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NEAR-OPTIMAL CONTROL OF A STOCHASTIC SICA MODEL WITH IMPRECISE PARAMETERS

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Dedicated to the memory of Professor Jack Warga on the occasion of his 100th birthday

Abstract. An adequate near-optimal control problem for a stochastic SICA (Susceptible–Infected–Chronic–AIDS) compartmental epidemic model for HIV transmission with imprecise parameters is formulated and investigated. We prove some estimates for the state and co-state variables of the stochastic system. The established inequalities are then used to prove a necessary and a sufficient condition for near-optimal control with imprecise parameters. The proofs involve several mathematical and stochastic tools, including the Burkholder–Davis–Gundy inequality.

Keywords. Burkholder-Davis-Gundy inequality; Imprecise parameters; Stochastic Pontryagin's maximum principle; Near-optimal control.

1. Introduction

Despite the advance of medical knowledge and technology, infectious diseases remain a growing threat to mankind. Indeed, various kinds of infectious diseases, including hepatitis C, HIV/AIDS and COVID-19, spread around the globe. Therefore, decision-makers and public health systems build up different strategies to curb the spread of diseases. Mathematical modeling is nowadays an important tool in analyzing the growth and in controlling pandemics [1, 2].

The infection by human immunodeficiency virus (HIV) is still a major public issue, where no cure or vaccine exists for the acquired immunodeficiency syndrome (AIDS). However, antiretroviral (ART) treatment improves health, prolongs life, and diminishes the risk of HIV infection. Several mathematical models have been proposed for HIV/AIDS transmission dynamics with ART control measures, see, e.g., [3, 15] and the references therein. Here we are mainly motivated by [4], where a stochastic SICA (Susceptible–Infected–Chronic–AIDS) compartmental

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epidemic model for HIV transmission is proposed and investigated; and by [13], where SICA modeling is shown to be a very useful tool with respect to real applications of HIV/AIDS.

The SICA model was introduced by Silva and Torres in 2015, as a sub-model of a general Tuberculosis and HIV/AIDS co-infection problem [12]. After that, it has been extensively used to investigate HIV/AIDS, in different settings and contexts, using fractional-order derivatives [14], stochasticity [4] and discrete-time operators [16], and adjusted to different HIV/AIDS epidemics, as those in Cape Verde [13] and Morocco [7].

In [4], Djordjevic, Silva and Torres proposed a stochastic SICA model by means of white noise (Brownian motion with positive intensity) due to environment fluctuations that perturb the coefficient rate of transmission β into $\beta+\delta B_t$ [8, 17, 18]. The stochastic SICA model of Djordjevic et al. has no control and was given as follows [4]:

$$
\begin{cases}
dS(t)=[\Lambda-\beta\left(I(t)+\eta_C C(t)+\eta_A A(t)\right)S(t)-\mu S(t)]\,dt\\
\qquad\qquad -\delta\left(I(t)+\eta_C C(t)+\eta_A A(t)\right)S(t)dB(t),\\
dI(t)=[\beta\left(I(t)+\eta_C C(t)+\eta_A A(t)\right)S(t)-\varepsilon_3 I(t)+\alpha A(t)+\omega C(t)]\,dt\\
\qquad\qquad +\delta\left(I(t)+\eta_C C(t)+\eta_A A(t)\right)S(t)dB(t),\\
dC(t)=[\phi I(t)-\varepsilon_2 C(t)]\,dt,\\
dA(t)=[eI(t)-\varepsilon_1 A(t)]\,dt,
\end{cases}
\tag{1.1}
$$

with $\varepsilon_1=\alpha+\mu+d$, $\varepsilon_2=\omega+\mu$ and $\varepsilon_3=e+\phi+\mu$, and where the meaning of the parameters Λ, β, μ, η_A, η_C, ϕ, e, α, ω, C and d is given in Table 1.

TABLE 1. Parameters of the stochastic SICA model (1.1) and their meaning.

Parameters	Meaning
Λ	Recruitment rate
β	Transmission rate
μ	Natural death rate
η_A	Relative infectiousness of individuals with AIDS symptoms
η_C	Partial restoration of immune function of individuals with HIV infection that use ART correctly
ϕ	HIV treatment rate for I individuals
e	Default treatment rate for I individuals
α	AIDS treatment rate
ω	Default treatment rate for C individuals
d	AIDS induced death rate

Thus, in SICA modeling, the human population is subdivided into four exhaustively and mutually exclusive compartments: susceptible individuals (S); HIV-infected individuals (I) with no clinical symptoms of AIDS (the virus is living or developing in the individuals but without producing symptoms or only mild ones) but able to transmit HIV to other individuals; HIV-infected individuals under ART treatment (the so called chronic stage) with a viral load remaining low (C); and HIV-infected individuals with AIDS clinical symptoms (A). The total population at time t, denoted by $N(t)$, is given by $N(t)=S(t)+I(t)+C(t)+A(t)$.

In Section 2, we apply to the SICA model (1.1) a meaningful control u. The Hamiltonian function is then introduced and we end up by recalling the Pontryagin maximum principle. Motivated by [6, 10], our main results are then given in Section 3, where we replace known biological parameters by imprecise ones, taking into account all possible environment perturbations. Near-optimal control is considered and, with the help of different mathematical techniques, like the inequalities of Cauchy–Schwartz, Hölder, and Burkholder–Davis–Gundy, Itô's formula, convexity, and Ekeland's variational principle, we prove estimates for the state (Lemmas 3.6 and 3.7) and co-state variables of the system (Lemmas 3.8 and 3.9), a necessary (Theorem 3.11) and a sufficient condition for near-optimal control (Theorem 3.12). We end with Section 4 of conclusion.

2. Optimal control of the stochastic SICA model

To obtain an adequate stochastic controlled SICA model, we introduce a control into (1.1) using the procedure presented in [10] for the stochastic SIRS model. In concrete, we propose the following dynamical control system:

$$\begin{cases} dS(t) = \left[\Lambda - \beta\left(I(t) + \eta_C C(t) + \eta_A A(t)\right) S(t) - \mu S(t)\right] dt \\ \qquad\qquad - \delta\left(I(t) + \eta_C C(t) + \eta_A A(t)\right) S(t) dB(t), \\ dI(t) = \left[\beta\left(I(t) + \eta_C C(t) + \eta_A A(t)\right) S(t) - \varepsilon_3 I(t) + \alpha A(t) + \omega C(t) - \dfrac{mu(t)I(t)}{1+\gamma I(t)}\right] dt \\ \qquad\qquad + \delta\left(I(t) + \eta_C C(t) + \eta_A A(t)\right) S(t) dB(t), \\ dC(t) = \left[\phi I(t) - \varepsilon_2 C(t) + \dfrac{mu(t)I(t)}{1+\gamma I(t)}\right] dt, \\ dA(t) = \left[e I(t) - \varepsilon_1 A(t)\right] dt. \end{cases} \tag{2.1}$$

Here m represents the adjustment coefficient of the control. To simplify the writing and the analysis, we define the vector $x = (x_1, x_2, x_3, x_4)$ as

$$x(t) := \left(S(t), I(t), C(t), A(t)\right)$$

and

$$\begin{aligned} f_1(x) &= \Lambda - \beta\left(x_2 + \eta_C x_3 + \eta_A x_4\right) x_1 - \mu x_1, \\ f_2(x,u) &= \beta\left(x_2 + \eta_C x_3 + \eta_A x_4\right) x_1 - \varepsilon_3 x_2 + \alpha x_4 + \omega x_3 - \frac{mux_2}{1+\gamma x_2}, \\ f_3(x,u) &= \phi x_2 - \varepsilon_2 x_3 + \frac{mux_2}{1+\gamma x_2}, \\ f_4(x) &= e x_2 - \varepsilon_1 x_4, \\ \sigma_1(x) &= -\sigma_2(x) = -\delta\left(x_2 + \eta_C x_3 + \eta_A x_4\right) x_1. \end{aligned} \tag{2.2}$$

With these notations, we write our SICA control system (2.1) as

$$\begin{cases} dx_1(t) = f_1(x(t))\, dt + \sigma_1(x(t))\, dB_t, \\ dx_2(t) = f_2(x(t), u(t))\, dt + \sigma_2(x(t))\, dB_t, \\ dx_3(t) = f_3(x(t), u(t))\, dt, \\ dx_4(t) = f_4(x(t))\, dt. \end{cases} \tag{2.3}$$

Now, we introduce the stochastic objective/cost functional $J(u)$ as

$$J(u) = E\left\{\int_0^T L(x(t), u(t))\, dt + h(x(T))\right\} \tag{2.4}$$

with $L: \mathbb{R}^4 \times \mathbb{R} \longrightarrow \mathbb{R}$ and $h: \mathbb{R}^4 \to \mathbb{R}$ assumed to be continuously differentiable.

