View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Archivio istituzionale della ricerca - Politecnico di Milano

NON UNDERCUTTING CONDITIONS IN INTERNAL GEARS

Giovanni C. Mimmi
Dip. di Meccanica Strutturale
Università degli studi di Pavia
Via Ferrata 1
I-27100 Pavia, Italy
Fax +39-02-70638377
e-mail mimmi@ipv36.unipv.it

Paolo E. Pennacchi
Dip. di Meccanica Strutturale
Università degli studi di Pavia
Via Ferrata 1
I-27100 Pavia, Italy

ABSTRACT
Many contributions regarding internal gear theory exist in literature. They mainly consider the problem of undercutting by means of analytical methods applied to specific and limited cases. The present paper deals with a general method showing the analytical condition for avoiding undercutting by the use of the concept of the limit curve. In particular the analytical determination of the limit curve allows the designer to obtain significant graphical representations of the design limits.

1. INTRODUCTION
The internal gears presented here are sometimes also called internal gears with circular profile, to differentiate them from those with an involute profile. They are employed as rotors in positive displacement machines, both as motors (hydraulic and pneumatic, micro-motors) and pumps.
As stated by eminent researchers in this field, among whom only [1] will be cited, one of the most important problems in the gear theory is the study of the conjugate profiles and of the conditions needed in order to avoid interference between the profiles. In fact, the conjugate profile concept represents a local condition, which applies only to the contact point, while it does not guarantee anything other than this point. Therefore, the profiles can mesh elsewhere too, causing undercutting and interference between the profiles.
The traditional method for the determination of the conditions of non-undercutting in circular profile internal gears is presented in some of Colbourne's studies. This method is based on geometrical considerations relative to the particular case examined [2, 3]. However, this fact is common to all the traditional methods for studying the conditions of non-undercutting. In fact it is enough to consider the methods developed for involute gears, which are sometimes based on graphical methods too [4, 5, 6]. The latter have the advantage of being easily interpretable, but are not very useful, especially in parametrical design. On the contrary, the method presented in this paper, which is based on Litvin's theory [7], on one hand is exact, due to its analytical formulation, on the other hand, it can be represented in an expressive graphical way. Moreover, since this method is easily integrable with design software, it allows for the parametric design of the profiles and for their verification.
The study presented here requires us to trace the profiles of the outer and the inner gears by means of their analytical expression, as presented in the first part of the paper. Then it is shown in detail how to determine the analytical expression of the limit curve. Finally a representation of the geometrical meaning of this curve is given and its efficacy as a design tool is stressed.

2. PROFILE TRACING
The kinematic pair analyzed is characterized by two circular pitch curves Γ_{p1} and Γ_{p2}, internally tangent to each other (Figure 1). Practically speaking, they are a special type of gear in which the

outer gear has few teeth, or rather lobes in this case, with a circular profile [8, 9, 10] while the inner gear has one lobe less than the outer one. The index 1 is relative to the inner gear, the index 2 to the outer one.

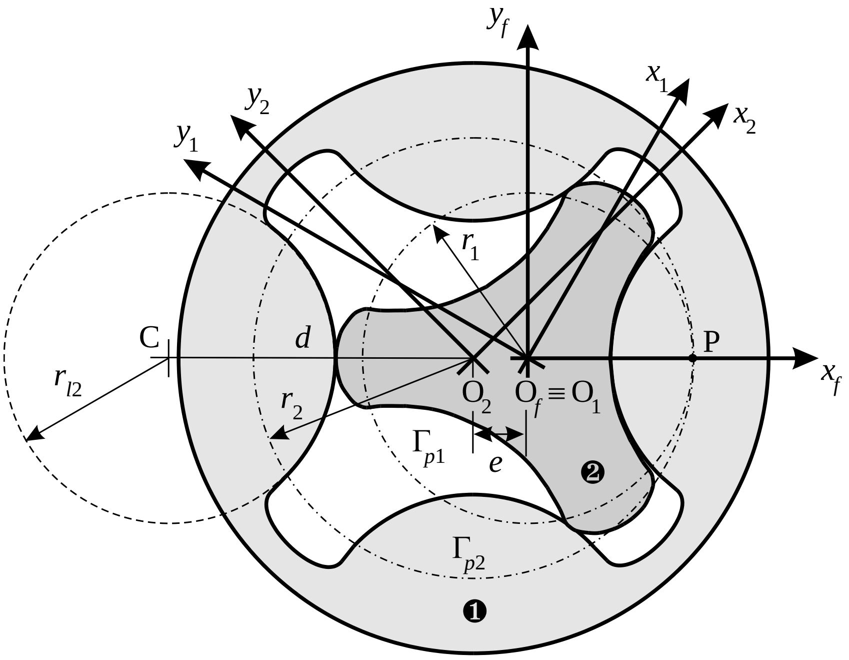

Figure 1. Kinematic pair, pitch circles, reference systems and pitch point

It is possible to propose a modular design for the gears considered and to reduce the number of the design parameters: if e indicates the distance between the centers O_1 and O_2 of the two gears, the pitch radii are equal to the product of the lobe number and the distance between the centers, that is:

