$. The eigenvalues are either 0 or 1 and the number of non-zero eigenvalues equals its rank, so the trace of $\mathbf{X}$ is p and $\mathrm{E}(h_{ii}) = p/n$. Elements h_{ii} that are more than 2 or 3 times the expected value are problematic [8]. BI estimators include the Mallows-type [28] and the Schweppe-type [13] GM-estimators that bound in different ways the influence of position.

A simple Schweppe-based version of GM-estimators, implemented in this paper, is based on the use of M-estimators working with adjusted or *studentized* residuals $\overline{r}_i$, using leverage and Huber's ρ function

$$\hat{\boldsymbol{\theta}}_{GM} = \min_{\boldsymbol{\theta}} \left\{ \sum_{i=1}^{n} \rho\left(\frac{\overline{r}_i}{\hat{\sigma}} \right) \right\} \tag{29}$$

where the adjusted residuals are

$$\overline{r}_i = \frac{r_i}{\sqrt{1-h_{ii}}} \tag{30}$$

$\hat{\sigma}$ is a standardized robust estimation of the scale parameter of the adjusted residuals.

Unfortunately BI estimators have a breakdown-point not greater than $\varepsilon_n^* = 1/p$, so they could give good results only for a small number of parameters.

5 QUALITY OF ESTIMATION

The correlation between the measured data and the estimated one is used as index of the quality of the estimation. The Pearson product-moment correlation coefficient ρ is a common measure of the correlation between two random variables X and Y with expected values μ_X and μ_Y and standard deviations σ_X and σ_Y. It is widely used as a measure of the strength of linear dependence between two variables, giving a value in $-1,+1$

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{E\left[\; X - \mu_X \quad Y - \mu_Y \;\right]}{\sigma_X \sigma_Y} \tag{31}$$

where σ_{XY} is the covariance. The unitary value is the desired ideal condition, meaning a good linear regression estimate.

The Pearson correlation works well under the hypothesis of normal distribution of two variables. When data is affect by outliers a robust measure of the correlation should be used like as the *Biweight Midcorrelation* [30]. Let

$$U_i = \frac{X_i - \mathrm{med}_X}{9 \cdot \mathrm{MAD}_X} \qquad V_i = \frac{Y_i - \mathrm{med}_Y}{9 \cdot \mathrm{MAD}_Y} \tag{32}$$

The sample *biweight midcovariance* between X and Y is given by

$$s_{bXY} = \frac{n\sum_i a_i \left(X_i - \mathrm{med}_X\right)\left(1-U_i^2\right)^2 b_i \left(Y_i - \mathrm{med}_Y\right)\left(1-V_i^2\right)^2}{\left[\sum_i a_i \left(1-U_i^2\right)\left(1-5\cdot U_i^2\right)\right]\left[\sum_i b_i \left(1-V_i^2\right)\left(1-5\cdot V_i^2\right)\right]} \tag{33}$$

where

$$\begin{aligned} &a_i = 1 \quad \text{if} \quad -1 \le U_i \le 1 \quad \text{otherwise} \quad a_i = 0 \\ &b_i = 1 \quad \text{if} \quad -1 \le V_i \le 1 \quad \text{otherwise} \quad b_i = 0 \end{aligned} \tag{34}$$

An estimate of the biweight midcorrelation between X and Y is given by

$$r_b = \frac{s_{bXY}}{s_{bX}\, s_{bY}} \tag{35}$$

where s_{bX} and s_{bY} are the biweight midvariances for the X and Y scores.

In this case the correlation is analyzed between the measured vibration $\mathbf{y}$ and the estimated one $\hat{\mathbf{y}}$.

Supposing to know the true value $\bar{\boldsymbol{\theta}}$ of predictors $\boldsymbol{\theta}$, it could be possible to evaluate the effectiveness of the estimation by means of the *global relative error* χ_G between the vector of known values and the estimated one using the Euclidean norm that for complex number is correctly given by

$$\chi_G = \sqrt{\frac{\left[\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right]^{*\mathrm{T}}\left[\hat{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\right]}{\bar{\boldsymbol{\theta}}^{*\mathrm{T}}\bar{\boldsymbol{\theta}}}} \tag{36}$$

The same index could be evaluated for the *w*th balancing mass (*w*th *relative error)*

$$\chi_w = \sqrt{\frac{\left[\hat{\theta}_{\mathrm{w}} - \bar{\theta}_{\mathrm{w}}\right]^{*\mathrm{T}}\left[\hat{\theta}_{\mathrm{w}} - \bar{\theta}_{\mathrm{w}}\right]}{\bar{\theta}_{\mathrm{w}}^{*\,\mathrm{T}}\bar{\theta}_{\mathrm{w}}}} \qquad w = 1,\ldots,n_b \tag{37}$$

6 CASE STUDY

The behavior of all high breakdown-point and bounded-influence estimators here introduced has been tested in the identification of the balancing masses of a rotating machine: the test-rig of Politecnico di Milano (PdM).

The test-rig, shown in Figure 1, is composed of two rigidly coupled steel shafts, driven by a variable speed electric motor and supported by four elliptical-shaped oil film bearings (labelled as A, B, C and D). Rotor train is about 2 m long and has a mass of about 90 kg. The rotors have three critical speeds within the operating speed range of 0-6000 rpm.

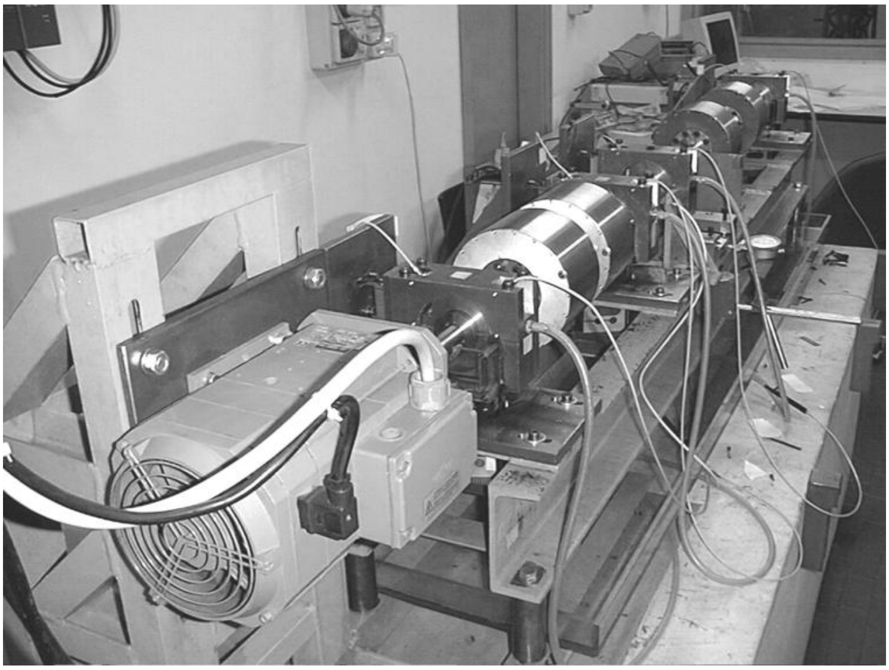

Figure 1 PdM test-rig.

