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CHARACTERISTIC, ADMITTANCE AND MATCHING POLYNOMIALS OF AN ANTIREGULAR GRAPH

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An antiregular graph is a simple graph with the maximum number of vertices with different degrees. In this paper we study the characteristic polynomial, the admittance (or Laplacian) polynomial and the matching polynomial of a connected antiregular graph. For these polynomials we obtain recurrences and explicit formulas. We also obtain some spectral properties. In particular, we prove an interlacing property for the eigenvalues and we give some bounds for the energy.

1. INTRODUCTION

Any graph on n vertices, with $n \geq 2$, has at least two vertices with the same degree. The graphs with at most two vertices with the same degree are called *antiregular* [**12**, **13**], *maximally nonregular* [**20**] or *quasiperfect* [**2**, **14**, **17**]. For any positive integer n there exists only one connected antiregular graph on n vertices, denoted by A_n (see Figure 1 for some examples). Similarly there exists only one non-connected antiregular graph on n vertices, given by the complementary graph $\overline{A}_n$ of A_n. In A_n the two vertices with the same degree are those with degree $\lfloor n/2 \rfloor$.

Antiregular graphs can be described in several ways. For instance, if G_n is the graph with vertices 1, 2, . . . , n where the vertex i is adjacent to the vertex j exactly when i is even and $i > j$, then $A_n = G_n$ when n is even and $A_n = \overline{G}_n$ when n is odd. Moreover, in A_n two vertices of degree d and e, respectively, are adjacent if and only if $d + e \geq n$ (see [**14**]). This simple property gives an iterative procedure to construct the graphs A_n, starting from the complete graph K_1 on one vertex. Indeed A_{n+1} can be obtained from A_n by adding a new vertex and connecting it to all vertices with degree greater then $\lfloor n/2 \rfloor$ and to one of the two vertices with the same degree $\lfloor n/2 \rfloor$. The new node will have degree

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$\lfloor (n+1)/2 \rfloor$ and hence it will be one of the two vertices with the same degree. As an immediate consequence, we have that there always exists a vertex $v \in A_{n+1}$ such that $A_{n+1} \setminus v = A_n$.

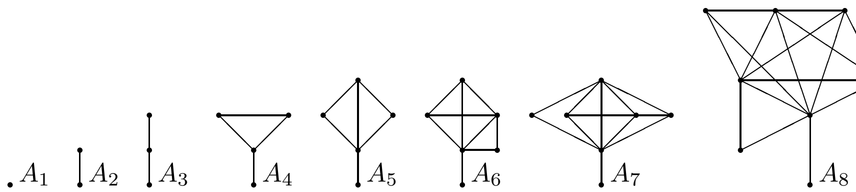

Figure 1. The first few connected antiregular graphs.

Let G_1 and G_2 be two simple graphs. The *sum* $G_1 + G_2$ is defined as their disjoint union of the two graphs, while the *complete sum* (or *join*) $G_1 \boxplus G_2$ is defined as the graph obtained from $G_1 + G_2$ joining every vertex of G_1 to every vertex of G_2 . The *complement* $\overline{G}$ of a graph G is the simple graph on the same vertex set where two distinct vertices are adjacent exactly when they are not adjacent in G . These operations are related by the identity

$$\overline{G_1 \boxplus G_2} = \overline{G}_1 + \overline{G}_2 \,. \tag{1}$$

The connected antiregular graphs A_n can also be defined by means of the recurrence

$$A_{n+1} = K_1 \boxplus \overline{A}_n \tag{2}$$

with the initial condition $A_0 = K_0$. Moreover, from identities (2) and (1), we also have the (second order) recurrence

$$A_{n+2} = K_1 \boxplus (K_1 + A_n) \tag{3}$$

(sketched in Figure 2) and the identity

$$\overline{A}_{n+1} = K_1 + A_n \,. \tag{4}$$

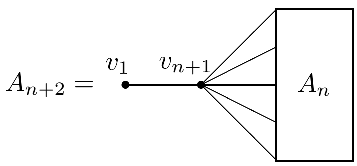

Figure 2. Recurrent structure of the connected antiregular graphs.

Antiregular graphs have several properties [**2**, **12**, **13**, **14**, **17**, **20**]. In this paper we will study the *characteristic polynomial*, the *admittance* (or *Laplacian*) *polynomial* and the *matching polynomial* of connected antiregular graphs. First we obtain two recurrences for the characteristic polynomials. Then we obtain their generating series and we show that they can be expressed in terms of *Chebyshev polynomials* of the first and the second kind. Moreover, we obtain an explicit form for the admittance polynomial and we show that it can be expressed in terms of the Stirling numbers of the first kind. Finally, we study the matching polynomial and also in this case we give a recurrence, a generating series and some explicit expressions.

2. CHARACTERISTIC POLYNOMIAL

The *adjacency matrix* of a simple graph G is a matrix with rows and columns labelled by the vertices of G, with 1 or 0 in position (i,j) according to whether the vertices v_i and v_j are adjacent or not. The *characteristic polynomial* of G is the characteristic polynomial of its adjacency matrix [**4**, p. 12], and here it will be denoted by $\varphi(G;\lambda)$. For the characteristic polynomial of the sum of two graphs we have the identity

$$\varphi(G_1+G_2;\lambda)=\varphi(G_1;\lambda)\,\varphi(G_2;\lambda)\,. \tag{5}$$

There is also a formula, due to Cvetković, for the characteristic polynomial of the complete sum of two arbitrary graphs. Here, however, we only need the following particular case

$$\varphi(G\boxplus K_1;\lambda)=(\lambda+1)\,\varphi(G;\lambda)+(-1)^n\varphi(\overline{G};-\lambda-1)\,. \tag{6}$$

2.1. RECURRENCES

For simplicity we will write $\varphi_n(\lambda)=\varphi(A_n;\lambda)$ for the characteristic polynomial of the antiregular graph A_n.

Theorem 1. *The characteristic polynomials of the antiregular graphs satisfy the second order non-linear recurrence*

$$\varphi_{n+2}(\lambda)=(-1)^n\varphi_{n+1}(-\lambda-1)+(\lambda^2+\lambda)\,\varphi_n(\lambda)\,. \tag{7}$$

and the fourth order linear recurrence

$$\varphi_{n+4}(\lambda)=(2\lambda^2+2\lambda-1)\varphi_{n+2}(\lambda)-(\lambda^2+\lambda)^2\,\varphi_n(\lambda)\,. \tag{8}$$

Proof. From recurrence (2), and identity (6), we have

$$\varphi(A_{n+1};\lambda)=\varphi(\overline{A}_n\boxplus K_1;\lambda)=(\lambda+1)\,\varphi(\overline{A}_n;\lambda)-(-1)^n\varphi(A_n;-\lambda-1)\,.$$

Similarly, from (2) and the properties of the characteristic polynomial, we have

$$\varphi(\overline{A}_{n+1};\lambda)=\varphi(A_n+K_1;\lambda)=\varphi(A_n;\lambda)\,\varphi(K_1;\lambda)=\lambda\,\varphi(A_n;\lambda)\,.$$

