mathematics

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Regional Controllability and Minimum Energy Control of Delayed Caputo Fractional-Order Linear Systems

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Abstract: We study the regional controllability problem for delayed fractional control systems through the use of the standard Caputo derivative. First, we recall several fundamental results and introduce the family of fractional-order systems under consideration. Afterward, we formulate the notion of regional controllability for fractional systems with control delays and give some of their important properties. Our main method consists of defining an attainable set, which allows us to prove exact and weak controllability. Moreover, the main results include not only those of controllability but also a powerful Hilbert uniqueness method, which allows us to solve the minimum energy optimal control problem. More precisely, an explicit control is obtained that drives the system from an initial given state to a desired regional state with minimum energy. Two examples are given to illustrate the obtained theoretical results.

Keywords: regional controllability; fractional-order systems; Caputo derivatives; control delays; optimal control; minimum energy

MSC: 26A33; 49J20; 93B05

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Citation: Karite, T.; Khazari, A.; Torres, D.F.M. Regional Controllability and Minimum Energy Control of Delayed Caputo Fractional-Order Linear Systems. *Mathematics* **2022**, *10*, 4813. https://doi.org/10.3390/math10244813

Academic Editors: Alexandra M.S.F. Galhano, António Lopes and Carlo Cattani

Received: 11 November 2022
Accepted: 14 December 2022
Published: 18 December 2022

1. Introduction

The celebrated letter addressed by L'Hopital to Leibniz about the possibilities that can be obtained when the order n of the derivative is a fraction $1/2$, revolutionized calculus and marked the birth of fractional calculus [1]. Since its beginnings, fractional calculus has attracted many great mathematicians, who directly or indirectly contributed to its development [2]. Today, many researchers consider fractional calculus an important tool for solving different problems in various fields, e.g., physics, thermodynamics, chemistry, biology, classical and quantum mechanics, viscoelasticity, finance, engineering, signal and image processing, and automatics and control [3–5].

Let Ω be a bounded subset of $\mathbb{R}^n$ with a regular boundary $\partial\Omega$, the final time be $\tau > 0$, $Q = \Omega \times [0,\tau]$, and $\Sigma = \partial\Omega \times [0,\tau]$. We then consider the system

$$\begin{cases} {}_0^C\mathbb{D}_t^r z(x,t) = \mathcal{A}z(x,t) + \mathfrak{B}\mathfrak{u}(t-h), \quad t \geq 0, \\ z(x,0) = z_0(x), \\ \mathfrak{u}(t) = \varphi(t), \quad -h \leq t \leq 0, \end{cases} \tag{1}$$

where ${}_0^C\mathbb{D}_t^r$ denotes the left-sided Caputo fractional order derivative of order $r \in (0,1)$ [6,7]. Note that z is a function of two parameters but the derivative is an operator that acts on t. The linear operator $\mathcal{A}$ is an infinitesimal generator of a C_0-semi-group $(\mathcal{T}(t))_{t\geq 0}$ on the Hilbert state space $L^2(\Omega)$ [8,9]. Here, $h > 0$ is the time control delay and $\varphi(t)$ is the initial control function. In the sequel, we have $z(x,t) \in \mathcal{Z} = L^2(0,\tau;L^2(\Omega))$ and the control $u \in \mathfrak{U} = L^2(0,\tau;\mathbb{R}^p)$. The initial state $z_0 \in L^2(\Omega)$ and the linear control operator $\mathfrak{B} : \mathbb{R}^p \to L^2(\Omega)$, which might be unbounded, depend on the number p and the structure of the actuators.

The notion of controllability seeks to find a command or control that brings the system under study from an initial state to a desired final state. This is generally difficult to achieve, in particular, for fractional order diffusion systems. This explains why a large number of scholars have been investigating control problems using the notion of "regional controllability". This concept was first introduced by El Jai et al. in 1995 [10] for parabolic systems and was then extended to the case of hyperbolic systems [11]. The concept is widely used to investigate problems where the target of interest is not fully specified as a state and relates only to a smaller internal region ω of the system domain Ω. It is especially crucial when it comes to real-world problems since the transfer costs are lower in a regional case, for instance, in the case of wildfires, where the main purpose is to control it in a smaller region and one tries to minimize the costs [12–14].

In various processes, future states are dependent on the current and previous states of the system, which implies that the models describing these processes should include delays, either in the state or control variables or both. If the delays are in the inputs, we are faced with systems with delayed commands. Due to the number of mathematical models describing dynamical systems with delays in the controls, solving controllability problems for such systems is of great importance. In particular, controllability problems for linear continuous-time fractional systems with a delayed control have been the subject of several works [15–18]. However, it should be noted that the majority of research in this area deals with the global case, that is, controllability is treated on the whole evolution domain. Here, we are interested in studying the concept in a specific region $\omega \in \Omega$.

Fractional delayed differential equations are equations involving fractional derivatives and delays. Unlike ordinary derivatives, they are nonlocal derivatives by nature and are able to model memory effects. Indeed, time delays express the history of a past state [19]. Many real-world problems can be modeled more accurately by including fractional derivatives and delays in a specific subregion ω of the whole evolution domain of the system Ω. For instance, when it comes to modeling several epidemiological problems, regional controllability of fractional delayed differential systems can be more plausible. In the case of monitoring glucose rates, fractional-order models provide a reasonable rate of movement of glucose from the blood into the environment [20,21].

Over the years, numerous mathematicians, utilizing their own notations and approaches, have defined different types of fractional derivatives and integrals. In this paper, we treat the controllability problem of a fractional diffusion equation in the sense of Caputo with a delay in the control. Recent works were expanded to solve optimal control problems with delays by combining conformable and Caputo–Fabrizio fractional derivatives via artificial neural networks [22]. Here, we define the regional controllability in the exact and weak senses; we give the necessary and sufficient conditions under which the system is controllable and we obtain the control that minimizes the energy cost functional.

The rest of this paper is structured as follows. Some definitions and fundamentals of fractional calculus are given in Section 2. In Section 3, a definition of regional fractional controllability for delayed systems is given and a necessary and sufficient condition to verify it is proved. Our main findings on controllability and optimal control are then formulated and proved in Section 4. In Section 5, we provide illustrative examples for cases of both a bounded and an unbounded control operator. We conclude with Section 6, providing a summary of the main conclusions and some insightful open questions that still deserve in-depth investigations.

2. Preliminary Results

We begin with some definitions, properties, and known results of fractional calculus that are used to study System (1). In particular, we recall the two more standard notions for fractional derivatives: the concept of the solution to System (1) and the fractional Green formula. In what follows, $g : \mathbb{R}^+ \to \mathbb{R}$ is a given function.

Definition 1 (Caputo fractional derivatives; see, e.g., [6])**.** *The Caputo fractional derivative of order $r > 0$ of a function $g : [0, \infty) \to \mathbb{R}$ is defined as*

$$ {}_0^C\mathbb{D}_t^r g(t) := {}_0I_t^{n-r} g^{(n)}(t) := \frac{1}{\Gamma(n-r)} \int_0^t (t-\sigma)^{n-r-1} g^{(n)}(\sigma) d\sigma, \tag{2} $$

where $n = -[-r]$, provided the right side is pointwise, is defined on $\mathbb{R}^+$ and ${}_0I_t^{n-r}$ is the left-sided Riemann–Liouville fractional integral of order $n - r > 0$ defined by

$$ {}_0I_t^{n-r} g(t) := \frac{1}{\Gamma(n-r)} \int_0^t (t-\sigma)^{n-r-1} g(\sigma) d\sigma, \quad t > 0. \tag{3} $$

Definition 2 (Riemann–Liouville fractional derivatives; see, e.g., [23–25])**.** *The left- and right-hand sides of the Riemann–Liouville fractional derivatives of order r of function g are expressed by*

$$ {}_0\mathbb{D}_t^r g(t) := \left(\frac{d}{dt}\right)^n {}_0I_t^{n-r} g(t) := \frac{1}{\Gamma(n-r)} \frac{d^n}{dt^n} \int_0^t (t-\sigma)^{n-r-1} g(\sigma) d\sigma, \quad t > 0, \tag{4} $$

and

$$ {}_t\mathbb{D}_\tau^r g(t) := \frac{1}{\Gamma(n-r)} \left(-\frac{d}{dt}\right)^n \int_t^\tau (\sigma - t)^{n-r-1} g(\sigma) d\sigma, \quad t < \tau, \tag{5} $$

respectively, where $r \in (n-1, n)$, $n \in \mathbb{N}$.