Let $U \subseteq \mathbb{R}$ be a given bounded nonempty closed set. A control $u: [0,T] \longrightarrow U$ is called admissible if it is an F_t-adapted process with values in U. The set of all admissible controls is denoted by U_{ad}.

Our optimal control problem consists to find an admissible control that minimizes the objective functional $J(u)$ subject to the control system (2.3) and a given initial condition $x(0) = x_0$.

Associated to our stochastic optimal control problem, we define the stochastic Hamiltonian function by

$$H(x, u, p, q) = \langle f(x, u), p\rangle + \langle \sigma, q\rangle - L(x, u), \tag{2.5}$$

where $\sigma = (\sigma_1, \sigma_2)$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product.

The stochastic Pontryagin's Maximum Principle [11, 17] asserts that if u^* is an optimal control and x^* is the corresponding optimal trajectory, then there exists nontrivial multipliers p and q such that the Hamiltonian system

$$\begin{cases} dx^*(t) = \dfrac{\partial H}{\partial p}(x^*(t), u^*(t), p(t), q(t))\, dt + \sigma(x^*(t))\, dB_t, \\ dp_i(t) = -\dfrac{\partial H}{\partial x_i}(x^*(t), u^*(t), p(t), q(t))\, dt + q_i(t) dB_t, \quad i = 1,2,3,4, \end{cases} \tag{2.6}$$

holds together with the maximality condition

$$H(x^*(t), u^*(t), p(t), q(t)) = \max_{v \in U} H(x^*(t), v, p(t), q(t)) \tag{2.7}$$

and the initial state and terminal costate conditions

$$x^*(0) = x_0, \quad p_i(T) = -\frac{\partial h}{\partial x_i}(x^*(T)). \tag{2.8}$$

Note that in our problem the diffusion term σ does not depend on the control u.

In Section 3, we transform (2.1) into a near-optimal control problem via the incorporation of imprecise parameters.

3. Near-optimal control with imprecise parameters

In the majority of available mathematical epidemic models, the parameter values are assumed to be precisely known. Nevertheless, in real applications, one needs to take into account the influence of numerous uncertainties. This motivate us to include here, for the first time in the literature of SICA modeling, imprecise parameters into the stochastic SICA model and to consider the problem of near-optimal control. Before that, we need some preliminary notions.

3.1. **Preliminaries.**

To make it explicitly that in our optimal control problem we begin at time $t = 0$ with the given initial state $x(0) = x_0$, from now on we denote the cost functional $J(u)$ defined in (2.4) by $J(0, x_0, u)$. Moreover, the minimum of functional (2.4) is denoted by $V(0, x_0)$, that is,

$$V(0, x_0) = \min_{u \in U_{ad}} J(0, x_0, u).$$

Function V is known in the literature as the value function.

We recall some known definitions available in the literature: see, e.g., [9, 10].

Definition 3.1 (optimal control)**.** An admissible control u^* is called optimal if

$$J(0,x_0,u^*) = V(0,x_0).$$

Definition 3.2 (ε-optimal control)**.** Let $\varepsilon > 0$. An admissible control u^ε is called ε-optimal if

$$\mid J(0,x_0,u^\varepsilon) - V(0,x_0) \mid \leq \varepsilon.$$

Definition 3.3 (near-optimal control)**.** Consider a family of admissible controls $\{u^\varepsilon\}$ parameterized by $\varepsilon > 0$ and let u^ε be any element in this family. We say that u^ε is a near-optimal control if

$$\mid J(0,x_0,u^\varepsilon) - V(0,x_0) \mid \leq \delta(\varepsilon)$$

holds for a sufficient small $\varepsilon > 0$, where δ is a function of ε satisfying $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The estimate $\delta(\varepsilon)$ is called an error bound. If $\delta(\varepsilon) = r\varepsilon^\omega$, for some $r, \omega > 0$, then u^ε is said to be a near-optimal control of order ε^ω.

Definition 3.4 (interval numbers)**.** An interval number A is represented by a closed interval $[a_l, a_u]$ defined by

$$A = [a_l, a_u] = \{x \in \mathbb{R} : a_l \leq x \leq a_u\},$$

where a_l and a_u are the lower and the upper limits of the interval number, respectively. We represent an interval $[a,b]$ by the so called interval-valued function, which is given by

$$h(k) = a^{1-k} b^k, \quad k \in [0,1].$$

Remark 3.5. The sum, difference, product and division of two interval numbers are also interval numbers.

3.2. **The stochastic SICA control model with imprecise parameters.** We assume that the stochastic SICA model has some biological imprecise parameters. The uncertain parameters are described by interval numbers. After replacing each parameter ζ with an imprecise one, $\zeta_k = (\zeta_l)^{1-k}(\zeta_u)^k \in [\zeta_l, \zeta_u]$ for $k \in [0,1]$, our control system (2.1) becomes:

$$\begin{cases} dS(t) = \left[\Lambda_k - \beta_k \left(I(t) + (\eta_C)_k C(t) + (\eta_A)_k A(t)\right) S(t) - \mu_k S(t)\right] dt \\ \qquad\qquad - \delta_k \left(I(t) + (\eta_C)_k C(t) + (\eta_A)_k A(t)\right) S(t) dB(t), \\ dI(t) = \Bigg[\beta_k \left(I(t) + (\eta_C)_k C(t) + (\eta_A)_k A(t)\right) S(t) - (\varepsilon_3)_k I(t) \\ \qquad\qquad + \alpha_k A(t) + \omega_k C(t) - \dfrac{m_k u(t) I(t)}{1 + \gamma_k I(t)} \Bigg] dt \\ \qquad\qquad + \delta_k \left(I(t) + (\eta_C)_k C(t) + (\eta_A)_k A(t)\right) S(t) dB(t), \\ dC(t) = \left[\phi_k I(t) - (\varepsilon_2)_k C(t) + \dfrac{m_k u(t) I(t)}{1 + \gamma_k I(t)} \right] dt, \\ dA(t) = \left[e_k I(t) - (\varepsilon_1)_k A(t) \right] dt. \end{cases} \tag{3.1}$$

Let u and $u' \in U_{ad}$. Set the following metric on $U_{ad}[0,T]$:

$$d(u,u') = E \,\mathrm{meas}\{t \in [0,T] : u(t) \neq u'(t)\}, \tag{3.2}$$

where "meas" represents the Lebesgue measure. Note that since U is closed, it follows that U_{ad} is a complete metric space under d.

Next we prove estimates of the state and co-state variables. Let

$$\Omega = \left\{ (S(t), I(t), C(t), A(t)) \in \mathbb{R}^4_+ : \frac{\Lambda_k}{\mu_k + d_k} \leq N(t) \leq \frac{\Lambda_k}{\mu_k} \right\} \subset \mathbb{R}^4_+, \tag{3.3}$$

where $N(t) = S(t) + I(t) + C(t) + A(t)$. Shortly, our Lemma 3.6 asserts that the trajectories of (3.1) will enter and remain in Ω with probability 1.