$$\begin{aligned} r_1 &= n_1\, e \\ r_2 &= n_2\, e \end{aligned} \tag{1}$$

The other two design parameters are the distance d (Figure 1) between the center O_2 of the outer gear and the center C of the circumference whose arc defines the profile of the lobe and the radius r_{l2} of the same circumference. This can be expressed as a function of the distance e between the centers:

$$\begin{aligned} d &= h e \\ r_{l2} &= k\, e \end{aligned} \tag{2}$$

Therefore the number of the design parameters is reduced to five, four of which are dimensionless: the number of lobes n_1 and n_2, the ratios h and k, and the distance e between the centers.
Nevertheless, from the point of view of the kinematic analysis presented here, the dimensionless parameters will not be used and the explicit values of the distance and of the radii will be adopted in order to simplify the calculations.

2.1. Reference systems adopted

Three reference systems will be used (Figure 1): a fixed reference system S_f, that can be considered as rigidly connected to the frame of the pair with its origin in the center of the inner gear, and two mobile reference systems S_1 and S_2, rigidly connected to the two gears, whose origins coincide with the center of the respective gears. The position and the orientation of the reference systems S_1 and S_2 are defined by the rotation angles ϕ_1 and ϕ_2, between which exists the constant gear ratio given by:

$$\frac{\phi_2}{\phi_1} = \frac{r_1}{r_2} = \frac{n_1}{n_2} \tag{3}$$

For the coordinate transformation from one reference system to another, the following transformation matrixes in homogeneous coordinates are defined:

$$\mathbf{M}_{f2} = \begin{bmatrix} \cos\phi_2 & -\sin\phi_2 & 0 & -e \\ \sin\phi_2 & \cos\phi_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_{2f} = \begin{bmatrix} \cos\phi_2 & \sin\phi_2 & 0 & e\cos\phi_2 \\ -\sin\phi_2 & \cos\phi_2 & 0 & -e\sin\phi_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$\mathbf{M}_{f1} = \begin{bmatrix} \cos\phi_1 & -\sin\phi_1 & 0 & 0 \\ \sin\phi_1 & \cos\phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M}_{1f} = \begin{bmatrix} \cos\phi_1 & \sin\phi_1 & 0 & 0 \\ -\sin\phi_1 & \cos\phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$\mathbf{M}_{12} = \begin{bmatrix} \cos(\phi_1-\phi_2) & \sin(\phi_1-\phi_2) & 0 & -e\cos\phi_1 \\ -\sin(\phi_1-\phi_2) & \cos(\phi_1-\phi_2) & 0 & e\sin\phi_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{M}_{21} = \begin{bmatrix} \cos(\phi_1-\phi_2) & -\sin(\phi_1-\phi_2) & 0 & e\cos\phi_2 \\ \sin(\phi_1-\phi_2) & \cos(\phi_1-\phi_2) & 0 & -e\sin\phi_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

The outer gear is composed of n_2 equal circular lobes with a suitable fillet; each lobe can be represented in a parametric form in the system S_2, by means of a regular line Γ_2:

$$\Gamma_2^{(2)}: \begin{cases} x_2^{(2)} = r_{l2}\cos\theta + d \\ y_2^{(2)} = r_{l2}\sin\theta \end{cases} \quad \theta \in E \tag{7}$$

The notation $x_2^{(2)}$ indicates the cartesian component x of the curve Γ_2 in the reference system S_2. The reference system index will be omitted when unnecessary. The lobe profile can be conveniently represented in vector notation by vector $\mathbf{r}_2$ as well. In the following, both the cartesian notation and the homogeneous coordinate matrix notation will be indifferently adopted, therefore:

$$\Gamma_2^{(2)}: \mathbf{r}_2^{(2)} = x_2^{(2)}\,\mathbf{i} + y_2^{(2)}\,\mathbf{j} = \left(r_{l2}\cos\theta + d\right)\mathbf{i} + r_{l2}\sin\theta\,\mathbf{j} \quad \leftrightarrow \quad \mathbf{r}_2^{(2)} = \begin{bmatrix} x_2^{(2)} \\ y_2^{(2)} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_{l2}\cos\theta + d \\ r_{l2}\sin\theta \\ 0 \\ 1 \end{bmatrix} \tag{8}$$