Definition 3 (Mittag–Leffler function; see, e.g., [26])**.** *The generalized Mittag–Leffler function is defined by*

$$ E_{r,s}(y) := \sum_{i=1}^{\infty} \frac{y^i}{\Gamma(ri+s)}, \quad \mathrm{Re}(r) > 0, \quad s, y \in \mathbb{C}. \tag{6} $$

Definition 4 (Three-parameter Mittag–Leffler function [7])**.** *The Prabhakar generalized Mittag–Leffler function is given by*

$$ E_{\alpha,\beta}^{\gamma}(y) := \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n) y^n}{n! \Gamma(\alpha n + \beta)}, \quad \mathrm{Re}(\alpha) > 0, \quad \alpha, \beta, \gamma \in \mathbb{C}. \tag{7} $$

Definition 5 (See, e.g., [27])**.** *For any given $F(x,t) \in \mathcal{Z}$, $0 < r < 1$, a function $z(x,t) \in \mathcal{Z}$ is said to be the general solution of*

$$ \begin{cases} {}_0^C\mathbb{D}_t^r z(x,t) = \mathcal{A} z(x,t) + F(x,t), & t \geq 0, \\ z(x,0) = z_0(x), \end{cases} $$

and is expressed by

$$ z(x,t) = \mathfrak{R}_r(t) z_0(x) + \int_0^t (t-\sigma)^{r-1} \mathcal{S}_r(t-\sigma) F(\sigma) d\sigma, $$

where

$$ \mathfrak{R}_r(t) = \int_0^\infty \Phi_r(\alpha) \mathcal{T}(t^r \alpha) d\alpha, \tag{8} $$

and

$$\mathcal{S}_r(t) = r\int_0^\infty \alpha\Phi_r(\alpha)\mathcal{T}(t^r\alpha)d\alpha. \tag{9}$$

Here, $\{\mathcal{T}(t)\}_{t\geq 0}$ *is the strongly continuous semigroup generated by operator* $\mathcal{A}$*,*

$$\Phi_r(\alpha) = \frac{1}{r}\alpha^{-1-1/r}\psi_r(\alpha^{-1/r}),$$

and ψ_r *is the probability density function defined by*

$$\psi_r(\alpha) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\alpha^{-rn-1}\frac{\Gamma(nr+1)}{n!}\sin(n\pi r), \quad \alpha\in(0,\infty). \tag{10}$$

Remark 1. *The density function given by* (10) *satisfies the properties*

$$\int_0^\infty e^{-v\alpha}\psi_r(\alpha)d\alpha = e^{-v^r}, \qquad \int_0^\infty \psi_r(\alpha)d\alpha = 1, \quad r\in(0,1), \tag{11}$$

and

$$\int_0^\infty \alpha^\nu\Phi_r(\alpha)d\alpha = \frac{\Gamma(1+\nu)}{\Gamma(1+r\nu)}, \quad \nu\geq 0. \tag{12}$$

The following hypotheses are used in our results:

(H_1) The control operator $\mathfrak{B}$ is dense and $\mathfrak{B}^*$ exists;

(H_2) $(\mathfrak{B}\mathcal{S}_r(t))^*$ exists and $(\mathfrak{B}\mathcal{S}_r(t))^* = \mathcal{S}_r^*(t)\mathfrak{B}^*$.

Note that (H_1) and (H_2) hold when $\mathfrak{B}$ is bounded and linear. Throughout this paper, we use $z(x,t)$ for the state of the system. Next, we introduce the notion of a mild solution of System (1), using for it the notation $z_{\mathfrak{u}}(x,t)$.

Definition 6 (Mild solution of System (1) [28])**.** *We say that a function* $z_{\mathfrak{u}}(x,t)\in\mathcal{Z}$ *is a mild solution of System* (1) *if it satisfies*

$$z_{\mathfrak{u}}(x,t) = \mathfrak{R}_r(t)z_0(x) + \int_0^{t-h}(t-\sigma-h)^{r-1}\mathcal{S}_r(t-\sigma-h)\mathfrak{B}\mathfrak{u}(\sigma)d\sigma + \int_{-h}^{0}(t-\sigma-h)^{r-1}\mathcal{S}_r(t-\sigma-h)\mathfrak{B}\varphi(\sigma)d\sigma. \tag{13}$$

We define $\mathcal{H}: L^2(0,\tau-h;\mathbb{R}^p)\to L^2(\Omega)$ by

$$\mathcal{H}\mathfrak{u} = \int_0^{\tau-h}(\tau-\sigma-h)^{r-1}\mathcal{S}_r(\tau-\sigma-h)\mathfrak{B}\mathfrak{u}(\sigma)d\sigma, \quad \text{for all} \quad \mathfrak{u}\in L^2(0,\tau-h;\mathbb{R}^p). \tag{14}$$

Assume that (H_1)–(H_2) hold and $(\mathcal{T}^*(t))_{t\geq 0}$ is a semigroup generated by $\mathcal{A}^*$ on $L^2(\Omega)$, which is strong and continuous. For $\nu\in L^2(\Omega)$, we have

$$\begin{aligned}\langle\mathcal{H}\mathfrak{u},\nu\rangle &= \left\langle\int_0^{\tau-h}(\tau-\sigma-h)^{r-1}\mathcal{S}_r(\tau-\sigma-h)\mathfrak{B}\mathfrak{u}(\sigma)d\sigma,\nu\right\rangle_{L^2(\Omega)}\\ &= \int_0^{\tau-h}\left\langle(\tau-\sigma-h)^{r-1}\mathcal{S}_r(\tau-\sigma-h)\mathfrak{B}\mathfrak{u}(\sigma),\nu\right\rangle_{L^2(\Omega)}d\sigma\\ &= \int_0^{\tau-h}\left\langle\mathfrak{u}(\sigma),\mathfrak{B}^*(\tau-\sigma-h)^{r-1}\mathcal{S}_r^*(\tau-\sigma-h)\nu\right\rangle_{\mathfrak{U}}d\sigma\\ &= \langle\mathfrak{u},\mathcal{H}^*\nu\rangle,\end{aligned} \tag{15}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on the vector space and

$$\mathcal{S}_r^*(t) = r\int_0^\infty \alpha\Phi_r(\alpha)\mathcal{T}^*(t^r\alpha)d\alpha.$$

Then, one has

$$\mathcal{H}^*\nu = \mathfrak{B}^*(\tau-\sigma-h)^{r-1}\mathcal{S}_r^*(\tau-\sigma-h)\nu, \quad \text{for all} \quad \nu \in L^2(\Omega). \tag{16}$$

Let $\mathcal{H}$ be defined as in (14) and let us define the operator $\mathfrak{L}_\varphi$ in such a way that

$$\begin{aligned} \mathfrak{L}_\varphi : L^2(-h,0;L^2(\Omega)) &\longrightarrow L^2(\Omega) \\ \varphi &\longmapsto \int_{-h}^{0}(\tau-\sigma-h)^{r-1}\mathcal{S}_r(\tau-\sigma-h)\mathfrak{B}\varphi(\sigma)d\sigma. \end{aligned} \tag{17}$$

Following the same steps in the computation of $\mathcal{H}^*$, we obtain

$$\mathfrak{L}_\varphi^* v = \mathfrak{B}^*(\tau-\sigma-h)^{r-1}\mathcal{S}_r^*(\tau-\sigma-h)v, \quad \text{for all} \quad v \in L^2(\Omega).$$

Remark 2. *The solutions of* (1) *are considered in the weak sense.*

The subsequent lemmas are necessary to demonstrate our main results: Lemma 1 is used in the proof of Theorem 1, whereas Lemma 3 is useful for proving Theorem 2.