Lemma 3.6. *For any $\theta \geq 0$ and $u \in U_{ad}$, we have*

$$E \sup_{0 \leq t \leq r} (\mid S(t) \mid^\theta + \mid I(t) \mid^\theta + \mid C(t) \mid^\theta + \mid A(t) \mid^\theta) \leq R,$$

where R is an imprecise parameter depending only on θ.

Proof. Adding member to member all equations of the system (3.1), we have

$$\frac{dN(t)}{dt} = \Lambda_k - \mu_k N(t) - d_k A(t).$$

Because $d_k A(t) \geq 0$, it follows that

$$\frac{dN(t)}{dt} \leq \Lambda_k - \mu_k N(t).$$

Multiplying by $\exp(u_k t)$ both sides of this inequality, we obtain that

$$\exp(u_k t) \frac{dN(t)}{dt} \leq \Lambda_k \exp(u_k t) - \mu_k \exp(u_k t) N(t).$$

An integration by parts between 0 and t leads to

$$N(t) \exp(u_k t) - N(0) - \int_0^t N(s) \exp(u_k s) ds \leq \frac{\Lambda_k}{\mu_k} (exp(u_k t) - 1) - \int_0^t \mu_k \exp(u_k s) N(s) ds.$$

Equivalently, we have

$$N(t) \leq \frac{\Lambda_k}{\mu_k} (1 - \exp(-u_k t)) + N(0) \exp(-u_k t)$$

and thus

$$\limsup_{t \to +\infty} N(t) \leq \frac{\Lambda_k}{\mu_k}.$$

We also have

$$\frac{dN(t)}{dt} \geq \Lambda_k - (\mu_k + d_k) N(t)$$

and, following the same arguments as before, we obtain

$$\liminf_{t \to +\infty} N(t) \geq \frac{\Lambda_k}{\mu_k + d_k}.$$

We conclude that

$$\frac{\Lambda_k}{\mu_k + d_k} \leq \liminf_{t \to +\infty} N(t) \leq \limsup_{t \to +\infty} N(t) \leq \frac{\Lambda_k}{\mu_k}.$$

This means that all solutions $S(t)$, $I(t)$, $C(t)$, and $A(t)$ of our model (3.1) are almost surely bounded over the positively invariant bounded set (3.3). □

In what follows, we use the notation $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) = (S(t), I(t), C(t), A(t))$ for our system (3.1).

Lemma 3.7. *Let $\theta \geq 0$ and $0 < k < 1$ with $k\theta < 1$. If $u, u' \in U_{ad}$ and x and x' are the corresponding state trajectories, then there exists an imprecise parameter $R = R(\theta, k)$ such that*

$$\sum_{i=1}^{4} E \sup_{0 \leq t \leq T} | x_i(t) - x_i'(t) |^{2\theta} \leq R d(u(t), u'(t))^{k\theta}. \tag{3.4}$$

Proof. Let us first suppose that $\theta \geq 1$. From system (3.1), and Hölder's inequality, for $\theta \geq 1$, we have

$$\begin{aligned} E \sup_{0 \leq t \leq r} | x_1(t) - x_1'(t) |^{2\theta} \leq RE \int_0^r &[(\beta_k^{2\theta} + \delta_k^{2\theta}) \mid (x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4) x_4 \\ &- ((x_2' + (\eta_C)_k x_3' + (\eta_A)_k x'_4)) x_4 |^{2\theta} + (\mu_k)^{2\theta} | x_1(t) - {x_1}'(t) |^{2\theta}] dt \\ \leq RE \int_0^r &\sum_{i=1}^{4} | x_i - {x'}_i |^{2\theta} \, dt. \end{aligned} \tag{3.5}$$

Since

$$\begin{aligned} E \int_0^r \chi_{u(t) \neq u'(t)} dt &\leq \left(E \int_0^r dt \right)^{1-k\theta} \left(E \int_0^r \chi_{u(t) \neq u'(t)} dt \right)^{k\theta} \\ &\leq R(\theta, k) \left(E \, \text{meas}\{t / u(t) \neq u'(t)\} \right)^{k\theta} \\ &\leq R d(u, u')^{k\theta}, \end{aligned}$$

we also have

$$\begin{aligned} E \sup_{0 \leq t \leq r} | x_2(t) - x_2'(t) |^{2\theta} \leq RE \int_0^r &\Bigg[(\beta_k^{2\theta} + \delta^{2\theta}) \mid (x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4) x_4 \\ &- ((x_2' + (\eta_C)_k x_3' + (\eta_A)_k x'_4)) x_4 |^{2\theta} \\ &+ (\varepsilon_3)_k^{2\theta} | x_2(t) - {x'}_2(t) |^{2\theta} + \alpha_k^{2\theta} | x_4(t) - x_4'(t) |^{2\theta} \\ &+ \omega_k^{2\theta} | x_3(t) - {x'}_3(t) |^{2\theta} \\ &+ m_k^{2\theta} \left| \frac{u(t) x_2(t)}{1 + \eta_k x_2} - \frac{u'(t) x_2'(t)}{1 + \eta_k x_2'} \right|^{2\theta} \Bigg] dt \\ \leq RE \int_0^r &\sum_{i=1}^{4} | x_i - {x'}_i |^{2\theta} + RE \int_0^r \chi_{u(t) \neq u'(t)} dt \\ \leq R &\left[E \int_0^r \sum_{i=1}^{4} | x_i - {x'}_i |^{2\theta} \, dt + d(u, u')^{k\theta} \right]. \end{aligned} \tag{3.6}$$

From the Hölder's inequality under the hypothesis $k\theta < 1$, we also find that

$$E \sup_{0 \leq t \leq r} | x_3(t) - x_3'(t) |^{2\theta} \leq R \left[E \int_0^r \sum_{i=1}^{4} | x_i - {x'}_i |^{2\theta} \, dt + d(u, u')^{k\theta} \right] \tag{3.7}$$

and

$$E\sup_{0\le t\le r}\mid x_4(t)-x_4'(t)\mid^{2\theta}\le R\left[E\int_0^r\sum_{i=1}^4\mid x_i-x'_i\mid^{2\theta}dt+d(u,u')^{k\theta}\right]. \tag{3.8}$$

Combining the last four inequalities (3.5)–(3.8), we have

$$\sum_{i=1}^4 E\sup_{0\le t\le r}\mid x_i(t)-x_i'(t)\mid^{2\theta}\le R\left[E\int_0^r\sum_{i=1}^4\mid x_i-x'_i\mid^{2\theta}dt+d(u,u')^{k\theta}\right].$$

By using Gronwall's inequality, we conclude that (3.4) holds.

To prove the desired result in the case $0\le\theta<1$, we apply the Cauchy–Schwartz inequality for obtaining

$$\begin{aligned}\sum_{i=1}^4 E\sup_{0\le t\le r}\mid x_i(t)-x_i'(t)\mid^{2\theta}&\le\sum_{i=1}^4\left[E\sup_{0\le t\le r}\mid x_i(t)-x_i'(t)\mid^{2}\right]^\theta\\&\le\left[Rd(u,u')^k\right]^\theta\\&\le R^\theta d(u,u')^{k\theta},\end{aligned}$$

where R is the imprecise parameter. The proof is complete. □

Now we prove estimates for the co-state variables.

Lemma 3.8. *Let p_i be the co-state variables given by the stochastic Pontryagin's maximum principle. Then,*

$$\sum_{i=1}^4 E\sup_{0\le t\le T}\mid p_i(t)\mid^2+\sum_{i=1}^2 E\int_0^T\mid q_i(t)\mid^2 dt\le R,$$

where R is an imprecise parameter.