The rotor system is mounted on a flexible steel foundation that has several natural frequencies in the operating speed range of the rotor. Two proximity probes in each bearing measure the relative shaft displacements and two accelerometers on each bearing housing measure its vibrations. The absolute vibration of the shaft is calculated by adding the relative displacement measured by the proximity probes to the absolute bearing housing displacement, which is obtained integrating twice the acceleration measured by the accelerometers. One run-down test was previously performed in order to store a reference vibration data without unbalancing masses.

Two known unbalancing masses are applied on both shafts of the rotor, at model nodes #9 (on the short shaft) and #35 (on the long shaft) as reported in Table 1.

Table 1. Nodes of the measuring, balancing and unbalancing planes in PdM test-rig case.

PLANES	*NODES*	*EQ. MASS*	*PHASE*
Measuring	#4 , #17 , #25 , #44		
Balancing	#9 , #35		
1st unbalance	#9	3.6e-4 kg m	-90°
2nd unbalance	#35	3.6e-4 kg m	-90°

The balancing planes considered are the same of the unbalances. Using the stored reference vibrations, the experimental additional vibrations due to the applied unbalancing masses, are obtained in the speed range 504-3001 rpm and are reported, for the sake of brevity, for the first (node #4, brg. A) and the third measuring plane (node #25, brg. C) in Figure 2 and Figure 3 respectively.

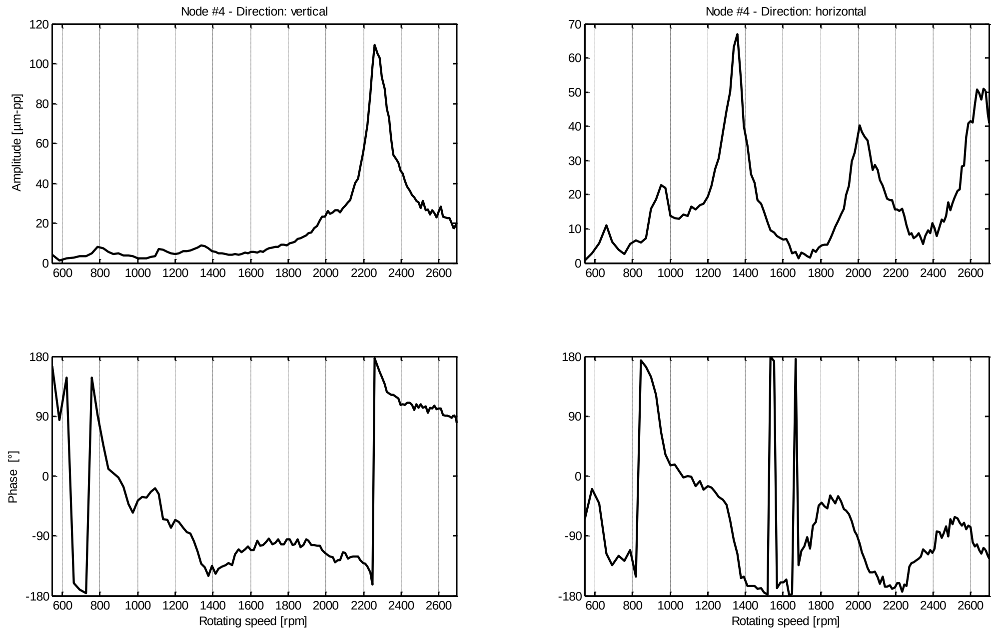

Figure 2 Experimentally measured additional vibrations at first measuring plane (node #4, brg. A) in presence of two known unbalances in PdM test-rig.

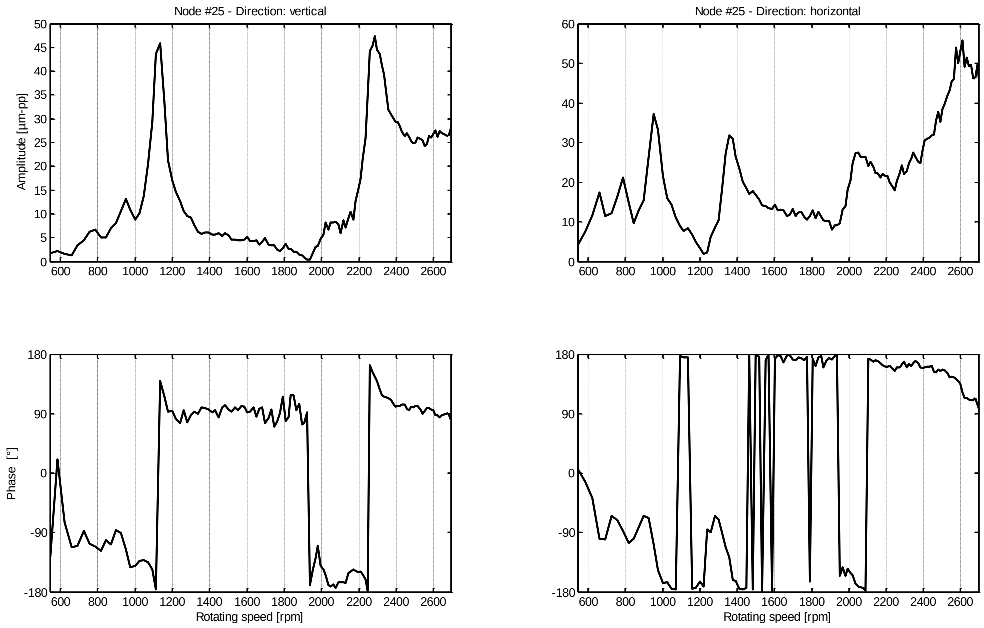

Figure 3 Experimentally measured additional vibrations at third measuring plane (node #25, brg. C) in presence of two known unbalances in PdM test-rig.

The model matrix $\mathbf{X}$ necessary for the estimation process is built by means of a finite element model of the rotor. The FE model, composed of 46 beam elements, as shown in Figure 4, has been tuned and the stiffness and damping coefficients of the bearings determined with accuracy, as described in [6]. The foundation has been modelled by means of a modal representation.

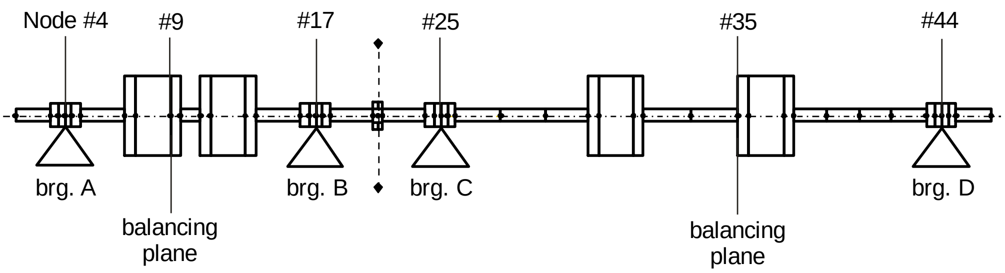

Figure 4 Finite element model of the PdM rotor test-rig.