In order to obtain the profiles of the other lobes of the outer gear, the curve Γ_2 will be repeated for n_2 times with a rotation of $2\pi/n_2$. However, for the analysis it is sufficient to consider only the lobe given by equation (8). So, the effective profile for tracing the inner gear is given in the reference system S_f by:

$$\mathbf{r}_2^{(f)} = \mathbf{M}_{f2}\,\mathbf{r}_2^{(2)} \tag{9}$$

2.2. Equation of meshing

With reference to the general theory of the planar conjugate profiles, as reported in [7], the determination of the profile conjugate to Γ_2 is obtained by the equation of meshing, which represents the necessary condition for the existence of the conjugate profile. Starting from equation (8), it is easy to note that the normal to the profile represented by $\mathbf{r}_2$ can always be defined, in fact, the relation:

$$\frac{\partial \mathbf{r}_2}{\partial \theta} \neq \mathbf{0} \quad \rightarrow \begin{cases} \dfrac{\partial x_2}{\partial \theta} = -r_{l2} \sin\theta \neq 0 \\ \dfrac{\partial y_2}{\partial \theta} = r_{l2} \cos\theta \neq 0 \end{cases} \tag{10}$$

is true since the two partial derivatives are never simultaneously equal to zero. Therefore the normal vector $\mathbf{N}_2^{(2)}$ to Γ_2 is equal to:

$$\mathbf{N}_2^{(2)} = \frac{\partial \mathbf{r}_2}{\partial \theta} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x_2}{\partial \theta} & \dfrac{\partial y_2}{\partial \theta} & 0 \\ 0 & 0 & 1 \end{vmatrix} = -r_{l2} \sin\theta \,\mathbf{i} + r_{l2} \cos\theta \,\mathbf{j} \tag{11}$$

The equation of meshing results as:

$$f(\theta, \phi_2) = \left(\frac{\partial \mathbf{r}_2}{\partial \theta} \times \mathbf{k} \right) \cdot \mathbf{v}_2^{(21)} = 0 \tag{12}$$

In this form the equation of meshing represents the condition of orthogonality between the normal to the profile Γ_2 and the direction of the relative velocity between two general points M′ and M′′ of the profiles when they are going to coincide with the contact point M.

The equation of meshing also has a meaningful geometrical interpretation. It represents the condition necessary for the normal $\mathbf{N}_2$ - in the contact point M between the two conjugate profiles - to pass through the pitch point P of the motion (Figure 2). With reference to Figure 1, the pitch point $\mathrm{P}^{(f)}$ has coordinates (r_1, 0, 0, 1), in the reference system S_f, while in the system S_2 it results as:

$$\mathrm{P}^{(2)} = \mathbf{M}_{2f}\, \mathrm{P}^{(f)} = \left(e\cos\phi_2 + r_1 \cos\phi_2 ,\, -e\sin\phi_2 - r_1 \sin\phi_2 ,\, 0, 1 \right) \tag{13}$$

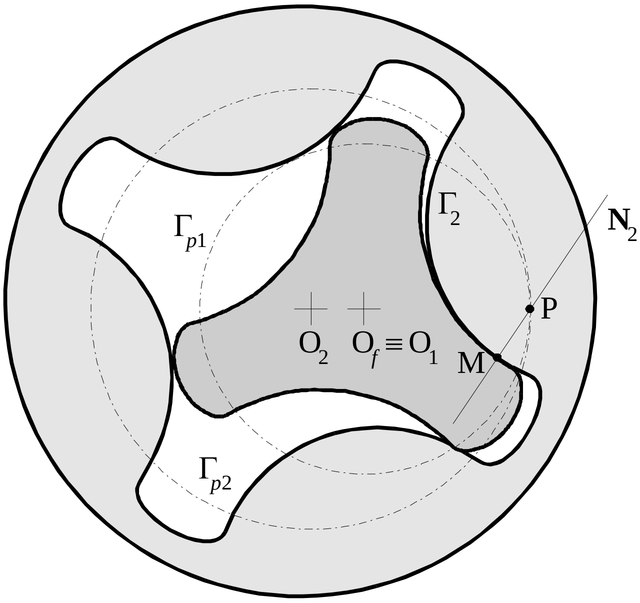

Figure 2. Geometrical interpretation of the equation of meshing.