Proof. From Pontryagin's maximum principle, we have

$$dp_1(t)=-\partial_{x_1}H(x^*,u^*,p,q)dt+q_1(t)dB_t=-b_1dt+q_1dB_t,$$

so that

$$p_1(t)+\int_t^T q_1(s)dB_s=p_1(T)+\int_t^T b_1(s)ds.$$

By Lemma 3.6 $x^*(t)\in\Omega$ and one has, for all $t\ge 0$,

$$\begin{aligned}E\mid p_1(t)\mid^2+E\left\{\int_0^T\mid q_1(s)\mid^2ds\right\}&\le RE\mid p_1(T)\mid^2+R(T-t)E\left\{\int_0^T\mid b_1(s)\mid^2ds\right\}\\\le RE\mid p_1(T)\mid^2+R(T-t)\sum_{i=1}^4E\left\{\int_0^T\mid p_i(s)\mid^2ds\right\}&+R(T-t)\sum_{i=1}^2E\left\{\int_0^T\mid q_i(s)\mid^2ds\right\}.\end{aligned}$$

Similarly, we obtain the same inequalities for $i\in\{2,3,4\}$. By adding member to member, we have

$$\begin{aligned}&\sum_{i=1}^4E\mid p_i(t)\mid^2+\sum_{i=1}^2E\int_t^T\mid q_i(s)\mid^2ds\\&\le RE\mid p_i(T)\mid^2+R(T-t)\sum_{i=1}^4E\left\{\int_t^T\mid p_i(s)\mid^2ds\right\}+R(T-t)\sum_{i=1}^2E\left\{\int_t^T\mid q_i(s)\mid^2ds\right\}.\end{aligned}$$

Then,

$$\sum_{i=1}^{4} E \mid p_i(t) \mid^2 + \frac{1}{2}\sum_{i=1}^{2} E\left\{\int_t^T \mid q_i(s) \mid^2 ds\right\}$$
$$\leq \sum_{i=1}^{4} RE \mid p_i(T) \mid^2 + R(T-t)\sum_{i=1}^{4} E\left\{\int_t^T \mid p_i(s) \mid^2 ds\right\},$$

where $t \in [T-\varepsilon, T]$ and $\varepsilon = \frac{1}{R}$. Using Gronwall's inequality, we obtain that

$$\sum_{i=1}^{4} \sup_{0\leq t\leq T} E \mid p_i(t) \mid^2 \leq R, \quad \sum_{i=1}^{4} E\left\{\int_t^T \mid q_i(s) \mid^2 ds\right\} \leq R. \tag{3.9}$$

We also obtain the same result over $[T-2\varepsilon, T]$, $[T-3\varepsilon, T]$, and so on. Repeating for a finite number of steps, the expected estimate emerges for any $t \in [0,T]$. Furthermore, from

$$p_1(t) = p_1(T) + \int_t^T b_1(s)ds - \int_0^T q_1(s)dB_s + \int_0^t q_1(s)dB_s$$

and the elementary inequality

$$\mid m_1 + m_2 + m_3 + m_4 \mid^n \leq 4^n \left(\mid m_1 \mid^n + \mid m_2 \mid^n + \mid m_3 \mid^n + \mid m_4 \mid^n\right), \tag{3.10}$$

valid for any $n > 0$, we have

$$\mid p_1(t) \mid^2 \leq R\Bigg[\mid p_1(T) \mid^2 + \int_0^T \left(\sum_{i=1}^{4} \mid p_i(s) \mid^2\right) ds + \int_0^T \left(\sum_{i=1}^{2} \mid q_i(s) \mid^2\right) ds$$
$$+ \left(\int_0^T q_1(s)dB\right)^2 + \left(\int_0^t q_1(s)dB\right)^2\Bigg].$$

Analogous statements are established for p_2, p_3, and p_4. Next, by addition, we have

$$\sum_{i=1}^{4} \mid p_i(t) \mid^2 \leq R\Bigg[\sum_{i=1}^{4} \mid p_i(T) \mid^2 + \int_0^T \left(\sum_{i=1}^{4} \mid p_i(s) \mid^2\right) ds$$
$$+ \int_0^T \left(\sum_{i=1}^{2} \mid q_i(s) \mid^2\right) ds + \sum_{i=1}^{2} \left(\int_0^T q_i(s)dB\right)^2 + \sum_{i=1}^{2} \left(\int_0^t q_i(s)dB\right)^2\Bigg].$$

The Burkholder–Davis–Gundy inequality [8] leads to

$$\sum_{i=1}^{4} E \sup_{0\leq t\leq T} \mid p_i(t) \mid^2 \leq R\Bigg[\sum_{i=1}^{4} \mid p_i(T) \mid^2 + \sum_{i=1}^{4} E\left\{\int_0^T \sup_{0\leq v\leq s} \mid p_i(v) \mid^2 ds\right\}$$
$$+ \sum_{i=1}^{2} E\left\{\int_0^T \sup_{0\leq v\leq s} \mid q_i(v) \mid^2 ds\right\}\Bigg]$$

and the establishment of our desired result is achieved by applying Gronwall's inequality. □

The following assumptions will be used in our next results:

(S_1) For all $0 \leq t \leq T$, the partial derivatives L_S, L_I, L_C, L_A, and h_S and h_I are continuous, and there exists an imprecise parameter R such that

$$\mid L_S + L_I + L_C + L_A \mid \leq R(1 + \mid S(t) \mid + \mid I(t) \mid + \mid C(t) \mid + \mid A(t) \mid).$$

Moreover,

$$(1+\mid S(t)\mid)^{-1}\mid h_S\mid+(1+\mid I(t)\mid)^{-1}\mid h_I\mid\leq R.$$

(S_2) If $x(t),x'(t)\in\mathbb{R}^4_+$ for any $0\leq t\leq T$, $u,u'\in U_{ad}$, and function $u\mapsto L(x,u)$ is differentiable, then there exists an imprecise parameter R such that

$$\mid L(x(t),u(t))-L(x(t),u'(t))\mid+\mid L_u(x(t),u(t))-L_u(x(t),u'(t))\mid\leq R\mid u(t)-u'(t)\mid.$$

Moreover, if h is differentiable, then

$$\sum_{i=0}^{4}\mid h_{x_i}(x(t))-h_{x_i}(x'(t))\mid\leq R\sum_{i=0}^{4}\mid x_i(t)-x_i'(t)\mid.$$

Lemma 3.9. *Let (S_1) and (S_2) hold. For any $1<\eta<2$ and $0<k<1$ satisfying $(1+k)\eta<2$, there exists a constant $R=R(\eta,k)$ such that, for any $u,u'\in U_{ad}$ along with the corresponding trajectories x,x' and adjoint multipliers $(p,q),(p',q')$,*

$$\sum_{i=1}^{4}E\left\{\int_0^T\mid p_i(t)-p_i'(t)\mid^\eta dt\right\}+\sum_{i=1}^{4}E\left\{\int_0^T\mid q_i(t)-q_i'(t)\mid^\eta dt\right\}\leq Rd(u,u')^{\frac{k\eta}{2}}.$$