Therefore we have

$$\begin{aligned}\varphi(A_{n+2};\lambda)&=(\lambda+1)\,\varphi(\overline{A}_{n+1};\lambda)-(-1)^{n+1}\varphi(A_{n+1};-\lambda-1)\\&=(\lambda+1)\lambda\,\varphi(A_n;\lambda)+(-1)^n\varphi(A_{n+1};-\lambda-1)\,,\end{aligned}$$

that is we have (7). Now, substituting $n+1$ to n in (7) and then simplifying by using (7) itself, it is easy to obtain the identity

$$\varphi_{n+3}(\lambda)=(\lambda^2+\lambda-1)\varphi_{n+1}(\lambda)-(-1)^n(\lambda^2+\lambda)\,\varphi_n(-\lambda-1)\,.$$

Again, substituting $n+1$ to n in this last equation and then simplifying by using (7), it easily follows recurrence (8). □

The first few polynomials $\varphi_n(\lambda)$ are

$$\begin{aligned}
\varphi_0(\lambda) &= 1\\
\varphi_1(\lambda) &= \lambda\\
\varphi_2(\lambda) &= \lambda^2 - 1\\
\varphi_3(\lambda) &= \lambda^3 - 2\lambda\\
\varphi_4(\lambda) &= \lambda^4 - 4\lambda^2 - 2\lambda + 1\\
\varphi_5(\lambda) &= \lambda^5 - 6\lambda^3 - 4\lambda^2 + 2\lambda\\
\varphi_6(\lambda) &= \lambda^6 - 9\lambda^4 - 10\lambda^3 + 3\lambda^2 + 4\lambda - 1\\
\varphi_7(\lambda) &= \lambda^7 - 12\lambda^5 - 16\lambda^4 + 4\lambda^3 + 8\lambda^2 - 2\lambda\\
\varphi_8(\lambda) &= \lambda^8 - 16\lambda^6 - 28\lambda^5 + 2\lambda^4 + 24\lambda^3 + 2\lambda^2 - 6\lambda + 1\\
\varphi_9(\lambda) &= \lambda^9 - 20\lambda^7 - 40\lambda^6 + 40\lambda^4 + 6\lambda^3 - 12\lambda^2 + 2\lambda
\end{aligned}$$

2.2. GENERATING SERIES AND EXPLICIT EXPRESSIONS

From recurrence (8) it follows straightforwardly that the generating series for the characteristic polynomials is

$$\varphi(\lambda;t) = \sum_{n\geq 0} \varphi_n(\lambda)\, t^n = \frac{1 + \lambda t - (\lambda^2 + 2\lambda)t^2 - (\lambda^3 + 2\lambda^2 + \lambda)t^3}{1 - (2\lambda^2 + 2\lambda - 1)t^2 + (\lambda^2 + \lambda)^2 t^4}\,. \tag{9}$$

Let us now consider the polynomials $f_n(\lambda) = \varphi_{2n}(\lambda)$ and $g_n(\lambda) = \varphi_{2n+1}(\lambda)$. Since series (9) can be written as

$$\varphi(\lambda;t) = \frac{1 - (\lambda^2 + 2\lambda)t^2}{1 - (2\lambda^2 + 2\lambda - 1)t^2 + (\lambda^2 + \lambda)^2 t^4} + \lambda t\, \frac{1 - (\lambda + 1)^2 t^2}{1 - (2\lambda^2 + 2\lambda - 1)t^2 + (\lambda^2 + \lambda)^2 t^4}$$

it immediately follows that

$$f(\lambda;t) = \sum_{n\geq 0} f_n(\lambda)\, t^n = \frac{1 - (\lambda^2 + 2\lambda)t}{1 - (2\lambda^2 + 2\lambda - 1)t + (\lambda^2 + \lambda)^2 t^2} \tag{10}$$

$$g(\lambda;t) = \sum_{n\geq 0} g_n(\lambda)\, t^n = \frac{\lambda - \lambda(\lambda + 1)^2 t}{1 - (2\lambda^2 + 2\lambda - 1)t + (\lambda^2 + \lambda)^2 t^2}\,. \tag{11}$$

Theorem 2. *The characteristic polynomials of the connected antiregular graphs admit the explicit expressions*

$$\varphi_{2n}(\lambda) = \sum_{k=0}^{n} \binom{n+k}{2k} (-1)^k \left(\frac{2k}{n+k}\,\lambda + 1\right) (\lambda^2 + \lambda)^{n-k}$$

$$\varphi_{2n+1}(\lambda)=\sum_{k=0}^{n}\binom{n+k}{2k}(-1)^k\left(\lambda-\frac{n-k}{2k+1}\right)(\lambda^2+\lambda)^{n-k}\,.$$

Proof. These identities come from series (10) and (11), by using the expansion

$$\frac{1}{1-(2\lambda^2+2\lambda-1)t+(\lambda^2+\lambda)^2t^2}=\frac{1}{(1-(\lambda^2+\lambda)t)^2+t}$$
$$=\frac{1}{(1-(\lambda^2+\lambda)t)^2}\,\frac{1}{1+\dfrac{t}{(1-(\lambda^2+\lambda)t)^2}}=\sum_{k\geq 0}\frac{(-1)^kt^k}{(1-(\lambda^2+\lambda)t)^{2k+2}}\,.\qquad\square$$

Remark. From Theorem 2 it easily follows that $\varphi_{2n}(0)=(-1)^n$ (for $n\geq 1$), $\varphi_{2n}(-1)=0$, $\varphi_{2n+1}(0)=0$, $\varphi_{2n+1}(-1)=(-1)^{n+1}$. Notice that -1 is never a root of $\varphi_{2n+1}(\lambda)$.

Theorem 3. *The characteristic polynomials of the connected antiregular graphs can be expressed in terms of the Chebyshev polynomials as follows*:

$$\varphi_{2n}(\lambda)=\frac{2(\lambda^2+2\lambda)(\lambda^2+\lambda)^n}{2\lambda^2+2\lambda-1}\,T_n(H(\lambda))-\frac{(2\lambda+1)(\lambda^2+\lambda)^n}{2\lambda^2+2\lambda-1}\,U_n(H(\lambda))$$

$$\varphi_{2n+1}(\lambda)=\frac{2\lambda(\lambda+1)^2(\lambda^2+\lambda)^n}{2\lambda^2+2\lambda-1}\,T_n(H(\lambda))-\frac{\lambda(2\lambda+3)(\lambda^2+\lambda)^n}{2\lambda^2+2\lambda-1}\,U_n(H(\lambda))$$

where

$$H(\lambda)=\frac{2\lambda^2+2\lambda-1}{2(\lambda^2+\lambda)}\,.$$

Proof. Since the generating series for the Chebyshev polynomials are

$$T(x;t)=\sum_{n\geq 0}T_n(x)\,t^n=\frac{1-xt}{1-2xt+t^2}\,,\quad U(x;t)=\sum_{n\geq 0}U_n(x)\,t^n=\frac{1}{1-2xt+t^2}\,,$$

from (10) and (11) it follows straightforwardly that

$$f(\lambda;t)=\frac{2(\lambda^2+2\lambda)}{2\lambda^2+2\lambda-1}\,T\big(H(\lambda);(\lambda^2+\lambda)t\big)-\frac{2\lambda+1}{2\lambda^2+2\lambda-1}\,U\big(H(\lambda);(\lambda^2+\lambda)t\big)$$

$$g(\lambda;t)=\frac{2\lambda(\lambda+1)^2}{2\lambda^2+2\lambda-1}\,T\big(H(\lambda);(\lambda^2+\lambda)t\big)-\frac{\lambda(2\lambda+3)}{2\lambda^2+2\lambda-1}\,U\big(H(\lambda);(\lambda^2+\lambda)t\big)\,.$$

Finally, expanding these series, we have the claimed identities. $\square$

Theorem 4. *The characteristic polynomials of the connected antiregular graphs admit the explicit expressions*

$$\varphi_{2n}(x)=(\lambda+1)\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\frac{n}{n-k}\,(-1)^k\,(n\lambda-n+k)H_{nk}(\lambda)\tag{12}$$

$$\varphi_{2n+1}(x) = \lambda \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \frac{n}{n-k} (-1)^k (n\lambda^2 + 2k\lambda - 2n + 3k) H_{nk}(\lambda) \tag{13}$$

where $H_{nk}(\lambda) = (2\lambda^2 + 2\lambda - 1)^{n-2k-1}(\lambda^2 + \lambda)^{2k}$.