Lemma 1 (See [29,30])**.** *The operators $\mathfrak{R}_r(t)$ and $\mathcal{S}_r(t)$ are bounded and linear. Moreover, for every $z \in L^2(\Omega)$, we have*

$$\|\mathfrak{R}_r(t)z\| \le M\|z\|, \qquad \text{and} \qquad \|\mathcal{S}_r(t)z\| \le \frac{rM}{\Gamma(1+r)}\|z\|. \tag{18}$$

Lemma 2 (See, e.g., [31])**.** *If the reflection operator $\mathcal{Q}$ on $[0,\tau]$ is defined for a differentiable and integrable function g by*

$$\mathcal{Q}g(t) := g(\tau - t), \tag{19}$$

then it satisfies the properties

$$\begin{aligned} \mathcal{Q}_0 I_t^r g(t) &= {}_t I_\tau^r \mathcal{Q} g(t), & \mathcal{Q}_0 \mathbb{D}_t^r g(t) &= {}_t\mathbb{D}_\tau^r \mathcal{Q} g(t), \\ {}_0 I_t^r \mathcal{Q} g(t) &= \mathcal{Q}_t I_\tau^r g(t), & {}_0\mathbb{D}_t^r \mathcal{Q} g(t) &= \mathcal{Q}_t \mathbb{D}_\tau^r g(t). \end{aligned} \tag{20}$$

In the following lemmas, we recall the integration by parts and the fractional Green formulas that relate the left-sided Caputo derivatives with the right-sided Riemann–Liouville derivatives.

Lemma 3 (Fractional integration by parts formula; see, e.g., [32])**.** *For $t \in [0,\tau]$ and $r \in (n-1,n)$, $n \in \mathbb{N}$, the integration by parts relation,*

$$\int_0^\tau f(t)\,{}_0^C\mathbb{D}_t^r g(t)dt = \sum_{i=0}^{n-1}(-1)^{n-1-i}\left[g^i(t)\,{}_t\mathbb{D}_\tau^{r-1-i}f(t)\right]_0^\tau + (-1)^n\int_0^\tau g(t)\,{}_t\mathbb{D}_\tau^r f(t)dt \tag{21}$$

holds.

Remark 3. *If $0 < r < 1$, we obtain from Lemma 3*

$$\int_0^\tau f(t)\,{}_0^C\mathbb{D}_t^r g(t)dt = \left[g(t)\,{}_t I_\tau^{1-r} f(t)\right]_0^\tau - \int_0^\tau g(t)\,{}_t\mathbb{D}_\tau^r f(t)dt. \tag{22}$$

Lemma 4 (Fractional Green formula; see, e.g., [24,33])**.** *Let* $0 < r \leq 1$ *and* $t \in [0, \tau]$. *Then,*

$$\begin{aligned}\int_0^\tau \int_\Omega & \left({}_0^C\mathbb{D}_t^r z(x,t) - \mathcal{A}z(x,t)\right)\phi(x,t)dxdt \\ &= \int_0^\tau \int_\Omega z(x,t)\left(-{}_0\mathbb{D}_\tau^r \phi(x,t) - \mathcal{A}^*\phi(x,t)\right)dxdt \\ &\quad + \int_{\partial\Omega} z(x,\tau)_t I_\tau^{1-r}\phi(x,\tau)d\Gamma - \int_{\partial\Omega} z(x,0)_t I_\tau^{1-r}\phi(x,0)d\Gamma \\ &\quad + \int_0^\tau \int_{\partial\Omega} z(x,t)\frac{\partial\phi(x,t)}{\partial\nu_\mathcal{A}}d\Gamma dt - \int_0^\tau \int_{\partial\Omega} \frac{\partial z(x,t)}{\partial\nu_\mathcal{A}}\phi(x,t)d\Gamma,\end{aligned} \tag{23}$$

for any $\phi \in C^\infty(\overline{Q})$.

As a corollary of Lemma 4, the following result can be derived.

Corollary 1. *Let* $0 < r < 1$. *Then, for any* $\phi \in \mathcal{C}^\infty(\overline{Q})$ *such that* $\phi(x,\tau) = 0$ *in* Ω *and* $\phi = 0$ *on* Σ*, we obtain*

$$\begin{aligned}\int_0^\tau \int_\Omega & \left({}_0^C\mathbb{D}_t^r z(x,t) - \mathcal{A}z(x,t)\right)\phi(x,t)\mathrm{d}x\mathrm{d}t \\ &= -\int_\Omega z(x,0)_t I_\tau^{1-r}\phi(x,0)\mathrm{d}x + \int_0^\tau \int_{\partial\Omega} z(x,t)\frac{\partial\phi(x,t)}{\partial\nu_\mathcal{A}}\mathrm{d}\Gamma\mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega z(x,t)\left(-{}_0\mathbb{D}_\tau^r \phi(x,t) - \mathcal{A}^*\phi(x,t)\right)\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{24}$$

3. Regional Fractional Controllability

Let ω be a given region and a subset of Ω with a positive Lebesgue measure. The projection operator on ω is denoted by the restriction mapping

$$\begin{aligned} P_\omega : L^2(\Omega) &\longrightarrow L^2(\omega) \\ y &\longmapsto P_\omega y = y|_\omega. \end{aligned} \tag{25}$$

Definition 7 (Regional exact controllability at time τ)**.** *We say that System* (1) *is* ω*-exactly controllable at time* τ *if, for any* $z_d \in L^2(\omega)$*, there exists a control* $\mathfrak{u} \in \mathfrak{U}$ *such that*

$$P_\omega z_\mathfrak{u}(x,\tau) = z_d. \tag{26}$$

Definition 8 (Regional weak controllability at time τ)**.** *We say that System* (1) *is* ω*-weakly controllable at time* τ *if, for every* $z_d \in L^2(\omega)$*, given* $\varepsilon > 0$*, there is a control* $\mathfrak{u} \in \mathfrak{U}$ *such that*

$$\|P_\omega z_\mathfrak{u}(x,\tau) - z_d\|_{L^2(\omega)} \leq \varepsilon. \tag{27}$$

Remark 4. *It is equivalent saying that System* (1) *is regionally exactly (resp. regionally weakly) controllable or that System* (1) *is* ω*-exactly (resp.* ω*-weakly) controllable.*

Taking into account that System (1) is linear, for $\mathfrak{u} \in \mathfrak{U}$, let us consider the attainable set $\mathbb{A}(t)$ in $L^2(\Omega)$ defined by

$$\mathbb{A}(t) = \left\{a(\cdot,t) \in L^2(\Omega) \,|\, a(x,t) = \mathcal{H}\mathfrak{u} + \mathfrak{L}_\varphi \varphi\right\}, \tag{28}$$

where operators $\mathcal{H}$ and $\mathfrak{L}_\varphi$ are defined by Equations (14) and (17), respectively. The following result holds.