Proof. Let $\bar{p}_i(t)=p_i(t)-p_i'(t)$, $i\in\{1,2,3,4\}$, and $\bar{q}_i(t)=q_i(t)-q_i'(t)$, $i\in\{1,2\}$. It follows from (2.6) that

$$\begin{cases}d\bar{p}_1(t)=-[(-\beta_k(x_2+(\eta_C)_kx_3+(\eta_A)_kx_4)-\mu_k)\bar{p}_1+(\beta_k(x_2+(\eta_C)_kx_3+(\eta_A)_kx_4)-\mu_k)\bar{p}_2\\ \quad-\delta_k(x_2+(\eta_C)_kx_3+(\eta_A)_kx_4-\mu_k)\bar{q}_1+\delta_k(x_2+(\eta_C)_kx_3+(\eta_A)_kx_4)\bar{q}_2+\bar{f}_1]dt+\bar{q}_1dB,\\ d\bar{p}_2(t)=-[-\beta_kx_1\bar{p}_1+(\beta_kx_1-\varepsilon_3-\frac{m_ku}{(1+\delta_kx_2)^2})\bar{p}_2\\ \quad+\left(\phi+\frac{m_ku}{(1+\gamma_kx_2)^2}\bar{p}_3+e\bar{p}_4-\delta x_1\right)\bar{q}_1+\delta_kx_1\bar{q}_2+\bar{f}_2]dt+\bar{q}_2dB,\\ d\bar{p}_3(t)=-[-\beta_k(\eta_C)_kx_1\bar{p}_1+\beta_k(\eta_C)_kx_1\bar{p}_2-\varepsilon_2\bar{p}_3-\delta_k(\eta_C)_kx_1\bar{q}_1+\delta_k\eta_Cx_1\bar{q}_2+\bar{f}_3]dt,\\ d\bar{p}_4(t)=-[-\beta_k(\eta_A)_kx_1\bar{p}_1+\beta_k(\eta_A)_kx_1\bar{p}_2-(\varepsilon_1)_k\bar{p}_4-\delta_k(\eta_A)_kx_1\bar{q}_1+\delta_k(\eta_A)_kx_1\bar{q}_2+\bar{f}_4]dt,\end{cases}$$

where

$$\begin{cases}\bar{f}_1=\beta_k[(x_2+(\eta_C)_kx_3+(\eta_A)_kx_4)-(x_2'+(\eta_C)_kx_3'+(\eta_A)_kx_4')](p_2'-p_1')\\ \quad-(\sigma)_k\left[(x_2+(\eta_C)_kx_3+(\eta_A)_kx_4)-(x_2'+(\eta_C)_kx_3'+(\eta_A)_kx_4')\right](q_2'-q_1')\\ \quad-L_{x_1}(x,u)+L_{x_1}(x',u'),\\ \bar{f}_2=(\beta)_k(x_1-x_1')(p_2'-p_1')+\left[\frac{m_ku}{1+(\eta)_kx_2}-\frac{(m_ku')_k}{1+(\eta)_kx_2'}\right](p_3'-p_2')\\ \quad+(\delta)_k(x_1-x_1')(q_2'-q_1')-L_{x_2}(x,u)+L_{x_2}(x',u'),\\ \bar{f}_3=(\beta)_k(\eta_C)_k(x_1-x_1')(p_2'-p_1')+(\delta)_k(\eta_C)_k(x_1-x_1')(q_2'-q_1')-L_{x_3}(x,u)+L_{x_3}(x',u'),\\ \bar{f}_4=(\beta)_k(\eta_A)_k(x_1-x_1')(p_2'-p_1')+(\delta)_k(\eta_A)_k(x_1-x_1')(q_2'-q_1')-L_{x_4}(x,u)+L_{x_4}(x',u').\end{cases}$$

Let $\phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t))$ be the solution of the following linear stochastic differential equation:

$$\begin{cases} d\phi_1(t) = \Big[-(\beta)_k(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4)\phi_1 - (\beta)_k x_1\phi_2 - (\beta)_k(\eta_C)_k x_1\phi_3 - \beta_k(\eta_A)_k x_1\phi_4 \\ \qquad + \mid \bar{p}_1 \mid^{\eta-1} \operatorname{sgn}(\bar{p}_1)\Big]dt + [-(\delta)_k(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4)\phi_1 \\ \qquad - (\delta)_k x_1\phi_2 - (\sigma)_k(\eta_C)_k x_1\phi_3 - (\delta)_k(\eta_A)_k x_1\phi_4 + \mid \bar{q}_1 \mid^{\eta-1} \operatorname{sgn}(\bar{q}_1)]dB, \\ d\phi_2(t) = [(\beta)_k(x_2 + (\eta_C)_k x_3 + (\eta_A)_k x_4 - (\mu)_k)\phi_1 + ((\beta)_k x_1 - (\varepsilon_3)_k - \dfrac{m_k u}{(1+(\gamma)_k x_2)^2}\phi_2 \\ \qquad + (\beta)_k(\eta_C)_k x_1\phi_3 + \beta_k(\eta_A)_k x_1\phi_4 + \mid \bar{p}_2 \mid^{\eta-1} \operatorname{sgn}(\bar{p}_2)]dt + [(\delta)_k(x_2 + (\eta_C)_k x_3 \\ \qquad + (\eta_A)_k x_4)\phi_1 + (\delta)_k x_1\phi_2 + (\delta)_k(\eta_C)_k x_1\phi_3 + (\delta)_k(\eta_A)_k x_1\phi_4 + \mid \bar{q}_2 \mid^{\eta-1} \operatorname{sgn}(\bar{q}_2)]dB, \\ d\phi_3(t) = [\phi_k + \dfrac{m_k u}{(1+(\gamma)_k x_2)^2} - (\varepsilon_2)_k\phi_3 + \mid \bar{p}_3 \mid^{\eta-1} \operatorname{sgn}(\bar{p}_3)]dt, \\ d\phi_4(t) = [e_k\phi_2 - (\varepsilon)_k\phi_4 + \mid \bar{p}_4 \mid^{\eta-1} \operatorname{sgn}(\bar{p}_4)]dt. \end{cases} \tag{3.11}$$

Lemma 3.8 and hypothesis (S_1) demonstrate the existence and uniqueness of solution to system (3.11). Using the Cauchy-Schwartz inequality, we obtain the following statement:

$$\sum_{i=1}^{4} E \sup_{0\le t\le T} \mid \phi_i(t) \mid^{\eta_1} \le \sum_{i=1}^{4} E\int_0^T \mid \bar{p}_i \mid^{\eta} dt + \sum_{i=1}^{2} E\int_0^T \mid \bar{q}_i \mid^{\eta} dt,$$

where $\eta_1 > 2$ and $\frac{1}{\eta_1} + \frac{1}{\eta} = 1$. Set $V(\bar{p}, \phi) = \sum_{i=1}^{4} \bar{p}_i\phi_i(t)$. Using Itô's formula, we have

$$\begin{aligned} &\sum_{i=1}^{4} E\int_0^T \mid \bar{p}_i \mid^{\eta} dt + \sum_{i=1}^{2} E\int_0^T \mid \bar{q}_i \mid^{\eta} dt \\ &\quad = -\sum_{i=1}^{4} E\int_0^T \phi_i\bar{f}_i dt + \sum_{i=1}^{4} E\phi_i(T)\left[h_{x_i}(x(T)) - h_{x_i}(x'(T))\right] \\ &\quad \le R\sum_{i=1}^{4}\left(E\int_0^T \mid \bar{f}_i \mid^{\eta} dt\right)^{\frac{1}{\eta}}\left(E\int_0^T \mid \phi_i \mid^{\eta_1} dt\right)^{\frac{1}{\eta_1}} \\ &\qquad + R\sum_{i=1}^{4}\left(E\left[h_{x_i}(x(T)) - h_{x_i}(x'(T))\right]^{\eta}\right)^{\frac{1}{\eta}} \times (E \mid \phi_i(T) \mid^{\eta_1})^{\frac{1}{\eta_1}} \\ &\quad \le R\left(\sum_{i=1}^{4} E\int_0^T \mid \bar{p}_i \mid^{\eta} dt + \sum_{i=1}^{2} E\int_0^T \mid \bar{q}_i \mid^{\eta} dt\right)^{\frac{1}{\eta_1}} \\ &\qquad \times\left(\sum_{i=1}^{4}\left(E\int_0^T \mid \bar{f}_i \mid^{\eta} dt\right)^{\frac{1}{\eta}} + \sum_{i=1}^{4}\left(E \mid h_{x_i}(x(T) - h_{x_i}(x'(T)) \mid^{\eta})^{\frac{1}{\eta}}\right)\right). \end{aligned}$$