Proof. These identities derive straightforwardly from Theorem 3 and the following well known expressions of the Chebyshev polynomials [**15**]:

$$T_n(x) = \frac{1}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \frac{n}{n-k} (-1)^k (2x)^{n-2k}$$

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} (-1)^k (2x)^{n-2k} . \qquad \square$$

Similarly, we have

Theorem 5. *The characteristic polynomials of the connected antiregular graphs admit the explicit expressions*

$$\varphi_{2n}(\lambda) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{2(2k+1)\lambda^2 - 2(n-4k+1)\lambda - n - 1}{2k+1} (-1)^k h_{nk}(\lambda)$$

$$\varphi_{2n+1}(\lambda) = \frac{\lambda}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \frac{2(2k+1)\lambda^2 - 2(n-4k-1)\lambda - 3n + 4k - 1}{2k+1} (-1)^k h_{nk}(\lambda)$$

where $h_{nk}(\lambda) = (2\lambda^2 + 2\lambda - 1)^{n-2k-1}(4\lambda^2 + 4\lambda - 1)^k$.

Proof. It is sufficient to use Theorem 3 and the expressions:

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k}(x^2-1)^k , \quad U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} x^{n-2k}(x^2-1)^k . \qquad \square$$

2.3. EIGENVALUES

As usual, let $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ be the vector of length n with the entry equal to 1 in position i . Then we have

Theorem 6. *For every* $n > 0$ *,* $\lambda = -1$ *is a simple eigenvalue of* A_{2n} *and the associated eigenspace is* $V_{-1} = \langle(-\mathbf{e}_n, \mathbf{e}_1)\rangle = \langle(0, \dots, 0, -1, 1, 0, \dots, 0)\rangle$ *. For every* $n \geq 0$ *,* $\lambda = 0$ *is an eigenvalue of* A_{2n+1} *and the associated eigenspace is* $V_0 = \langle(\mathbf{e}_n, -1, \mathbf{0})\rangle = \langle(0, \dots, 0, -1, 1, 0, 0, \dots, 0)\rangle$.

Proof. From (12) it follows that $\lambda = -1$ is a simple root of $\varphi_{2n}(\lambda)$ for every $n > 0$, while from (13) it follows that $\lambda = 0$ is a simple root of $\varphi_{2n+1}(\lambda)$ for every $n \geq 0$. To obtain the associated eigenspaces we use the adjacency matrices. Ordering the vertices according to their degree, the adjacency matrix of A_n turns out to be the matrix $\mathcal{A}_n = [a_{ij}]_{i,j=1}^n$ where $a_{ij} = 1$ when $i \neq j$ and $i + j > n$, and $a_{ij} = 0$ otherwise. Hence we have the matrices

$$\mathcal{A}_{2n} = \begin{bmatrix} O & T \\ T & U - I \end{bmatrix} \qquad \text{and} \qquad \mathcal{A}_{2n+1} = \begin{bmatrix} O & \mathbf{0} & T \\ \mathbf{0}_T & 0 & \mathbf{u}_T \\ T & \mathbf{u} & U - I \end{bmatrix}$$

where O is the $n \times n$ null matrix, T is the $n \times n$ matrix with all entries equal to 1 below or on the secondary diagonal and with all other entries equal to 0, U is the $n \times n$ matrix with all entries equal to 1, $\mathbf{0}$ is the zero column vector of length n and $\mathbf{u}$ is the column vector of length n with all entries equal to 1. Now, considering $\mathbf{e}_i$ as a column vector of length n, we have $T\mathbf{e}_1 = \mathbf{e}_n$, $T\mathbf{e}_n = \mathbf{u}$, $U\mathbf{e}_1 = \mathbf{u}$. At this point it is straightforward to prove that

$$\begin{bmatrix} \mathbf{e}_n \\ -\mathbf{e}_1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \mathbf{e}_n \\ -1 \\ \mathbf{0} \end{bmatrix}$$

are eigenvectors of $\mathcal{A}_{2n}$ and $\mathcal{A}_{2n+1}$ with respect to the eigenvalues $\lambda = -1$ and $\lambda = 0$. □

Theorem 7. *The determinant of the adjacency matrix $\mathcal{A}_n$ of the antiregular graph A_n is*

$$\det \mathcal{A}_n = \begin{cases} (-1)^{n/2} & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd.} \end{cases}$$

Proof. Since $\det \mathcal{A}_n = (-1)^n \varphi_n(0)$, from (9) it follows that the generating series for these numbers is $\varphi(0; -t) = (1 + t^2)^{-1}$. □

Theorem 8. *If λ_n^* is the maximum eigenvalue of A_n then*

$$\frac{2}{n} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \leq \lambda_n^* \leq \frac{n \lfloor n/2 \rfloor}{1 + \lfloor n/2 \rfloor}.$$

Proof. For every graph G with n vertices, m edges, vertex degrees $d_1, \ldots, d_n$, chromatic index $\chi(G)$ and maximum eigenvalue μ, we have the inequalities $2m/n \leq \mu \leq \max_{1 \leq i \leq n} d_i$ [**6**, Theorem 5.8, p. 43] and $\chi(G) \geq n/(n - \mu)$ [**6**, Theorem 3.19, p. 92]. Here $m = \lfloor n/2 \rfloor \lceil n/2 \rceil$ (see [**14**]), $\max_{1 \leq i \leq n} d_i = n - 1$ and $\chi(A_n) = \lceil (n+1)/2 \rceil = \lfloor n/2 \rfloor + 1$ (see [**14**]). □

The polynomials $f_n(\lambda)$ and $g_n(\lambda)$ admit a sort of interlacing property. Specifically we have

Theorem 9. *The polynomial $f_n(\lambda)$ has $2n$ distinct real roots, n negative and n positive, while the polynomial $g_n(\lambda)$ has $2n + 1$ distinct real roots, n negative, one zero and n positive. Moreover, if $r_{-n} < \cdots < r_{-2} < -1 < r_1 < \cdots < r_n$ are* *the roots of* $f_n(\lambda)$ *and* $s_{-n-1} < \cdots < s_{-2} < -1 < s_1 < \cdots < s_{n+1}$ *are the roots of* $f_{n+1}(\lambda)$ *then*

$$s_{-n-1} < r_{-n} < s_{-n} < \cdots < r_{-2} < s_{-2} < -1 < s_1 < r_1 < \cdots < s_n < r_n < s_{n+1}\,.$$

Similarly, if $r_{-n} < \cdots < r_{-1} < 0 < r_1 < \cdots < r_n$ *are the roots of* $g_n(\lambda)$ *and* $s_{-n-1} < \cdots < s_{-1} < 0 < s_1 < \cdots < s_{n+1}$ *are the roots of* $g_{n+1}(\lambda)$ *then*

$$s_{-n-1} < r_{-n} < s_{-n} < \cdots < r_{-1} < s_{-1} < 0 < s_1 < r_1 < \cdots < s_n < r_n < s_{n+1}\,.$$

Proof. The proof is similar for both the sequences of polynomials. Here we prove the part concerning the polynomials $g_n(\lambda)$ and we leave to the reader the part concerning the other polynomials. First of all, notice that $g_n(\lambda)$ has degree $2n+1$ and always admit 0 as root (Theorem 6). Moreover, from series (11) we have the recurrence

$$g_{n+2}(\lambda) = (2\lambda^2 + 2\lambda - 1)g_{n+1}(\lambda) - (\lambda^2+\lambda)^2 g_n(\lambda) \tag{14}$$

with the initial conditions $g_0(\lambda) = \lambda$ and $g_1(\lambda) = \lambda^3 - 2\lambda$. For the first few polynomials the theorem can be verified directly. Now we proceed by induction on n. Suppose that the theorem is true for a given n. To prove that it is true also for $n+1$ we will consider the following intervals.