Therefore the equation of meshing can be written as follows:

$$f(\theta,\phi_2)=0 \quad \rightarrow \quad \frac{x_{P^{(2)}}-x_2^{(2)}}{N_{x2}^{(2)}}=\frac{y_{P^{(2)}}-y_2^{(2)}}{N_{y2}^{(2)}} \quad \rightarrow$$
$$f(\theta,\phi_2)=-d\sin\theta+e\sin(\phi_2+\theta)+r_1\sin(\phi_2+\theta)=0 \qquad (14)$$

Note in (14) that the equation of meshing is not a function of r_{l2}, that is, of the lobe radius. This can be explained by considering the peculiarity of the circular profile where the normals in corresponding points of similar circles are radial anyhow [9] and independent from the circle radius. This observation also permits profile tracing by means of a simplified method described in [9].

The equation of meshing (14) expresses a relationship between the rotation angle ϕ_2 of the profile and the parameter θ, by defining in this way the coordinates of the contact point between the tracing profile Γ_2 and the conjugated profile Γ_1. So the trajectory of the contact point in the system S_1 is given by the following equations:

$$\mathbf{r}_1=x_1\,\mathbf{i}+y_1\,\mathbf{j}=\mathbf{M}_{12}\,\mathbf{r}_2\,,\quad f(\theta,\phi_2)=0 \qquad (15)$$

which allow us to envelop and trace the profile (Figure 1 and Figure 2).

3. LIMIT CURVE DEFINITION

The method for determining and tracing the limit curve is presented in this paragraph. The concept of limit curve, introduced by [7], allows us to verify not only the presence of interference between the conjugate profiles, but also to exactly determine the points of the lobe of the outer gear that will cause undercutting on the internal gear.

3.1. General considerations on undercutting

Let us consider the velocity in the contact point of the profiles. The motion of this point can be regarded as composed of a transfer motion with the gear i, whose velocity is indicated by $\mathbf{v}_{tr}^{(i)}$, and a relative motion along profile Γ_i, whose velocity is $\mathbf{v}_{r}^{(i)}$. Inasmuch as the considered profiles are always in contact, the resulting velocity of the contact point is equal for both gears. Therefore:

$$\mathbf{v} = \mathbf{v}_{tr}^{(1)} + \mathbf{v}_r^{(1)} = \mathbf{v}_{tr}^{(2)} + \mathbf{v}_r^{(2)} \tag{16}$$

From the previous equation, the relative velocity $\mathbf{v}_r^{(1)}$ can be expressed as:

$$\mathbf{v}_r^{(1)} = \mathbf{v}_{tr}^{(2)} - \mathbf{v}_{tr}^{(1)} + \mathbf{v}_r^{(2)} = \mathbf{v}_r^{(2)} + \mathbf{v}^{(21)} \tag{17}$$

where $\mathbf{v}^{(21)}$ is the sliding velocity.
The presence of undercutting on the profile Γ_1 of the inner gear is determined by the presence of singular points. From a kinematic point of view, the identification of these points can be made by considering that in these points the relative velocity $\mathbf{v}_r^{(1)}$ of the point along the profile becomes zero [7], due to the presence of angular points or loops, i.e. for equation (17):

$$\mathbf{v}_r^{(2)} + \mathbf{v}^{(21)} = 0 \tag{18}$$

By considering the equation (8) of vector $\mathbf{r}_2$, which represents profile Γ_2, equation (18) can be rewritten, by calculating $\mathbf{v}_r^{(2)}$, as:

$$\frac{\partial \mathbf{r}_2}{\partial \theta}\frac{d\theta}{dt} = -\mathbf{v}^{(21)} \tag{19}$$

It is now necessary to calculate the sliding velocity $\mathbf{v}^{(21)}$ of equation (19). First of all, the angular velocity vectors of the two gears are:

$$\begin{aligned} \boldsymbol{\omega}_1 &= \omega_1 \mathbf{k} \\ \boldsymbol{\omega}_2 &= \omega_2 \mathbf{k} \end{aligned} \tag{20}$$

and the position vector $\mathbf{R}_2^{(2)}$ of the origin O_f of the fixed reference system S_f in respect to the origin O_2 of system S_2, is:

$$\mathbf{R}_2^{(2)} = e\cos\phi_2\,\mathbf{i} - e\sin\phi_2\,\mathbf{j} \tag{21}$$