From elementary inequality (3.10), we have

$$\sum_{i=1}^{4} E\int_0^T |\bar{p}_i|^\eta\, dt + \sum_{i=1}^{2} E\int_0^T |\bar{q}_i|^\eta\, dt \leq R\left[\sum_{i=1}^{4} E\int_0^T |\bar{f}_i|^\eta\, dt + \sum_{i=1}^{4} E\,|h_{x_i}(x(T)) - h_{x_i}(x'(T))|^\eta\right].$$

Using (S_2) and Lemma 3.7, we obtain that

$$\sum_{i=1}^{4} E\,|h_{x_i}(x(T)) - h_{x_i}(x'(T))|^\eta \leq R^\eta \sum_{i=1}^{4} E\,|x_i(t) - x_i'(t)|^\eta] \leq Rd(u,u')^{\frac{k\eta}{2}}.$$

It follows from the Cauchy–Schwartz inequality that

$$\begin{aligned} E\int_0^T |\bar{f}_i|^\eta\, dt \leq R\Bigg[& E\int_0^T |x_2 - x_2'|^\eta \left(\sum_{i=1}^{4} |p_i'|^\eta + \sum_{i=1}^{2} |q_i'|^\eta\right) dt \\ & + E\int_0^T |L_{x_1}(x,u) - L_{x_1}(x',u')|^\eta\, dt\Bigg] \\ \leq R & \left(E\int_0^T |x_2 - x_2'|^{\frac{2\eta}{2-\eta}}\, dt\right)^{1-\frac{\eta}{2}} \\ & \times \left[\left(\sum_{i=1}^{2}\int_0^T |p_i'|^2\, dt\right)^{\frac{\eta}{2}} + \left(\sum_{i=1}^{2}\int_0^T |q_i'|^2\, dt\right)^{\frac{\eta}{2}}\right] + Rd(u,u')^{\frac{k\eta}{2}}. \end{aligned}$$

Observe that $\frac{2\eta}{1-\eta} < 1$, $1 - \frac{\eta}{2} > \frac{k\eta}{2}$, and $d(u,u') < 1$. It comes from (3.8) and (3.9) that

$$E\left[\int_0^T |\bar{f}_1|^\eta\, dt\right] \leq d(u,u')^{\frac{k\eta}{2}}.$$

Similarly,

$$\sum_{i=1}^{4} E\int_0^T |\bar{f}_i|^\eta\, dt \leq Rd(u,u')^{\frac{k\eta}{2}}.$$

It results that

$$\sum_{i=1}^{4} E\int_0^T |\bar{p}_i|^\eta\, dt + \sum_{i=1}^{2} E\int_0^T |\bar{q}_i|^\eta\, dt \leq Rd(u,u')^{\frac{k\eta}{2}}$$

and the proof is complete. □

3.3. **Necessary condition for near-optimal control.** Our necessary condition for the near-optimal control of system (3.1) makes use of the following classical result.

Lemma 3.10 (Ekeland's variational principle [5])**.** *Let (Q,d) be a complete metric space and $F : Q \to \mathbb{R}$ be a lower-semi continuous function bounded from below. For any $\varepsilon > 0$, assume that $u^\varepsilon \in Q$ satisfies*

$$F(u^\varepsilon) \leq \inf_{u\in Q} F(u) + \varepsilon. \tag{3.12}$$

Then, there exists a $u^\lambda \in Q$, $\lambda > 0$, *such that for all* $u \in Q$ *one has*

$$F(u^\lambda) \leq F(u^\varepsilon), \quad d(u^\lambda, u^\varepsilon) \leq \lambda, \quad F(u^\lambda) \leq F(u) + \frac{\varepsilon}{\lambda} d(u^\lambda, u^\varepsilon).$$

Theorem 3.11. *Let* (S_1) *and* (S_2) *hold,* $x \mapsto L(x,u)$ *and* $x \mapsto h(x)$ *be convex almost surely, and* $(p^\varepsilon(t),\ q^\varepsilon(t))$ *be the solution of the adjoint equation under control* u^ε. *Then there exists an imprecise parameter* R *such that, for any* $\eta \in [0,1]$, $\varepsilon > 0$ *and any* ε*-optimal pair* $(x^\varepsilon(t), u^\varepsilon(t))$, *the following condition holds:*

$$\begin{aligned}
\inf_{u \in U_{ad}} E\left\{ \int_0^T \left(\frac{m_k u(t) I^\varepsilon(t)}{1+\gamma_k I^\varepsilon(t)} (p_3^\varepsilon(t) - p_2^\varepsilon(t)) + L(x^\varepsilon(t), u(t)) \right) dt \right\} \\
\geq E\left\{ \int_0^T \left(\frac{m_k u^\varepsilon(t) I^\varepsilon(t)}{1+\gamma_k I^\varepsilon(t)} (p_3^\varepsilon(t) - p_2^\varepsilon(t)) + L(x^\varepsilon(t), u^\varepsilon(t)) \right) dt \right\} - R\varepsilon^{\frac{\eta}{3}}.
\end{aligned}$$

Proof. The function $J(x_0,\cdot) : U_{ad} \to \mathbb{R}$ is continuous under the metric d defined by (3.2). Applying Lemma 3.10, and taking $\lambda = \varepsilon^{\frac{2}{3}}$, there exists an admissible pair $(\bar{x}^\varepsilon(t), \bar{u}^\varepsilon(t))$ such that

$$d(u^\varepsilon(t), \bar{u}^\varepsilon(t)) \leq \varepsilon^{\frac{2}{3}}, \tag{3.13}$$

$$J(0, x_0, \bar{u}^\varepsilon(t)) \leq \bar{J}(0, x_0, u) \tag{3.14}$$

for all $u \in U_{ad}[0,T]$, where

$$\bar{J}(0,x_0,u) = J(0,x_0,u) + \varepsilon^{\frac{1}{3}} d(u^\varepsilon, \bar{u}^\varepsilon). \tag{3.15}$$

This is equivalent to the fact that $(\bar{x}^\varepsilon(t), \bar{u}^\varepsilon(t))$ is an ε-optimal pair for the system (3.1) under the cost functional (2.4). Moreover, a necessary condition for $(\bar{x}^\varepsilon(t), \bar{u}^\varepsilon(t))$ is deduced by the following setting. Let $\bar{t} \in [0,T]$, $\rho > 0$ and $u \in U_{ad}[0,T]$. We define $u^\rho(t) = u(t)$ if $t \in [\bar{t}, \bar{t}+\rho]$, and $u^\rho(t) = \bar{u}^\varepsilon(t)$ if $t \in [0,T] \backslash [\bar{t}, \bar{t}+\rho]$. We deduce from equations (3.14) and (3.15) that

$$\bar{J}(0,x_0,\bar{u}^\varepsilon) \leq \bar{J}(0,x_0,u^\rho) \tag{3.16}$$

and

$$d(u^\rho, \bar{u}^\varepsilon) \leq \varepsilon^{\frac{2}{3}}. \tag{3.17}$$

It comes from (3.16), (3.17), and Taylor's expansion that

$$\begin{aligned}
-\rho\varepsilon^{\frac{1}{3}} &\leq J(0,x_0,u^\rho) - J(0,x_0,\bar{u}^\varepsilon) \\
&= E\int_0^T [L(x^\rho, u^\rho) - L(\bar{x}^\varepsilon, \bar{u}^\varepsilon)]\, dt + E[h(x^\rho(T)) - h(\bar{x}^\varepsilon(T))] \\
&\leq \sum_{i=1}^4 E\left\{ \int_0^T L_{x_i}(\bar{x}^\varepsilon, u^\rho) \left(x_i^\rho - \bar{x}_i^\varepsilon\right) dt \right\} \\
&\quad + E\left\{ \int_{\bar{t}}^{\bar{t}+\rho} [L(\bar{x}^\varepsilon, u) - L(\bar{x}^\varepsilon, \bar{u}^\varepsilon)] dt \right\} \\
&\quad + \sum_{i=1}^4 E\left[h_{x_i}(\bar{x}^\varepsilon(T))(x_i^\rho(T) - \bar{x}_i^\varepsilon(t)) \right] + o(\rho).
\end{aligned} \tag{3.18}$$