Starting from experimental vibrations due to the applied unbalances in the measuring planes, the aim is to identify the balancing masses in terms of absolute values and phases placed with a fixed eccentricity respect to the rotating axis, by means of different robust estimators. The result of the estimation process is the identification of the masses in the balancing planes that reproduce the unbalance fault. In other words, the rotor balancing is realized with masses of equal absolute value and with a phase rotated of 180° respect to the estimated ones. In this way, knowing the unbalancing masses, it is easy to evaluate the robustness and the effectiveness of the estimators.

The estimator behaviour is evaluated in terms of identified balancing masses, used weights at different rotating speed, relative residuals δ (eq. (21)) and relative errors χ (eq. (36) and (37)), when the estimated balancing masses are applied. Also the biweight midcorrelation is analyzed as index of the quality of estimation.

The results of estimation by means of different robust estimators and the known applied balancing masses are reported in Table 2.

Table 2. Estimated balancing masses in PdM test-rig case.

Estimator	*Balancing Plane 1 Node #9*			*Balancing Plane 2 Node #35*			*Global Rel. Error* χ_G *[%]*	*Rel. Res.* δ	*Correlation*	
	Amp. [kg m] 10^{-4}	*Phase [°]*	*Rel. Error* χ_1 *[%]*	*Amp. [kg m]* 10^{-4}	*Phase [°]*	*Rel. Error* χ_2 *[%]*			*Pearson*	*Biw. midcorr.*
ACTUAL	**3.60**	**-90.0**		**3.60**	**-90.0**					
LS	3.82	-94.6	10.3	4.81	-99.7	39.0	28.5	0.5842	0.6472	0.4863
MINIMAX	6.65	-89.4	84.8	2.34	-173.5	113.0	99	0.7577	0.5136	0.1426
M-HUBER	3.52	-89.2	2.5	4.47	-91.5	24.5	17.4	0.5939	0.6472	0.4917
LMS	2.59	-68.6	42.1	3.11	-83.8	16.8	32.1	0.6956	0.6437	0.5761
LTS	2.86	-79.4	26.2	4.39	-90.6	22.0	24.2	0.6287	0.6423	0.4252
S	2.93	-78.1	26.3	3.88	-83.4	14.1%	21.1	0.6441	0.6469	0.5623
MM	3.53	-84.6	9.5	4.34	-88.5	20.8	16.2%	0.6049	0.6472	0.3070
GM	3.58	-89.4	1.1%	4.55	-93.6	27.5	19.4	0.5912	0.6472	0.3083

Similar balancing masses are obtained with M-based estimators (M-Huber, MM and GM). In general, considering the known values of the unbalances, it is possible to observe an improvement of the estimation, both for the amplitudes and the phases of all robust methods respect to the LS one. Regarding the phase values, the matching is very good for M-based estimators. Assuming the relative error index of eq. (37), the best estimation (highlighted in Table 2) for the first balancing mass is given by the GM-estimator (relative error $\chi_1 = 1.1\%$), whereas S-estimator is the best one for the second mass (relative error $\chi_2 = 14.1\%$). Considering the global relative error, the best estimation is given by the MM-estimator ($\chi_G = 16.2\%$).

The presence of outliers is indicated by the high values of the relative residuals index δ (about 60% for all the methods) and by the low values of the correlation coefficient (a unitary value indicates a perfect correspondence between the estimation and the experimental data). The Pearson correlation coefficient is practically the same for all the implemented methods except for the minimax one, even if considerable differences appear on the estimated balancing masses among the methods. Looking only to the values of the correlation obtained by means of the robust biweight midcorrelation, the best estimations seem to be obtained by the LMS and the S estimators. All robust methods downweight observations at critical speeds as appears on the graph of the weights used by each algorithm as shown in Figure 5 and Figure 6 for the first and the third measuring plane respectively.

These weights are those used in the last step of the IRLS algorithm implemented (exact or approximated) in almost of all proposed methods except for the LTS estimator, in which they correspond to the neglected observations in the trimmed sum of residuals. The effect of the weights to the estimation process is observable in the estimated response amplitudes of Figure 7 where both the experimentally measured response and the estimated ones are reported, for the sake of brevity only for the first measuring plane (node #4, brg. A). Estimated responses are obtained by means of the model applying the estimated balancing masses. It is possible to observe a considerable difference between measured and estimated responses at critical speed especially for the vertical direction of the measuring plane. In this case, this difference could be symptom of some inaccuracies of the model. In this sense each method, working on residuals, downweight observations where this difference is important. It is possible to observe that the critical speed frequency is well identified by the model, whereas the damping is lightly inaccurate. From this point of view, S-estimators give good estimation when the reference model is inaccurate, downweighting indeed observations in a wide range across critical speed where the mentioned difference is considerable. As expected, the estimated response of the minimax method is the one that well approximates the measured response at critical speeds. In this way this method is able to compensate the inaccuracy of the model at critical speed, reducing the worst residual, but wrongly estimating, on the contrary, the balancing mass. The responses of other robust methods are very similar and differ only for the maximum amplitude of vibrations at critical speeds.

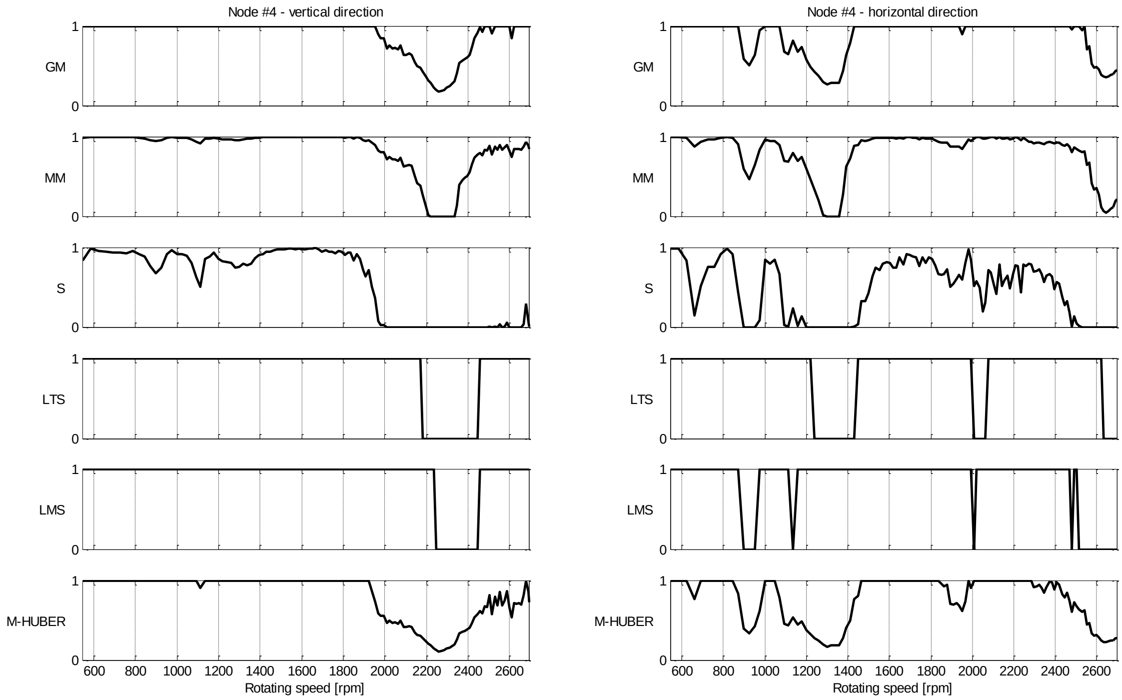

Figure 5 Estimator weights in PdM test-rig case for the first measuring plane (node #4, brg. A).