By considering that the relation of the gear ratio (3) for the rotations is valid also for the angular velocity, the two transfer velocity, in the reference system S_2, can be expressed as follows:

$$\mathbf{v}_{tr}^{(2)} = \boldsymbol{\omega}_2 \times \mathbf{r}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_2 \\ x_2 & y_2 & 0 \end{vmatrix} = -y_2\,\omega_2\,\mathbf{i} + x_2\,\omega_2\,\mathbf{j} = -r_{l2}\frac{r_1}{r_2}\omega_1 \sin\theta\,\mathbf{i} + \frac{r_1}{r_2}\omega_1\left(d + r_{l2}\cos\theta\right)\mathbf{j} \tag{22}$$

$$\mathbf{v}_{tr}^{(1)} = \boldsymbol{\omega}_1 \times \mathbf{r}_2 + \mathbf{R}_2 \times \boldsymbol{\omega}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_1 \\ x_1 & y_1 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e\cos\phi_2 & -e\sin\phi_2 & 0 \\ 0 & 0 & \omega_1 \end{vmatrix} =$$

$$= -\left(y_1 + e\sin\phi_2\right)\omega_1\,\mathbf{i} + \left(x_1 - e\cos\phi_2\right)\omega_1\,\mathbf{j} = -\left(r_{l2}\sin\theta + e\sin\phi_2\right)\omega_1\,\mathbf{i} + \left(d + r_{l2}\cos\theta - e\cos\phi_2\right)\omega_1\,\mathbf{j} \tag{23}$$

The sliding velocity $\mathbf{v}^{(21)}$ results as:

$$\mathbf{v}^{(21)} = \mathbf{v}_{tr}^{(2)} - \mathbf{v}_{tr}^{(1)} =$$
$$= \left(r_{l2} \sin\theta + e \sin\phi_2 - r_{l2} \frac{r_1}{r_2} \sin\theta \right) \omega_1 \, \mathbf{i} + \left(\frac{r_1}{r_2} d + \frac{r_1}{r_2} r_{l2} \cos\theta - d - r_{l2} \cos\theta + e \cos\phi_2 \right) \omega_1 \, \mathbf{j} \tag{24}$$

Now, all the elements which are needed to evaluate equation (19) are available but it is also necessary to consider the derivative of the equation of meshing (14) to evaluate the time dependence between the motion parameter ϕ_2 and the parameter θ [7]. Finally system (25) is obtained:

$$\begin{cases} \dfrac{\partial \mathbf{r}_2}{\partial \theta} \dfrac{d\theta}{dt} = -\mathbf{v}^{(21)} \\ \dfrac{\partial f}{\partial \theta} \dfrac{d\theta}{dt} = -\dfrac{\partial f}{\partial \phi_2} \dfrac{d\phi_2}{dt} \end{cases} \rightarrow \begin{cases} \dfrac{\partial x_2}{\partial \theta} \dfrac{d\theta}{dt} = -v_x^{(21)} \\ \dfrac{\partial y_2}{\partial \theta} \dfrac{d\theta}{dt} = -v_y^{(21)} \\ \dfrac{\partial f}{\partial \theta} \dfrac{d\theta}{dt} = -\dfrac{\partial f}{\partial \phi_2} \dfrac{d\phi_2}{dt} \end{cases} \tag{25}$$

System (25) is a linear system of three equations which have the only unknown $d\theta/dt$. For a solution to exist, the rank of coefficient matrix **A** of system (25), given by:

$$\mathbf{A} = \begin{bmatrix} \dfrac{\partial x_2}{\partial \theta} & -v_x^{(21)} \\ \dfrac{\partial y_2}{\partial \theta} & -v_y^{(21)} \\ \dfrac{\partial f}{\partial \theta} & -\dfrac{\partial f}{\partial \phi_2} \dfrac{d\phi_2}{dt} \end{bmatrix} \tag{26}$$

has to be equal to 1. Therefore, all the minors of rank 2 extracted from matrix (26) must have their determinants Δ_1, Δ_2 and Δ_3 equal to zero:

$$\Delta_1 = \begin{vmatrix} \dfrac{\partial x_2}{\partial \theta} & -v_x^{(21)} \\ \dfrac{\partial y_2}{\partial \theta} & -v_y^{(21)} \end{vmatrix} = 0 \tag{27}$$

$$\Delta_2 = \begin{vmatrix} \dfrac{\partial x_2}{\partial \theta} & -v_x^{(21)} \\ \dfrac{\partial f}{\partial \theta} & -\dfrac{\partial f}{\partial \phi_2} \dfrac{d\phi_2}{dt} \end{vmatrix} = 0 \tag{28}$$

$$\Delta_3 = \begin{vmatrix} \dfrac{\partial y_2}{\partial \theta} & -v_y^{(21)} \\ \dfrac{\partial f}{\partial \theta} & -\dfrac{\partial f}{\partial \phi_2} \dfrac{d\phi_2}{dt} \end{vmatrix} = 0 \tag{29}$$

Equation (27) is immediately verified since it again represents the equation of meshing in the form reported in equation (12), in fact:

$$\Delta_1 = -\frac{\partial x_2}{\partial \theta} v_y^{(21)} + \frac{\partial y_2}{\partial \theta} v_x^{(21)} = \begin{vmatrix} \frac{\partial x_2}{\partial \theta} & \frac{\partial y_2}{\partial \theta} & 0 \\ 0 & 0 & 1 \\ v_x^{(21)} & v_y^{(21)} & 0 \end{vmatrix} = \left(\frac{\partial \mathbf{r}_2}{\partial \theta} \times \mathbf{k} \right) \cdot \mathbf{v}^{(21)} \tag{30}$$