The Itô formula applied on $\sum_{i=1}^4 \bar{p}_i^\varepsilon (x_i^\rho - \bar{x}_i^\varepsilon)$ and the use of Lemmas 3.6 and 3.8 yield

$$\sum_{i=1}^4 E\left[h_{x_i}(x^\rho(T))(x_i^\rho(T) - \bar{x}_i^\varepsilon(T)) \right] \leq E\left\{ \int_{\bar{t}}^{\bar{t}+\rho} [(u^\rho - \bar{u}^\varepsilon)(\bar{p}_3^\varepsilon) - \bar{p}_1^\varepsilon + (u^\rho - \bar{u}^\varepsilon)\bar{p}_3^\varepsilon]\, dt \right\}.$$

Subsequently,

$$
\begin{aligned}
-\rho\varepsilon^{\frac{1}{3}} \leq\ & J(0,x_0,u^{\rho}) - J(0,x_0,\bar{u}^{\varepsilon}) \\
& + E\left\{\int_{\bar{t}}^{\bar{t}+\rho} [L(\bar{x}^{\rho},u) - L(\bar{x}^{\varepsilon},\bar{u}^{\varepsilon})]\,dt\right\} \\
& + E\left\{\int_{\bar{t}}^{\bar{t}+\rho} [(u(t)-\bar{u}^{\varepsilon})(t)]\,[(\bar{p}_3(t)-\bar{p}_1(t)) + (u(t)-\bar{u}^{\varepsilon})\,\bar{p}_3^{\varepsilon}]\,dt\right\} + o(\rho).
\end{aligned}
\tag{3.19}
$$

Dividing by ρ and letting $\rho \to 0$, we have

$$
\begin{aligned}
-\varepsilon^{\frac{1}{3}} \leq\ & E\,[L(\bar{x}^{\varepsilon}(t),u(\bar{t})) - L(\bar{x}^{\varepsilon}(t),\bar{u}^{\varepsilon}(t))] \\
& + E\,[(u(\bar{t})-\bar{u}^{\varepsilon}(t))\,((\bar{p}_3(t)-\bar{p}_1(t)) + (u(t)-\bar{u}^{\varepsilon}))\,\bar{p}_3^{\varepsilon}(t)]\,.
\end{aligned}
\tag{3.20}
$$

We estimate the following variation:

$$
\begin{aligned}
& E\left\{\int_0^T [(u^{\rho}(t)-\bar{u}^{\varepsilon}(t))\bar{p}_3^{\varepsilon}(t) - (u^{\rho}(t)-u^{\varepsilon}(t))\,p_3^{\varepsilon}(t)]\,dt\right\} \\
&= E\left\{\int_0^T (\bar{p}_3^{\varepsilon}(t)-p_3^{\varepsilon})\,(u^{\rho}(t)-u^{\varepsilon}(t))\,dt\right\} + \left\{\int_0^T (p_3^{\varepsilon}(t)\,(u^{\varepsilon}(t)-\bar{u}^{\varepsilon})\,(t))\,dt\right\} \\
&= I_1 + I_2.
\end{aligned}
$$

Using Lemma 3.9, we conclude that, for $0<k<1$ and $1<\eta<2$ verifying $(1+k)\eta<2$,

$$
\begin{aligned}
I_1 &\leq \left(E\int_0^T |\,\bar{p}_3^{\varepsilon}(t)-p_3^{\varepsilon}(t)\,|^{\eta}\,dt\right)^{\frac{1}{\eta}} \left(E\int_0^T |\,u^{\rho}(t)-u^{\varepsilon}(t)\,|^{\frac{\eta}{\eta-1}}\,dt\right)^{\frac{\eta-1}{\eta}} \\
&\leq R\left(d\,(u^{\rho},u^{\varepsilon})^{\frac{k\eta}{2}}\right)^{\frac{1}{\eta}} \left(E\int_0^T \left(|\,u^{\rho}(t)\,|^{\frac{\eta}{\eta-1}} + |\,u^{\varepsilon}(t)\,|^{\frac{\eta}{\eta-1}}\right)dt\right)^{\frac{\eta-1}{\eta}} \\
&\leq R\varepsilon^{\frac{k}{3}}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\leq R\left(E\int_0^T |\,p_3^{\varepsilon}(t)\,|^2\,dt\right)^{\frac{1}{2}} \left(E\int_0^T |\,u^{\varepsilon}(t)-\bar{u}^{\varepsilon}(t)\,|^2\,\chi_{u^{\varepsilon}(t)\neq\bar{u}^{\varepsilon}(t)}(t)dt\right)^{\frac{1}{2}} \\
&\leq R\left(E\int_0^T \left(|\,u^{\varepsilon}(t)\,|^4 + |\,\bar{u}^{\varepsilon}(t)\,|^4\right)dt\right)^{\frac{1}{4}} \left(E\int_0^T \chi_{u^{\varepsilon}(t)\neq\bar{u}^{\varepsilon}(t)}(t)dt\right)^{\frac{1}{4}}.
\end{aligned}
$$

Thus,

$$
E\int_0^T [(u^{\rho}(t)-\bar{u}^{\varepsilon}(t))\,(\bar{p}_3^{\varepsilon}(t) - (\bar{p}_1^{\varepsilon}(t)) - (u^{\rho}(t)-u^{\varepsilon}(t))\,p_3^{\varepsilon}(t))]\,dt \leq \varepsilon^{\frac{k}{3}}. \tag{3.21}
$$

With a similar argument, we obtain that

$$
\begin{aligned}
& E\left\{\int_0^T [(u^{\rho}(t)-\bar{u}^{\varepsilon}(t))\,(\bar{p}_3^{\varepsilon}(t)\bar{p}_1^{\varepsilon}(t)) - u^{\rho}(t)-u^{\varepsilon}(t)\,(p_3^{\varepsilon}(t)-p_1^{\varepsilon}(t))]\,dt\right\} \\
& + E\left\{\int_0^T [L(\bar{x}^{\varepsilon}(t),u^{\rho}(t)) - L(\bar{x}^{\varepsilon}(t),\bar{u}^{\varepsilon}(t))] - [L\,(x^{\varepsilon}(t),u^{\rho}(t)) - L\,(x^{\varepsilon}(t),u^{\varepsilon}(t))]\,dt\right\} \\
& \leq \varepsilon^{\frac{k}{3}}.
\end{aligned}
\tag{3.22}
$$

We obtain the desired result from the above inequalities. □

3.4. **Sufficient condition for near-optimal control.** Besides (S_1) and (S_2), now we also impose a further hypothesis:

(S_3) The set U where the control takes values is convex.