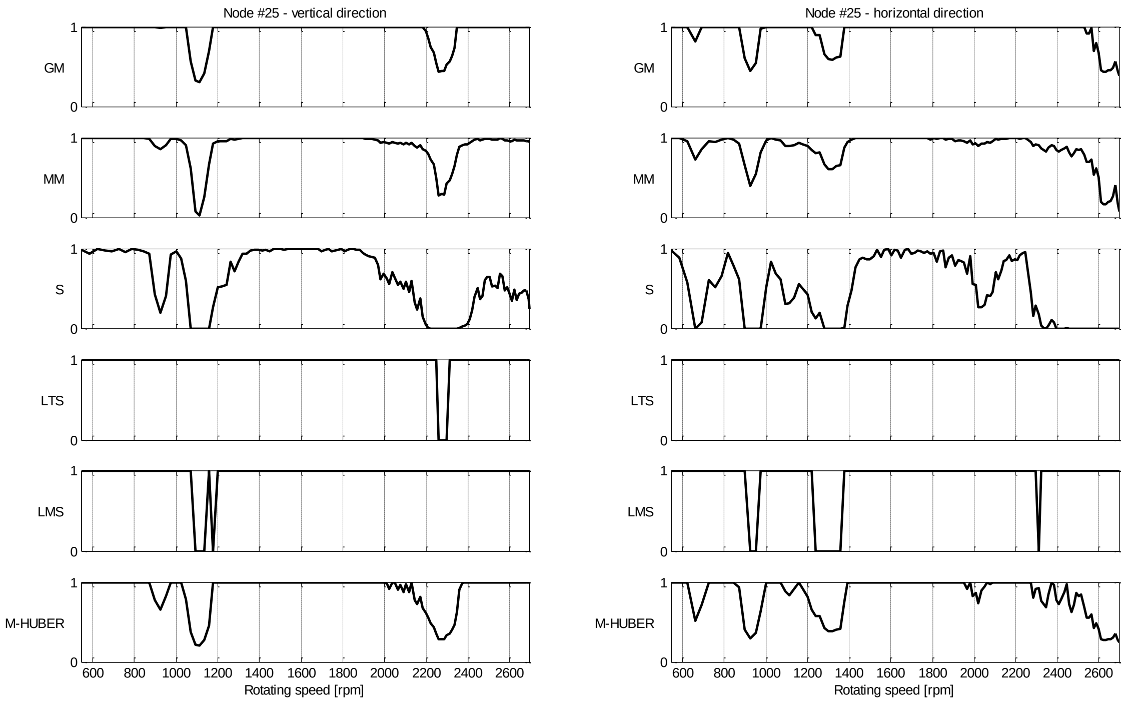

Figure 6 Estimator weights in PdM test-rig case for the third measuring plane (node #25, brg. C).

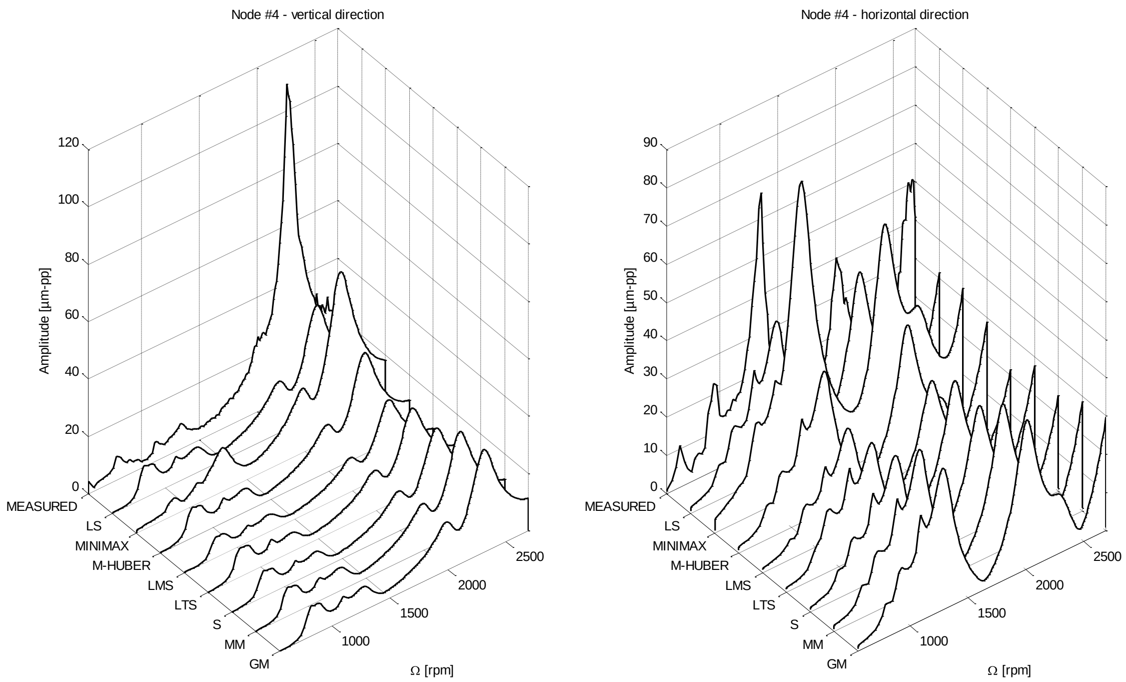

Figure 7 Experimentally measured and estimated responses (applying the estimated balancing masses) for the first measuring plane (node #4), in PdM test-rig case.

7 CONCLUDING REMARKS

The application of several robust regression methods to rotor balancing has been here investigated. The experimental case of the PdM test-rig, is analyzed. Two known unbalancing masses are applied in the same place of the selected balancing planes. In this way the aim was to perfectly identify the balancing masses equal to those applied as faults. Knowing the true value of the unbalances, the robustness and the effectiveness of robust methods are so analyzed. In the experimental case, all high breakdown-point and bounded-influence methods allow the improvement of the estimation with respect to M-estimator and classical least squares.