1. $I = (s_{-i-1}, s_{-i})$ with $1 \leq i \leq n$. Since $g_{n+1}(s_{-i-1})g_n(s_{-i}) < 0$ by induction hypothesis, from recurrence (14) it follows at once that

$$\begin{aligned} & g_{n+2}(s_{-i-1})g_{n+2}(s_{-i}) \\ & \quad = (s_{-i-1}^2 - s_{-i-1})^2(s_{-i}^2 - s_{-i})^2 g_{n+1}(s_{-i-1})g_n(s_{-i}) < 0\,. \end{aligned}$$

   Hence, by continuity, $g_{n+2}(\lambda)$ has at least a root inside the interval I.

2. $I = (s_i, s_{i+1})$ with $1 \leq i \leq n$. Exactly as in the previous case, it can be proved that $g_{n+2}(\lambda)$ has at least a root in I.

3. $I = (-\infty, s_{-n-1})$. From (14) we have the identity

$$g_{n+2}(s_{-n-1}) = -(s_{-n-1}^2 - s_{-n-1})^2 g_n(s_{-n-1})\,.$$

   Having odd degree, $g_n(\lambda)$ tends to $-\infty$ as $\lambda \to -\infty$. Since r_{-n} is the smallest root of $g_n(\lambda)$ and $s_{-n-1} < r_{-n}$, it follows that $g_n(s_{-n-1}) < 0$ and consequently that $g_{n+2}(s_{-n-1}) > 0$. Hence, since $g_{n+2}(\lambda) \to -\infty$ as $\lambda \to -\infty$, it follows that $g_{n+2}(\lambda)$ has at least a root in I.

4. $I = (s_{n+1}, +\infty)$. From (14) we obtain

$$g_{n+2}(s_{n+1}) = -(s_{n+1}^2 - s_{n+1})^2 g_n(s_{n+1})\,.$$

   Clearly $g_n(\lambda) \to +\infty$ as $\lambda \to +\infty$. Moreover, since r_n is the greatest root of $g_n(\lambda)$ and $r_n < s_{n+1}$, we have $g_n(s_{n+1}) > 0$ and $g_{n+2}(s_{n+1}) < 0$. Since $g_{n+2}(\lambda) \to +\infty$ as $\lambda \to +\infty$, it follows that $g_{n+2}(\lambda)$ has at least a root in I.

5. $I = (s_{-1}, 0)$. Since from (14) we have $g_{n+2}(s_{-1}) = -(s_{-1}^2 - s_{-1})^2 g_n(s_{-1})$, $g_{n+2}(s_{-1})$ and $g_n(s_{-1})$ have discordant signs. Moreover, always from (14), we have $g'_{n+2}(0) = -g'_{n+1}(0)$. This implies that $g_{n+2}(\lambda)$ and $g_n(\lambda)$ have slopes with the same sign in $\lambda = 0$. So, finally, we can conclude that $g_{n+2}(\lambda)$ has a root also in this interval.

6. $I = (0, s_1)$. As in case 5, it can be proved that $g_{n+2}(\lambda)$ has at least a root in I.

We have found at least $2n+5$ roots of $g_{n+2}(\lambda)$ (including 0 which is always a root). Since $g_{n+2}(\lambda)$ has degree $2n+5$, these are exactly all its roots. □

Theorem 9 immediately imply

Theorem 10. *The antiregular graph A_{2n} has $2n$ simple real eigenvalues: n negative and n positive. The antiregular graph A_{2n+1} has $2n+1$ simple real eigenvalues: n negative, one zero and n positive.*

Let $p(x)$ be a polynomial of degree n with real roots $r_1 \le r_2 \le \cdots \le r_n$ and let $q(x)$ be a polynomial of degree $n{+}1$ with real roots $s_1 \le s_2 \le \cdots \le s_{n+1}$. They have the *interlacing property* when $s_1 \le r_1 \le s_2 \le r_2 \le \cdots \le s_n \le r_n \le s_{n+1}$. A sequence $\{p_n(x)\}_n$ of polynomials is a *Sturm sequence* when every $p_n(x)$ is a real polynomial of degree n with n real distinct roots, and $p_n(x)$ and $p_{n+1}(x)$ have the interlacing property for every $n \in \mathbb{N}$.

Theorem 11. *The characteristic polynomials of the connected antiregular graphs form a Sturm sequence.*

Proof. Clearly $\varphi_n(\lambda)$ is a real polynomial of degree n with all real roots. Moreover, Theorem 10 says that all these roots are simple. Finally, since for every graph G the characteristic polynomials $\varphi(G;\lambda)$ and $\varphi(G \setminus v;\lambda)$ (where v is any vertex of G) have the interlacing property [**8**, p. 29] and since there always exists a vertex $v \in A_{n+1}$ such that $A_{n+1} \setminus v = A_n$ (as we recalled in the introduction), also the sequence $\{\varphi_n(\lambda)\}_n$ has the interlacing property. □

The *energy* of a graph G is defined as the sum of the absolute values of all its eigenvalues (see, for instance, [**1**, **9**, **10**]). For the first few values of n we have $E(A_1) = 0$, $E(A_2) = 2$, $E(A_2) = 2\sqrt{2}$. More generally, we have

Theorem 12. *For every $n \in \mathbb{N}$, the energy of the antiregular graph A_n satisfies the inequalities*

$$2\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil} \le E(A_n) \le 2\left\lfloor \frac{n}{2} \right\rfloor \sqrt{\left\lceil \frac{n}{2} \right\rceil}\,.$$

Proof. For every graph G with m edges, t triangles, q quadrilaterals, n_+ positive eigenvalues and n_- negative eigenvalues, we have the inequalities (see [**3**] and formula (5.17) in [**5**])

$$2\sqrt{m} \le E(G) \le \sqrt{2m(n_+ + n_-) - (n_+ - n_-)^2 \left(\frac{q}{2m} - \frac{9t^2}{m^2}\right)}\,.$$

For the antiregular graph A_n we have $n_+ = n_- = \lfloor n/2 \rfloor$ (Theorem 10) and $m = \lfloor n/2 \rfloor \lceil n/2 \rceil$. □

REMARK. As observed by I. GUTMAN in a personal communication, the upper bound for the energy can be slightly improved by using the KOOLEN-MOULTON inequality [**11**]

$$E(G) \leq \lambda^* + \sqrt{(n-1)(2m-(\lambda^*)^2)}$$

which holds for any graph G with n vertices, m edges and greatest eigenvalue λ^*. Since this inequality continues to be valid also when λ^* is replaced by its lower bound, Theorem 8 implies

$$E(A_n) \leq \frac{2}{n} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + \sqrt{2(n-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{2}{n^2} \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil\right)}\,.$$

3. ADMITTANCE POLYNOMIAL

Another classical polynomial associated to a graph G is the *admittance polynomial*, that is the characteristic polynomial $C(G;\lambda) = |\lambda I - C|$ of the *admittance matrix* (*Laplacian matrix*) $C = D - A$, where A is the adjacency matrix and D is the degree matrix (the diagonal matrix with the vertex degrees as diagonal entries).

Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, respectively. Then the admittance polynomial for their sum and their complete sum is given by (formulae (2.12) and (2.18) in [**4**]):

$$\begin{aligned} C(G_1 + G_2;\lambda) &= C(G_1;\lambda)C(G_2;\lambda) \\ C(G_1 \boxplus G_2;\lambda) &= \frac{\lambda(\lambda - n_1 - n_2)}{(\lambda - n_1)(\lambda - n_2)}\, C(G_1;\lambda - n_2)C(G_2;\lambda - n_1)\,. \end{aligned}$$

These formulae are what we need to determinate the admittance polynomial for the antiregular graphs A_n. Indeed, using recurrence (3), we have

$$\begin{aligned} C(A_{n+2};\lambda) &= C(K_1 \boxplus (K_1 + A_n);\lambda) \\ &= \frac{\lambda(\lambda - n - 2)}{(\lambda - 1)(\lambda - n - 1)}\, C(K_1;\lambda - n - 1)C(K_1 + A_n;\lambda - 1)\,. \end{aligned}$$

Since $C(K_1;\lambda) = \lambda$ we have

$$C(A_{n+2};\lambda) = \frac{\lambda(\lambda - n - 2)}{\lambda - 1}\, C(K_1;\lambda - 1)C(A_n;\lambda - 1)\,.$$

Hence, writing for simplicity $C_n(\lambda) = C(A_n;\lambda)$, we have

$$C_{n+2}(\lambda) = \lambda(\lambda - n - 2)\, C_n(\lambda - 1)\,. \tag{15}$$

Now iterating recurrence (15) we can easily obtain the identity

$$C_n(\lambda) = \lambda^{\underline{k}} (\lambda - n)^{\overline{k}}\, C_{n-2k}(\lambda - k)$$

where $\lambda^{\underline{k}} = \lambda(\lambda-1)\cdots(\lambda-k+1)$ are the *falling factorial polynomials* and $\lambda^{\overline{k}} = \lambda(\lambda+1)\cdots(\lambda+k-1)$ are the *raising factorial polynomials.* Since $C_0(\lambda) = 1$ and $C_1(\lambda) = \lambda$, it follows that

$$C_{2n}(\lambda) = \lambda^{\underline{n}}\,(\lambda-2n)^{\overline{n}}\,C_0(\lambda-n) = \lambda^{\underline{n}}\,(\lambda-2n)^{\overline{n}} = \frac{\lambda^{\underline{2n+1}}}{\lambda-n}$$

$$C_{2n+1}(\lambda) = \lambda^{\underline{n}}\,(\lambda-2n-1)^{\overline{n}}\,C_1(\lambda-n) = \lambda^{\underline{n+1}}\,(\lambda-2n-1)^{\overline{n}} = \frac{\lambda^{\underline{2n+2}}}{\lambda-n-1}\,.$$

Consequently, we have the identity

$$C_n(\lambda) = \frac{\lambda^{\underline{n+1}}}{\lambda-\lceil n/2\rceil}\,. \tag{16}$$

This result can also be obtained by using MERRIS's theorem [**12**,p. 184] observing that A_n is a threshold graph.

At this point we can also obtain the *Kel'mans polynomial* $B_n(\lambda) = B(A_n;\lambda)$ (see (1.17) in [**4**]). Indeed

$$B_n(\lambda) = \frac{(-1)^n}{n}\,C_n(-\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}{\lambda+\lceil n/2\rceil}\,.$$

In particular, since $t(G) = B(G;0)/n$ (see [**4**, p. 39]), we reobtain the number of all spanning trees of A_n calculated in [**17**], namely

$$t(A_n) = \frac{1}{n}\,B_n(0) = \frac{(n-1)!}{\lceil n/2\rceil}\,.$$

The coefficients of the admittance polynomial of a graph G can be interpreted combinatorially using SACHS's theorem [**4**, p. 38]. Indeed, we have

$$C(G;\lambda) = \sum_{k=0}^{n}(-1)^{n-k}C_k(G)\,\lambda^k \qquad \text{with} \qquad C_k(G) = \sum_{F\in\mathcal{F}_k}\gamma(F)$$

where $\mathcal{F}_k$ is the set of all spanning forest of G with k connected components, and $\gamma(F) = n_1\cdots n_k$ whenever F is a forest with k connected components with n_1, $\dots$, n_k vertices, respectively.

Let now $C_{nk} = C_k(A_n)$. From the above interpretation in follows at once that $C_{n1} = t(A_n) = (n-1)!/\lceil n/2\rceil$, $C_{n,n-1} = 2\lfloor n/2\rfloor\lceil n/2\rceil$ and $C_{nn} = 1$. From (16) we can obtain an explicit form for all coefficients. First of all recall that the falling factorials can be expanded as

$$x^{\underline{n}} = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^{n-k}x^k$$

where the coefficients $\begin{bmatrix}n\\k\end{bmatrix}$ are the *Stirling numbers of the first kind* [**7**]. Hence writing

$$C_n(\lambda) = -\frac{1}{\lceil n/2\rceil}\,\frac{1}{1-\lambda/\lceil n/2\rceil}\,\lambda^{\underline{n+1}} = -\frac{1}{\lceil n/2\rceil}\,\frac{1}{1-\lambda/\lceil n/2\rceil}\,\lambda^{\underline{n+1}}$$

and then expanding the product, we obtain the identity (for $n, k \geq 1$):

$$C_{nk} = \sum_{j=0}^{k-1} \begin{bmatrix} n+1 \\ j+1 \end{bmatrix} \frac{(-1)^{k-j-1}}{\lceil n/2 \rceil^{j-k}}.$$

The first few values of these coefficients are showed in the following matrix:

$$C = [C_{nk}]_{n,k\geq 0} = \begin{bmatrix} 1 \\ 0 & 1 \\ 0 & 2 & 1 \\ 0 & 3 & 4 & 1 \\ 0 & 12 & 19 & 8 & 1 \\ 0 & 40 & 78 & 49 & 12 & 1 \\ 0 & 240 & 508 & 372 & 121 & 18 & 1 \\ 0 & 1260 & 2952 & 2545 & 1056 & 226 & 24 & 1 \\ 0 & 10080 & 24876 & 23312 & 10993 & 2864 & 418 & 32 & 1 \\ \dots \end{bmatrix}$$

From (16) we can also obtain a recurrence for the coefficients C_{nk}. Indeed from such an identity we immediately have the equation

$$\left(\lambda - \left\lceil \frac{n+1}{2} \right\rceil\right) C_{n+1}(\lambda) = \left(\lambda - \left\lceil \frac{n}{2} \right\rceil\right) (\lambda - n - 1) C_n(\lambda)$$

which yields the recurrence

$$\left\lceil \frac{n+1}{2} \right\rceil C_{n+1,k+2} = \left\lceil \frac{n}{2} \right\rceil (n+1) C_{n,k+2} + \left(n + 1 + \left\lceil \frac{n}{2} \right\rceil\right) C_{n,k+1} + C_{nk} - C_{n+1,k+1}.$$

4. MATCHINGS

A *matching* of a graph is a set of independent edges, i.e. a set of edges with no vertices in common. A matching is *perfect* when it spans the graph, i.e. when every vertex of the graph lies exactly in one edge of the matching. A matching is *quasi-perfect* when it contains all vertices except one. Clearly, a graph with an odd number of vertices cannot have a perfect matching and a graph with an even number of vertices cannot have a quasi-perfect matching.