The expansions of the other two equations give:

$$\begin{aligned} \Delta_2(r_{l2}, \theta, \phi_2) &= -\frac{\partial f}{\partial \phi_2}\frac{d\phi_2}{dt}\frac{\partial x_2}{\partial \theta} + v_x^{(21)}\frac{\partial f}{\partial \theta} = \\ &= r_{l2}\frac{r_1}{r_2}(e+r_1)\cos\theta\sin(\phi_2+\theta) + \left(-d\cos\theta + e\cos(\phi_2+\theta) + r_1\cos(\phi_2+\theta)\right) \\ &\quad \left(e\sin\phi_2 + r_{l2}\sin\theta - r_{l2}\frac{r_1}{r_2}\sin\theta\right) = 0 \end{aligned} \tag{31}$$

$$\begin{aligned} \Delta_3(r_{l2}, \theta, \phi_2) &= -\frac{\partial f}{\partial \phi_2}\frac{d\phi_2}{dt}\frac{\partial y_2}{\partial \theta} + v_y^{(21)}\frac{\partial f}{\partial \theta} = \\ &= r_{l2}\frac{r_1}{r_2}(e+r_1)\cos\theta\cos(\phi_2+\theta) + \frac{1}{r_2}\left(d\,r_1 - d\,r_2 + e\,r_2\cos\phi_2 + r_{l2}\,r_1\cos\theta - r_{l2}\,r_2\cos\theta\right) \\ &\quad [-d\cos\theta + e\cos(\phi_2+\theta) + r_1\cos(\phi_2+\theta)] = 0 \end{aligned} \tag{32}$$

Note that the previous equations (31) and (32) are not functions of the angular velocity ω_1, as can be predicted since the limit has to be related only to the design parameter and to the kinematic pair configuration.

Equations (31) and (32) have to be solved as functions of r_{l2} for each value of ϕ_2 and θ, by taking into account that these two parameters are not independent because they are related by equation (14). It is possible to verify that the values of $\bar{r}_{l2}(\theta, \phi_2)$ obtained from equation (31), also satisfy equation (32). The limit curve Γ_l can be defined as:

$$\Gamma_l^{(2)} : \begin{cases} x_{l2}^{(2)} = \bar{r}_{l2}(\theta,\phi_2)\cos\theta + d \\ y_{l2}^{(2)} = \bar{r}_{l2}(\theta,\phi_2)\sin\theta \end{cases} \tag{33}$$

and plotted with the profile of the outer gear.

3.2. Geometrical interpretation

As follows from what has been stated in the previous paragraph, the limit curve is the locus of the points that may cause interference on the conjugated gear. Note that the limit curve is not a function of the radius of curvature of the lobe of the outer gear, but depends on the other design parameters. Therefore, once the limit curve is determined and plotted, if Γ_l does not intersect Γ_2, then the obtained profile Γ_1 does not present undercutting (Figure 3).

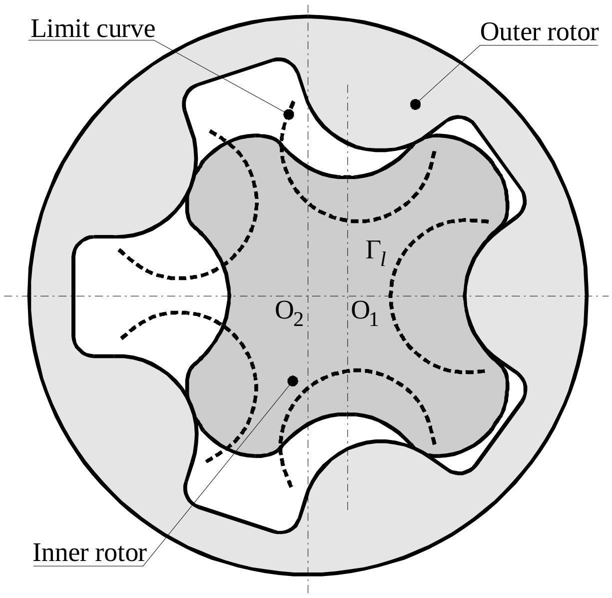

Figure 3. The limit curve does not intersect the outer gear profile that correctly envelops the inner gear.

This fact explains the definition of Γ_l as limit curve, since if the lobe is inside the curve itself, the conjugate gear does not have undercut. Otherwise the conjugate profile is undercut (Figure 4) when intersection is present.