In the ideal case, if it is known that the unbalances are placed in the balancing planes, M, LMS, MM and S estimator well identify the modulus and the phases. Furthermore if it is known that the system is affected by consistent data corruptions or if the associate model has some inaccuracies, the S, LMS and MM (in descending order) estimators give better results. Consistent data corruptions could come from a systematic error in the measured data, or due to different acquisition conditions. For instance the behaviour of rotating machines in warm and cold thermal conditions is different and measured data could be collected in a mixed way neglecting this fact. Inaccuracies or inadequacy of the model could be overcame and identified by these robust estimators. From another point of view the residuals could be indifferently given both by error of the measured response or in the estimated one.

In a real application, where it is not obviously possible to know a priori the true value of unbalances, the estimation process should be done with different approaches. The analysis of the weights used by each method could identify some critical point or zone in the response diagram. Afterwards the analysis of estimated response and measured one could indicate if the error come from measured data or from model inaccuracies or inadequacy.

The information given by the biweight midcorrelation could be used as a selection rule of the estimate, when different estimates are available. In the considered case of the test-rig, the biweight midcorrelation index suggests to use (in descending order) LMS, S and M-estimators.

Anyhow, the use of robust estimators allows successful both unbalance identification and automatic selection of the weights without an expert's knowledge.

The behaviour of the robust S, LMS and MM estimators in a particular case when the model is only inaccurate or only inadequate should be well investigated.

The best estimator or a selection rule could be identified by the analysis of different case studies.

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Table1

PLANES	*NODES*	*EQ. MASS*	*PHASE*
Measuring	#4 , #17 , #25 , #44		
Balancing	#9 , #35		
1st unbalance	#9	3.6e-4 kg m	-90°
2nd unbalance	#35	3.6e-4 kg m	-90°

Table2

Estimator	*Balancing Plane 1 Node #9*			*Balancing Plane 2 Node #35*			*Global Rel. Error χ_G [%]*	*Rel. Res. δ*	*Correlation*	
	Amp. [kg m] 10^{-4}	*Phase [°]*	*Rel. Error χ_1 [%]*	*Amp. [kg m] 10^{-4}*	*Phase [°]*	*Rel. Error χ_2 [%]*			*Pearson*	*Biw. midcorr.*
ACTUAL	**3.60**	**-90.0**		**3.60**	**-90.0**					
LS	3.82	-94.6	10.3	4.81	-99.7	39.0	28.5	0.5842	0.6472	0.4863
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Figure1