4.1. MATCHING POLYNOMIAL

The *matching polynomial* of a graph G with n vertices is defined by

$$M(G; x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{n-k} m_k(G)\, x^{n-2k}$$

where $m_k(G)$ is the number of all matchings of G made of k independent edges [**5**, **8**]. These polynomials satisfy the following properties (formulae (4.2), (4.8) and (4.9) in [**5**]).

1. For every graph G_1 and G_2 we have the identity

$$M(G_1 + G_2; x) = M(G_1; x)M(G_2; x)\,. \tag{17}$$

2. For every vertex v of G, let $G \setminus v$ be the graph obtained deleting v and all edges incident to it. Then

$$M(G; x) = xM(G \setminus v; x) - \sum_{u \text{ adj } v} M(G \setminus u; x)\,. \tag{18}$$

3. For every graph G, the derivative of the matching polynomial is

$$M'(G; x) = \sum_{v \in V(G)} M(G \setminus v; x)\,. \tag{19}$$

From these properties we immediately have

Theorem 13. *For every graph* G

$$M(K_1 \boxplus G; x) = xM(G; x) - M'(G; x)\,. \tag{20}$$

Proof. Let v be the vertex corresponding to K_1 in the graph $K_1 \boxplus G$. Since it is adjacent to every vertex in G, from (18) we have

$$M(K_1 \boxplus G; x) = xM(G; x) - \sum_{u \in V(G)} M(G \setminus u; x)$$

and from (19) we obtain at once (20). □

For simplicity we will write $M_n(x) = M(A_n; x)$. Hence we have

Theorem 14. *The matching polynomials of the connected antiregular graphs satisfy the recurrence*

$$M_{n+2}(x) = (x^2 - 1)M_n(x) - xM_n'(x)\,. \tag{21}$$

Proof. Using recurrence (3) and then properties (17) and (20), we have

$$\begin{aligned} M_{n+2}(x) &= M(K_1 \boxplus (K_1 + A_n); x) \\ &= xM(K_1 + A_n; x) - M'(K_1 + A_n; x) \\ &= xM(K_1; x)M(A_n; x) - \frac{\mathrm{d}}{\mathrm{d}x}[M(K_1; x)M(A_n; x)] \\ &= x^2 M_n(x) - \frac{\mathrm{d}}{\mathrm{d}x}[xM_n(x)] \\ &= x^2 M_n(x) - M_n(x) - xM_n'(x) \end{aligned}$$

where we have used the fact that $M(K_1; x) = x$. □

The first few polynomials $M_n(x)$ are

$$\begin{aligned}
&M_0(x) = 1\\
&M_1(x) = x\\
&M_2(x) = x^2 - 1\\
&M_3(x) = x^3 - 2x\\
&M_4(x) = x^4 - 4x^2 + 1\\
&M_5(x) = x^5 - 6x^3 + 4x\\
&M_6(x) = x^6 - 9x^4 + 13x^2 - 1\\
&M_7(x) = x^7 - 12x^5 + 28x^3 - 8x\\
&M_8(x) = x^8 - 16x^6 + 58x^4 - 40x^2 + 1\\
&M_9(x) = x^9 - 20x^7 + 100x^5 - 120x^3 + 16x\\
&M_{10}(x) = x^{10} - 25x^8 + 170x^6 - 330x^4 + 121x^2 - 1\,.
\end{aligned}$$

Theorem 15. *The matching polynomial $M_n(x)$ has n real roots. When $n \geq 3$ all the roots are contained in the interval $(-2\sqrt{n-2}, 2\sqrt{n-2}\,)$.*

Proof. As it is well known, the roots of a matching polynomial are always real and lie in the interval $(-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1})$ whenever the maximum valency Δ of the graph is greater that one [**8**, Corollary 1.2, p. 97]. Here $\Delta = n-1$ and $\Delta > 1$ when $n > 2$. □

Theorem 16. *The matching polynomials of the connected antiregular graphs from a Sturm sequence.*

Proof. The polynomial $M_n(x)$ has degree n by definition and has n real simple roots for next Theorem 20. Moreover, since for any graph G and for every vertex v of G the matching polynomials $m(G;x)$ and $m(G \setminus v;x)$ have the interlacing property [**5**, Theorem 4.13, p. 111], and since there always exits a vertex $v \in A_{n+1}$ such that $A_{n+1} \setminus v = A_n$, it follows at once that $M_n(x)$ and $M_{n+1}(x)$ have the interlacing property. □

4.2. ENUMERATION OF MATCHINGS

Let us now consider the polynomials $a_n(x) = M_{2n}(x)$ and $b_n(x) = M_{2n+1}(x)$. From recurrence (21) it follows at once that

$$\begin{cases} a_{n+1}(x) = (x^2-1)a_n(x) - xa_n'(x) \\ a_0(x) = 1 \end{cases} \tag{22}$$

$$\begin{cases} b_{n+1}(x) = (x^2-1)b_n(x) - xb_n'(x) \\ b_0(x) = x\,. \end{cases} \tag{23}$$

Theorem 17. *The exponential generating series for the polynomials* $a_n(x)$ *and* $b_n(x)$ *are*

$$a(x;t) = \sum_{n\geq 0} a_n(x)\frac{t^n}{n!} = \mathrm{e}^{-t}\,\mathrm{e}^{-\frac{x^2}{2}(\mathrm{e}^{-2t}-1)} \tag{24}$$

$$b(x;t) = \sum_{n\geq 0} b_n(x)\frac{t^n}{n!} = x\mathrm{e}^{-2t}\,\mathrm{e}^{-\frac{x^2}{2}(\mathrm{e}^{-2t}-1)}\,. \tag{25}$$

Proof. From (22) and (23) it follows that the polynomials $a_n(x)$ and $b_n(x)$ satisfy the same recurrence $f_{n+1}(x) = (x^2-1)f_n(x) - xf_n'(x)$, which is equivalent to the partial differential equation

$$x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} = (x^2-1)f$$

where $f = f(x,t)$ is the exponential generating series of the polynomials $f_n(x)$. By means of the classical method of Lagrange's auxiliary system, it is straightforward to obtain the general solution

$$f(x,t) = \frac{1}{x}\,\mathrm{e}^{x^2/2}\,\varphi\left(x\mathrm{e}^{-t}\right)$$

where $\varphi(u)$ is an arbitrary series. Now, using the initial values given in (22) and (23), we obtain $\varphi(u) = u\mathrm{e}^{-u^2/2}$ in the first case and $\varphi(u) = u^2\mathrm{e}^{-u^2/2}$ in the second case. Hence (24) and (25) follows at once. □