Note that the same design parameters (r_1 = 4, r_2 = 5, d = 6) have been used in Figure 3 and Figure 4, with the exception of r_{l2} (equal to 2 in the first case and to 3 in the second case), so the limit curve is the same.

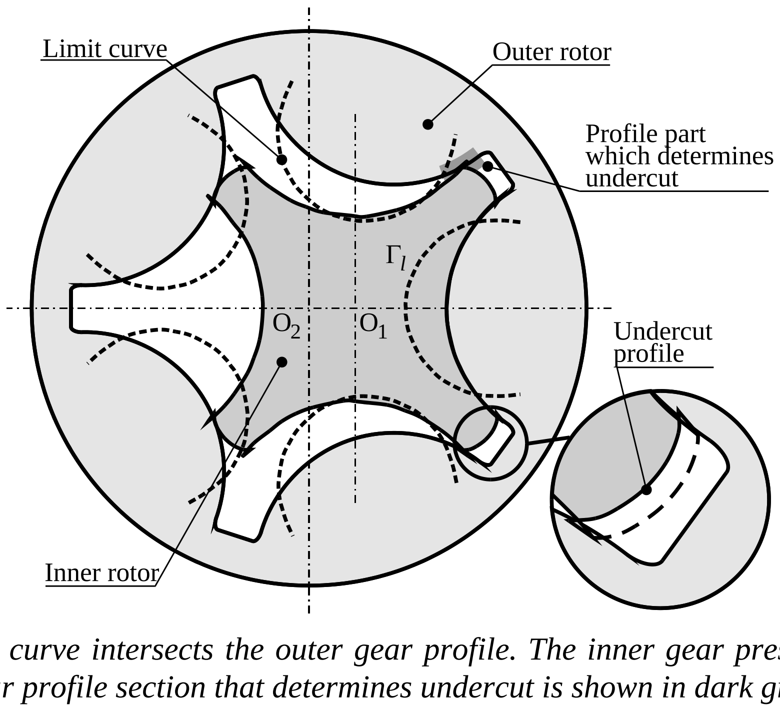

Figure 4. The limit curve intersects the outer gear profile. The inner gear presents undercut. The outer gear profile section that determines undercut is shown in dark gray.

Moreover, note that the part of the profile Γ_2 which lies outside the limit curve, corresponds to the undercut part (Figure 4) and that the starting point of undercutting exactly corresponds to the intersection point. This fact can be observed by means of a suitable rotation of the gears, which permits the intersection point between Γ_2 and Γ_l to coincide with the contact point between Γ_2 and Γ_1 (Figure 5).

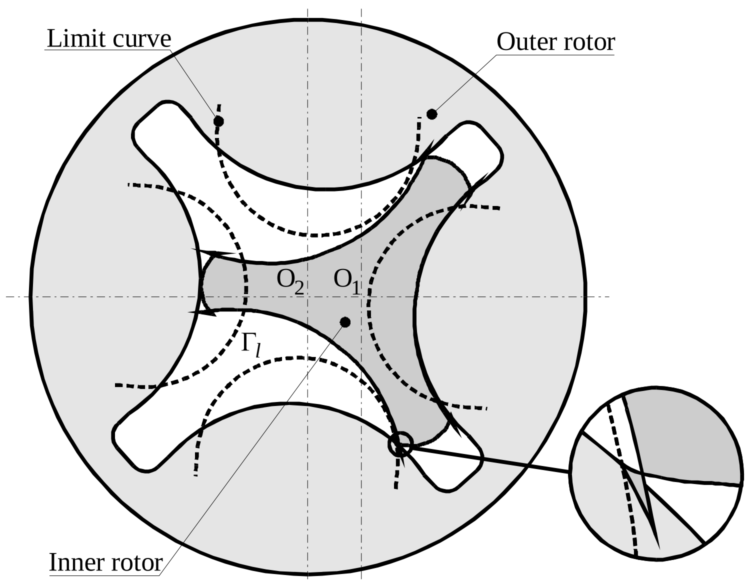

Figure 5. The outer gear profile section that determines undercut is external to the intersection with the limit curve.

These considerations allow us to evaluate the maximum value of the lobe radius r_{l2}, which does not produce undercutting on profile Γ_1 in the design phase. In particular, this value is the minimum value of the solutions of equations (31) and (32):

$$r_{l2limit} = \min_{\theta,\phi_2} \bar{r}_{l2}(\theta,\phi_2) \quad (34)$$

A special case with the lobe radius of the outer gear equal to the limit value given by equation (34) is shown in Figure 6. Note that the intersection between the limit curve and the lobe profile is exactly at the limit of the effective profile for the lobe envelopement. At this point there are two possible options depending on the design limitations. If it is possible to freely choose the value of r_{l2}, in that case a value less than the limit given by equation (34) has to be chosen. Otherwise, if the radius of the outer gear is stated and is greater than the value given by equation (34), the part of the lobe outside the limit curve has to be eliminated. However, this would cause incomplete enveloping of the inner gear profile and therefore it would be necessary to modify the outer gear profile, not by using a plain circular profile but rather a profile composed of different arcs.