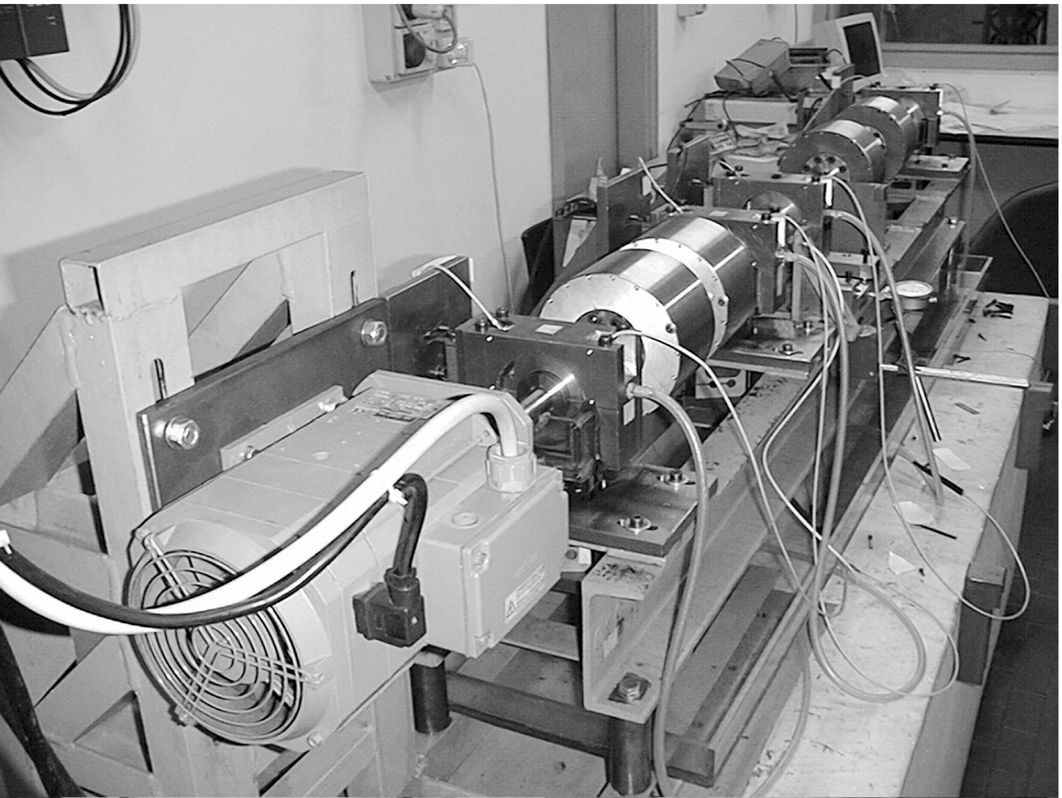
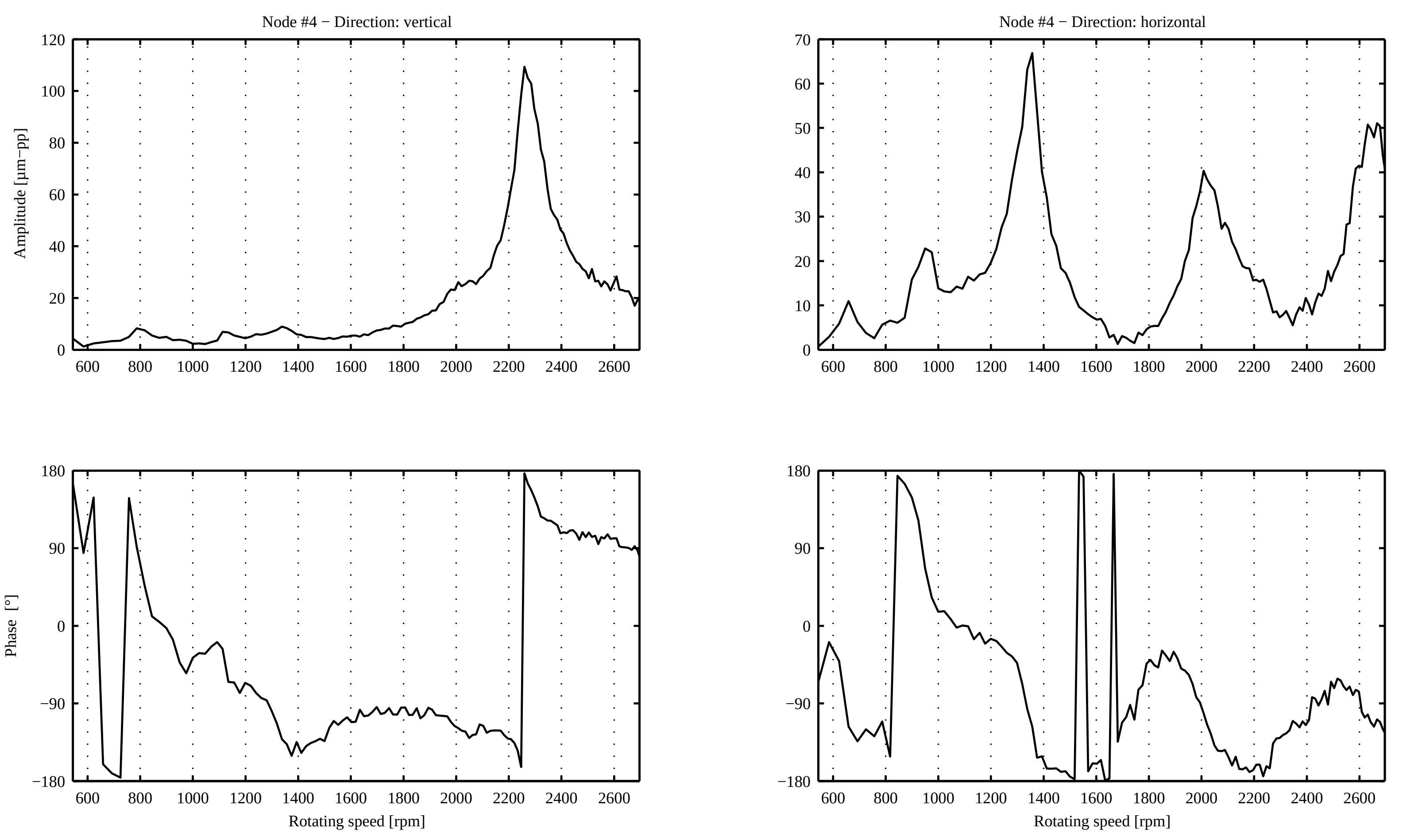
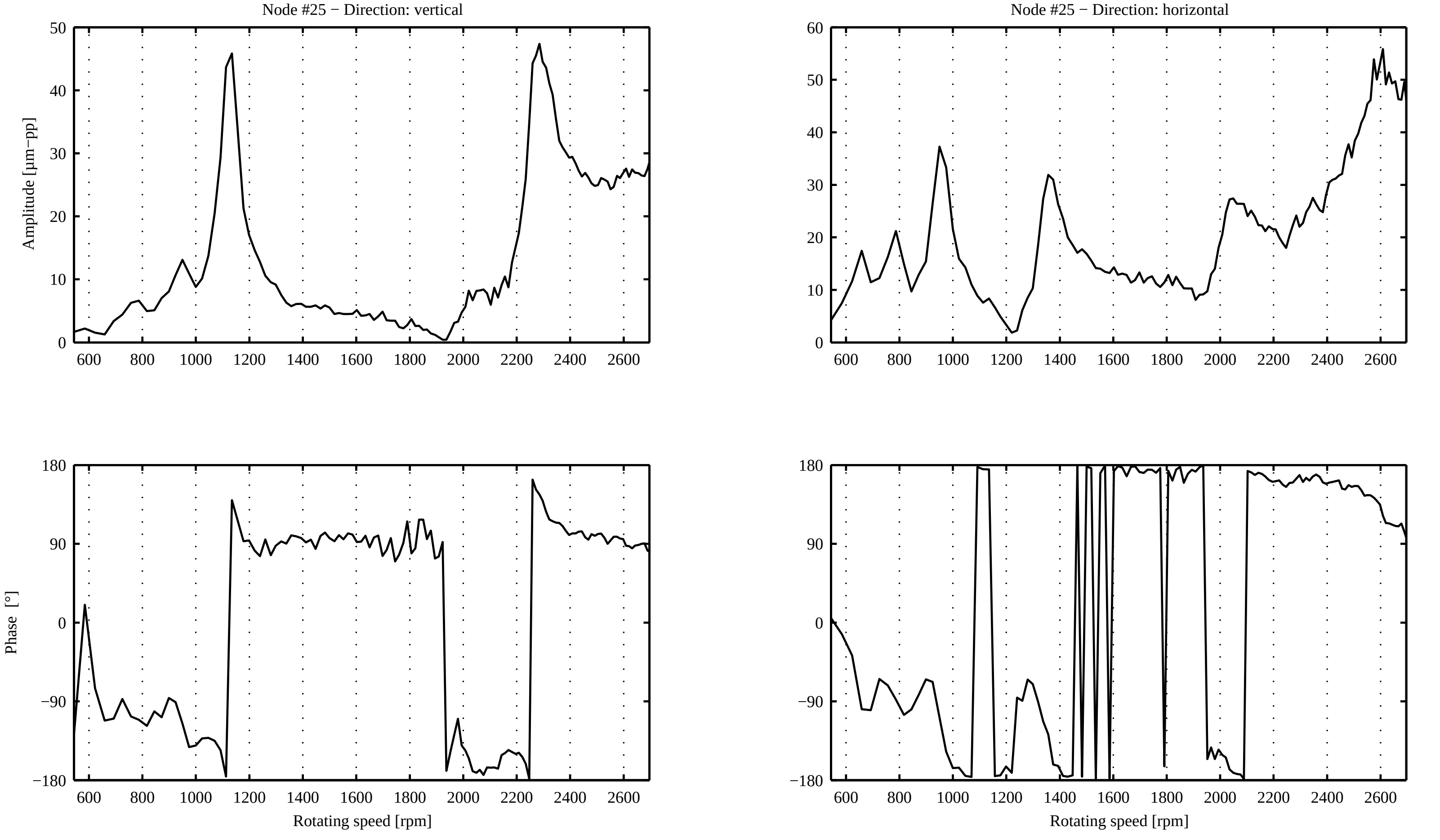


Figure2



Node #4 − Direction: vertical
Node #4 − Direction: horizontal
Amplitude [µm−pp]
Phase [°]
Rotating speed [rpm]
Rotating speed [rpm]
0
20
40
60
80
100
120
0
10
20
30
40
50
60
70
−180
−90
0
90
180
600
800
1000
1200
1400
1600
1800
2000
2200
2400
2600

Figure3



Node #25 − Direction: vertical
Node #25 − Direction: horizontal
Amplitude [μm−pp]
Phase [°]
Rotating speed [rpm]
Rotating speed [rpm]