Theorem 1 (Necessary and sufficient conditions for regional controllability)**.** *For any given* $\tau > 0$*, System* (1) *is regionally exactly (resp. regionally approximately) controllable if, and only if,*

$$P_\omega \mathbb{A}(\tau) = L^2(\omega) \qquad (\textit{resp. } \overline{P_\omega \mathbb{A}(\tau)} = L^2(\omega)).$$

Proof. We prove the approximate controllability case. Using the same proof as in [34,35] and assuming that $\mathfrak{u} \equiv 0$ for all t, System (1) admits a unique solution

$$\mathcal{X}(z_0)(x,t) \in \mathcal{Z}, \qquad \text{such that} \qquad \mathcal{X}(z_0)(x,t) = \mathfrak{R}_r(t) z_0(x). \tag{29}$$

Then, using Lemma 1, $\exists c > 0$ that satisfies $\|\mathcal{X}(z_0)\|_{L^2(\Omega)} \le c\|z_0\|_{L^2(\Omega)}$. Hence, (29) is well defined. For every $\nu \in L^2(\omega)$, since $P_\omega \mathcal{X}(z_0)(\cdot,\tau) \in L^2(\omega)$, we obtain

$$(\nu - P_\omega \mathcal{X}(z_0))(\cdot,\tau) \in L^2(\omega),$$

and

$$\begin{aligned} \|P_\omega z(x,\tau) - \nu\|_{L^2(\omega)} &= \|P_\omega z(x,\tau) + P_\omega \mathcal{X}(z_0)(x,\tau) - (P_\omega \mathcal{X}(z_0) - \nu)(x,\tau)\|_{L^2(\omega)} \\ &= \|(P_\omega z - P_\omega \mathcal{X}(z_0))(x,\tau) - (\nu - P_\omega \mathcal{X}(z_0))(x,\tau)\|_{L^2(\omega)} \\ &= \|P_\omega a(x,\tau) - (\nu - P_\omega \mathcal{X}(z_0))(x,\tau)\|_{L^2(\omega)}. \end{aligned}$$

If $\overline{P_\omega \mathbb{A}(\tau)} = L^2(\omega)$, for any $\varepsilon > 0$, we can find that $\mathfrak{u} \in \mathfrak{U}$ satisfies

$$\|P_\omega a(\cdot,\tau) - (\nu - P_\omega \mathcal{X}(z_0))(\cdot,\tau)\| \le \varepsilon,$$

where $a(\cdot,\cdot)$ is an element of the attainable set (28). This implies that $\|P_\omega z_{\mathfrak{u}}(\cdot,\tau) - \nu\| \le \varepsilon$, where $z_{\mathfrak{u}}(\cdot,\tau) = \mathcal{X}(z_0)(\cdot,\tau) + a(\cdot,\tau)$ is the mild solution of System (1). Then, System (1) is ω-weakly controllable at time τ.

On the other hand, for a given $\tau > 0$, System (1) is ω-weakly controllable if for any $z_d \in L^2(\omega)$, given $\varepsilon > 0$, there is a control $\mathfrak{u} \in \mathfrak{U}$ such that

$$\begin{aligned} \|P_\omega z_{\mathfrak{u}}(x,\tau) - z_d\|_{L^2(\omega)} &= \|P_\omega z_{\mathfrak{u}}(\cdot,\tau) + P_\omega \mathcal{X}(z_0)(\cdot,\tau) - P_\omega \mathcal{X}(z_0)(\cdot,\tau) - z_d\|_{L^2(\omega)} \\ &= \|P_\omega z_{\mathfrak{u}}(\cdot,\tau) - P_\omega \mathcal{X}(z_0)(\cdot,\tau) - (z_d - P_\omega \mathcal{X}(z_0)(\cdot,\tau))\|_{L^2(\omega)} \\ &= \|P_\omega a(\cdot,\tau) - (z_d - P_\omega \mathcal{X}(z_0)(\cdot,\tau))\|_{L^2(\omega)} \\ &\le \varepsilon. \end{aligned}$$

One has $(P_\omega z_{\mathfrak{u}}(\cdot,\tau) - P_\omega \mathcal{X}(z_0)(\cdot,\tau)) \in P_\omega \mathbb{A}(\tau)$. Then, $(z_d - P_\omega \mathcal{X}(z_0)(\cdot,\tau)) \in L^2(\omega)$. Thus $\overline{P_\omega \mathbb{A}(\tau)} = L^2(\omega)$. □

Proposition 1. *The following properties are equivalent:*

(1) *System (1) is ω-exactly controllable;*
(2) $\operatorname{Im}(P_\omega(\mathcal{H} + \mathfrak{L}_\varphi)) = L^2(\omega)$;
(3) $\operatorname{Ker}(P_\omega) + \operatorname{Im}(\mathcal{H} + \mathfrak{L}_\varphi) = L^2(\Omega)$.

Proof. (1) $\Leftrightarrow$ (2). Suppose that System (1) is ω-exactly controllable. Then, there exists $z_d \in L^2(\omega)$ such that $P_\omega z_{\mathfrak{u}}(x,\tau) = z_d$, which is equivalent to

$$P_\omega \mathfrak{R}_r z_0 + P_\omega \mathcal{H}\mathfrak{u} + P_\omega \mathfrak{L}_\varphi \varphi = z_d.$$

For $z_0 = 0$, we have $P_\omega \mathcal{H}\mathfrak{u} + P_\omega \mathfrak{L}_\varphi \varphi = z_d \Leftrightarrow \operatorname{Im}(P_\omega \mathcal{H}) + \operatorname{Im}(P_\omega \mathfrak{L}_\varphi) = L^2(\omega)$.

(2) $\Rightarrow$ (3). For every $z \in L^2(\omega)$, we designate by $\tilde{z}$ the prolongation of z to $L^2(\Omega)$. Given $\operatorname{Im}(P_\omega \mathcal{H}) + \operatorname{Im}(P_\omega \mathfrak{L}_\varphi) = L^2(\omega)$, there is a control $\mathfrak{u} \in \mathfrak{U}$, $\varphi \in L^2(-h,0;L^2(\Omega))$, and $z_1 \in \operatorname{Ker}(P_\omega)$, such that $\tilde{z} = z_1 + \mathcal{H}\mathfrak{u} + \mathfrak{L}_\varphi \varphi$.

(3) $\Rightarrow$ (2). For every $\tilde{z} \in L^2(\Omega)$, it follows from (3) that $\tilde{z} = z_1 + z_2 + z_3$, where $z_1 \in \operatorname{Ker}(P_\omega)$, $z_2 \in \operatorname{Im}\mathcal{H}$, and $z_3 \in \operatorname{Im}\mathfrak{L}$. Then, there exists $\mathfrak{u} \in \mathfrak{U}$ such that $\mathcal{H}\mathfrak{u} = z_2$ and $\varphi \in L^2(-h,0;L^2(\Omega))$ such that $\mathfrak{L}_\varphi \varphi = z_3$. Hence, it follows from (25) that

$$\operatorname{Im}(P_\omega \mathcal{H}) + \operatorname{Im}(P_\omega \mathfrak{L}_\varphi) = L^2(\omega).$$

The proof is complete. □

Proposition 2. *The following properties are equivalent:*

(1) *System* (1) *is ω-weakly controllable;*

(2) $\overline{\mathrm{Im}\big(P_\omega(\mathcal{H}+\mathfrak{L}_\varphi)\big)} = L^2(\omega)$*;*

(3) $\mathrm{Ker}(P_\omega) + \overline{\mathrm{Im}\big(\mathcal{H}+\mathfrak{L}_\varphi\big)} = L^2(\Omega)$.

Proof. The proof that (1)⇔ (2)⇔ (3) is similar to the proof of Proposition 1 and is left to the reader. □

We give an illustrative example of the application of our results.