Since $a(x;t) = \mathrm{e}^t b(x;t)/x$, or equivalently $b(x;t) = x\mathrm{e}^{-t}a(x;t)$, we have the identities

$$a_n(x) = \frac{1}{x}\sum_{k=0}^{n}\binom{n}{k}b_k(x) \qquad \text{and} \qquad b_n(x) = x\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}a_k(x)\,.$$

Notice that the polynomials $a_n(x)$ and $b_n(x)$ are both *actuarial polynomials* [**16**]. Specifically, since the actuarial polynomials $a_n^{(\beta)}(x)$ have exponential generating series

$$\sum_{n\geq 0} a_n^{(\beta)}(x)\,\frac{t^n}{n!} = \mathrm{e}^{\beta t}\,\mathrm{e}^{-x(\mathrm{e}^t-1)}$$

it follows that $a_n(x) = (-2)^n\,a_n^{(1/2)}(x^2/2)$ and $b_n(x) = (-2)^n\,a_n^{(1)}(x^2/2)$. Moreover the polynomials $a_n(x)$ and $b_n(x)$ can be expressed in terms of *Stirling polynomials* and of *modified Stirling polynomials* [**16**], respectively defined by

$$S_n(x) = \sum_{k=0}^{n}\left\{{n \atop k}\right\}x^k \qquad \text{and} \qquad \widehat{S}_n(x) = \frac{1}{x}\,S_{n+1}(x) = \sum_{k=0}^{n}\left\{{n+1 \atop k+1}\right\}x^k$$

where the coefficients $\left\{ {n \atop k} \right\}$ are the STIRLING numbers of the second kind [**7**]. Since their exponential generating series are

$$S(x;t) = \sum_{n\geq 0} S_n(x)\,\frac{t^n}{n!} = \mathrm{e}^{x(\mathrm{e}^t-1)} \qquad \text{and} \qquad \widehat{S}(x;t) = \sum_{n\geq 0} \widehat{S}_n(x)\,\frac{t^n}{n!} = \mathrm{e}^t\,\mathrm{e}^{x(\mathrm{e}^t-1)}$$

it immediately follows that

$$a(x;t) = \mathrm{e}^{-t}\,S(-x^2/2;-2t) \qquad \text{and} \qquad b(x;t) = x\widehat{S}(-x^2/2;-2t)$$

and hence we have the identities

$$a_n(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} 2^k\, S_n\left(-\frac{x^2}{2}\right) \qquad \text{and} \qquad b_n(x) = (-2)^n\, x\,\widehat{S}_n\left(-\frac{x^2}{2}\right).$$

In particular, if we write

$$a_n(x) = \sum_{k=0}^{n} (-1)^{n-k} a_{nk} x^{2k} \qquad \text{and} \qquad b_n(x) = \sum_{k=0}^{n} (-1)^{n-k} b_{nk} x^{2k+1}$$

then we have the following explicit expressions

$$a_{nk} = \sum_{j=k}^{n} \binom{n}{j} \left\{ {j+1 \atop k+1} \right\} (-1)^{n-j} 2^{j-k} \qquad \text{and} \qquad b_{nk} = \left\{ {n+1 \atop k+1} \right\} 2^{n-k}. \tag{26}$$

Consider now the matrices $A = [a_{nk}]_{n,k\geq 0}$ and $B = [b_{nk}]_{n,k\geq 0}$, that is

$$A = \begin{bmatrix} 1 \\ 1 & 1 \\ 1 & 4 & 1 \\ 1 & 13 & 9 & 1 \\ 1 & 40 & 58 & 16 & 1 \\ 1 & 121 & 330 & 170 & 25 & 1 \\ \cdots \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 & 1 \\ 4 & 6 & 1 \\ 8 & 28 & 12 & 1 \\ 16 & 120 & 100 & 20 & 1 \\ 32 & 496 & 720 & 260 & 30 & 1 \\ \cdots \end{bmatrix}.$$

The coefficients of the matrix A appear in [**18**] as sequence A039755 (or A039756), while the coefficients of the matrix B appear as sequence A075497. See also [**19**].

Notice that A_{2n} has exactly one perfect matching ($m_{2n,2n} = a_{n0} = 1$), while A_{2n+1} has 2^n quasi-perfect matching ($m_{2n+1,2n} = b_{n0} = 2^n$).

Theorem 18. *The coefficients* $a_{nk} = m_{2n,n-k}$ *and* $b_{nk} = m_{2n+1,n-k}$ *satisfy the recurrences*

$$a_{n+1,k+1} = a_{nk} + (2k+3)a_{n,k+1} \quad \textit{and} \quad b_{n+1,k+1} = b_{nk} + 2(k+2)b_{n,k+1}. \tag{27}$$

Proof. The recurrence for the coefficients a_{nk} can be deduced from (22), while the recurrence for the coefficients b_{nk} can be easily deduced from the one for the Stirling numbers of the second kind [**7**]. □

The coefficients a_{nk} and b_{nk} can also be expressed in terms of the *homogeneous symmetric functions* $\left(\!\binom{x_1,\dots,x_n}{k}\!\right)$ defined by the ordinary generating series

$$\sum_{n\geq 0}\left(\!\binom{x_0,x_1,\dots,x_k}{n}\!\right)t^n=\frac{1}{(1-x_0t)(1-x_1t)\cdots(1-x_kt)}\,. \tag{28}$$

Theorem 19. *The ordinary generating series of the coefficients* a_{nk} *and* b_{nk} *are*

$$\sum_{n,k\geq 0}a_{nk}x^kt^n=\sum_{k\geq 0}\frac{x^kt^k}{(1-t)(1-3t)\cdots(1-(2k+1)t)} \tag{29}$$

$$\sum_{n,k\geq 0}b_{nk}x^kt^n=\sum_{k\geq 0}\frac{x^kt^k}{(1-2t)(1-4t)\cdots(1-(2k+2)t)} \tag{30}$$

Moreover

$$a_{nk}=\left(\!\binom{1,3,\dots,2k+1}{n-k}\!\right)\qquad and \qquad b_{nk}=\left(\!\binom{1,2,\dots,k+1}{n-k}\!\right)2^{n-k}\,. \tag{31}$$

Proof. If $A_k(t)=\sum_{n\geq k}a_{nk}\,t^n$ and $B_k(t)=\sum_{n\geq k}b_{nk}\,t^n$ are the ordinary generating series for the columns of the matrices A and B, from (27) it easily follows that

$$A_{k+1}(t)=\frac{t}{1-(2k+3)t}\,A_k(x)\qquad\text{and}\qquad B_{k+1}(t)=\frac{t}{1-2(k+2)t}\,B_k(x)\,.$$

Since $a_{n0}=1$ and $b_{n0}=2^n$, we have

$$A_0(t)=\sum_{n\geq 0}a_{n0}t^n=\frac{1}{1-t}\qquad\text{and}\qquad B_0(t)=\sum_{n\geq 0}b_{n0}t^n=\frac{1}{1-2t}\,,$$

and hence

$$A_k(t)=\frac{t^k}{(1-t)(1-3t)\cdots(1-(2k+1)t)}$$

$$B_k(t)=\frac{t^k}{(1-2t)(1-4t)\cdots(1-(2k+2)t)}\,.$$

So we have (29) and (30), and from (28) we have (31). □

Theorem 20. *The polynomial* $a_n(x)$ *has* $2n$ *simple real roots,* n *negative and* n *positive, while the polynomial* $b_n(x)$ *has* $2n+1$ *simple real roots,* n *negative, one zero and* n *positive. Moreover, if* $r_{-n}<\cdots<r_{-1}<r_1<\cdots<r_n$ *are the roots of* $a_n(x)$ *and* $s_{-n-1}<\cdots<s_{-1}<s_1<\cdots<s_{n+1}$ *are the roots of* $a_{n+1}(x)$ *then*

$$s_{-n-1}<r_{-n}<s_{-n}<\cdots<r_{-1}<s_{-1}<0<s_1<r_1<\cdots s_n<r_n<s_{n+1}\,.$$