Figure4

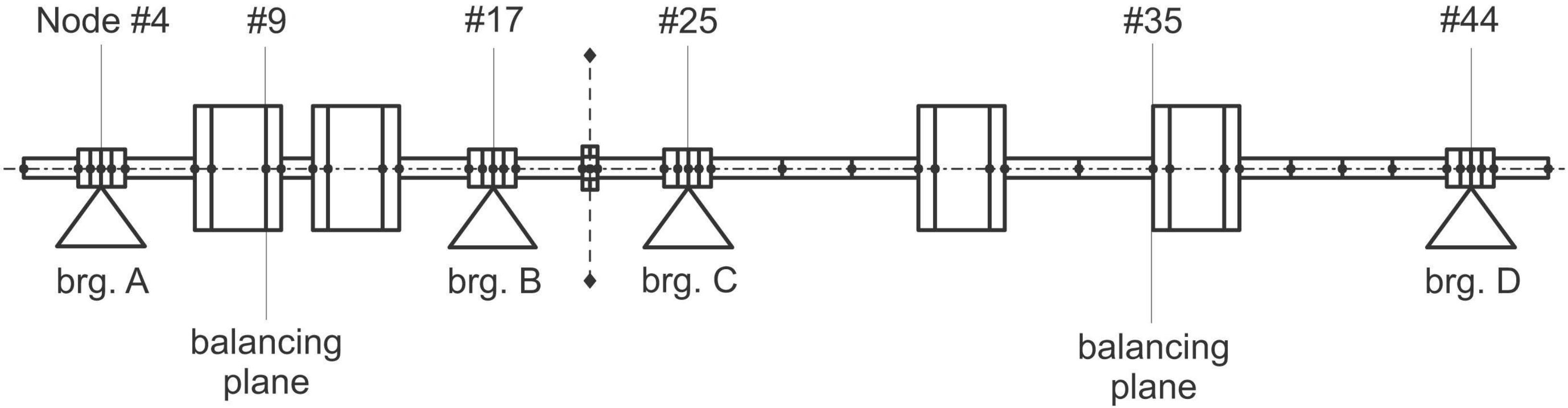
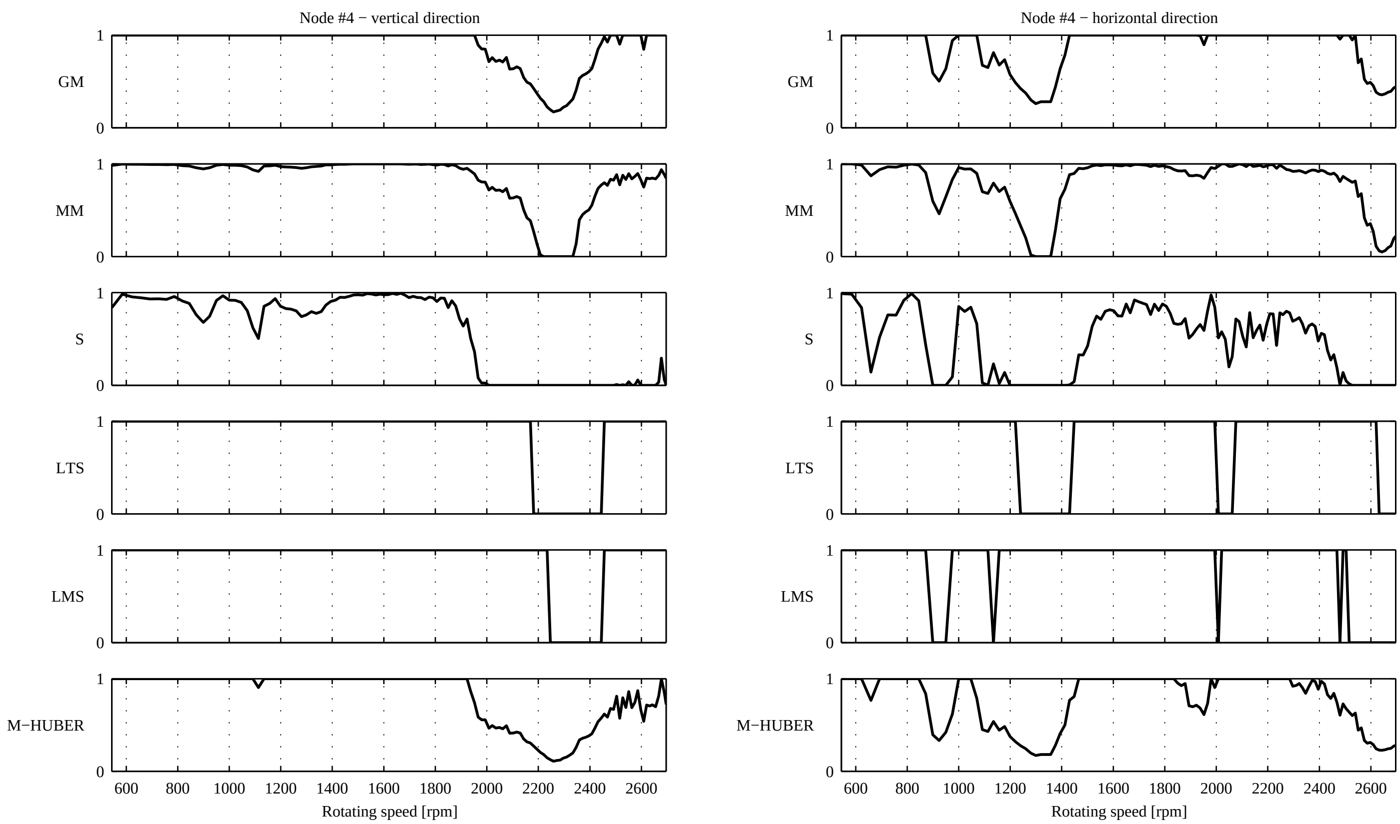

Node #4
#9
#17
#25
#35
#44
brg. A
brg. B
brg. C
brg. D
balancing plane
balancing plane

Figure5



Node #4 − vertical direction
Node #4 − horizontal direction
GM
MM
S
LTS
LMS
M−HUBER
1
0
600
800
1000
1200
1400
1600
1800
2000
2200
2400
2600
Rotating speed [rpm]