Example 1. *Consider the time fractional order differential system with a zonal actuator governed by the state equation*

$$\begin{cases} {}^C_0\mathbb{D}^r_t z(x,t) = \Delta z(x,t) + P_{[\beta_1,\beta_2]}\mathrm{u}(t-h) & \text{in } [0,1]\times[0,\tau-h], \\ z(0) = z_0, \\ \mathrm{u}(t) = \varphi(t), \end{cases} \tag{30}$$

where $0 < r < 1$, $\mathfrak{B}\mathrm{u} = P_{[\beta_1,\beta_2]}\mathrm{u}$ *and* $0 \le \beta_1 \le \beta_2 \le 1$. *Moreover, since* $\mathcal{A} = \dfrac{\partial^2 \cdot}{\partial x^2}$ *is a self-adjoint operator, we find that the eigenvalues of the operator* $\mathcal{A}$ *are given by* $v_i = -i^2\pi^2$ *and its eigenfunctions by* $\zeta_i(x) = \sqrt{2}\sin(i\pi x)$. *The uniformly continuous semigroup generated by* $\mathcal{A}$ *is*

$$\Xi(t)z(x,t) = \sum_{i=1}^{\infty} e^{(v_i t)}(z,\zeta_i)_{L^2(0,1)}\zeta_i(x).$$

It implies

$$\mathcal{S}_r(t)z(x,t) = r\int_0^\infty \theta\phi_r(\theta)\Xi(t^r\theta)z(x,t)d\theta = r\sum_{i=1}^{\infty} E^2_{r,r+1}(v_i t^r)(z,\zeta_i)_{L^2(0,1)}\zeta_i(x),$$

and we obtain that

$$\begin{aligned} \big(\mathcal{H}+\mathfrak{L}_\varphi\big)^* z(x,t) &= 2\Big[\mathfrak{B}^*(\tau-\sigma-h)^{r-1}\mathcal{S}^*_r(\tau-\sigma-h)z\Big](x,t) \\ &= 2r\mathfrak{B}^*(\tau-\sigma-h)^{r-1}\sum_{i=1}^{\infty} E^2_{r,r+1}(v_i(\tau-\sigma-h)^r)(z,\zeta_i)_{L^2(0,1)}\zeta_i(x) \\ &= 2r(\tau-\sigma-h)^{r-1}\sum_{i=1}^{\infty} E^2_{r,r+1}(v_i(\tau-\sigma-h)^r)(z,\zeta_i)_{L^2(0,1)}\int_{\beta_1}^{\beta_2}\zeta_i(x)dx, \end{aligned} \tag{31}$$

whereas from $\int_{\beta_1}^{\beta_2}\zeta_i(x)dx = \dfrac{\sqrt{2}}{i\pi}\sin\dfrac{i\pi(\beta_1+\beta_2)}{2}\sin\dfrac{i\pi(\beta_1-\beta_2)}{2}$ *we obtain* $\mathrm{Ker}\big(\mathcal{H}+\mathfrak{L}_\varphi\big)^* \neq \{0\}$, *i.e.,* $\mathrm{Im}\big(\mathcal{H}+\mathfrak{L}_\varphi\big) \neq L^2(\omega)$, *which means that System* (30) *is not controllable on* $\Omega = [0,1]$. *Let* $[\beta_1 = 0, \beta_2 = 1/3]$. *We then have*

$$\begin{aligned} \int_0^{1/3}\zeta_i(x)dx &= \int_0^{1/3}\sqrt{2}\sin(i\pi x)dx \\ &= \sqrt{2}\Big[-\frac{\cos(i\pi x)}{i\pi}\Big]_0^{1/3} \\ &= \frac{\sqrt{2}}{i\pi}(1-\cos(i\pi/3)). \end{aligned}$$

If $\omega = [1/3, 2/3] \subset [0,1]$, *then*

$$\begin{aligned}(\mathcal{H}+\mathfrak{L}_{\varphi})^* P_{\omega}^*(P_{\omega}\zeta_j) &= 2r(\tau-\sigma-h)^{r-1}\sum_{k=1}^{\infty} E_{r,r+1}^2(v_k(\tau-\sigma-h)^r)\langle\zeta_i,\zeta_j\rangle_{L^2(\omega)}\int_0^{1/3}\zeta_i(x)dx \\ &= \sum_{k=1}^{\infty}\frac{rE_{r,r+1}^2(v_k(\tau-\sigma-h)^r)}{2(\tau-\sigma-h)^{1-r}}\langle\zeta_i,\zeta_j\rangle_{L^2(\omega)}\int_0^{1/3}\zeta_i(x)dx \\ &= \sum_{k=1}^{\infty}\frac{rE_{r,r+1}^2(v_k(\tau-\sigma-h)^r)}{2(\tau-\sigma-h)^{1-r}}\int_{1/3}^{2/3}\zeta_i(x)\zeta_j(x)dx\frac{\sqrt{2}}{i\pi}(1-\cos(i\pi/2)) \\ &= \sum_{k=1}^{\infty}\frac{rE_{r,r+1}^2(v_k(\tau-\sigma-h)^r)}{\sqrt{2}i\pi(\tau-\sigma-h)^{1-r}}\int_{1/3}^{2/3}\zeta_i(x)\zeta_j(x)dx[1-\cos(i\pi/2)] \\ &\neq 0.\end{aligned} \tag{32}$$

We conclude that the state ζ_j *is reachable on* ω.

4. Optimal Control with a Regional Target

Fractional optimal control is a rapidly developing topic (see, for instance, [36–39]). This section is motivated by the results of [10,40–44] and is devoted to the proof that the steering control is a minimizer of a suitable optimal control problem. For this, we use an extended version of the Hilbert uniqueness method (HUM) first introduced by Lions [45,46].

Let F be a closed subspace of $L^2(\Omega)$. Our extended optimal control problem consists of seeking a minimum-norm control that drives the system to F at time τ. More precisely, we consider

$$\inf_{\mathfrak{u}} \mathcal{J}(\mathfrak{u}) = \inf_{\mathfrak{u}}\left\{\int_0^{\tau}\frac{1}{2}\|\mathfrak{u}(t)\|^2 dt \quad : \quad \mathfrak{u}\in\mathfrak{U}_{ad}\right\}, \tag{33}$$

where $\mathfrak{U}_{ad} = \{\mathfrak{u}\in\mathfrak{U} \quad | \quad P_{\omega}z_{\mathfrak{u}}(\cdot,\tau) - z_d \in F\}$, and the set

$$F^{\circ} = \left\{f\in L^2(\Omega) \quad | \quad f=0 \quad \text{in } \Omega\setminus\omega\right\}.$$

For $\psi_0 \in F^{\circ}$, we consider the system

$$\begin{cases} {}_t^C\mathbb{D}_{\tau}^r \mathcal{Q}\psi(t) = -\mathcal{A}^*\mathcal{Q}\psi(t), \\ \lim\limits_{t\to\tau^-} {}_tI_{\tau}^{1-r}\mathcal{Q}\psi(t) = \psi_0 \subset L^2(\Omega), \end{cases} \tag{34}$$

in $L^2(\Omega)$ and let

$$\|\psi_0\|_{F^{\circ}}^2 = \int_0^{\tau}\|\mathfrak{B}^*(\tau-\sigma-h)^{r-1}\mathcal{S}_r^*(\tau-\sigma-h)P_{\omega}^*\psi(\sigma)\|^2 d\sigma, \tag{35}$$

which is a semi-norm on F°.

Using Lemma 2, we can rewrite (34) as

$$\begin{cases} {}_0^C\mathbb{D}_t^r\psi(t) = -\mathcal{A}^*\psi(t), \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-r}\psi(t) = \psi_0 \subset L^2(\Omega), \end{cases} \tag{36}$$

with the solution given by $\psi(t) = -t^{r-1}K_r^*\psi_0$.