Theorem 3.12. *Suppose that hypothesis (S_1), (S_2), and (S_3) hold. Let $(x^\varepsilon(t),u^\varepsilon(t))$ be an admissible pair and $(p^\varepsilon(t),q^\varepsilon(t))$ be the solution of the adjoint equation corresponding to $(x^\varepsilon(t),u^\varepsilon(t))$. Assume that $(x,u)\mapsto H(t,x,u,p,q)$ and $x\mapsto h(x)$ are convex almost surely. If, for some $\varepsilon>0$,*

$$E\int_0^T \left(\frac{m_k u^\varepsilon(t)x_2^\varepsilon(t)}{1+\gamma_k x_2^\varepsilon(t)}(p_3^\varepsilon-p_2^\varepsilon)+L(x^\varepsilon(t),u^\varepsilon(t))\right)dt \leq \inf_{u\in U_{ad}} E\int_0^T \frac{m_k u(t)x_2^\varepsilon(t)}{1+\gamma_k x_2^\varepsilon(t)}(p_3^\varepsilon-p_2^\varepsilon)+L(x^\varepsilon(t),u(t))\,dt+\varepsilon,$$

then $J(0,x_0,u^\varepsilon)\leq \inf_{u\in U_{ad}} J(0,x_0,u)+R\varepsilon^{\frac{1}{3}}$.

Proof. From the definition of the Hamiltonian function (2.5), we have

$$J(0,x_0,u^\varepsilon)-J(0,x_0,u)=I_1(t)+I_2(t)+I_3(t) \tag{3.23}$$

with

$$\begin{aligned} I_1 &=E\int_0^T [H(t,x^\varepsilon(t),u^\varepsilon(t),p^\varepsilon(t),q^\varepsilon(t))-H(t,x(t),u(t),p^\varepsilon(t),q^\varepsilon(t))]dt,\\ I_2 &=E[h(x^\varepsilon(T))-h(x(T))],\\ I_3 &=E\int_0^T \Big[(f^\top(x^\varepsilon(t),u^\varepsilon(t))-f^\top(x(t),u(t)))p^\varepsilon(t)\\ &\qquad +(\sigma^\top(x^\varepsilon(t),u^\varepsilon(t))-\sigma^\top(x(t),u(t)))q^\varepsilon(t)\Big]dt. \end{aligned} \tag{3.24}$$

Using the convexity of H, we obtain that

$$I_1\leq \sum_{i=1}^4 E\int_0^T H_{x_i}(t,x^\varepsilon(t),u^\varepsilon(t),p^\varepsilon(t),q^\varepsilon(t))(x_i^\varepsilon(t)-x_i(t))\,dt + E\int_0^T H_u(t,x(t),u(t),p^\varepsilon(t),q^\varepsilon(t))(u^\varepsilon(t)-u(t))\,dt. \tag{3.25}$$

Similarly,

$$I_2\leq \sum_{i=1}^4 E(h_{x_i}(x^\varepsilon(T)-x_i(T))). \tag{3.26}$$

The Itô's formula acting on $\sum_{i=1}^{4} p_i^{\varepsilon}(t)(x_i^{\varepsilon}(t)-x_i(t))$ yields

$$\begin{aligned}\sum_{i=1}^{4} E\left(h_{x_i}\left(x^{\varepsilon}(T)-x_i(T)\right)\right) = &-\sum_{i=1}^{4} E\int_0^T \left(x_i^{\varepsilon}(t)-x_i(t)\right) H_{x_i}\left(t,x^{\varepsilon}(t),u^{\varepsilon}(t),p^{\varepsilon}(t),q^{\varepsilon}(t)\right) dt\\ &+\sum_{i=1}^{4} E\int_0^T p_i^{\varepsilon}(t) \mid f_i\left(x^{\varepsilon}(t),u^{\varepsilon}(t)\right)-f_i\left(x(t),u(t)\right) \mid dt\\ &+\sum_{i=1}^{4} E\int_0^T q_i^{\varepsilon}(t) \mid \sigma_i(x^{\varepsilon}(t))-\sigma(x(t)) \mid dt.\end{aligned}$$

Hence,

$$\begin{aligned}I_3 = &\sum_{i=1}^{4} E\left(h_{x_i}(x^{\varepsilon}(T))(x_i^{\varepsilon}(T)-x_i(T))\right)\\ &+\sum_{i=1}^{4} E\int_0^T \left(x_i^{\varepsilon}(t)-x_i(t)\right)\cdot H_{x_i}\left(t,x^{\varepsilon}(t),u^{\varepsilon}(t),p^{\varepsilon}(t),q^{\varepsilon}(t)\right) dt. \quad (3.27)\end{aligned}$$

Replacing equations (3.25), (3.26) and (3.27) into (3.23), we obtain that

$$J(0,x_0,u^{\varepsilon})-J(0,x_0,u) \leq E\int_0^T H_u\left(t,x^{\varepsilon}(t),u^{\varepsilon}(t),p^{\varepsilon}(t),q^{\varepsilon}(t)\right)\left(u^{\varepsilon}(t)-u(t)\right) dt. \quad (3.28)$$

To finish the proof, we need to estimate the right-hand side of (3.28). Consider the metric $\bar{d}$ on U_{ad} defined by

$$\bar{d}(u,u') = E\int_0^T y^{\varepsilon}(t) \mid u(t)-u'(t) \mid dt, \quad (3.29)$$

where

$$y^{\varepsilon}(t) = 1+\sum_{i=1}^{4} \mid p_i^{\varepsilon}(t) \mid + \sum_{i=1}^{2} \mid q_i^{\varepsilon}(t) \mid. \quad (3.30)$$

It is straightforward to state that $\bar{d}$ is a complete metric as a weighted L^1 norm. Let functional $F(\cdot): U_{ad} \to \mathbb{R}$ be defined by

$$F(u^{\varepsilon}) = E\int_0^T H\left(t,x^{\varepsilon}(t),u^{\varepsilon}(t),p^{\varepsilon}(t),q^{\varepsilon}(t)\right) dt. \quad (3.31)$$

Using (S_2), we see that $F(\cdot)$ is continuous with respect to the metric $\bar{d}$. Thus, from Lemma 3.10, there exists a $\bar{u}^{\varepsilon} \in U_{ad}$ such that

$$\begin{aligned}&\bar{d}(u^{\varepsilon},\bar{u}^{\varepsilon}) \leq \varepsilon^{\frac{1}{2}},\\ &F(\bar{u}^{\varepsilon}) \leq F(u)+\varepsilon^{\frac{1}{2}}\bar{d}(u^{\varepsilon},\bar{u}^{\varepsilon}),\end{aligned} \quad (3.32)$$

for any $u \in U_{ad}$. Replacing (3.31) into (3.32), it follows from the differentiability of function H with respect to u and hypothesis (S_1) that

$$H_u\left(t,x^{\varepsilon}(t),u^{\varepsilon}(t),p^{\varepsilon}(t),q^{\varepsilon}(t)\right)+\lambda_1^{\varepsilon}(t) = 0 \quad (3.33)$$

with

$$\lambda_1^{\varepsilon}(t) \in \left[-\varepsilon^{\frac{1}{2}}y^{\varepsilon}(t), \varepsilon^{\frac{1}{2}}y^{\varepsilon}(t)\right]. \quad (3.34)$$

We conclude that

$$| H_u(t,x^{\varepsilon}(t),u^{\varepsilon}(t),p^{\varepsilon}(t),q^{\varepsilon}(t)) | = | \lambda_1^{\varepsilon}(t) | \leq 2\varepsilon^{\frac{1}{2}} |y^{\varepsilon}(t)|. \tag{3.35}$$

The result follows from (3.28), (3.30), (3.35), Lemma 3.8, and Hölder's inequality. □

4. Conclusion

Parameter values are usually considered to be precisely known in epidemic mathematical modeling. However, often they are imprecise due to various uncertainties. Therefore, here we proposed the near-optimal control of a stochastic epidemic SICA model with imprecise parameters. By using some mathematical inequalities, namely Cauchy-Schwartz's, Gronwall's and Burkholder–Davis–Gundy inequalities, we proved some estimates of the state and co-state variables in order to investigate necessary and sufficient conditions of near-optimality.

As future work, we plan to develop numerical methods for near-optimal control of the stochastic SICA model with imprecise parameters. This is under investigation and will be addressed elsewhere.

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