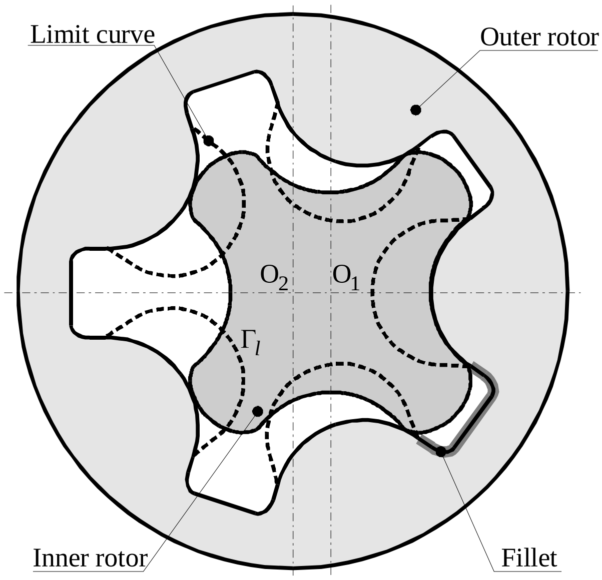

Figure 6. Outer gear lobe with radius equal to the limit radius.

4. CONCLUSIONS

The present paper gives a contribution to the internal gear theory by analytically determining the limits on the design parameters for avoiding undercutting by using the limit curve concept. The following topics will be covered in detail: (i) analytical model of the gears, (ii) analytical determination of the limit curve and (iii) geometrical interpretation of the obtained results, by also including some examples of the technique employed.

References

[1] Litvin, F.L. (1995) **Applied theory of Gearing: State of the Art**. *ASME Journal of Mechanical Design*, Vol.**117**, No.2, June 1995, pp. 128-134.

[2] Colbourne, J.R. (1975) **Gear Shape and Theoretical Flow Rate in Internal Gear Pumps**. *Transaction of the Canadian Society for Mechanical Engineering CSME*, Vol.**3**, No.4, pp.215-223.

[3] Colbourne, J.R. (1976) **Reduction of Contact Stress in Internal Pumps**. *ASME Journal of Engineering for Industry*, Vol.**98**, series B, No.4, pp.1296-1300.

[4] Buckingham, E. (1963) **Analytical Mechanics of Gears**. Dover, New York.

[5] Henriot, G. (1993) **Manuale Pratico degli Ingranaggi**. Tecniche Nuove, Milano.

[6] Ruggeri, G. (1986) **Problemi Speciali di Meccanica**. Spiegel, Milano.

[7] Litvin, F.L. (1994) **Gear Geometry and Applied Theory**. Prentice Hall, Englewood Cliffs.

[8] Mimmi G. and Pennacchi P. (1997a) **Involute Gear Pumps versus Lobe Pumps: a Comparison**. *ASME Journal of Mechanical Design,* Vol.**119**, No.4, December 1997, pp. 458-465.

[9] Mimmi G. and Pennacchi P. (1997b) **Internal Lobe Pump Design**. *Transactions of the Canadian Society for Mechanical Engineering CSME*, Vol.**21**, No.2, pp. 109-122.

[10] Mimmi G. and Pennacchi P. (1997c) **Rotor Design and Optimization in Internal Lobe Pumps**. *Applied Mechanics Reviews*, Vol.**50**, No.11, part 2, November 1997, pp. S133-S141.

Nomenclature

A - system coefficient matrix;
d - distance between the center of the outer gear and the center of the circle whose arc is the profile of the lobe;
e - distance between the centers;
f - equation of meshing;
h - ratio d/e;
k - ratio r_{l2}/e;
M - coordinate transformation matrix;
N - normal vector to a profile;
n - number of lobes;
P - pitch point;
R - position vector;
r - profile vector;
r - pitch radius;
r_{l2} - radius of the lobe;
S - reference system;
$\mathbf{v}_{tr}$ - transfer velocity;
$\mathbf{v}_r$ - relative velocity;
$\mathbf{v}^{(21)}$ - sliding velocity;
Γ - curve;
Δ - determinant;
ϕ - rotation angle;
θ - parameter;
$\boldsymbol{\omega}$ - angular velocity.