Theorem 2. *If* $\mathfrak{u}$ *spans* $\mathfrak{U}$, *then* $\overline{z_{\mathfrak{u}}(x,\tau)} = L^2(\Omega)$.

Proof. Take $z(x,0) = z_0(x) = 0$ and suppose that $z_{\mathfrak{u}}(x,\tau)$ is not dense in $L^2(\Omega)$. Consequently, there is $\psi_0 \in L^2(\Omega)$, $\psi_0 \neq 0$, such that

$$\langle z_{\mathfrak{u}}(x,\tau), \psi_0 \rangle = 0, \qquad \forall \mathfrak{u} \in \mathfrak{U}. \tag{37}$$

By multiplying both sides of (34) by $z(t)$ and then integrating it over Q, we obtain

$$\begin{aligned} \int_\Omega \int_0^\tau z(x,t) {}_0^C\mathbb{D}_t^r \mathcal{Q}\psi(t) dt dx &= \int_0^\tau \langle z(x,t), -\mathcal{A}^* \mathcal{Q}\psi(t) \rangle_\Omega dt \\ &= -\int_0^\tau \langle \mathcal{A} z(x,t), \mathcal{Q}\psi(t) \rangle_\Omega dt. \end{aligned} \tag{38}$$

From Lemma 3, we have

$$\begin{aligned} \int_\Omega \int_0^\tau z(x,t) {}_0^C\mathbb{D}_t^r \mathcal{Q}\psi(t) dt dx &= \int_\Omega \Big[z(x,t) {}_t I_\tau^{1-r} \mathcal{Q}\psi(t) \Big]_0^\tau - \int_\Omega \int_0^\tau \mathcal{Q}\psi(t) {}_0^C\mathbb{D}_t^r z(x,t) dt dx \\ &= \Big\langle z(x,\tau), \lim_{t\to\tau} {}_t I_\tau^{1-r} \mathcal{Q}\psi(t) \Big\rangle_\Omega - \Big\langle z(x,0), \lim_{t\to 0} {}_t I_\tau^{1-r} \mathcal{Q}\psi(t) \Big\rangle_\Omega \\ &\quad - \int_0^\tau \Big\langle \mathcal{Q}\psi(t), {}_0^C\mathbb{D}_t^r z(x,t) \Big\rangle_\Omega dt \\ &= \langle z(x,\tau), \psi_0 \rangle - \int_0^\tau \Big\langle \mathcal{Q}\psi(t), {}_0^C\mathbb{D}_t^r z(x,t) \Big\rangle_\Omega dt \\ &= \langle z(x,\tau), \psi_0 \rangle - \int_0^\tau \langle \mathcal{Q}\psi(t), \mathcal{A} z(x,t) + \mathfrak{B}\mathfrak{u}(t-h) \rangle dt. \end{aligned} \tag{39}$$

From Equations (38) and (39), one has

$$\langle z(x,\tau), \psi_0 \rangle = \int_0^\tau \langle \mathcal{Q}\psi(t), \mathfrak{B}\mathfrak{u}(t-h) \rangle dt. \tag{40}$$

Using (37), we have

$$\int_0^\tau \langle \mathcal{Q}\psi(t), \mathfrak{B}\mathfrak{u}(t-h) \rangle dt = 0 \Leftrightarrow \mathcal{Q}\psi(t) = \psi(\tau - t) \equiv 0, \quad \text{in } L^2(\Omega), \ \forall t \in [0,\tau]. \tag{41}$$

Then $\psi_0 = 0$, which is a contradiction. The proof is complete. □

To proceed with the HUM approach, we first need to prove that the semi-norm $||\cdot||$ on F° in (35) is a norm. We prove the next results.

Lemma 5. *Assuming that* (H_1)–(H_2) *hold,* (35) *defines a norm of* F° *when System* (1) *is* ω*-weakly controllable.*

Proof. Suppose that system (1) is ω-weakly controllable. Then, $\mathrm{Ker}((\mathcal{H} + \mathfrak{L}_\varphi)^* P_\omega^*) = \{0\}$, that is,

$$\mathfrak{B}^*(\tau - \sigma - h)^{r-1} \mathcal{S}_r^*(\tau - \sigma - h) P_\omega^* \psi = 0 \Longrightarrow \psi = 0.$$

Therefore, for every $\psi_0 \in F^\circ$, it follows that

$$\begin{aligned} \|\psi_0\|_{F^\circ} &= \int_0^\tau \|\mathfrak{B}^*(\tau - t - h)^{r-1} \mathcal{S}_r^*(\tau - t - h) P_\omega^* \psi(t)\|^2 dt = 0 \\ &\Leftrightarrow \mathfrak{B}^*(\tau - t - h)^{r-1} \mathcal{S}_r^*(\tau - t - h) P_\omega^* \psi(t) = 0. \end{aligned}$$

Then, (35) is a norm. □

Furthermore, let us define an operator $\mathcal{M} : F^{\circ *} \to F^\circ$ by

$$\mathcal{M} f = \mathcal{P}(\phi(\tau)), \tag{42}$$

where $\mathcal{P} = P_\omega^* P_\omega$ and ϕ is defined by

$$\begin{cases} {}_0^C\mathbb{D}_t^r \phi(t) = \mathcal{A}\phi(t) + \mathfrak{B}\mathfrak{B}^*(\tau - t - h)^{r-1}\mathcal{S}_r^*(\tau - t - h)\psi(t), \\ \phi(0) = z_0. \end{cases} \tag{43}$$

We then decompose $\mathcal{M}$ as

$$\mathcal{M}f = \mathcal{P}(\phi_1(\tau) + \phi_2(\tau)),$$

where

$$\begin{cases} {}_0^C\mathbb{D}_t^r \phi_1(t) = \mathcal{A}\phi_1(t), \\ \phi_1(0) = z_0, \end{cases} \tag{44}$$

and

$$\begin{cases} {}_0^C\mathbb{D}_t^r \phi_2(t) = \mathcal{A}\phi_2(t) + \mathfrak{B}\mathfrak{B}^*(\tau - t - h)^{r-1}\mathcal{S}_r^*(\tau - t - h)\psi(t), \quad t \in [0, \tau - h], \\ \phi_2(0) = 0. \end{cases} \tag{45}$$

Let

$$\Lambda\psi_0 = P_\omega \mathcal{P}(\phi_2(\tau)). \tag{46}$$

Then, the regional controllability problem leads to the resolution of the equation

$$\Lambda\psi_0 = z_d - P_\omega \mathcal{P}(\phi_1(\tau)). \tag{47}$$

For any $f \in (F^\circ)^*$ and $g \in F$, by Holder's inequality we have that

$$\begin{aligned} \langle \Lambda f, g \rangle &= \int_\Omega P_\omega \int_0^\tau (\tau - \sigma - h)^{r-1}\mathcal{S}_r(\tau - \sigma - h)\mathfrak{B}\mathfrak{B}^*(\tau - \sigma - h)^{r-1} \\ &\quad \times \mathcal{S}_r^*(\tau - \sigma - h)P_\omega^* f(\sigma) d\sigma g(x) dx \\ &\leq \int_0^\tau \| \mathfrak{B}^*(\tau - \sigma - h)^{r-1}\mathcal{S}_r^*(\tau - \sigma - h)P_\omega^* f(\sigma) \|^2 d\sigma \| g \| \\ &\leq \| f \| \| g \|, \end{aligned}$$

and $\| \Lambda f \| \leq \| f \|$. Moreover, for any $f \in (F^\circ)^*$, we obtain

$$\begin{aligned} &\langle \Lambda\psi_0, \psi_0 \rangle_{F^{\circ *}, F^\circ} = \langle P_\omega \mathcal{P}(\phi_2(\tau)), \psi_0 \rangle \\ &= \left\langle \int_0^\tau P_\omega(\tau - \sigma - h)^{r-1}\mathcal{S}_r(\tau - \sigma - h)\mathfrak{B}\mathfrak{B}^*(\tau - \sigma - h)^{r-1}\mathcal{S}_r^*(\tau - \sigma - h)P_\omega^*\psi_0(\sigma) d\sigma, \psi_0(\sigma) \right\rangle \\ &= \int_0^\tau \left\langle \mathfrak{B}^*(\tau - \sigma - h)^{r-1}\mathcal{S}_r^*(\tau - \sigma - h)P_\omega^*\psi_0(\sigma), \mathfrak{B}^*(\tau - \sigma - h)^{r-1}\mathcal{S}_r(\tau - \sigma - h)P_\omega^*\psi_0(\sigma) \right\rangle d\sigma \\ &= \int_0^\tau \|\mathfrak{B}^*(\tau - \sigma - h)^{r-1}\mathcal{S}_r^*(\tau - \sigma - h)P_\omega^*\psi_0(\sigma)\|^2 d\sigma \\ &= \|\psi_0\|_{F^\circ}^2. \end{aligned}$$