Figure6

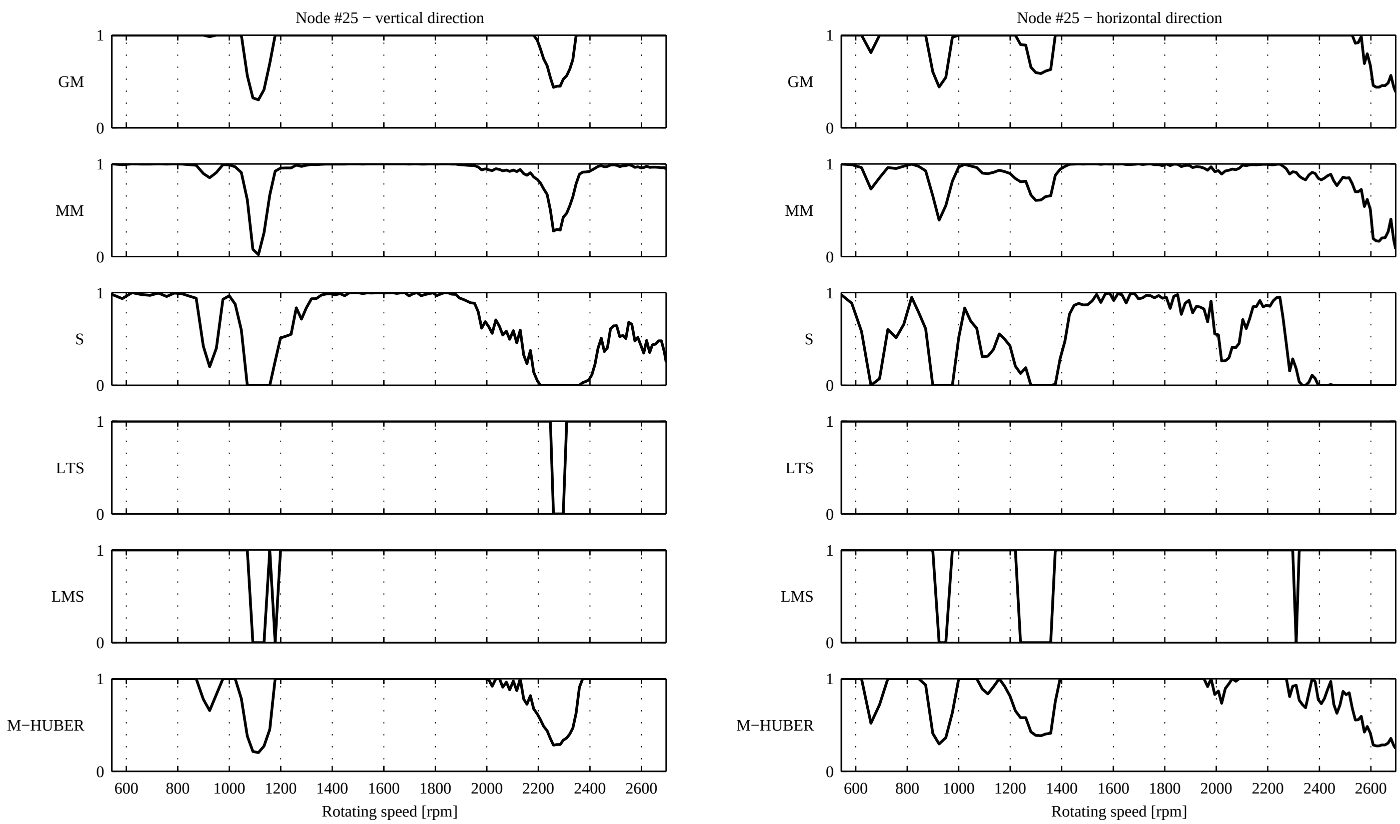
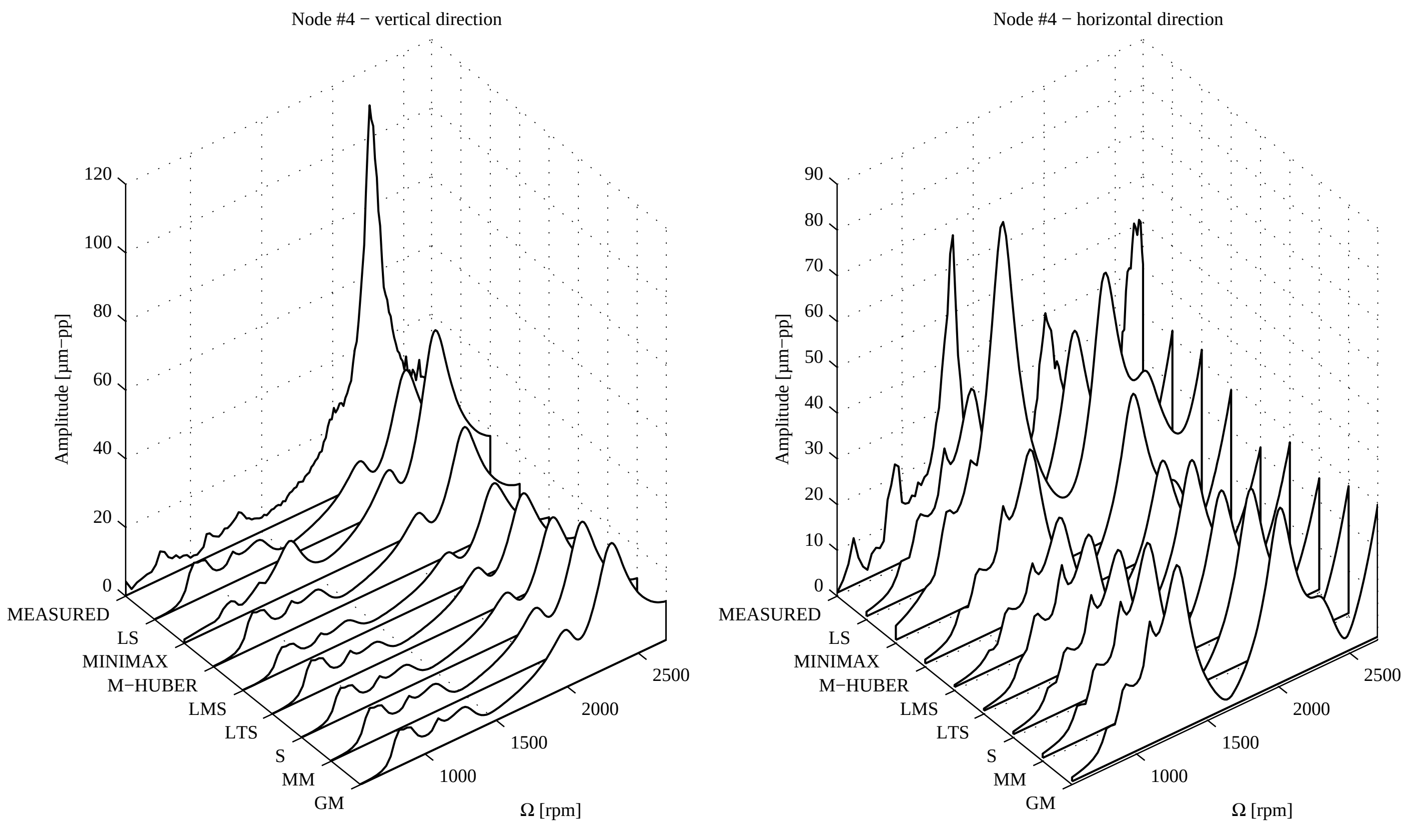

Node #25 − vertical direction
Node #25 − horizontal direction
GM
MM
S
LTS
LMS
M−HUBER
1
0
600
800
1000
1200
1400
1600
1800
2000
2200
2400
2600
Rotating speed [rpm]

Figure7



Node #4 − vertical direction
Amplitude [μm−pp]
120
100
80
60
40
20
0
MEASURED
LS
MINIMAX
M−HUBER
LMS
LTS
S
MM
GM
1000
1500
2000
2500
Ω [rpm]
Node #4 − horizontal direction
Amplitude [μm−pp]
90
80
70
60
50
40
30
20
10
0
MEASURED
LS
MINIMAX
M−HUBER
LMS
LTS
S
MM
GM
1000
1500
2000
2500
Ω [rpm]

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