Similarly, if $r_{-n} < \cdots < r_{-1} < 0 < r_1 < \cdots < r_n$ *are the roots of* $b_n(x)$ *and* $s_{-n-1} < \cdots < s_{-1} < 0 < s_1 < \cdots < s_{n+1}$ *are the roots of* $b_{n+1}(x)$ *then*

$$s_{-n-1} < r_{-n} < s_{-n} < \cdots < r_{-1} < s_{-1} < 0 < s_1 < r_1 < \cdots s_n < r_n < s_{n+1} .$$

Proof. Let us consider the polynomial sequence $\{f_n(x)\}_n$ where $f_n(x) = a_n(x)$ or $f_n(x) = b_n(x)$. In both cases we have the recurrence $f_{n+1}(x) = (x^2-1)f_n(x) - xf_n'(x)$. The theorem is immediately verified for the first cases. Then we proceed by induction on n. We suppose that the property is true for n. Let $r_{-n} < \cdots < r_{-1} < r_1 < \cdots < r_n$ be the (non zero) roots of $f_n(x)$. To prove the property for $n+1$ we will consider the following intervals.

1. $I = (r_{-i-1}, r_{-i})$ or $I = (r_i, r_{i+1})$ with $i = 1, 2, \ldots, n-1$, and $I = (r_{-1}, 0)$ and $I = (0, r_1)$, when $f_n(x) = b_n(x)$. In all cases, if $I = (a, b)$ we have $f_{n+1}(a)f_{n+1}(b) = abf_n'(a)f_n'(b) < 0$ since a and b always have the same sign, and $f_n'(a)f_n'(b) < 0$ (being a and b two consecutive zeros of $f_n(x)$). Hence, by continuity, I contains at least a root of $f_{n+1}(x)$.

2. $I = (-\infty, r_{-n})$. In this case $a_{n+1}(r_{-n}) = -r_{-n}a_n'(r_{-n}) < 0$ since $r_{-n} < 0$ and $a_n'(r_{-n}) < 0$ (since $a_n(x) \to \infty$ as $x \to -\infty$) and $b_{n+1}(r_{-n}) = -r_{-n}b_n'(r_{-n}) > 0$ since $r_{-n} < 0$ but $b_n'(r_{-n}) > 0$ (since $b_n(x) \to -\infty$ as $x \to -\infty$). Hence I contains at least one root of $f_{n+1}(x)$ (since $a_{n+1}(x) \to \infty$ and $b_{n+1}(x) \to -\infty$ as $x \to -\infty$).

3. $I = (r_n, +\infty)$. As in case 2, I contains at least a root of $f_{n+1}(x)$.

4. $I = (r_{-1}, 0)$, when $f_n(x) = a_n(x)$. Since $a_k(0) = (-1)^k$ for every $k \in \mathbb{N}$, we have $a_n(0)a_{n+1}(0) < 0$. Moreover, the sign of $a_n'(r_{-1})$ is equal to the sign of $a_n(0)$. Consequently, since $a_{n+1}(r_{-1}) = -r_{-1}a_n'(r_{-1})$, it follows that $a_{n+1}(r_{-1})$ and $a_{n+1}(0)$ have discordant signs. Hence I contains at least a root of $a_{n+1}(x)$

5. $I = (0, r_1)$, when $f_n(x) = a_n(x)$. As in case 4, it follows that I contains at least a root of $a_{n+1}(x)$.

We have obtained at least $2n+2$ roots for $a_{n+1}(x)$ and at least $2n+3$ roots (including 0) for $b_{n+1}(x)$. Since $a_{n+1}(x)$ has degree $2n+2$ and $b_{n+1}(x)$ has degree $2n+3$, it follows that these are all their roots. □

A sequence $\{a_0, a_1, \ldots, a_n\}$ of real numbers is *unimodal* when there exists an index k such that $a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$, and is *log-concave* when $a_i^2 \geq a_{i-1}a_{i+1}$ for every $i \geq 1$. A log-concave sequence of positive numbers is always unimodal. Moreover a sequence $\{a_0, a_1, \ldots, a_n\}$ is unimodal (log-concave) if and only if is unimodal (log-concave) the reverted sequence $\{a_n, \ldots, a_1, a_0\}$.

Theorem 21. *All rows of* A *and* B, *that is all sequences* $\{a_{n0}, a_{n1}, \ldots, a_{nn}\}$ *and* $\{b_{n0}, b_{n1}, \ldots, b_{nn}\}$, *are log-concave and unimodal.*

Proof. For any graph G the sequence $\{m_k(G)\}_k$ is log-concave (and consequently unimodal) [**8**, Corollary 3.3, p. 101]. Hence, for every $n \in \mathbb{N}$, the sequence $\{m_{nk}\}_k$

is log-concave and unimodal and consequently this is true also for the reverted sequences $\{a_{nk}\}_k = \{m_{2n,n-k}\}_k$ and $\{b_{nk}\}_k = \{m_{2n+1,n-k}\}_k$. □

Let M_n be the total number of matching in A_n . The first values of these numbers are: 1, 1, 2, 3, 6, 11, 24, 49, 116, 257, 648, 1539, 4088, 10299, 28640.

Theorem 23. *The numbers M_n satisfy the identities*

$$M_{2n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} M_{2k+1} \qquad \text{and} \qquad M_{2n+1} = \sum_{k=0}^{n} \binom{n}{k} M_{2k} .$$

Proof. First of all notice that

$$M_{2n} = \sum_{k=0}^{n} m_{2n,k} = \sum_{k=0}^{n} a_{nk} \qquad \text{and} \qquad M_{2n+1} = \sum_{k=0}^{n} m_{2n+1,k} = \sum_{k=0}^{n} b_{nk} . \tag{32}$$

Hence, from (26) it follows at once that $M_{2n+1} = \sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\} 2^{n-k}$ and then

$$M_{2n} = \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{n}{j} \left\{ {j+1 \atop k+1} \right\} (-1)^{n-j} 2^{j-k} = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \sum_{k=0}^{j} \left\{ {j+1 \atop k+1} \right\} 2^{j-k}$$

which is the first identity in (32). The second identity follows by inverting the first one. □

From (24) and (25) we can deduce the following exponential generating series

$$\sum_{n\geq 0} M_{2n} \frac{t^n}{n!} = \mathrm{e}^{(\mathrm{e}^t - 1 - 2t)/2} \qquad \text{and} \qquad \sum_{n\geq 0} M_{2n+1} \frac{t^n}{n!} = \mathrm{e}^{(\mathrm{e}^t - 1 - 4t)/2}$$

while from (29) and (30) we have the following ordinary generating series

$$\begin{aligned} \sum_{n\geq 0} M_{2n}\, t^n &= \sum_{k\geq 0} \frac{t^k}{(1-t)(1-3t)\cdots(1-(2k+1)t)} \\ \sum_{n\geq 0} M_{2n+1}\, t^n &= \sum_{k\geq 0} \frac{t^k}{(1-2t)(1-4t)\cdots(1-(2k+2)t)} . \end{aligned}$$

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