Consequently, if System (1) is ω-weakly controllable at τ, we find that $\psi_0 = 0$. From the uniqueness of $\mathcal{M}$, we find that Λ defined in (46) is an isomorphism.

Theorem 3. *If System* (1) *is ω-weakly controllable, for any $z_d \in L^2(\omega)$,* (47) *has a unique solution $\psi_0 \in F^\circ$ and the control*

$$\mathfrak{u}^*(t) = \begin{cases} \mathfrak{B}^*(\tau - t - h)^{r-1}\mathcal{S}_r^*(\tau - t - h)P_\omega^*\psi(t), & 0 \leq t \leq \tau - h, \\ \varphi(t), & \tau - h \leq t \leq \tau, \end{cases} \tag{48}$$

steers the system to z_d in ω. Moreover, $\mathfrak{u}^*$ solves the minimization optimal control problem in (33).

Proof. If System (1) is ω-weakly controllable, (35) is a norm. Let us consider the completion of F° regarding the norm in (35) and let us denote it again by F°. Now, we prove that (47) admits a unique solution in F°. For any $\psi_0 \in F^\circ$, one has

$$\langle \Lambda\psi_0, \psi_0 \rangle_{F^{\circ *}, F^\circ} = \langle P_\omega \mathcal{P}(\phi_1(\tau)), \psi_0 \rangle = \|\psi_0\|^2_{F^\circ}.$$

Using Theorems 1.1 and 2.1 in [45], one can see that (47) has a unique solution. Moreover, by setting $\mathfrak{u} = \mathfrak{u}^*$ in System (1), we have $P_\omega z_{\mathfrak{u}^*}(x,\tau) = z_d$. For $\mathfrak{u}$ and v in $\mathfrak{U}_{ad}$, one has $P_\omega z_{\mathfrak{u}}(x,\tau) = P_\omega z_v(x,\tau) = z_d$ so $P_\omega[z_{\mathfrak{u}}(x,\tau) - z_v(x,\tau)] = 0$. We can easily find, for any $\psi_0 \in F^\circ$, that

$$\begin{aligned}
&\langle \psi_0, P_\omega[z_{\mathfrak{u}}(x,\tau) - z_v(x,\tau)] \rangle = 0 \\
&\Leftrightarrow \langle \psi_0, P_\omega \int_0^{\tau-h} (\tau-\sigma-h)^{r-1}\mathcal{S}_r(\tau-\sigma-h)\mathfrak{B}[\mathfrak{u}(\sigma) - v(\sigma)]d\sigma \rangle = 0 \\
&\Leftrightarrow \int_0^{\tau-h} \langle \mathfrak{B}^*(\tau-\sigma-h)^{r-1}\mathcal{S}_r^*(\tau-\sigma-h)P_\omega^*\psi_0, \mathfrak{u}(\sigma) - v(\sigma)\rangle d\sigma = 0,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{J}'(\mathfrak{u})(\mathfrak{u}-v) &= \int_0^{\tau-h} \langle \mathfrak{u}(\sigma), \mathfrak{u}(\sigma) - v(\sigma)\rangle d\sigma \\
&= \int_0^{\tau-h} \langle \mathfrak{B}^*(\tau-\sigma-h)^{r-1}\mathcal{S}_r^*(\tau-\sigma-h)P_\omega^*\psi_0, \mathfrak{u}(\sigma) - v(\sigma)\rangle d\sigma \\
&= 0.
\end{aligned}$$

Because $\mathfrak{U}_{ad}$ is convex, by using Theorem 1.3 in [45], we establish the optimality of $\mathfrak{u}^*$. □

5. Examples

We provide two illustrative examples for cases of both a bounded (Section 5.1) and an unbounded control operator (Section 5.2).

5.1. Example 1: Zonal Actuator

We consider the system

$$\begin{cases}
{}_0^C\mathbb{D}_t^{0.3} z(x,t) = \Delta z(x,t) + P_{[\beta_1,\beta_2]}\mathfrak{u}(t-h), \quad [0,1]\times[0,\tau-h], \\
z(x,0) = 0, \\
\mathfrak{u}(t) = \varphi(t), \quad -h \le t \le 0,
\end{cases} \tag{49}$$

with a fractional order $r = 0.3$. Here, the control operator $\mathfrak{B}$ is bounded, $[\beta_1, \beta_2] = [0, 1/2]$, $\mathcal{A} = \Delta = \dfrac{\partial^2 \cdot}{\partial x^2}$, and the semigroup $(\mathcal{T}(t))_{t\ge 0}$ is given by

$$\mathcal{T}(t)z(x,t) = \sum_{i=1}^{\infty} e^{v_i t}(z,\zeta_i)\zeta_i(x), \qquad x \in [0,1],$$

where $v_i = -i^2\pi^2$ and $\zeta_i(x) = \sqrt{2}\sin(i\pi x)$, $i = 1,2,\ldots$ Then, $(\mathcal{T}(t))_{t\ge 0}$ is uniformly bounded. Moreover, one has

$$
\begin{aligned}
\mathcal{S}_{0.3}(t)z(x,t) &= 0.3\int_0^\infty \theta\Phi_{0.3}(\theta)\mathcal{T}(t^{0.3}\theta)z(x,t)d\theta \\
&= 0.3\int_0^\infty \theta\Phi_{0.3}(\theta)\sum_{i=1}^{\infty} e^{v_i t^{0.3}\theta}(z,\zeta_i)\zeta_i(x)d\theta \\
&= 0.3\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\int_0^\infty \frac{\left(v_i t^{0.3}\right)^n}{n!}\theta^{n+1}\Phi_{0.3}(\theta)d\theta(z,\zeta_i)\zeta_i(x) \\
&= 0.3\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{\left(v_i t^{0.3}\right)^n}{n!}\frac{\Gamma(n+2)}{\Gamma(1+0.3n+0.3)}(z,\zeta_i)\zeta_i(x) \\
&= 0.3\sum_{i=1}^{\infty} E^2_{0.3,1.3}\left(v_i t^{0.3}\right)(z,\zeta_i)\zeta_i(x).
\end{aligned} \tag{50}
$$

Similarly, we have

$$
\mathfrak{R}_{0.3}(t)z(x,t) = \sum_{i=1}^{\infty}(z,\zeta_i)E_{0.3,1}\left(v_i t^{0.3}\right)\zeta_i(x). \tag{51}
$$

Let $\omega = [0.25, 0.75]$. We can easily verify that System (49) is $[0.25, 0.75]$-controllable using the same arguments in (32) and, by Theorem 3, we obtain

$$
\|f\|_{F^\circ} = \int_0^{\tau-h} \|0.3(\tau-\sigma-h)^{-0.7}\sum_{i=1}^{\infty}E^2_{0.3,1.3}\left(v_i(\tau-\sigma-h)^{0.3}\right)(z,\zeta_i)P^*_\omega\int_0^{1/2} f(x)dx\|^2 d\sigma,
$$

which defines a norm on F°. Moreover, $\Lambda f = P_\omega\mathcal{P}(\phi_2(\tau))$ is an isomorphism and we obtain the control

$$
u^*(t) = 0.3(\tau-\sigma-h)^{-0.7}\sum_{i=1}^{\infty}E^2_{0.3,1.3}\left(v_i(\tau-\sigma-h)^{0.3}\right)(z,\zeta_i)P^*_\omega\int_0^{1/2} f(x)dx,
$$

steering system in (49) from $z_0(x)$ to z_d with minimum energy.

5.2. Example 2: Pointwise Actuator

Let us now consider the same system as in Example 1 with the control operator $\mathfrak{B} = \delta(x-b)$, where $0 < b < 1$ is the control action point. The system is given by

$$
\begin{cases}
{}^C_0\mathbb{D}^{0.3}_t z(x,t) = \Delta z(x,t) + \delta(x-b)u(t-h), & [0,1]\times[0,\tau-h], \\
z(x,0) = 0, \\
u(t) = \varphi(t), & -h \le t \le 0,
\end{cases} \tag{52}
$$

with δ the impulse function defined by

$$
\begin{cases}
\delta(t) = 0, \quad \text{for} \quad t \neq 0, \\
\int_{-\infty}^{+\infty}\delta(t)dt = 1.
\end{cases} \tag{53}
$$

Here the operator $\mathfrak{B}$ is unbounded. Using Equations (50) and (51), one has

$$
\begin{aligned}
(\mathcal{H}+\mathfrak{L}_\varphi)^* z(x,t) &= 2\left[\mathfrak{B}^*(\tau-\sigma-h)^{r-1}\mathcal{S}^*_r(\tau-\sigma-h)z\right](x,t) \\
&= 0.6\sum_{i=1}^{\infty}\frac{E^2_{0.3,1.3}\left(v_i(\tau-\sigma-h)^{0.3}\right)(z,\zeta_i)_{L^2(0,1)}\zeta_i(b)}{(\tau-\sigma-h)^{0.7}}.
\end{aligned} \tag{54}
$$

If $b \in \mathbb{Q}$, System (52) is not Ω-controllable. Considering $\omega = [1/3, 3/4]$ and $b = 1/2$, System (52) is ω-controllable. Indeed,

$$(\mathcal{H}+\mathfrak{L}_\varphi)^* z(x,t) = 0.6\sum_{i=1}^{\infty} \frac{E^2_{0.3,1.3}\big(v_i(\tau-\sigma-h)^{0.3}\big)(z,\zeta_i)_{L^2(0,1)}\sqrt{2}\sin\Big(i\frac{\pi}{2}\Big)}{(\tau-\sigma-h)^{0.7}}$$
$$= 0.6\sqrt{2}\sum_{i=1}^{\infty} \frac{E^2_{0.3,1.3}\big(v_i(\tau-\sigma-h)^{0.3}\big)(z,\zeta_i)_{L^2(0,1)}\sin\Big(i\frac{\pi}{2}\Big)}{(\tau-\sigma-h)^{0.7}}.$$

Then,

$$(\mathcal{H}+\mathfrak{L}_\varphi)^* P^*_\omega(P_\omega\zeta_j) = 0.6\sqrt{2}\sum_{k=1}^{\infty} \frac{E^2_{0.3,1.3}\big(v_k(\tau-\sigma-h)^{0.3}\big)}{2(\tau-\sigma-h)^{0.7}}\langle\zeta_i,\zeta_j\rangle_{L^2(\omega)}\sin\Big(i\frac{\pi}{2}\Big)$$
$$= 0.6\sqrt{2}\sum_{k=1}^{\infty} \frac{E^2_{0.3,1.3}\big(v_k(\tau-\sigma-h)^{0.3}\big)}{2(\tau-\sigma-h)^{0.7}}\sin\Big(i\frac{\pi}{2}\Big)\int_{1/3}^{2/3}\zeta_i(x)\zeta_j(x)dx$$
$$\neq 0.$$

Moreover,

$$\begin{aligned}\|\psi_0\|_{F^\circ} &= \int_0^{\tau-h}\Big\|(\tau-\sigma-h)^{-0.7}\mathcal{S}^*_{0.3}(\tau-\sigma-h)P^*_{[1/3,3/4]}\psi(b)\Big\|^2 d\sigma \\ &= \int_0^{\tau-h}\left\|0.3\sum_{i=1}^{\infty}\frac{E^2_{0.3,1.3}\Big(v_i(\tau-\sigma-h)^{0.3}\Big)(z,\zeta_i)P^*_\omega\psi(1/2)}{(\tau-\sigma-h)^{0.7}}\right\|^2 d\sigma.\end{aligned} \tag{55}$$

From Lemma 5, (55) defines a norm on F° and (46) is an isomorphism from $F^{\circ *}$ to F°, where $\phi_2(\tau)$ is the solution of the system

$$\begin{cases} {}^C_0\mathbb{D}^{0.3}_t\phi_2(t) = \Delta\phi_2(t) + (\tau-t-h)^{-0.7}\mathcal{S}^*_{0.3}(\tau-t-h)\psi(b), & t\in[0,\tau-h], \\ \phi_2(0) = 0. \end{cases} \tag{56}$$

By Theorem 3, we obtain the control

$$u^*(t) = 0.3\sum_{i=1}^{\infty}\frac{E^2_{0.3,1.3}\Big(v_i(\tau-\sigma-h)^{0.3}\Big)(z,\zeta_i)P^*_\omega\psi(1/2)}{(\tau-\sigma-h)^{0.7}}d\sigma, \tag{57}$$

steering system in (52) to z_d, which is simultaneously the solution of the minimization problem in (33), where ψ is a solution of (47) and $\phi_1(\tau)$ is a solution of

$$\begin{cases} {}^C_0\mathbb{D}^{0.3}_t\phi_1(t) = \Delta\phi_1(t), \\ \phi_1(0) = z_0. \end{cases} \tag{58}$$

6. Conclusions

In this paper, we dealt with a fractional Caputo diffusion equation defined in (1). We studied regional controllability with a delay in the control. By defining an attainable set, we proved the exact and weak controllability of such a system. We also formulated a minimum optimal energy control problem subject to System (1) and computed its optimal control. The solution of the optimal control problem was obtained via an extension of the Hilbert uniqueness method.

In future work, we intend to extend the obtained results (i) to the case of fractional semi-linear systems with delays in either the control, state variables, or both; (ii) to the case of neutral evolution systems [47,48] by extending the notion of regional controllability to such systems; (iii) to the case of the complete controllability of nonlinear fractional neutral functional differential equations [49] with delays; and (iv) to the case of the regional stability of fractional delay systems [50]. Another line of research consists of developing the numerical part and providing suitable numerical simulations for real problems. This is under investigation and will be addressed elsewhere.

Author Contributions: Conceptualization, T.K., A.K. and D.F.M.T.; methodology, T.K., A.K. and D.F.M.T.; formal analysis, T.K., A.K. and D.F.M.T.; investigation, T.K., A.K. and D.F.M.T.; writing—original draft preparation, T.K., A.K. and D.F.M.T.; writing—review and editing, T.K., A.K. and D.F.M.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by The Portuguese Foundation for Science and Technology (FCT—Fundação para a Ciência e a Tecnologia), grant number UIDB/04106/2020 (CIDMA).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors are grateful to the anonymous reviewers for their comments and suggestions.

Conflicts of Interest: The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

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